A new flexible parking reservation scheme for the morning commute under limited parking supplies

Wei Wu[a,b], Wei Liu[b,c,*], Fangni Zhang[d], Vinayak Dixit[c]

[a] *Key Laboratory of Highway Engineering of Ministry of Education, Changsha University of Science & Technology, Changsha 410114, Hunan, China*

[b] *School of Computer Science and Engineering, University of New South Wales, Sydney, NSW 2052, Australia*

[c] *Research Centre for Integrated Transport Innovation, School of Civil and Environmental Engineering, University of New South Wales, Sydney, NSW 2052, Australia*

[d] *Department of Industrial and Manufacturing Systems Engineering, University of Hong Kong, Hong Kong, China*

Abstract

Recent studies proposed parking reservation schemes in a bi-modal transport network to manage parking competition and traffic congestion. This study proposes a new flexible parking reservation scheme, under which commuters' reservation can expire and those arriving later than the reservation expiration time can retain his reservation by paying additional late-for-reservation fees. Time-varying late-for-reservation fees are also examined. The proposed reservation scheme is more flexible and practical than those in the literature. Moreover, we found that, when compared to reservation scheme in the literature, the total social cost can be further reduced by an appropriately designed flexible reservation scheme in this paper. We also analytically and numerically quantify the efficiency gain of the proposed flexible parking reservation scheme.

Keywords: morning commute, limited parking supply, parking reservation, expiration time, flexible reservation.

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*Corresponding author: Email: wei.liu@unsw.edu.au (W. Liu)

1 Introduction

The availability of parking can considerably affect commuters' travel choices, as reported in the studies of, e.g., Qian et al. (2011) and Zhang et al. (2011). Shoup (2006) reported that 30% of traffic were cruising for a vacant parking space, based on empirical data of several cities (e.g., Chicago, San Francisco). Finding a parking space in the downtown areas of big cities becomes a difficult task for commuters. In this context, many studies combined driving and parking as an integrated problem, and proposed parking pricing/permit to manage the parking supply/operation and traffic congestion (e.g., Arnott et al., 1991; Zhang et al., 2005, 2011; Qian et al., 2012; Qian and Rajagopal, 2015; He et al., 2015; Liu et al., 2016; Zheng and Geroliminis, 2016; Lei and Ouyang, 2017; Jakob et al., 2018). In particular, cruising for parking and its negative impacts have been extensively studied (e.g., Anderson and De Palma, 2004; Arnott and Inci, 2006; Qian and Rajagopal, 2014; Inci and Lindsey, 2015; Liu and Geroliminis, 2016; Tian, 2016). Recently, a growing number of studies developed parking management strategies with an emphasis on network-wide applications (Lam et al., 2006; Li et al., 2008; Geroliminis, 2015; Boyles et al., 2015; Zhang et al., 2019a). More recently, there is a growing interest in modeling the parking problem for autonomous vehicles (Liu, 2018; Zhang et al., 2019b). For a more comprehensive review of parking studies, one may refer to e.g., Inci (2015).

Considering that the total parking supply in downtown areas can be insufficient, Zhang et al. (2011) examined how the inadequate parking supply might reshape the morning commute equilibrium. It is found that the morning peak will be pushed to start earlier due to competition for parking. Zhang et al. (2011) also introduced a parking permit distribution and trading scheme to eliminate the external cost arising from competition for parking spaces. Later, Yang et al. (2013) further considered the hybrid supply of reserved and unreserved parking spaces when the total parking supply in the downtown is insufficient. It is found that an appropriate combination supply of reserved and unreserved parking spaces can temporally smooth out traffic congestion and hence reduce the total system cost. In recent years, parking reservation has been proposed for parking management in many studies (e.g., Chen et al., 2015, 2016; Shao et al., 2016; Chen et al., 2019). Xiao et al. (2016), Xiao et al. (2019), and Su and Wang (2019) further extended Yang et al. (2013) by incorporating the ridesharing or ride-sourcing behaviors into the commuting problem given the limited parking supplies and examined the interaction between parking supply and carpooling. Xu et al. (2017) examined the allocation of road space to on-street parking for vacant ride-sourcing vehicles.

In Yang et al. (2013), the parking reservation is valid for the whole day, i.e., no matter when the commuter with a reserved parking space chooses to travel, that parking space will be reserved for him or her. Liu et al. (2014a) further considered the expirable parking reservation scheme, under which commuters with parking reservation have to arrive at the parking spaces before a predetermined expiration time, to mitigate both traffic congestion and competition for available parking spaces. In Liu et al. (2014a), the parking space is no longer reserved to the commuter once he or she arrives later than the expiration time, and they assumed that commuters with parking reservation will take advantage of the reservation and arrive before the expiration time. This consideration eases the analysis of the parking problem, but can be unrealistic in practice. This study considers a more flexible and operable case, which is termed as "a flexible expirable parking reservation scheme", under which a commuter with a parking reservation arriving later than the expiration time of the reservation can pay an additional late-for-reservation fee (can be constant or time-dependent) to remain his reservation valid. We explore and quantify how such a new scheme may affect travel patterns and potentially improve system traffic efficiency. In particular, we found that giving this flexibility to commuters, the total system cost can be further reduced. Efficiency of the flexible parking reservation scheme has been analytically examined. Moreover, we model departure time choices for both car and public transit commuters in this paper and discuss how insufficient parking supply might affect the bi-modal commuting patterns.

The rest of the paper is organized as follows. Section 2 presents the travel cost formulations and formulates the bi-modal equilibrium under insufficient parking supply and parking reservation schemes, where departure time choices are considered for both private car and public transit commuters. In Section 3, the new flexible parking reservation schemes are introduced and discussed in detail, and the efficiency of the flexible parking reservation schemes are evaluated and compared against the reservation schemes in the literature. Section 4 further introduces a time-varying late-for-reservation fee and quantify the potential efficiency gains. Section 5 illustrates the theoretical findings through numerical examples and Section 6 concludes the paper.

2 Morning commute with limited parking supplies

We consider a linear city depicted in Figure 1 with two travel modes: a transit line and a parallel highway with capacity-constrained bottleneck. Every day there is a total number of N commuters traveling from home to the Central Business District (CBD). Commuters have a desired arrival time t^* at the workplace. Early arrival at the workplace will be penalized.

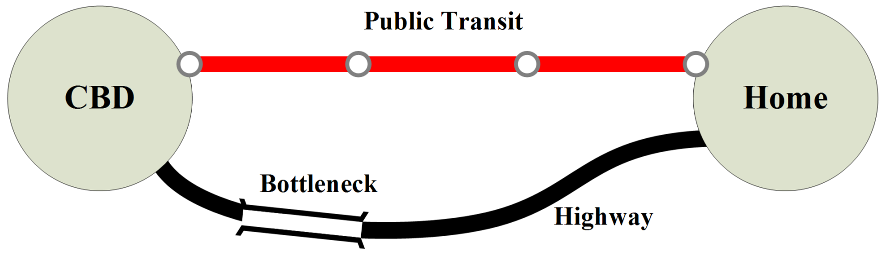


Figure 1: The two-mode network

Late arrival at the workplace is not allowed.[1] It is assumed that all parking spaces are located at the destination (i.e., CBD), and the walking time between parking spaces and workplace is ignored for simplicity. The numbers of car and transit commuters are denoted by N_a and N_b, respectively, where $N_a + N_b = N$.

2.1 Travel cost and bi-modal equilibrium

We now formulate the travel costs for car and transit modes. Travel cost for commuters includes the travel time cost, schedule delay cost (due to un-punctual arrival at the destination), and the monetary cost. For the private car commuters departing from home at time t, the travel cost can be written as

$$c_a(t) = \alpha \cdot T_a(t) + \beta \cdot [t^* - t - T_a(t)]^+ + \tau_a, \tag{1}$$

where $[\cdot]^+ = \max\{0, \cdot\}$, $T_a(t)$ is the travel time experienced by the commuters departing from home at time t, α is the value of time, β is the penalty for a unit time of early arrival at the destination for commuters, and τ_a is the monetary cost for the car mode, which is assumed to be the parking fee (note that the parking fee is assumed constant for all available parking spaces). It is assumed that $\alpha > \beta$, which is consistent with many empirical evidences in the literature. Since late arrival is not allowed, $t + T_a(t) \leq t^*$ holds.

In Eq. (1), the travel time $T_a(t)$ contains both the free-flow travel time and the queuing delay at the highway bottleneck, i.e., $T_a(t) = T_a + \frac{q(t)}{s}$, where T_a is the free-flow time for the highway, $q(t)$ is the queue length experienced by commuters departing from home at time t and s is the bottleneck capacity.

Given N_a and assuming there is sufficient parking supply in the CBD, at user equilibrium, all commuters should have identical travel cost, i.e., $\frac{dc_a(t)}{dt} = 0$ for $t \in [t_s^{u,a}, t_e^{u,a}]$, where $t_s^{u,a}$

[1] This paper aims to generate analytical insights regarding flexible parking reservation schemes and illustrate the potential efficiency gains from giving flexibility to commuters with a parking reservation. To ease the analysis and avoid tedious algebra, late arrival at the workplace is not considered. Similar treatments can be found in, e.g., Zhang et al. (2008), Xiao et al. (2013), and Liu et al. (2014b).

is the departure time of the first car commuter, and $t_e^{u,a}$ is the departure time of the last private car commuter. The departure rate $r_a(t)$ from home (as well as the arrival rate at the bottleneck) for car commuters is

$$r_a(t) = \frac{\alpha}{\alpha - \beta} s. \tag{2}$$

One can further derive that $t_s^{u,a} = t^* - T_a - \frac{N_a}{s}$ and $t_e^{u,a} = t^* - T_a - \frac{\beta}{\alpha}\frac{N_a}{s}$ at equilibrium. One may refer to Arnott et al. (1990) for similar derivations. The departure time choice problem has been modeled by many studies with different emphases (van den Berg and Verhoef, 2011; Liu et al., 2015b; Nie, 2015) since Vickrey (1969) proposed the first bottleneck model. The equilibrium travel cost of car commuters under given N_a is

$$p_a(N_a) = \alpha T_a + \beta \frac{N_a}{s} + \tau_a. \tag{3}$$

For transit commuters, following Wu and Huang (2014), the travel cost is assumed to contain the travel time cost, the schedule delay cost, the crowding cost, and the transit fare. For commuters departing from home at time t, the transit cost can be written as

$$c_b(t) = \alpha \cdot T_b + \beta \cdot [t^* - t - T_b]^+ + \theta \delta T_b r_b(t) + \tau_b, \tag{4}$$

where $[\cdot]^+$, α, β, and t^* follow those in Eq. (1), and T_b is the constant transit travel time, θ is a cost parameter for crowding, δ is the transit headway during the peak for transit services, $r_b(t)$ is the departure rate of transit commuters at time t, and τ_b is the transit fare. The crowding cost is proportional to the number of commuters in the transit vehicle and the transit headway. Different from Wu and Huang (2014), the late arrival at the destination is not allowed in this study.

Given the number of transit commuters N_b, we can derive that the first transit commuter departs from home at $t_s^{u,b} = t^* - T_b - \sqrt{\frac{2\theta\delta T_b N_b}{\beta}}$ and the last transit commuter departs from home at $t_e^{u,b} = t^* - T_b$. For transit commuters, the departure rate from home is

$$r_b(t) = \frac{\beta \left(t - t_s^{u,b}\right)}{\theta \delta T_b}. \tag{5}$$

The equilibrium travel cost of transit commuters can be written as

$$p_b(N_b) = \alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b N_b}. \tag{6}$$

We now turn to discuss the bi-modal equilibrium. We consider an interior bi-modal equilibrium, where both car and transit modes are used by some commuters, i.e., $N_a > 0$ and

$N_b > 0$. Based on Eq. (3) and Eq. (6), it can be verified that $\alpha T_a+\tau_a < \alpha T_b+\tau_b+\sqrt{2\beta\lambda\delta T_b N}$ and $\alpha T_a + \beta \frac{N}{s} + \tau_a > \alpha T_b + \tau_b$ will ensure an interior bi-modal equilibrium. Following this consideration, at equilibrium $p_a(N_a) = p_b(N_b)$, where $N_a + N_b = N$. It is straightforward to find that the equilibrium solution is unique since $p_a(N_a)$ in Eq. (3) and $p_b(N_b)$ in Eq. (6) are strictly increasing with respect to N_a and N_b, respectively. The equilibrium solution is further denoted as N_a^* and N_b^*, and the corresponding cost is denoted as $c^* = p_a(N_a^*) = p_b(N_b^*)$.

2.2 Bi-modal equilibrium under limited parking supplies

We now discuss the bi-modal equilibrium under insufficient parking, i.e., the parking supply m is less than the equilibrium car demand N_a^* (i.e., $m < N_a^*$). Similar to Zhang et al. (2011), at equilibrium, $N_a = m$ and $N_b = N - m$. The equilibrium travel cost will be equal to $p_b(N-m)$, which is larger than $c^* = p_b(N_b^*)$ since $N - m > N_b^*$. It implies that due to the constrained parking supply, the travel cost of commuters is larger when compared to the case with sufficient parking supply. Different from Zhang et al. (2011), this paper models departure time choices for both private car and public transit commuters, which is further analyzed below.

