# FOURIER-COSINE METHOD FOR FINITE-TIME GERBER-SHIU FUNCTIONS*

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**Abstract.** In this article, we provide the first systematic numerical study on, via the popular Fourier-cosine (COS) method, finite-time Gerber-Shiu functions with the risk process being driven by a generic Lévy subordinator. These functions play a major role in modern actuarial science, and there are still many open problems left behind such as the one here of looking for a universal effective numerical scheme for them. By extending the celebrated Ballot theorem to the continuous setting, we first derive an explicit integral expression for these functions, with an arbitrary penalty, in terms of their infinite-time counterpart. As is common in actuarial or financial practice, an advanced knowledge of the characteristic function of the driving Lévy Process facilitates the applicants of the Fourier-cosine method to this integral expression. Under some mild and practically feasible assumptions, a comprehensive and rigorous (yet demanding) error analysis is provided; indeed, up to an arbitrarily chosen error tolerance level, the numerical scheme is linear in computational complexity which can even reach the theoretically fastest possible rate of 3; all of these are the most effective records of the contemporary state of the art in actuarial science. Finally, the effectiveness of our approximation method is illustrated through different representative numerical experiments, some of them, such as those driven by Gamma and Generalized Stable Processes, are even achieved for the first time in the literature, due to the limitations of most common existing approaches, and we shall discuss more in this article.

**Key words.** Lévy subordinator; Gerber-Shiu functions; Fourier-cosine method; Numerical integration; Algebraic index; Gibbs Phenomenon.

**AMS subject classifications.** 68Q25, 68R10, 68U05

## 1. Preliminaries.

### 1.1. Background.

Since the pioneer work of [22], study on Gerber-Shiu functions has attracted numerous research efforts, and it has now become one of the most representative research directions in actuarial science and quantitative finance. The main philosophy behind the theory is to consider three important quantities once at a time, namely: (i) the time of ruin, (ii) the surplus before the time of ruin, and (iii) the deficit at ruin. Particularly, the first-step analysis was adopted in [22] to derive a defective renewal equation, from which explicit solutions could be obtained under the classical risk model with exponential claim sizes. Traditionally, Gerber-Shiu functions, being expected discounted penalty functions, are used to evaluate the overall financial performance of an insurance company before going bankrupt. For a systematic study on Gerber-Shiu risk theory, one can refer to [3, 29, 49]. More precisely, let $\{R_t\}_{t\geq 0}$ be the surplus process of an insurance company, and $\tau$ be the random time of ruin, then the Gerber-Shiu function, denoted by $\varphi$, is defined by:

$$\varphi(u) := \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{[0,\infty)}(\tau)|R_0 = u], \tag{1.1}$$

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where $R_{\tau-}$ is the surplus just before $\tau$, $|R_\tau|$ is the deficit at the time of ruin and $\kappa(x,y)$ represents a non-negative penalty when the company bankrupts. Here, $\mathbb{1}$ denotes an indicator function and $\delta$ is a given positive constant representing the interest rate incurred. The most representative and straightforward use of Gerber-Shiu functions is the ruin probability; indeed, setting $\kappa(x,y) \equiv 1$ and $\delta = 0$, the Gerber-Shiu function becomes $\phi(u) = \mathbb{E}[\mathbb{1}_{[0,\infty)}(\tau)|R_0 = u] = \mathbb{P}(\tau < \infty|R_0 = u)$, which is exactly the ruin probability of an insurance company with an initial surplus $u$, which has been widely studied, see [3, 7] and the references therein. Alternatively, by setting $\delta > 0$ and $\kappa(x,y) = 1$, the Gerber-Shiu function can also be treated as the Laplace transform of the time of ruin $\tau$. Generally, $\delta$ is interpreted as a discount rate, and $\kappa(R_{\tau-}, |R_\tau|)$ as the penalty of the bankruptcy, which arrives at the natural application of $\varphi(u)$, the expected discounted penalty function, see [4] for an application in optimal dividend problems. Apart from the natural applications in actuarial science, if we interpret $\kappa(x,y)$ as a payoff function, the Gerber-Shiu function can be also connected to the pricing of options, see [21]. Further applications in finance have been found in the literature, for instance, optimal capital structure problems are considered in [10], and [24] studied pricing credit default swaps via the Gerber-Shiu theory.