In particular, the car commuters have to depart from home earlier due to the competition for the limited parking supply. The departure/arrival equilibrium at the car traffic side can be depicted as that in Figure 2(a) (where the blue, red and black solid lines represent departures from home, arrivals at the queue, and arrivals at the destination, respectively, the dotted lines are used to illustrate m' to be introduced shortly), where $m' = \frac{P_b(N-m)-\alpha T_a-\tau_a}{\beta}s$ and $m' > N_a^*$. Moreover, it can be verified that $t_s^{u,a} = t^* - T_a - \frac{m'}{s}$ and $t_e^{u,a} = t^* - T_a - \frac{m'}{s} + \left(1 - \frac{\beta}{\alpha}\right)\frac{m}{s}$.

The departure/arrival choice equilibrium of transit commuters under insufficient parking supply $m < N_a^*$ is depicted in Figure 2(b), where the blue, red and black lines represent departures from home, departures from the transit stop, and arrivals at the destination, respectively. Moreover, the first public transit commuter in the peak will depart at time $t_s^{u,b} = t^* - T_b - \sqrt{\frac{2\theta\delta T_b(N-m)}{\beta}}$ and the last public transit commuter will depart at time $t_e^{u,b} = t^* - T_b$.

Proposition 2.1. *Given insufficient parking supply $m < N_a^*$, when m decreases, i.e., less parking, (i) the earliest private car and public transit commuters will depart earlier (the peaks for highway and public transit both start earlier), i.e., $t_s^{u,a} = t^* - T_a - \frac{m'}{s}$ decreases and $t_s^{u,b} = t^* - T_b - \sqrt{\frac{2\theta\delta T_b(N-m)}{\beta}}$ decreases; (ii) the marginal change of m will result in the same change in the peak start times $t_s^{u,a}$ and $t_s^{u,b}$, i.e., $\frac{dt_s^{u,a}}{dm} = \frac{dt_s^{u,b}}{dm}$; (iii) the last private car commuter will depart earlier while the last transit commuter will depart at the same time, i.e., $t_e^{u,a}$ decreases and $t_e^{u,b}$ remains constant; (iv) $t_e^{u,a} - t_s^{u,a}$ decreases and $t_e^{u,b} - t_s^{u,b}$*

increases; (v) the departure rate from home for car commuters in Eq. (2) remains constant while departure rate from home for transit commuters in Eq. (5) at $t_e^{u,b}$ increases.

Proposition 2.1 can be readily verified by examining the first derivatives of $t_s^{u,a}$, $t_s^{u,b}$, $t_e^{u,a}$, and $t_e^{u,b}$ with respect to m and the departure rates for car commuters and transit commuters in Eq. (2) and Eq. (5). Proposition 2.1(i) means that under a more severe parking limitation (a smaller m), the peaks for private car traffic side and public transit side will both be pushed to start earlier (due to more severe parking competition). Proposition 2.1(ii) further says that a marginal change in parking supply m would result in equal marginal changes in the peak start times for both private car traffic side and public transit side. Proposition 2.1(iii) and Proposition 2.1(iv) indicate that the peak length for private car traffic side decreases and peak length for transit side increases under a smaller m. Proposition 2.1(v) further indicates that, besides an earlier peak, a more severe parking limitation can result in temporally more intensive transit passenger departure during a certain time interval within the peak at the bi-modal equilibrium. The above results are consistent with the observation that under a smaller m, there will be less private car commuters and more public transit commuters.

The total user cost (TUC) can be written as:

$$TUC = N \cdot p_b(N - m). \tag{7}$$

The total social cost (TSC) under a limited parking supply equals the total user cost (TUC) minus the money transfers, i.e., the parking fees and public transit fares. Then the total social cost (TSC) can be written as:

$$TSC = N \cdot p_b(N - m) - (N - m)\tau_b - m\tau_a. \tag{8}$$

2.3 Inflexible expirable parking reservation scheme

We now introduce the parking reservation scheme proposed in Liu et al. (2014a), which is termed as "inflexible expirable parking reservation scheme" (later we will introduce the "flexible" scheme, i.e., allowing people to be late for the parking reservation). Note that different parking reservation schemes to be discussed under a given parking supply m only manage private car traffic and the traffic pattern at the public transit side remains the same as those in Figure 2(b).

In particular, under such an inflexible reservation scheme, commuters with a reservation have to arrive at the parking spaces before a predetermined expiration time. Otherwise the parking spaces will be no longer reserved to them. To take advantage of the reservations,

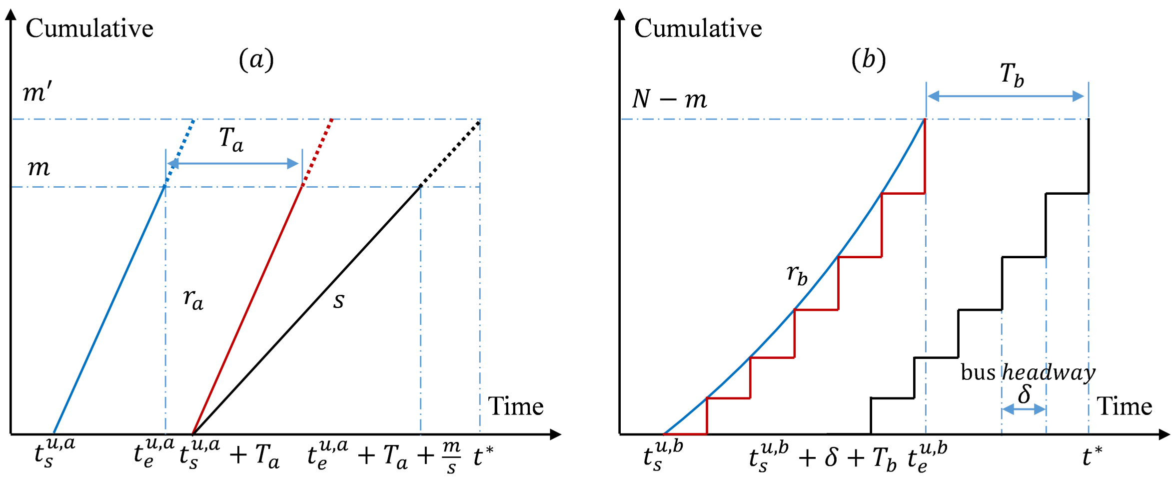


Figure 2: The departure/arrival equilibrium for car and public transit commuters under parking limitation

commuters with reservations need to arrive at the parking spaces in time so as to minimize their costs given that the reservation scheme is appropriately designed. To ease the presentation, we denote those with a reservation as "r-commuter" and those without a reservation as "u-commuter".

We consider a general parking reservation scheme where reservations expire at n different time points. The total number of parking spaces for reservation is $m^r \leq m$ (note that $m < N_a^*$, i.e., parking is insufficient). The number of parking spaces open for competition (unreserved) is $m^u = m - m^r$. The number of reserved parking spaces associated with the i^{th} expiration time t_i^r is denoted by m_i^r, where $m_i^r > 0$ for $i = 1, 2, 3, ..., n$, $t_1^r < t_2^r < \cdots < t_n^r$, and $\sum_{i=1}^{n} m_i^r = m^r$. There are n groups of parking reservations in total.

This paper will not enumerate all the possible commuting equilibrium patterns under different expirable reservation schemes (different combinations of m_i^r and t_i^r). Instead, we only describe appropriately designed parking reservation schemes (i.e., relatively efficient), which are discussed as follows for $n = 1$ and $n \geq 2$. For the parking reservation schemes to be relatively efficient (i.e., tends to yield small total social cost), (i) the arrival of r-commuters (those with parking reservations) should be continuous (no capacity waste); (ii) the last r-commuter should arrive at the destination at t^* in order to reduce schedule delay cost; (iii) some parking spaces might have to be open for competition (but not reserved) in order to separate departures of r-commuters (those with reservation) and u-commuters (those without reservation) and temporally reduce congestion. Note that these principles still hold when we introduce the proposed flexible reservation schemes.

2.3.1 The expirable reservation scheme with $n = 1$

We start with an identical expiration time for all the reserved parking spaces, i.e., $n = 1$. The number of reserved parking spaces should not exceed the cumulative capacity of the highway bottleneck between $t_e^{u,a} + T_a + \frac{m^u}{s}$ and t_1^r, i.e., $m_1^r = m^r \le \left(t_1^r - t_e^{u,a} - T_a - \frac{\beta}{\alpha}\frac{m^u}{s}\right) s$, where $t_e^{u,a}$ is the departure time from home of the last u-commuter using car and $t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s}$ is the arrival time (at the CBD) of this u-commuter. This consideration ensures that all the commuters with a parking reservation can drive through the highway bottleneck before the expiration time. This is necessary for an optimal design of the parking reservation scheme. As discussed in Yang et al. (2013), the optimal expiration time should be set large enough if all the reserved spaces have an identical expiration time, i.e., $t_1^r \ge t^*$ in this paper. Under the optimal combination of m^u and m_1^r, the car departure/arrival equilibrium can be depicted as that in Figure 3, where there are two sub-peaks: one for u-commuters (blue solid lines: departure from home and arrival at bottleneck) and one for the r-commuters (red solid lines: departure from home and arrival at bottleneck). The black solid lines represent the arrivals at the destination. The exact values of m^u, m_1^r should be determined by minimizing the total system cost under the flow pattern depicted in Figure 3.[2]

Based on the bi-modal equilibrium conditions, the time points in Figure 3 can be derived, i.e., the departure time of the first car commuter without a parking reservation $t_s^{u,a} = t^* - T_a - \frac{p_b(N-m)-(\alpha T_a+\tau_a)}{\beta}$, the departure time of the last car commuter without a parking reservation $t_e^{u,a} = t^* - T_a - \frac{p_b(N-m)-(\alpha T_a+\tau_a)}{\beta} + \frac{m^u}{r_a}$, and the departure times of the first and last commuters with a parking reservation $t_s^r = t^* - T_a - \frac{m^r}{s}$ and $t_e^r = t^* - T_a - \frac{\beta}{\alpha}\frac{m^r}{s}$. To economize the notation, subscript/superscript a is omitted for the notation of the time points associated with car commuters with reservation (r-commuters).

We now further present the system efficiency metrics for the case with an identical expiration time for all the reserved parking spaces, i.e., $n = 1$. The total cost of all the car commuters with parking reservations can be written as

$$TC_a^r = m^r \cdot \left(\alpha T_a + \beta\frac{m^r}{s} + \tau_a\right). \tag{9}$$

[2]It is noteworthy that, when $m^u > 0$, in Figure 3 there exists a time interval $\left[t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s}, t^* - \frac{m^r}{s}\right]$ during which the bottleneck service capacity is wasted. Note that $m < N_a^*$ (insufficient parking) results in $t^* - \frac{m^r}{s} > t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s}$, i.e., parking competition due to limited supply forces commuters without reservation to depart much earlier than those with reservation at the bi-modal equilibrium. This capacity waste can still occur even if we introduce more advanced parking reservation schemes later on as long as $m < N_a^*$ and $m^u > 0$. However, there should be no capacity waste during the arrival of r-commuters under an efficient reservation scheme (as discussed earlier), i.e., the arrival is continuous between $t^* - \frac{m^r}{s}$ and t^* in Figure 3.

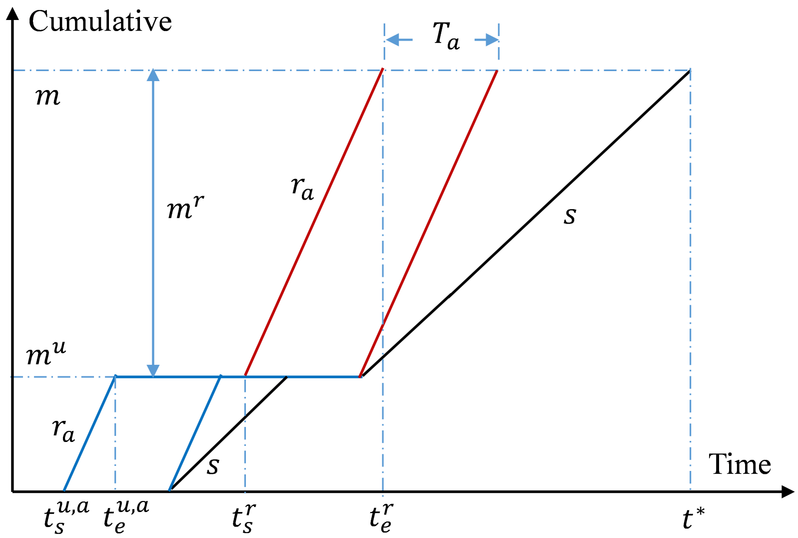


Figure 3: The departure/arrival equilibrium for car commuters under inflexible parking reservation ($n = 1$)

The total cost of the commuters without reservation, i.e., the public transit commuters and car commuters without parking reservations can be specified as:

$$TC_b^u = (N - m^r) \cdot p_b(N - m). \tag{10}$$

The total user cost under the inflexible expirable parking reservation scheme when $n = 1$ can be written as:

$$TUC_{in,1} = m^r \cdot \left(\alpha T_a + \beta \frac{m^r}{s} + \tau_a\right) + (N - m^r) \cdot p_b\left(N - m\right). \tag{11}$$

The total social cost (excluding the money transfers) under the inflexible expirable parking reservation scheme when $n = 1$ can be written as:

$$TSC_{in,1} = m^r \cdot \left(\alpha T_a + \beta \frac{m^r}{s} + \tau_a\right) + (N - m^r) \cdot p_b(N - m) - (N - m)\tau_b - m\tau_a. \tag{12}$$

By comparing Eq. (11) with Eq. (7), or comparing Eq. (12) with Eq. (8), the total cost saving, i.e., user cost saving or social cost saving under the inflexible expirable parking reservation scheme with $n = 1$ when compared that under no reservation scheme, can be given as follows:

$$CS_{in,1} = m^r \left[p_b(N - m) - p_a(m^r)\right]. \tag{13}$$

The optimal m_r under a given m to minimize the total social cost, or equivalently to minimize the total user cost, or equivalently, to maximize the total cost saving $CS_{in,1}$ is derived in Proposition 2.2.

Proposition 2.2. *Given $m < N_a^*$, the optimal number of reserved parking spaces m^r under the inflexible expirable parking reservation scheme with $n = 1$ to minimize Eq. (12) can be specified as:*

$$m^{r,*} = \begin{cases} m_{in}^{r,*} < m & p_b(N-m) < p_a(2m) \\ m & p_b(N-m) \geq p_a(2m) \end{cases} \tag{14}$$

where $m_{in}^{r,*} = \frac{s}{2\beta} \cdot (p_b(N-m) - \tau_a - \alpha T_a)$.

Proposition 2.2 can be verified by taking the first derivative of Eq. (12) with respect to m^r. Proposition 2.2 is further explained in the following. It can be verified that there is a unique $m_{c,1}$ such that $p_b(N - m_{c,1}) = p_a(2m_{c,1})$. Moreover, $m > m_{c,1} \Leftrightarrow p_b(N-m) < p_a(2m)$ and $m \leq m_{c,1} \Leftrightarrow p_b(N-m) \geq p_a(2m)$. Proposition 2.2 means that when $m > m_{c,1}$, $m^{r,*} = m_{in}^{r,*} < m$ and when $m \leq m_{c,1}$, $m^{r,*} = m$. It implies that when parking insufficiency is less severe (i.e., $m > m_{c,1}$), we should keep some parking open for competition (i.e., $m^{r,*} < m$ or $m^{u,*} = m - m^{r,*} > 0$) to separate car traffic of r-commuters and u-commuters so as to reduce congestion. Note that $m^{u,*} = m - m^{r,*} > 0$ also implies that in order to minimize total social cost, capacity waste between the arrivals of r-commuters and u-commuters occurs, where the capacity waste results in additional schedule delay cost for u-commuters. This means that the benefit from reducing traffic intensity (separating r-commuters and u-commuters) is more significant than saving schedule delay cost of u-commuters when parking limitation is less severe. Instead, when parking insufficiency is more severe (i.e., $m \leq m_{c,1}$), all parking spaces should be reserved to avoid large schedule delays due to severe parking competition (the capacity waste between arrivals of u-commuters and r-commuters does not occur since there is no u-commuters, i.e., an extreme case of Figure 3 where $m^u = 0$).

We now further discuss how different parameters in the studied bi-modal system are related to Proposition 2.2. In particular, based on Eq. (3), Eq. (6) and Eq. (14), one can identify that when $\alpha \downarrow$ (VOT is smaller), $T_b \downarrow$ (public transit is faster), $\tau_b \downarrow$ (public transit fare is cheaper), $\theta \downarrow$ (public transit crowding is less valued by commuters), $\delta \downarrow$ (public transit headway is smaller), $N \downarrow$ (total peak demand is smaller), $T_a \uparrow$ (free-flow time on highway is larger), and $\tau_a \uparrow$ (monetary cost for car mode is larger), $p_b(N-m) < p_a(2m)$ is more likely to hold and thus $m^{r,*} < m$, i.e., it is more likely that some parking spaces should be open for public competition in order to reduce congestion and minimize total social cost.

2.3.2 The expirable reservation scheme with $n \geq 2$

We now turn to the case with more than one group of reserved parking spaces with different reservation expiration times, i.e., $n \geq 2$, the reserved parking spaces are divided into at least two groups with different expiration times. To ease the analysis, it is assumed that the reserved parking m^r is evenly divided into n groups with different reservation expiration times. Therefore, $m_i^r = \frac{m^r}{n}$, where $i = 1, 2, ..., n$. In the case with $n = 1$ discussed in Section 2.3.1, $t_1^r \geq t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s} + \frac{m^r}{s}$, where $t_e^{u,a} = t^* - T_a - \frac{p_b(N-m)-(\alpha T_a+\tau_a)}{\beta} + \frac{\alpha-\beta}{\alpha}\frac{m^u}{s}$ is given in Section 2.3.1. Similarly, to ensure that commuters with parking reservation can drive through the highway bottleneck before the expiration time, the number of reserved parking spaces should not exceed the cumulative capacity of the highway bottleneck, which is summarized in Assumption 1.

Assumption 1. *The number of reserved parking spaces should not exceed the cumulative capacity of the highway bottleneck between t_{i-1}^r and t_i^r, i.e., $m_i^r \leq \left(t_i^r - t_{i-1}^r\right) s$ for $i \geq 2$, and $t_1^r \geq t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s} + \frac{m_1^r}{s}$, where $t_e^{u,a} = t^* - T_a - \frac{p_b(N-m)-(\alpha T_a+\tau_a)}{\beta} + \frac{\alpha-\beta}{\alpha}\frac{m^u}{s}$.*

Under Assumption 1, we can show that it is socially preferable to set $t_i^r = t^* - \frac{n-i}{n}\frac{m^r}{s}$ so that the schedule delay cost is minimized for the r-commuters (they arrive as close to t^* as possible), and there is no capacity waste, i.e., $\left(t_i^r - t_{i-1}^r\right) s = \frac{m^r}{n}$ and the arrivals of r-commuters are continuous. In this case, the departure/arrival equilibrium is depicted in Figure 4, where the blue solid lines represent departure from home and arrival at the bottleneck of u-commuters, the red solid lines represent departure from home and arrival at the bottleneck of r-commuters, and the black solid lines represent the arrivals at the destination. There are one small peak for u-commuters and n small peaks (each group has one) for r-commuters. Note that given $m < N_a^*$, i.e., insufficient parking supply, if $m^r < m$ and $m^u > 0$, capacity waste occurs between the arrivals of the last u-commuter and the first r-commuter, as shown in Figure 4. All the r-commuters will arrive at the parking before their expiration time to take advantage of the reserved space (in order to minimize their travel costs). The system efficiency metrics are further discussed in the following.

The travel cost of car commuters with a group or step i parking reservation, i.e., with a expiration time t_i^r can be written as:

$$p_{a,i} = \alpha T_a + \frac{n-i+1}{n} \cdot \frac{\beta m^r}{s} + \tau_a. \tag{15}$$

The total cost of all the commuters with reservation is the sum of the costs of users with reservations from reservation step 1 to step n, which is $\sum_{i=1}^{n} m_i^r \cdot p_{a,i}$. The total user cost is

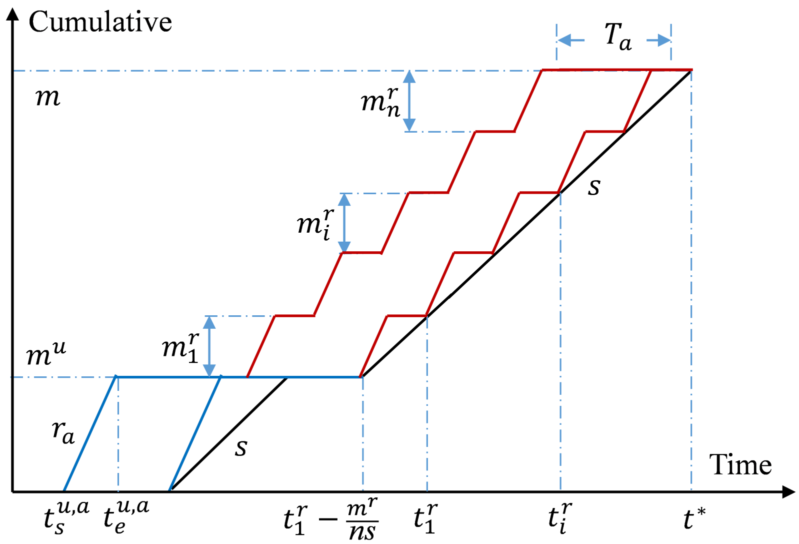


Figure 4: The departure/arrival equilibrium for car commuters under the inflexible reservation scheme ($n \geq 2$)

then

$$TUC_{in,n} = m^r \cdot \left(\alpha T_a + \frac{n+1}{2n}\frac{\beta m^r}{s} + \tau_a\right) + (N - m^r) \cdot p_b(N - m). \tag{16}$$

The total social cost can be written as

$$TSC_{in,n} = m^r \cdot \left(\alpha T_a + \frac{n+1}{2n}\frac{\beta m^r}{s} + \tau_a\right) + (N - m^r) \cdot p_b(N - m) - (N - m)\tau_b - m\tau_a. \tag{17}$$

We now discuss the efficiency gap between any two expirable reservation schemes with the same number of parking for reservation (i.e., the same total reserved parking supply m^r) but different numbers of parking reservation groups (e.g., $n_1 \neq n_2$), where different reservation groups involve, e.g., different reservation expiration times. We have the following result.

Proposition 2.3. *The total social cost gap between two inflexible reservation schemes with the same total reserved parking supply, i.e., the same m^r, but different numbers of reservation groups or steps, i.e., $n_1 \neq n_2$, can be expressed as follows:*

$$CS_{in,n_1,n_2} = TSC_{in,n_1} - TSC_{in,n_2} = \frac{n_2 - n_1}{2n_1 n_2}\frac{\beta (m^r)^2}{s} \tag{18}$$

Eq. (18) can be verified by comparing the total social costs under n_1 and n_2 steps in Eq. (17). It is evident that when $n_2 > n_1$, $CS_{in,n_1,n_2} > 0$, i.e., a more differentiated scheme is more efficient. Moreover, suppose $n_2 \geq n_1$, it can be verified that when $\frac{n_1}{n_2}$ is a constant, CS_{in,n_1,n_2} decreases with n_2. This means that the cost saving from further differentiating the expirable reservation ($n_2 \geq n_1$) is not proportional to the ratio of $\frac{n_1}{n_2}$. Instead, it is diminishing when both n_1 and n_2 increase and $\frac{n_1}{n_2}$ remains constant.

The optimal m^r to minimize the total social cost or the total user cost under a given m is derived in the following.

Proposition 2.4. *Given $m < N_a^*$, the optimal number of reserved parking spaces m^r under the inflexible expirable parking reservation scheme with n reservation groups or steps to minimize Eq. (17) can be specified as*

$$m^{r,*} = \begin{cases} m_{in,n}^{r,*} < m & p_b(N-m) < p_a(\frac{n+1}{n}m) \\ m & p_b(N-m) \geq p_a(\frac{n+1}{n}m) \end{cases} \tag{19}$$

where $m_{in,n}^{r,*} = \frac{n}{n+1} \cdot \frac{s}{\beta} \cdot (p_b(N-m) - \tau_a - \alpha T_a)$.

Proposition 2.4 can be verified by examining the first derivative of Eq. (17) with respect to m^r under a given m. Proposition 2.4 is a generalization of Proposition 2.2 to $n \geq 2$. Similar to Proposition 2.2, there exists a unique $m_{c,n}$ for a given n such that $p_b(N - m_{c,n}) = p_a\left(\frac{n+1}{n}m_{c,n}\right)$. When parking insufficiency is less severe, i.e., $m > m_{c,n}$ and thus $p_b(N-m) < p_a\left(\frac{n+1}{n}m\right)$, it is preferable to keep some parking open for competition (i.e., $m^{r,*} < m$) to separate car traffic of r-commuters and u-commuters in order to reduce congestion. Instead, when parking insufficiency is more severe, i.e., $m \leq m_{c,n}$ and thus $p_b(N-m) \geq p_a(\frac{n+1}{n}m)$, all parking should be reserved to avoid large schedule delays due to severe parking competition. Moreover, the discussions regarding how different parameters could affect the likelihood of $p_b(N-m) < p_a\left(\frac{n+1}{n}m\right)$ to occur will still hold. It is also noted that when $n \uparrow$, $p_b(N-m) < p_a\left(\frac{n+1}{n}m\right)$ is less likely to hold, which means that when we have more reservation steps in the reservation scheme, it is more likely that all parking spaces should be reserved to commuters in order to minimize total social cost. This also indicates that when we have more reservation steps in the reservation scheme, it is less likely that capacity waste between the arrivals of u-commuters and r-commuters will occur (refer to Figure 4).

Proposition 2.5. *For two inflexible reservation schemes with the same total parking supply m but different numbers of reservation steps, i.e., $n_1 \neq n_2$, if $n_1 < n_2$, $(m^{r,*})_1 \leq (m^{r,*})_2$, where $(m^{r,*})_1$ and $(m^{r,*})_2$ are given by Eq. (19) with $n = n_1$ and $n = n_2$, respectively.*

Proposition 2.5 can be verified by examining Eq. (19). It implies that when the number of reservation steps n increases (the reservation scheme is more differentiated), the optimal $m^{r,*}$ increases or at least does not decrease. This is because, when n increases, the expirable reservation scheme is more capable to temporally reduce traffic intensity and and thus reduce congestion, the marginal cost for an additional r-commuter is smaller due to the larger capability of the more advanced reservation scheme to reduce social cost.