Over the infinite-time horizon, researchers started with finding explicit solutions for the Gerber-Shiu functions under various settings. The works [37, 38] expressed the solution of the defective renewal equation derived in [22] in terms of the tail distribution of compound geometric random variables. An explicit expression can still be obtained in the Sparre Andersen risk model or a perturbed one, for example, see [23], [34], [30] and [33]. An alternative approach of deriving the explicit solution is to first transform the integral equation to a boundary value problem, and then to utilize symbolic techniques to solve for the integro-differential equation, for instance, see [1, 2], [41] and [42]. Moreover, [20] extended the theory to general Lévy subordinators. Here, the explicit solution often refers to an infinite series of convolutional products (see (A.1) for a representative example), however the high order convolutional product terms are very hard to compute directly, let alone analytically but also numerically. To this end, more recent efforts have been made for developing an efficient numerical evaluation of the Gerber-Shiu functions over the infinite-time horizon, [40] considered the approximation problem under the classical risk model via a functional approach; [47] proposed a nonparametric estimator of the Gerber-Shiu functions under a perturbed compound Poisson risk model; [48] and [52] proposed approximations by a Fourier-Sinc series and a Laguerre series expansion, respectively. As for the general Lévy risk model, [8] used the Fourier-cosine method, as first developed by [16], to obtain an efficient approximation.

On the other hand, in most practical considerations in finance, the planning time horizon is finite, therefore finite-time Gerber-Shiu functions defined by (notations are the same as those in (1.1))

$$\varphi(u,T) \coloneqq \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{[0,T]}(\tau)|R_0 = u], \tag{1.2}$$

should be more relevant in real world applications. However, the scope of research on finite-time Gerber-Shiu functions is still limited; particularly, numerical studies of the effective numerical schemes are only available in a few special classes of risk models with certain forms of the corresponding penalty function $\kappa$. For instance, [28] gave an implementable numerical scheme for a family of meromorphic processes. In [19], the authors demonstrated numerical examples for three carefully chosen penalty functions under the classical compound Poisson model. [24] used a double inverse

Fourier transform for the computation of the finite-time Gerber-Shiu functions leading to the pricing of credit default swaps, in which the penalty function relies only on the deficit at ruin (i.e., $\kappa(R_{\tau-}, |R_\tau|)$ in (1.2) reduces to $\kappa(|R_\tau|)$ ) such that the double Laplace transform can be explicitly obtained. Yet, on finding explicit expressions for the finite-time Gerber-Shiu functions, systematic results remain rare in the existing literature. One exception is the recent work of [35] under the classical compound Poisson model by solving the corresponding integro-differential equation, they obtained an integral solution for the finite-time Gerber-Shiu functions in terms of the infinite-time Gerber-Shiu functions with zero initial surplus as integrands; later, in [36], they further extended the work to a perturbed compound Poisson model. Nevertheless, their obtained expressions may be too complicated for implementable numerical computations since these contain terms of either finite-time (in [35]) or derivatives of infinite-time Gerber-Shiu functions (in [36]), both of which mostly possess no closed forms, and so they require extra numerical effort (even unstable due to the presence of the derivatives). It is worth mentioning that [6] and [32] studied a similar mathematical function but with the stopping time $\tau$ being replaced by a deterministic time $T$ in pricing barrier options, and they also investigated an efficient computation of a special case of (1.2) with $\kappa(x, y) = 1$ using the Wiener-Hopf factorization for the pricing of credit default swaps.

In this article, we discuss the numerical scheme against a Lévy subordinator for modelling the claim process, which certainly includes a Compound Poisson Process, a Gamma Process, and a Generalized Stable Process as special cases; particularly, there is no effective numerical approach on calibrating one against a Generalized Stable Process. To this end, we introduce the $T$-deferred Gerber-Shiu functions* defined as

$$\overline{\varphi}(u, T) := \varphi(u) - \varphi(u, T) = \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = u]. \tag{1.3}$$