3 Flexible expirable parking reservation scheme with a constant late-for-reservation fee

This section discusses the flexible expirable reservation scheme, where commuters are allowed to arrive later than the expiration time with an additional late fee or penalty. In particular, the additional fee for all the late-for-reservation users is assumed to be constant in this section. The time-varying late-for-reservation fee (also termed as late fee hereinafter) will be analyzed in the next section. The late r-commuters should not be too late, i.e., there is a maximum time gap that is allowed, which is denoted by Δ_i for r-commuters in group i. There will be m_i^r r-commuters in group i (the number of commuters is equal to the number of reserved parking spaces). These r-commuters can pass through the road bottleneck within a time interval of length $\frac{m_i^r}{s}$. To ensure efficient usage of road bottleneck service capacity, Δ_i should be no greater than $\frac{m_i^r}{s}$, i.e., $\Delta_i \leq \frac{m_i^r}{s}$. If $\Delta_i > \frac{m_i^r}{s}$, the bottleneck capacity for a certain time interval within the time length Δ_i will be wasted given that only the m_i^r r-commuters will pass the bottleneck during this time duration (with a length of Δ_i). For the group i where $i = 1, 2, \cdots, n$ with an expiration time t_i^r, let f_a^i denote the additional fee to be paid. Under the above flexible expirable parking reservation scheme, an r-commuter in group i with a reservation expiration time t_i^r can choose to arrive no later than t_i^r (labeled as O-type r-commuters, where "O" refers to on-time for parking reservation), and can also choose to arrive later than t_i^r but earlier than $t_i^r + \Delta_i$ and pay a late fee of f_a^i (labeled as L-type r-commuters, where "L" refers to late for the parking reservation).

Similar to Section 2.3, we do not enumerate all possible commuting patterns under different designs of the flexible expirable parking reservation scheme. Instead, we will focus on further reducing the total social cost by a properly designed flexible scheme (the three principles discussed in Section 2.3 still hold), which is governed by the following assumptions.

Assumption 2. *The number of reserved parking spaces in reservation group or step i should be equal to the cumulative capacity of the highway bottleneck between* $t_{i-1}^r + \Delta_{i-1}$ *and* $t_i^r + \Delta_i$, *i.e.,* $m_i^r = \left(t_i^r + \Delta_i - t_{i-1}^r - \Delta_{i-1}\right) s$, *for* $i \geq 2$. *Moreover,* $(t_1^r + \Delta_1) - \left(t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s}\right) \geq \frac{m_1^r}{s}$, *where* $t_e^{u,a} = t^* - T_a - \frac{p_b(N-m)-(\alpha T_a+\tau_a)}{\beta} + \frac{\alpha-\beta}{\alpha}\frac{m^u}{s}$.

Assumption 2 for the flexible expirable reservation scheme is an enhanced version of that in Assumption 1 for the inflexible reservation scheme. Firstly, since a buffer time

Δ_i with a fee is allowed for r-commuters with an expiration time of t_i^r, the buffer time is taken into account when calculating the cumulative capacity and determining whether it is sufficient for the "group i" r-commuters. Secondly, since it is socially preferable to not allow any capacity waste (during the arrival of r-commuters), therefore, we set $m_i^r = \left(t_i^r + \Delta_i - t_{i-1}^r - \Delta_{i-1}\right) s$, i.e., the m_i^r r-commuters in group $i \geq 2$ should arrive at the parking exactly between $t_{i-1}^r + \Delta_{i-1}$ and $t_i^r + \Delta_i$, and the m_1^r r-commuters in "group 1" should arrive between $\left(t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s}\right)$ and $t_1^r + \Delta_1$. Indeed, r-commuters in group 1 will arrive exactly between $t_1^r + \Delta_1 - \frac{m_1^r}{s}$ $(\geq t_e^{u,a} + T_a + \frac{\beta}{\alpha}\frac{m^u}{s})$ and $t_1^r + \Delta_1$ in order to minimize schedule delay cost.

Based on the above settings, for group i r-commuters, they are split into two sub-groups: (i) $(1-\lambda_i)m_i^r$, i.e., those arriving between $t_i^r - \frac{(1-\lambda_i)m_i^r}{s}$ and t_i^r; and (ii) $\lambda_i m_i^r$, i.e., those arriving between t_i^r and $t_i^r + \Delta_i$, where $\lambda_i = \frac{\Delta_i s}{m_i^r} \leq 1$. To minimize the total queuing delay for the r-commuters in group i, the first commuters among the $(1-\lambda_i)m_i^r$ and $\lambda_i m_i^r$ should encounter zero queuing delay. The late-for-reservation fee for $\lambda_i m_i^r$ is specified in Assumption 4, which is similar to the coarse toll in Arnott et al. (1990) for early arrival commuters.

Assumption 3. *For a flexible expirable reservation scheme with n reservation steps (or groups), we set $t_n^r + \Delta_n = t^*$, i.e., the last r-commuter can arrive at the destination just on-time.*

Assumption 3 is similar to $t_1^r \geq t^*$ for the expirable reservation scheme with $n = 1$ in Section 2.3.1. It simply ensures that the last r-commuter can arrive at the destination at time t^* and thus the total schedule delay cost of r-commuters can be reduced.

Assumption 4. *The late-for-reservation fee for a parking reservation with an expiration time t_i^r should be $f_a^i = \frac{\beta}{s}\left(1 - \frac{s\Delta_i}{m_i^r}\right) m_i^r = \frac{\beta}{s}\left(1 - \lambda_i\right) m_i^r$.*

Under the Assumptions 2-4 for the flexible expirable reservation scheme, we now can analyze the socially preferred flow patterns under such a reservation scheme.

3.1 Flexible parking reservation with $n = 1$

We first start with $n = 1$, the departure/arrival equilibrium for car commuters under the flexible expirable parking reservation scheme is shown in Figure 5, where the blue solid lines represent departure from home and arrival at the bottleneck for u-commuters, the red solid lines represent departure from home and arrival at the bottleneck for r-commuters, the black solid lines represent the arrivals at the destination. Note that for the transit commuters, the equilibrium flow pattern is the same as that in Figure 2(b).

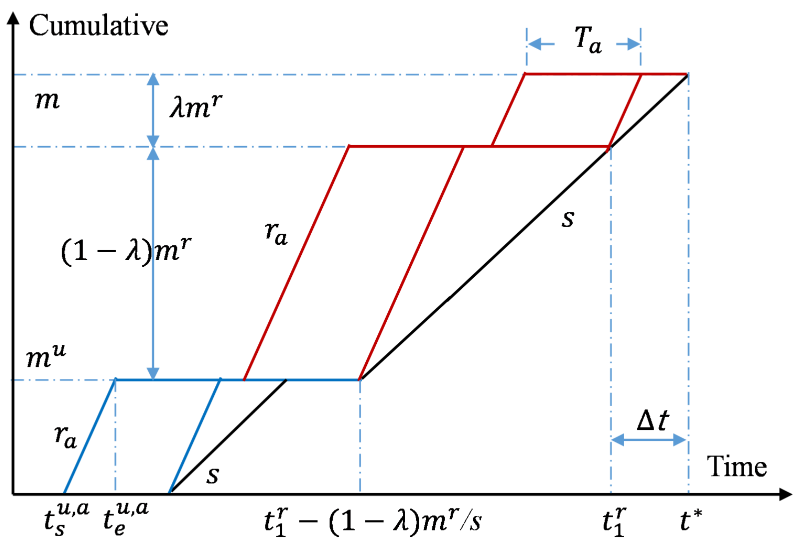


Figure 5: The departure/arrival equilibrium for car commuters with flexible parking reservation ($n = 1$)

We let $\Delta_1 = \Delta t$, where $\Delta t = \lambda \frac{m^r}{s}$ and $\lambda \in [0, 1]$. Moreover, based on Assumption 4, we have $f_a^1 = \frac{\beta}{s}(1-\lambda)\, m^r$. As shown in Figure 5, there are three small peaks associated with u-commuters choosing car mode, r-commuters choosing to be no later than the expiration time, and r-commuters choosing to be later than the expiration time. Similar to those under the inflexible reservation schemes, given $m < N_a^*$, i.e., insufficient parking supply, if $m^r < m$ and $m^u > 0$, capacity waste occurs between the arrivals of the last u-commuter and the first r-commuter, as shown in Figure 5. The efficiency metrics can be further determined based on the bi-modal equilibrium conditions. The cost of u-commuters is equal to $p_b(N - m)$. The travel cost of r-commuters arriving no later than t_i^r is

$$p_{a,1}^o = \alpha T_a + \beta \frac{m^r}{s} + \tau_a \tag{20}$$

The travel cost of r-commuters arriving later than t_i^r is

$$p_{a,1}^l = \alpha T_a + \lambda\beta \frac{m^r}{s} + \tau_a + f_a^1 \tag{21}$$

The total user cost is denoted by $TUC_{f,1}$, which is

$$TUC_{f,1} = (\alpha T_a + \tau_a)\, m^r + (\lambda^2 - \lambda + 1)\frac{\beta (m^r)^2}{s} + \lambda m^r f_a^1 + (N - m^r) \cdot p_b(N - m). \tag{22}$$

The total social cost $TSC_{f,1}$ excludes the monetary cost in the total user cost, which is

$$TSC_{f,1} = (\alpha T_a + \tau_a)\, m^r + (\lambda^2 - \lambda + 1)\frac{\beta (m^r)^2}{s} + (N - m^r) \cdot p_b(N - m) - (N - m)\tau_b - m\tau_a \quad (23)$$

Proposition 3.1. *If we can vary $\lambda = \frac{s\Delta t}{m^r}$ by changing Δt, the total social cost is minimized when $\lambda = \frac{1}{2}$.*

Proposition 3.1 can be verified by examining the first derivative of Eq. (23) with respect to λ. It indicates that we should split m^r into two equal sub-groups to minimize the social cost (as well as the queuing delay cost). Note that since r-commuters arriving at the parking later than t_1^r have to pay a late fee of $f_a^1 = \frac{\beta}{s}(1 - \lambda)\, m^r$, when varying Δt (as well as λ), the total user cost does not change (i.e., the fee replaces the queuing cost).

Proposition 3.2. *The total social cost reduction of the flexible expirable parking reservation scheme under $n = 1$ when compared to the inflexible expirable parking reservation scheme under $n = 1$ can be written as*

$$CS_{f,1} = TSC_{in,1} - TSC_{f,1} = \lambda(1 - \lambda)\frac{\beta (m^r)^2}{s}, \quad (24)$$

where $TSC_{in,1}$ and $TSC_{f,1}$ are given in Eq. (12) and Eq. (23), respectively. Moreover, the cost reduction is maximized when $\lambda = \frac{1}{2}$.

By comparing the total social cost in Eq. (23) with the total social cost in Eq. (12), Eq. (24) can be readily verified. Moreover, one can verify that $\lambda(1 - \lambda)$ is maximized at $\lambda = 0.5$, and so is $CS_{f,1}$. The optimal m^r under given m for the flexible expirable parking reservation scheme is further examined in Proposition 3.3.

Proposition 3.3. *Given $m < N_a^*$, the optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with $n = 1$ can be specified as:*

$$m^{r,*} = \begin{cases} m_{f,1}^{r,*} < m & p_b(N - m) < p_a(2m(\lambda^2 - \lambda + 1)) \\ m & p_b(N - m) \geq p_a(2m(\lambda^2 - \lambda + 1)) \end{cases} \quad (25)$$

where $m_{f,1}^{r,*} = \frac{s}{2\beta(\lambda^2 - \lambda + 1)} \cdot (p_b(N - m) - \tau_a - \alpha T_a)$.

Proposition 3.3 can be verified by examining the first derivative of the total social cost in Eq. (23) with respect to m^r. Proposition 3.3 is further explained in the following. It can be verified that there is only a unique $m_{cf,1}$ such that $p_b(N - m_{cf,1}) = p_a\left(2m_{cf,1}\left(\lambda^2 - \lambda + 1\right)\right)$, and moreover, $m > m_{cf,1} \Leftrightarrow p_b(N - m) < p_a\left(2m\left(\lambda^2 - \lambda + 1\right)\right)$ and $m \leq m_{cf,1} \Leftrightarrow p_b(N - m) \geq p_a\left(2m\left(\lambda^2 - \lambda + 1\right)\right)$. Proposition 3.3 indicates that when $m > m_{cf,1}$, $m^{r,*} = m_f^{r,*} < m$ and when $m \leq m_{cf,1}$, $m^{r,*} = m$. This means that when parking insufficiency is less severe (i.e., $m > m_{cf,1}$), it is preferable to keep some parking open for competition (i.e., $m^{r,*} < m$) to separate car traffic of r-commuters and u-commuters to reduce congestion (even if capacity waste between arrivals of r-commuters and u-commuter occurs). Instead, when parking insufficiency is more severe (i.e., $m \leq m_{cf,1}$), all parking should be reserved to avoid large schedule delays. These results are in line with Proposition 2.2 for the inflexible reservation scheme. We further compare the flexible and inflexible reservation schemes in the following.

Proposition 3.4. *The optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with $n = 1$ is no less than that under the inflexible expirable parking reservation scheme with $n = 1$, i.e., $m^{r,*}$ defined in Eq. (25) is no less than $m^{r,*}$ defined in Eq. (14).*

Note that $\lambda \in [0, 1]$, then $0.75 \leq (\lambda^2 - \lambda + 1) \leq 1$. By comparing Eq. (25) and Eq. (14) and noting that $m^{r,*}_{f,1} \geq m^{r,*}_{in,1}$, Proposition 3.4 can be verified. This is because, the flexible expirable parking reservation scheme is a more capable scheme than the inflexible expirable parking reservation scheme in terms of eliminating congestion (given the same number of steps, i.e., $n = 1$), and adding an additional r-commuter creates a smaller marginal cost under the flexible expirable parking reservation scheme.