We aim to relate these $T$-deferred Gerber-Shiu functions to the infinite-time Gerber-Shiu functions by conditioning on the random surplus level $U'$ at time $T$. By defining a new risk process starting from this initial surplus $U'$ and considering the conditional expectation given $U'$, we can obtain an integral expression for these $T$-deferred Gerber-Shiu functions in terms of the infinite-time Gerber-Shiu functions and the conditional probability density of $U'$ to be determined. To figure out the conditional probability density, we need to develop a continuous analogue of the Ballot Theorem (Generalized Ballot Theorem in Section SM1 of the Supplementary Materials) for the Lévy subordinator; its proof for the classical risk models had been given in Lemma 3.1 of [31]. Two different formulae for computing the finite-time ruin probabilities were obtained via two approaches in [31], they are respectively the Seal-type formula by the standard approach, and the PL-type formula obtained using pseudo-probability densities. In the present work, we avoid the pseudo-probability density method in order to ensure the numerical approximation is still valid for a very large amount of the initial surplus $u$; see also the work of [44] for the numerical instability of the PL-type formula even for a moderate size of $u$. To numerically solve for the infinite-time Gerber-Shiu functions, an efficient approach has been proposed in [8] which is based on the Fourier-cosine method; now, as an extension, we extend this well-received Fourier-cosine method to effectively compute the finite-time Gerber-Shiu functions numerically. As first introduced in [16], the Fourier-cosine method was to deal with European type options with a numerical scheme of linear complexity. With an indeterminate integrand function $f$ such that only its Fourier transform is

*See [35] and [36] for the definition.

known, the Fourier-cosine method provides an effective numerical method for evaluating $\int_\Gamma f(x)\mathrm{d}x$. Comparing it with the usual approach that first calculates the inverse Fourier transform, either analytically or numerically, and then substitutes this result back to the integral, the novel idea of the Fourier-cosine method is to directly incorporate the Fourier-cosine expansion of $f$ under the integration and to derive an approximation via Fubini's theorem, and hence avoids the complicated direct inverse Fourier transform. Under this Fourier-cosine scheme, up to a predetermined tolerance level, we show that the computational complexity is linear in the number of terms to be calculated, which is much faster than the traditional Monte Carlo method (when the Monte Carlo simulation can still be valid).

Furthermore, it is demonstrated in [17], [18], [51], [43] that this Fourier-cosine method is effective when pricing barrier options, Bermudan options, Asian options as well as other financial derivatives. In this present work, the efficiency of the Fourier-cosine method will be demonstrated again on computing the finite-time Gerber-Shiu functions, which is one of the pillars in the context of insurance and actuarial science.

The rest of this article is organized as follows. We first give a summary of our main formulae in Subsection 1.2, including the integral expressions and the approximations, but postpone the model setting in Subsection 1.3. Due to the fundamental difference in the analyses for the cases $u = 0$ and $u > 0$, we shall discuss them one by one. The simpler case $u = 0$ is discussed in Section 2. We construct an approximation in Subsection 2.2 and provide the corresponding error analysis in Section SM3 of the Supplementary Materials; in Subsection 2.3, several numerical examples are conducted to show the effectiveness of the Fourier-cosine method. Section 3 introduces the Fourier-cosine numerical scheme when the initial-surplus is positive, and also provides an effective approximation in Subsection 3.1; Subsection 3.2 gives more numerical illustrations in this new setting, based on which we can see the efficiency of the Fourier-cosine method. All of the proofs are given in the supplementary materials.

### 1.2. Main formulae.

We here first summarize the useful integral expressions for the finite-time Gerber-Shiu functions and the corresponding approximation formulae as follows:

(i) **Initial surplus** $u = 0$: the integral expression for the finite-time Gerber-Shiu function is given by (also see (2.6))

$$\varphi(0,T) = h_1(0) - e^{-\delta T}\left[\varphi(T)\mathbb{P}(L_T = 0) + [g_T * \varphi](T)\right], \tag{1.4}$$

and the corresponding approximation formula with a linear complexity is given by (also see (2.19))

$$\varphi(0,T) = h_1(0)\left(1 - e^{-\delta T}\right) - e^{-\delta T}{\sum_{k=0}^{K}}'\left[\mathbb{P}(L_T = 0)F_k^{(1)} - \frac{h_1(0)}{T}F_k^{(2)} + F_k^{(3)}\right]\chi_k(0,T) + \eta, \tag{1.5}$$

where the notations involved, e.g. the error term $\eta$, can be found in formula (2.19) in Section 2;