3.2 Flexible parking reservation with $n \geq 2$

We now consider $n \geq 2$. To ease the analysis, it is assumed that the maximum time gap allowed for each reservation step (associated with t^r_i) are identical, i.e., for $i = 1$ to n, $\Delta_i = \Delta t$. In addition, m^r is divided into n groups equally, i.e., $m^r_i = \frac{m^r}{n}$. Therefore, we only consider $0 \leq \Delta t \leq \frac{m^r}{ns}$. It follows that $\lambda_i = \frac{sn\Delta t}{m^r}$. To ease the presentation, we let $\lambda = \frac{sn\Delta t}{m^r}$, where $\lambda \in [0, 1]$. Similar to Section 3.1, the late fee $f^i_a = \frac{\beta}{s}(1 - \lambda)\frac{m^r}{n}$ (please refer to Assumption 4), which is denoted as f_a and is identical for each group of r-commuters. Note that the even division of m^r into n groups also makes the flexible reservation more comparable to the inflexible reservation scheme in Section 2.3.2.

Under Assumptions 2-4, the traffic flow pattern under the flexible expirable parking reservation scheme with n steps can be depicted as that in Figure 6, where the blue solid lines represent departure from home and arrival at the bottleneck for u-commuters, the red solid lines represent departure from home and arrival at the bottleneck for r-commuters, the black solid lines represent the arrivals at the destination. There are one small peak for u-commuters and $2n$ small peaks for r-commuters (for each group, there are two small peaks: one for r-commuters arriving at parking no later than the expiration time and one for those later than the expiration time). Similar to the flexible reservation scheme with $n = 1$, for $n \geq 2$, given $m < N_a^*$, i.e., insufficient parking supply, if $m^r < m$ and $m^u > 0$, capacity waste occurs between the arrivals of the last u-commuter and the first r-commuter, as shown in Figure 6. The number of steps for the flexible reservation scheme in Figure 6 is only illustrative. We take "group 1" as an example. $(1-\lambda)\frac{m^r}{n}$ r-commuters in this group will arrive at the parking on or before t_1^r, while the other r-commuters in this group will arrive at the parking later than t_1^r (but no later than $t_1^r + \Delta t$).

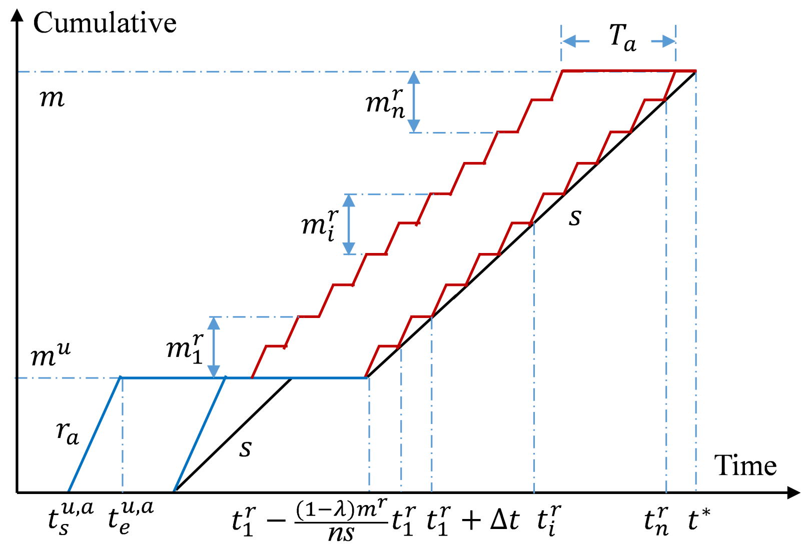


Figure 6: The departure/arrival equilibrium for car commuters with flexible parking reservation ($n \geq 2$)

The travel cost for r-commuters in group i arriving on or before t_i^r is

$$p_{a,i}^o = \alpha T_a + \frac{n-i+1}{n}\beta\frac{m^r}{s} + \tau_a \tag{26}$$

The travel cost for r-commuters arriving later than t_i^r is

$$p_{a,i}^l = \alpha T_a + \frac{\lambda+n-i}{n}\beta\frac{m^r}{s} + \tau_a + f_a \tag{27}$$

The total user cost the can be specified as:

$$TUC_{f,n} = m^r \cdot \left(\alpha T_a + \frac{2\lambda^2 - 2\lambda + n + 1}{2n}\frac{\beta m^r}{s} + \tau_a\right) + \lambda m^r f_a + (N - m^r)\cdot p_b(N-m) \tag{28}$$

The total social cost can be written as:

$$TSC_{f,n} = \alpha T_a m^r + \frac{2\lambda^2 - 2\lambda + n + 1}{2n}\beta\frac{(m^r)^2}{s} + (N - m^r)\cdot p_b(N-m) - (N-m)\tau_b - m^u\tau_a \quad (29)$$

With Eq. (29), we can quantify the efficiency gap between any two flexible expirable parking reservation schemes with n_1 steps and n_2 steps. This is summarized in the following.

Proposition 3.5. *The total social cost gap for two flexible reservation schemes with the same m^r but different numbers of reservation groups (or steps), i.e., n_1 and n_2, can be expressed as*

$$CS_{f,n_1,n_2} = TSC_{f,n_1} - TSC_{f,n_2} = \frac{(n_2 - n_1)(2\lambda^2 - 2\lambda + 1)}{2n_1 n_2}\frac{\beta(m^r)^2}{s} \quad (30)$$

Proposition 3.5 indicates that when $n_2 > n_1$, $CS_{f,n_1,n_2} > 0$, i.e., a more differentiated flexible reservation scheme is more efficient. Moreover, suppose $n_2 \geq n_1$, it can be verified that when $\frac{n_1}{n_2}$ is a constant, CS_{f,n_1,n_2} decreases with n_2. This means that the cost saving from further differentiating the flexible expirable reservation ($n_2 \geq n_1$) is not proportional to the ratio of $\frac{n_1}{n_2}$. Instead, it is diminishing when both n_1 and n_2 increase and $\frac{n_1}{n_2}$ remains constant. These observations are in line with those in Proposition 2.3. We now further compare the cost gap CS_{f,n_1,n_2} in Proposition 3.5 to the cost gap CS_{in,n_1,n_2} in Proposition 2.3, which is summarized below.

Proposition 3.6. *The cost gaps under the inflexible and flexible expirable parking reservation schemes, i.e. CS_{in,n_1,n_2} in Eq. (18) and CS_{f,n_1,n_2} in Eq. (30) satisfy the following*

$$0.5CS_{in,n_1,n_2} \leq CS_{f,n_1,n_2} \leq CS_{in,n_1,n_2}. \quad (31)$$

Proposition 3.6 can be verified based on Eq. (18) and Eq. (30), given that $\lambda \in [0,1]$ such that $0.5 \leq 2\lambda^2 - 2\lambda + 1 \leq 1$. Proposition 3.6 indicates that the efficiency gap between two flexible expirable reservation schemes with n_1 steps and n_2 steps is less than the gap for the two corresponding inflexible expirable reservation schemes. This is due to that given the same number of steps, the flexible expirable reservation scheme is more efficient and the potential for further efficiency improvement is smaller. Furthermore, $0.5CS_{in,n_1,n_2} = CS_{f,n_1,n_2}$ holds when $\lambda = 0.5$.

We now further compare the efficiency of the flexible expirable parking reservation scheme with n steps against the inflexible expirable parking reservation scheme with n steps, which is summarized in Proposition 3.7. Note that Proposition 3.7 is a generalization of Proposition 3.2 from one-step scheme to a n-step scheme.

Proposition 3.7. *The total social cost reduction of the flexible expirable parking reservation scheme with $n \geq 2$ steps when compared to the inflexible expirable parking reservation scheme with $n \geq 2$ can be written as*

$$CS^n_{f,n} = TSC_{in,n} - TSC_{f,n} = \frac{\lambda(1-\lambda)}{n}\frac{\beta(m^r)^2}{s}, \tag{32}$$

where $TSC_{in,n}$ and $TSC_{f,n}$ are given in Eq. (17) and Eq. (29), respectively. Moreover, the cost reduction is maximized when $\lambda = \frac{1}{2}$.

We now further discuss the optimal m_r to minimize the total social cost in Eq. (29) for the flexible expirable reservation scheme and compare it with that for the inflexible expirable reservation scheme.

Proposition 3.8. *Given $m < N_a^*$, the optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with n steps can be specified as*

$$m^{r,*} = \begin{cases} m^{r,*}_{f,n} & p_b(N-m) \leq p_a(\frac{m}{n}(2\lambda^2 - 2\lambda + n + 1)) \\ m & p_b(N-m) > p_a(\frac{m}{n}(2\lambda^2 - 2\lambda + n + 1)) \end{cases} \tag{33}$$

where $m^{r,*}_{f,n} = \frac{n}{2\lambda^2 - 2\lambda + n + 1} \cdot \frac{s}{\beta}(p_b(N-m) - \tau_a - \alpha T_a)$.

Proposition 3.9. *The optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with n steps is no less than that under the inflexible expirable parking reservation scheme with n, i.e., $m^{r,*}$ defined in Eq. (33) is no less than $m^{r,*}$ defined in Eq. (19).*

Proposition 3.10. *For two flexible reservation schemes with the same m but different numbers of steps, i.e., n_1 and n_2, if $n_1 < n_2$, $(m^{r,*})_1 \leq (m^{r,*})_2$, where $(m^{r,*})_1$ and $(m^{r,*})_2$ are given by Eq. (33) with $n = n_1$ and $n = n_2$, respectively.*

Proposition 3.8 and Proposition 3.9 generalize the results in Proposition 3.3 and Proposition 3.4 for the one-step flexible reservation scheme to the multi-step flexible reservation scheme. The reasoning and insights are similar, details of which are omitted. Proposition 3.10 extends the result in Proposition 2.5 for the inflexible reservation case into the flexible expirable reservation case. The reasoning and insights are also similar, details of which are omitted.

3.3 The optimal parking supply

We now further examine the optimal total parking supply m when the total reserved parking supply m^r under the flexible expirable parking reservation scheme is also optimized. The social cost (also can be considered as the opportunity cost) of a parking supply m is denoted as $\kappa(m)$, where $\kappa'(m) = \frac{d\kappa(m)}{dm} > 0$ and $\kappa''(m) = \frac{d^2\kappa(m)}{dm^2} > 0$. In this study, we focus on $m < N$, i.e., parking is insufficient. Therefore, we consider that $\kappa(N)$ will be extremely large, i.e., a soft constraint on parking supply (extremely large $\kappa'(N)$ and $\kappa''(N)$ as well).

The total system cost is the summation of the total social cost (related to travel) under the flexible parking reservation scheme with n steps (given in Eq. (29)) and the parking supply cost, which can be rewritten as a function of m and m^r, i.e.,

$$\begin{gathered} TS_{f,n}(m, m^r) = \kappa(m) + \alpha T_a m^r + \frac{2\lambda^2 - 2\lambda + n + 1}{2n}\beta\frac{(m^r)^2}{s} + \\ (N - m^r)\cdot\left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b(N-m)}\right) - (N-m)\tau_b - (m - m^r)\tau_a \end{gathered} \tag{34}$$

Note that $0 \leq m^r \leq m$. We then can derive the first derivatives of $TS_{f,n}$ with respect to m and m^r as follows:

$$\begin{gathered} \frac{\partial TS_{f,n}}{\partial m} = \kappa'(m) - (N - m^r)\sqrt{2\beta\theta\delta T_b}\frac{1}{\sqrt{N-m}} + \tau_b - \tau_a; \\ \frac{\partial TS_{f,n}}{\partial m^r} = \alpha T_a + \tau_a + \frac{2\lambda^2 - 2\lambda + n + 1}{n}\beta\frac{m^r}{s} - \left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b(N-m)}\right). \end{gathered} \tag{35}$$

It can be shown that when $m^r \to 0$, $\frac{\partial TS_{f,n}}{\partial m^r} < 0$, which means that $m^r \to 0$ is not optimal (this is consistent with Proposition 3.8).

When $m^r \to m$, if $\frac{\partial TS_{f,n}}{\partial m^r} \leq 0$, the optimal m^r should be equal to m, i.e., $m^{r,*} = m$. Considering an interior optimal m (note that $\kappa(m)$ is very large when m approaches N), we should have $\frac{\partial TS_{f,n}}{\partial m} = 0$. Given $m^{r,*} = m$ and with Eq. (35), we have

$$\kappa'(m) = \sqrt{2\beta\theta\delta T_b(N-m)} + \tau_a - \tau_b \tag{36}$$

The optimal parking supply m^* solves Eq. (36), where $\alpha T_a + \tau_a + \frac{2\lambda^2-2\lambda+n+1}{n}\beta\frac{m^*}{s} - \left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b(N-m^*)}\right) \leq 0$ should also hold (which ensures $\frac{\partial TS_{f,n}}{\partial m^r} \leq 0$).

When $m^r \to m$, if $\frac{\partial TS_{f,n}}{\partial m^r} > 0$, the optimal m^r should be less than m, i.e., $m^{r,*} < m$. Then, we have $\frac{\partial TS_{f,n}}{\partial m^r} = 0$ at $m^{r,*}$, and thus

$$m^{r,*} = \frac{ns}{\beta(2\lambda^2 - 2\lambda + n + 1)}\left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b(N-m)} - (\alpha T_a + \tau_a)\right). \tag{37}$$

where m is further determined by $\frac{\partial TS_{f,n}}{\partial m} = 0$ (similarly, considering an interior optimal m). Based on Eq. (37), $\frac{\partial TS_{f,n}}{\partial m} = 0$ and Eq. (35), the optimal parking supply m^* solves the following:

$$\kappa'(m) = (N - m^{r,*})\sqrt{2\beta\theta\delta T_b}\frac{1}{\sqrt{N-m}} + \tau_a - \tau_b. \tag{38}$$

where $m^{r,*}$ as a function of m is given in Eq. (37). Note that $\alpha T_a + \tau_a + \frac{2\lambda^2-2\lambda+n+1}{n}\beta\frac{m^{r,*}}{s} - \left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b(N-m^*)}\right) > 0$ should hold, i.e., $\frac{\partial TS_{f,n}}{\partial m^r} > 0$.

4 Flexible expirable parking reservation scheme with a time-dependent late-for-reservation fee

In the previous section, the flexible expirable parking reservation scheme with a constant fee for being late for the reservation is studied. In this section, a time-varying late fee for reservation is further studied, i.e., commuters later for reservation will pay a fee depending on how late they are. The other aspects/settings of the reservation schemes are identical to those in the previous section.

For those with a reservation associated with an expiration time t_i^r, the late fee is denoted by $f_a^i + \rho_i(t - t_i^r)$, where f_a^i is the constant late fee defined in the previous section and ρ_i is the late fee rate for a unit time of being late for the reservation.

To avoid tedious algebra, we assume that the late fee rate ρ_i for all the reservation steps (associated with t_i^r) are identical, i.e., for $i = 1$ to n, $\rho_i = \rho$. Then for the commuters associated with expiration t_i^r, his or her travel cost is

$$c_a(t) = \alpha \cdot T_a(t) + \beta \cdot [t^* - t - T_a(t)]^+ + \tau_a + f_a^i + \rho \cdot [t - t_i^r]^+ \tag{39}$$

Similar to Eq. (2), by setting $\frac{dc_a(t)}{dt} = 0$, the departure rate for those late for their reservation can be derived, which is

$$r_a'(t) = \frac{\alpha - \rho}{\alpha - \beta}s. \tag{40}$$

Note that the value of ρ should be set between zero and β, i.e., $\rho \in [0, \beta]$. When $\rho = 0$, $r_a'(t) = r_a(t)$, which is indeed the constant late fee case. When $\rho = \beta$, $r_a'(t) = s$, which means that the departure rate is identical to the bottleneck capacity. In this case, there is no queuing delay for these commuters late for their reservations.

4.1 The flexible reservation scheme with a time-dependent late-for-reservation fee and $n = 1$

In this section, the performance of flexible expirable parking reservation scheme with a time-varying late fee when $n = 1$ is analyzed and compared to the flexible reservation scheme with a constant late fee. Note that given the same Δt (the maximum allowed time for being late for the parking reservation, as introduced in Section 3), the individual travel costs of commuters under schemes with a time-dependent and constant late fee are identical. The main difference between the two schemes lies in the total social cost, i.e., the queuing delays (of r-commuters late for reservation) are replaced by the fees for being late for the reservation. This is similar to the time-dependent congestion pricing schemes discussed in Arnott et al. (1990).

We start with $n = 1$. The total social cost $TSC_{t,1}$ under the scheme with a time-varying late fee for being late for the reservation is

$$TCS_{t,1} = (\alpha T_a + \tau_a)\, m^r + (\lambda^2 - \lambda + 1)\frac{\beta(m^r)^2}{s} + (N - m^r) \cdot p_b(N - m) - (N - m)\tau_b - m\tau_a - F_a^1 \tag{41}$$

where F_a^1 is the total additional late-for-reservation fee paid by travelers, which is

$$F_a^1 = \int_0^{\frac{\lambda m^r}{r_a'}} \frac{\rho}{r_a'} t \mathrm{d}t = \frac{1}{2} \cdot \frac{\rho(\alpha - \beta)}{s(\alpha - \rho)}(\lambda m^r)^2 \tag{42}$$

Proposition 4.1. *The total social cost reduction of the flexible expirable parking reservation scheme with a time-dependent late fee under $n = 1$ when compared to that with a constant late fee under $n = 1$ can be written as*

$$CS_{t,1} = \frac{1}{2} \cdot \frac{\rho(\alpha - \beta)}{s(\alpha - \rho)}(\lambda m^r)^2 \tag{43}$$

Moreover, the cost reduction is maximized when $\rho = \beta$ and $\lambda = 1$.

By comparing the total social cost in Eq. (41) with that in Eq. (23), Eq. (43) can be readily verified. Moreover, one can verify that $\frac{\partial F_a^1}{\partial \rho} > 0$ and $\frac{\partial F_a^1}{\partial \lambda} > 0$, since $\rho \in [0, \beta]$ and $\lambda \in [0, 1]$. Therefore, the cost reduction is maximized when $\rho = \beta$ and $\lambda = 1$.

Proposition 4.2. *Given $m < N_a^*$, the optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with a time-dependent late fee under $n = 1$*

can be specified as:

$$m^{r,*} = \begin{cases} m^{r,*}_{t,1} & p_b(N-m) < p_a(m[2(\lambda^2-\lambda+1) - \frac{\rho(\alpha-\beta)\lambda^2}{\beta(\alpha-\rho)}]) \\ m & p_b(N-m) \geq p_a(m[2(\lambda^2-\lambda+1) - \frac{\rho(\alpha-\beta)\lambda^2}{\beta(\alpha-\rho)}]) \end{cases} \tag{44}$$

where $m^{r,*}_{f,1} = \frac{s}{2\beta(\lambda^2-\lambda+1)-\frac{\rho(\alpha-\beta)\lambda^2}{(\alpha-\rho)}} \cdot (p_b(N-m) - \tau_a - \alpha T_a)$.

Proposition 4.2 can be verified by examining the first derivative of the total social cost in Eq.(41) with respect to m^r. The reasoning and implications of Proposition 4.2 are similar to those in Proposition 3.3. Besides, it is noteworthy that the condition for $m^{r,*} = m$ in Proposition 4.2 is more relaxed than that in Proposition 3.3. This is because, the flexible reservation scheme is more capable of reducing congestion delays of r-commuters (those with reservations), and it is more likely that all parking spaces should be reserved and congestion delays of commuters can be reduced by a larger extent.

We further compare the reservation schemes with time-dependent and constant late fee in the following.

Proposition 4.3. *The optimal number of parking spaces for reservation m^r under the flexible expirable parking reservation scheme with a time-dependent late fee under $n = 1$ is no less than that with a constant late fee under $n = 1$, i.e., $m^{r,*}$ defined in Eq. (44) is no less than $m^{r,*}$ defined in Eq. (25).*

Note that $\rho \in [0, \beta]$, $\alpha > \beta$, $\alpha > \rho$ and $\lambda \in [0, 1]$. One then can derive that $\frac{\rho(\alpha-\beta)\lambda^2}{(\alpha-\rho)} > 0$. By comparing Eq. (44) and Eq. (25) and noting that $m^{r,*}_{t,1} \geq m^{r,*}_{f,1}$, Proposition 4.3 can be verified. This is because, the flexible expirable parking reservation scheme with a time-dependent late fee eliminates more congestion when compared to the constant late fee scheme and adding an additional r-commuter creates a smaller marginal cost.

4.2 The flexible reservation scheme with a time-dependent late-for-reservation fee and $n \geq 2$

We now consider $n \geq 2$. To be more compatible with the flexible schemes with a constant late fee, we adopt the same scheme setting as that of the flexible reservation scheme with a constant late fee and only replace the constant fee by a time-dependent fee. For example, for $i = 1$ to n, $\Delta_i = \Delta t$, m^r is divided into n groups equally, i.e., $m^r_i = \frac{m^r}{n}$, $\lambda = \frac{sn\Delta t}{m^r}$, $f^i_a = \frac{\beta}{s}(1-\lambda)\frac{m^r}{n}$. As stated before, for simplicity , we assume that the late-for-reservation fee rate ρ_i for each reservation steps (associated with t^r_i) are identical, i.e., for $i = 1$ to n, $\rho_i = \rho$.

The travel costs for r-commuters in "group i" arriving on or before t_i^r and after t_i^r and the total user cost will be all identical to those under the flexible reservation scheme with a constant late fee, which are given in Eq. (26), Eq. (27) and Eq. (28). With respect to the the total social cost, we have

$$TSC_{t,n} = TSC_{f,n} - F_a^n \tag{45}$$

where $TSC_{f,n}$ is the total social cost under the flexible expirable parking reservation scheme with a constant late fee, which is given in Eq. (29), F_a^n is the total additional late fee paid by travelers under the time-dependent late fee schemes under $n \geq 2$, which is

$$F_a^n = \frac{1}{2} \cdot \frac{\rho(\alpha - \beta)}{n \cdot s(\alpha - \rho)} (\lambda m^r)^2 \tag{46}$$

Proposition 4.4. *The total social cost gap for two flexible reservation schemes with a time-dependent late fee given the same m^r but different numbers of reservation groups or steps, i.e., n_1 and n_2, can be expressed as*

$$CS_{t,n_1,n_2} = \frac{(n_2 - n_1)\rho(\alpha - \beta)}{2n_1 n_2 \cdot s(\alpha - \rho)} (\lambda m^r)^2 \tag{47}$$

Proposition 4.4 indicates that when $n_2 > n_1$, $CS_{t,n_1,n_2} > 0$, i.e., a more differentiated flexible reservation scheme is more efficient. Also, suppose $n_2 \geq n_1$, it can be verified that when $\frac{n_1}{n_2}$ is a constant, CS_{f,n_1,n_2} decreases with n_2. This means that the cost saving from further differentiating the flexible expirable reservation ($n_2 \geq n_1$) is not proportional to the ratio of $\frac{n_1}{n_2}$. Instead, it is diminishing when both n_1 and n_2 increase and $\frac{n_1}{n_2}$ remains constant. These observations are in line with those in Proposition 3.5.

We now further discuss the optimal m_r to minimize the total social cost in Eq. (45) for the flexible expirable reservation scheme with a time-dependent late fee and compare it with that for the flexible expirable reservation scheme with a constant late fee.

Proposition 4.5. *Given $m < N_a^*$, the optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with a time-dependent late fee of n steps can be specified as*

$$m^{r,*} = \begin{cases} m_{t,n}^{r,*} & p_b(N - m) < p_a(\frac{m}{n}(2\lambda^2 - 2\lambda + n + 1 - \frac{\rho(\alpha-\beta)\lambda^2}{\beta(\alpha-\rho)})) \\ m & p_b(N - m) \geq p_a(\frac{m}{n}(2\lambda^2 - 2\lambda + n + 1 - \frac{\rho(\alpha-\beta)\lambda^2}{\beta(\alpha-\rho)}]) \end{cases} \tag{48}$$

where $m_{f,n}^{r,*} = \frac{ns}{\beta(2\lambda^2 - 2\lambda + n + 1) - \frac{\rho(\alpha-\beta)\lambda^2}{(\alpha-\rho)}} \cdot (p_b(N - m) - \tau_a - \alpha T_a)$.

Proposition 4.6. *The optimal number of parking for reservation m^r under the flexible expirable parking reservation scheme with a time-dependent late fee of n steps is no less than that under the flexible expirable parking reservation scheme with a constant late fee of n steps, i.e., $m^{r,*}$ defined in Eq. (48) is no less than $m^{r,*}$ defined in Eq. (33).*

Proposition 4.7. *For two flexible reservation schemes with a time-dependent late fee, suppose they have the same parking supply m but different numbers of steps, i.e., n_1 and n_2, if $n_1 < n_2$, $(m^{r,*})_1 \le (m^{r,*})_2$, where $(m^{r,*})_1$ and $(m^{r,*})_2$ are given by Eq. (48) with $n = n_1$ and $n = n_2$, respectively.*

Proposition 4.5 and Proposition 4.6 generalize the results in Proposition 4.2 and Proposition 4.3 for the one-step flexible reservation scheme with a time-dependent late-for-reservation fee into the case with multiple steps. The reasoning and interpretations are similar. Proposition 4.7 extends the result in Proposition 2.5 for the inflexible reservation scheme into the flexible expirable reservation case with time-varying late fees. The reasoning and insights are also similar.

4.3 The optimal parking supply

We now further examine the optimal parking supply m under different flexible expirable parking reservation schemes with time-dependent late-for-reservation fees, where m^r is also optimized. Similar to Section 3.3, the parking supply involves a social cost $\kappa(m)$, where $\kappa' > 0$ and $\kappa'' > 0$, and $\kappa(N)$ will be extremely large, as well as extremely large $\kappa'(N)$ and $\kappa''(N)$.

The total system cost is the summation of the total social cost (related to travel) under the flexible parking reservation scheme time-dependent late-for-reservation fees (given in Eq. (45)) and the parking supply cost, which can be rewritten as a function of m and m^r, i.e.,

$$\begin{aligned} TS_{t,n}(m, m^r) = \kappa(m) + \alpha T_a m^r + \frac{2\lambda^2 - 2\lambda + n + 1}{2n}\beta\frac{(m^r)^2}{s} + \\ (N - m^r)\cdot\left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b(N-m)}\right) - (N-m)\tau_b - (m - m^r)\tau_a \\ -\frac{1}{2}\cdot\frac{\rho(\alpha-\beta)}{n\cdot s(\alpha-\rho)}(\lambda m^r)^2, \end{aligned} \tag{49}$$

where $0 \le m^r \le m$. We then can derive the first derivatives of $TS_{t,n}$ with respect to m and

m^r as follows:

$$\frac{\partial TS_{t,n}}{\partial m} = \kappa'(m) - (N - m^r)\sqrt{2\beta\theta\delta T_b}\frac{1}{\sqrt{N-m}} + \tau_b - \tau_a;$$
$$\frac{\partial TS_{t,n}}{\partial m^r} = \alpha T_a + \tau_a + \frac{2\lambda^2 - 2\lambda + n + 1}{n}\beta\frac{m^r}{s} - \left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b (N-m)}\right) - \frac{\rho(\alpha-\beta)}{n\cdot s(\alpha-\rho)}(\lambda^2 m^r). \tag{50}$$

It can be shown that when $m^r \to 0$, $\frac{\partial TS_{t,n}}{\partial m^r} < 0$, which means that $m^r \to 0$ is not optimal (this is consistent with Proposition 4.5).

When $m^r \to m$, if $\frac{\partial TS_{t,n}}{\partial m^r} \leq 0$, the optimal m^r should be equal to m, i.e., $m^{r,*} = m$. Considering an interior optimal m (note that $\kappa(m)$ is very large when m approaches N), we should have $\frac{\partial TS_{t,n}}{\partial m} = 0$. Given $m^{r,*} = m$, we have

$$\kappa'(m) = \sqrt{2\beta\theta\delta T_b (N-m)} + \tau_a - \tau_b \tag{51}$$

The optimal parking supply m^* solves Eq. (51), which is similar to Eq. (36) for the flexible reservation with a constant late fee. Moreover, $\alpha T_a + \tau_a + \frac{2\lambda^2-2\lambda+n+1}{n}\beta\frac{m^*}{s} - \left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b (N-m^*)}\right) - \frac{\rho(\alpha-\beta)}{n\cdot s(\alpha-\rho)}(\lambda^2 m^*) \leq 0$ should hold (i.e., $\frac{\partial TS_{t,n}}{\partial m^r} \leq 0$). This condition is more relaxed than that for the flexible reservation with a constant late-for-reservation fee.

When $m^r \to m$, if $\frac{\partial TS_{t,n}}{\partial m^r} > 0$, the optimal m^r should be less than m, i.e., $m^{r,*} < m$. Then, we have $\frac{\partial TS_{t,n}}{\partial m^r} = 0$ at $m^{r,*}$, and thus

$$m^{r,*} = \frac{ns}{\beta(2\lambda^2 - 2\lambda + n + 1) - \frac{\rho(\alpha-\beta)\lambda^2}{(\alpha-\rho)}}\left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b (N-m)} - (\alpha T_a + \tau_a)\right), \tag{52}$$

where m is further determined by $\frac{\partial TS_{f,n}}{\partial m} = 0$ (similarly, considering an interior optimal m). Given the same m, it can be readily verified that $m^{r,*}$ in Eq. (52) is larger than $m^{r,*}$ in Eq. (37). This is consistent with Proposition 4.6. Based on Eq. (52), $\frac{\partial TS_{t,n}}{\partial m} = 0$ and Eq. (50), the optimal parking supply m^* solves the following:

$$\kappa'(m) = (N - m^{r,*})\sqrt{2\beta\theta\delta T_b}\frac{1}{\sqrt{N-m}} + \tau_a - \tau_b. \tag{53}$$

where $m^{r,*}$ as a function of m is given in Eq. (52). Moreover, $\alpha T_a + \tau_a + \frac{2\lambda^2-2\lambda+n+1}{n}\beta\frac{m^{r,*}}{s} - \left(\alpha T_b + \tau_b + \sqrt{2\beta\theta\delta T_b (N-m^*)}\right) - \frac{\rho(\alpha-\beta)}{n\cdot s(\alpha-\rho)}(\lambda^2 m^{r,*}) > 0$ should hold (i.e., $\frac{\partial TS_{t,n}}{\partial m^r} > 0$).

It is noteworthy that while Eq. (53) (under the flexible reservation scheme with time-varying late fees) is similar to Eq. (38) (under the flexible scheme with constant late fees), $m^{r,*}$ in Eq. (53) is different from that in Eq. (38), and thus the optimal total parking supply under the two schemes will be different.

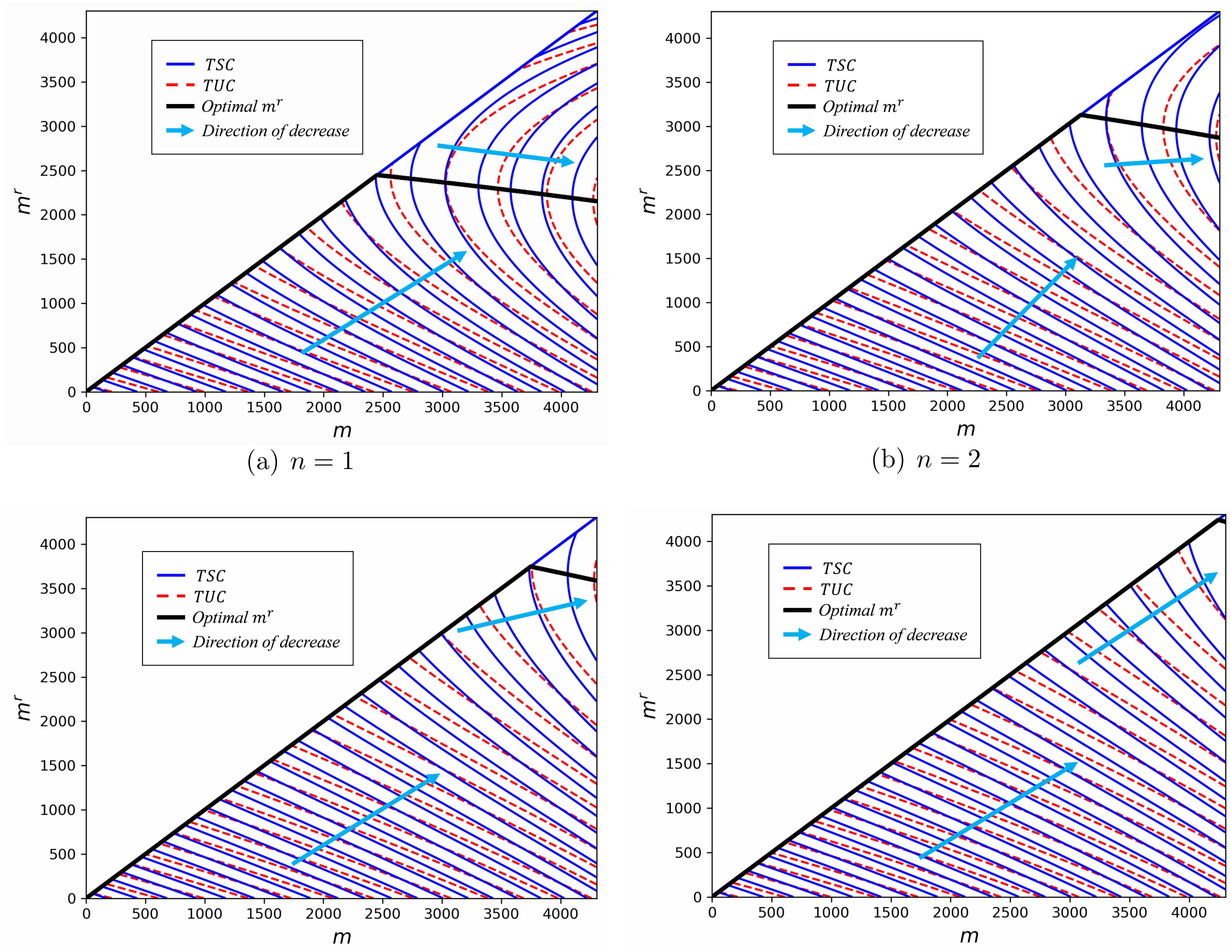
5 Numerical Studies

This section presents some numerical studies to illustrate the analysis in previous sections. Firstly, we show how different system efficiency metrics vary in the two-dimension domain of (m^r, m). Secondly, we examine how the total user cost and total social cost vary with the total parking supply m. Thirdly, the variation of total social cost with respect to the late-for-reservation fee rate ρ is explored.

We adopt the following value of time and early arrival penalty: $\alpha = 13.7$ (USD per hour) and $\beta = 6.4$ (USD per hour), which follow Liu et al. (2015a). Moreover, we assume that $s = 2000$ (veh/h); $N = 8000$; $T_a = 15$ minutes; $T_b = 45$ minutes; $\tau_a = 4$ (USD); $\tau_b = 2.5$ (USD); $\theta = 0.0004$; $\delta = 5$ (minutes); $\lambda = 0.5$. $\rho = 4.8$. By using these parameters, at the bi-modal equilibrium without parking space constraints, $N_a^* = 4304$, $N_b^* = 3696$.

Figure 7, Figure 8 and Figure 9 display the total social cost contours (the blue solid lines) and the total user cost contours (the red dashed lines) in the domain of (m, m^r) for the inflexible reservation scheme, the flexible reservation scheme with a constant late fee, and the flexible reservation scheme with a time-dependent late fee, respectively. The black solid lines and/or the green dashed lines correspond to the optimal m^r to minimize the total social cost or the total user cost. Four cases with $n = 1$, $n = 2$, $n = 5$ and $n = 50$ are examined for the three different reservation schemes.

Several observations from Figure 7, Figure 8 and Figure 9 are summarized below. Firstly, for the inflexible reservation schemes, the optimal m^r for both the total user cost and the total social cost are identical under given m. Secondly, for all the cases (with different values of n for both the inflexible and the two flexible schemes), when m is relatively small, the optimal m^r should be equal to m, i.e., all parking should be reserved. Instead, when m is relatively large, the optimal m^r should be less than m, i.e., some parking spaces should be open for public competition. Thirdly, when n increases ($1 \to 2 \to 5 \to 50$), the optimal m^r increases or at least does not decrease for the three schemes. Fourthly, by comparing Figure 7, Figure 8 and Figure 9, it can be seen that given the same n, the optimal m^r under the inflexible scheme is no greater than that under the flexible scheme with a constant late-for-reservation fee, and the optimal m^r under the the flexible scheme with a constant late-for-reservation fee is no greater than that under the flexible scheme with a time-dependent late-for-reservation fee. Fifthly, for the two flexible schemes with the same n, the m^r to minimize the social cost

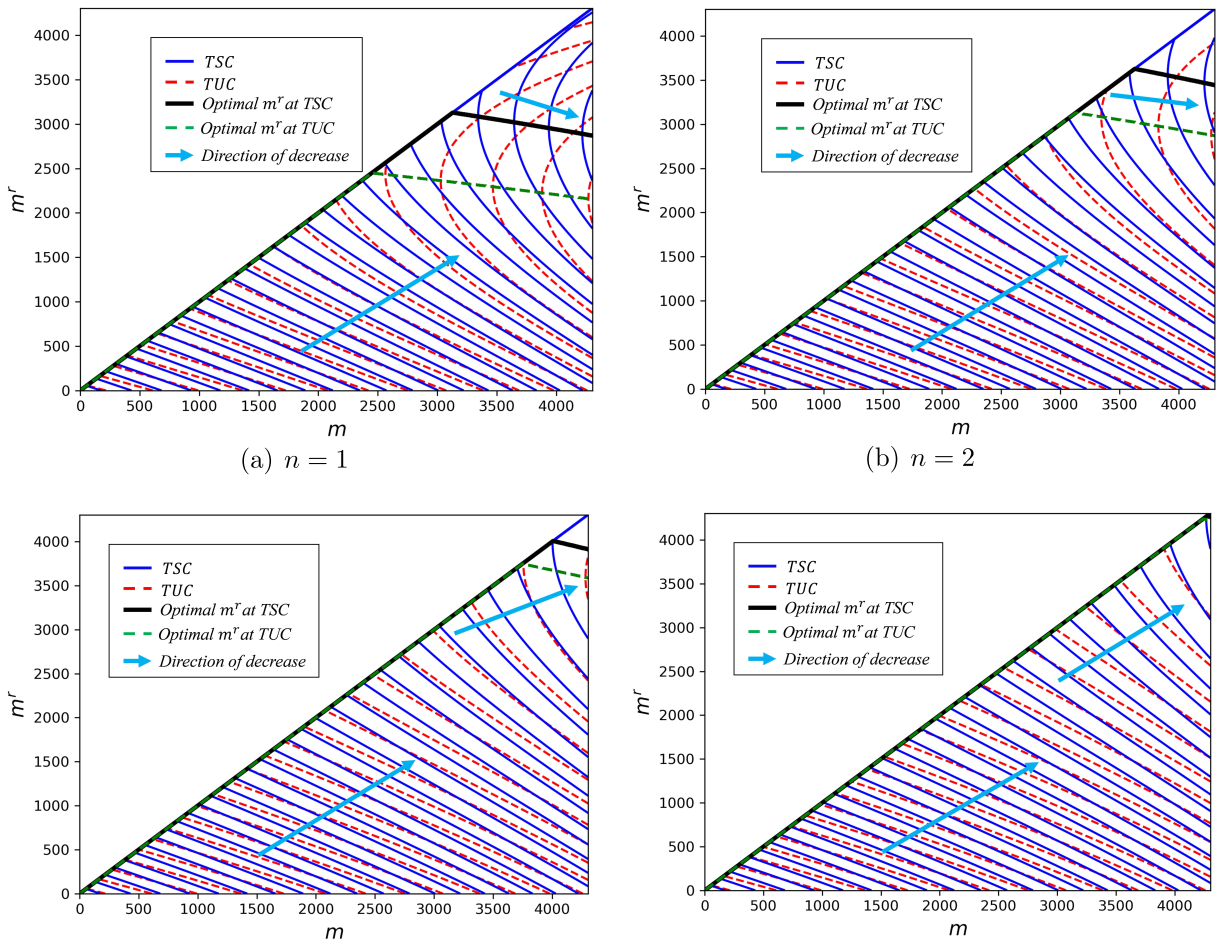


(a) $n = 1$ (b) $n = 2$ (c) $n = 5$ (d) $n = 50$

Figure 7: The variation of total user cost and the total social cost in the domain of (m, m^r) under inflexible expirable reservation schemes

is no less than that to minimize the user cost. These observations are consistent with the analytical results in Section 2, Section 3 and Section 4.

Figure 10 further displays how the total user cost and the total social cost vary with m, where $m^r = m$ is assumed. A few observations from Figure 10 are summarized as follows. Firstly, when m increases but is still relatively small, both TUC and TSC decrease under the two flexible expirable reservation schemes with constant or time-varying late fee at $n = 1, 2, 5, 50$. This is because increasing m at this stage induce little congestion on road (as the car traffic is bounded by the parking supply m) and crowding at the transit side can be reduced significantly. When m becomes larger, the TUC and TSC can increase with m. For example, for $n = 1$, the user cost and social cost reaches their minimums at $m = 3108$ and 4123 at the flexible scheme with a constant late-for-reservation fee, respectively. Secondly, when n increases, i.e., n varies according to $1 \to 2 \to 5 \to 50$, TUC and TSC decrease,



(a) $n = 1$

(b) $n = 2$

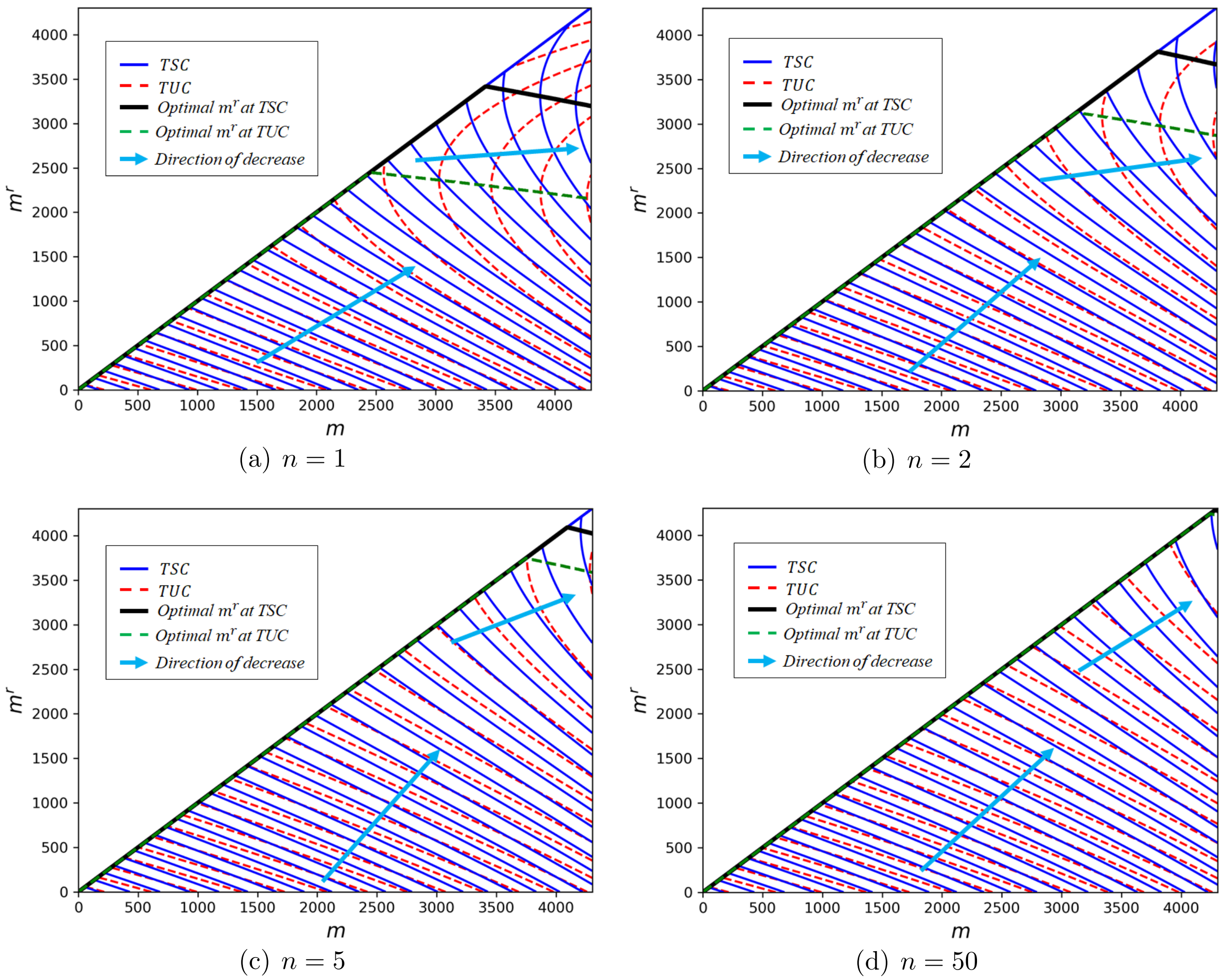
(c) $n = 5$

(d) $n = 50$

Figure 8: The variation of total user cost and the total social cost in the domain of (m, m^r) under flexible expirable reservation schemes with a constant late fee

i.e., a more differentiated scheme is more efficient. Thirdly, the total user costs under the schemes with a constant late-for-reservation fee and a time-dependent late-for-reservation fee are identical. The total social cost under the scheme with a constant late-for-reservation fee is no less than that under the scheme with a time-dependent late-for-reservation fee. Moreover, the optimal m under the scheme with a constant late-for-reservation fee to minimize the total social cost is no greater than that under the scheme with a time-dependent late-for-reservation fee. These observations are also consistent with the analytical results in Section 4.

Figure 11 illustrates the variation of total social cost with ρ (i.e., the rate of late-for-reservation fee and $\rho \in [0, \beta]$) under the flexible scheme with time-dependent late-for-reservation fee, where $m^r = m = 3500$ is assumed. A few observations are summarized. Firstly, the total social cost decreases with respect to ρ. This is because by increasing the late fee for private car commuters, the queuing delay for private car commuters will decrease.



(a) $n = 1$ (b) $n = 2$

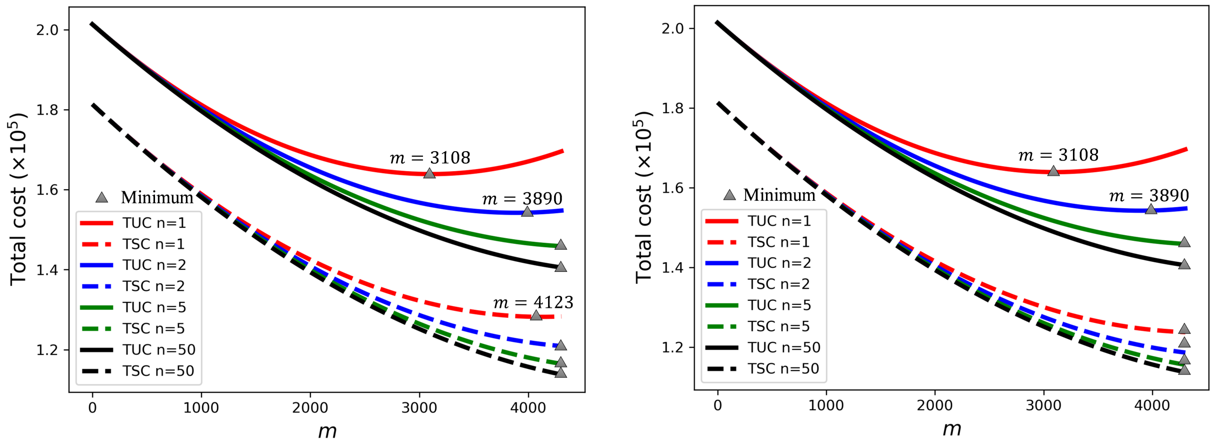
(c) $n = 5$ (d) $n = 50$

Figure 9: The variation of total user cost and the total social cost in the domain of (m, m^r) under flexible expirable reservation schemes with a time-varying late fee

Secondly, when n is relatively small, e.g., when $n = 1$, with the increase of ρ, the decrease of total social cost is more significantly than that when n is relatively large, e.g., $n = 50$. This is because when n is large, the total social cost and the queuing delay of car commuters are relatively small, and the potential to further decrease queuing delay is then limited.

6 Conclusion

This study proposes flexible expirable parking reservation schemes, under which commuters with a parking reservation arriving later than the reservation expiration time can retain the reservation with an additional fee. The flexible expirable parking reservation schemes with a constant late-for-reservation fee and a time-dependent late-for-reservation fee are examined



(a) Scheme of constant late fee

(b) Scheme with a time-varying late fee

Figure 10: Variation of total user cost and total social cost with m

and compared. These flexible schemes are more efficient than the inflexible schemes in Liu et al. (2014a). It indicates that allowing flexibility (to commuters with reservations) can yield efficiency gains.

To ease the analysis, late arrival at the workplace is not considered in this paper. If late arrival at the workplace is allowed, the equilibrium departure/arrival pattern of commuters will change since some commuters can be late and they now make a trade-off between late arrival penalty and queuing delay. The exact flexible parking reservation schemes to reduce total system cost will be different from those identified in this paper, and the exact efficiency gain from the proposed schemes will be slightly different. However, the central idea and insights from this paper will still hold, i.e., (i) allowing flexibility for parking reservation can help reduce total system cost (the flexible reservation scheme should be appropriately designed); (ii) a more differentiated reservation scheme (with more reservation groups or steps) is potentially more efficient (while it can be less practical); (iii) the efficiency gain from the proposed flexible reservation scheme will be bounded. A future study may incorporate late arrival at workplace under consideration of an integrated morning commuting, parking, and evening commuting pattern, which involves additional modeling complexity but might help generate new insights on parking problems.

This study can be further extended in several other ways. Firstly, a future study may consider incorporating the parking search process of a commuter without a parking reservation, similar to those in the study of Liu and Geroliminis (2016). Secondly, similar to Zhang et al. (2011), a parking permit distribution and trading scheme can be designed to implement the flexible expirable parking reservation scheme. It is expected that the parking permits with different expiration times and/or flexibility can have different market prices in

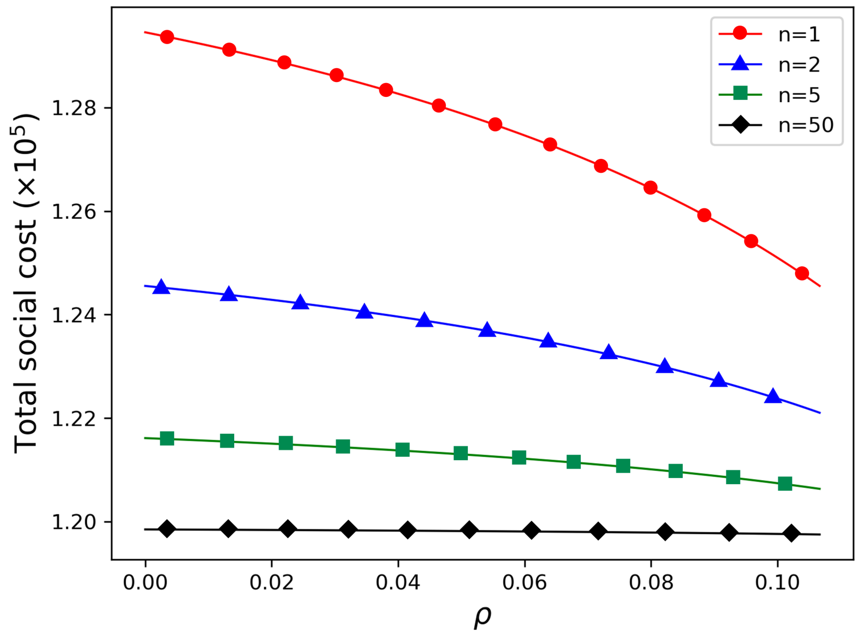


Figure 11: Variation of total social cost with ρ

the trading market of permits. Thirdly, it is of our interest to examine responsive transit services in the bi-modal problem such as those in Zhang et al. (2014, 2016) and jointly manage parking and transit services. Fourthly, a future study may incorporate parking information provision and examine how real-time information could affect commuters' decisions, similar to Li et al. (2012). Last but not least, the current study only considers a single origin-destination pair along the traffic corridor. A future study can further examine cases with multiple origin-destination pairs along the linear traffic corridor (Liu et al., 2009; Wu et al., 2020) or multiple traffic corridors (Liu et al., 2016).

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