(ii) **Initial surplus** $u > 0$: the integral expression for the finite-time Gerber-Shiu function is given by (also see (3.5))

$$\begin{aligned}\varphi(u,T) = \varphi(u) - e^{-\delta T}\Bigg[&\mathbb{P}\left(L_T = 0\right)\varphi(u+T) + [f_T * \varphi](u+T)\\ &- \int_0^T f_{T-z}(u+T-z)\Big(\mathbb{P}(L_z=0)\varphi(z) + [g_z*\varphi](z)\Big)\mathrm{d}z\Bigg],\end{aligned} \tag{1.6}$$

and the corresponding approximation formula with a linear complexity is given by (also see (3.9))

$$\begin{aligned}\varphi(u,T) = h_1(0) + {\sum_{k=0}^{K}}{}' F_k^{(1)}\chi_k(0,u) - e^{-\delta T}\Bigg\{&\mathbb{P}\left(L_T=0\right)\left[h_1(0) + {\sum_{k=0}^{K}}{}' F_k^{(1)}\chi_k(0,u+T)\right]\\ + {\sum_{k=0}^{K}}{}'\left[h_1(0)F_k^{(4)}(T) + F_k^{(5)}(T)\right]\chi_k(0,u+T) - &\int_0^T \frac{\left[\sum_{k=0}^{K}{}' F_k^{(6)}(T-z)\cos\frac{k\pi(u+T-z)}{a}\right]}{(u+T-z)^{n_0}}\\ \cdot \Bigg[h_1(0) + {\sum_{k=0}^{K}}{}'\Big(\mathbb{P}(L_z=0)F_k^{(1)} - h_1(0)F_k^{(2)}(z) + F_k^{(3)}(z)\Big)&\chi_k(0,z)\Bigg]\mathrm{d}z\Bigg\} + \varepsilon_3',\end{aligned} \tag{1.7}$$

where the notations involved, e.g. the error term $\varepsilon_3'$, can be found in Theorem 3.3.

**1.3. Model setting.** We now lay down the general model setting and introduce some useful notations. Let $\{R_t\}_{t\geq 0}$ be the surplus process of an insurance company defined by

$$R_t := u + t - L_t, \tag{1.8}$$

where $u \geq 0$ is the initial surplus, the claim size process $\{L_t\}_{t\geq 0}$ is modelled by a Lévy subordinator which consists of only positive jumps with $L_0 = 0$ and the mean of $L_t$ is finite, which is increasing in $t$, for all $t \geq 0$; see for an introduction to such a process in [13], [39], [46], [29] and the references therein. The premium rate is set to be 1 per unit time for simplicity, or we can adjust the time parameter to achieve this; to add a point, no matter how we accelerate the process by whatever constant multiple, $L_t$ still remains a Lévy Process, so for any constant premium rate, we only need to study the case when the premium rate is 1. The characteristic function of $L_t$ is given by

$$\mathbb{E}[\exp(i\omega L_t)] = \exp\left(t\int_{(0,\infty)}(e^{i\omega x}-1)\nu(\mathrm{d}x)\right) =: \exp(t\Lambda(\omega)), \tag{1.9}$$

where the Lévy measure $\nu$ is a Borel measure on $(0,\infty)$ with $\int_0^\infty(|x|^2\wedge 1)\nu(\mathrm{d}x) < \infty$. In the present work, we further assume the safety loading condition $\mu_\nu := \int_0^\infty x\nu(\mathrm{d}x) < 1$ (also see [3] and [25]) to avoid almost certain ruin. For each $t > 0$, the density function of $L_t$ is denoted by $f_t(x)$ for all $x \in (0,\infty)$ and we avoid defining $f_t(0)$, which could take infinity sometimes, for instance, it is the case of the "density"(actually a Dirac delta) of a compound Poisson distribution at 0. We also denote the survival function of $L_t$ by $S_t(x) = \int_x^\infty f_t(y)\,\mathrm{d}y$ for $x \in [0,\infty)$ and thus $S_t(0) = 1 - \mathbb{P}(L_t = 0)$.

The time at ruin is defined by $\tau(u) := \inf_{t\geq 0}\{t : R_t < 0\}$. By the zero-one law, note that $\tau(0) \neq 0$ almost surely. Hence the case $u = 0$ is a non-trivial one, which will be devoted for further discussion in Section 2.

Throughout this paper, we shall denote the Fourier transform of an arbitrary function $h : [0,\infty) \to \mathbb{R}$ by $\hat{h}(s) := \int_0^\infty h(x)e^{isx}\mathrm{d}x$.

**2. With zero initial surplus.** To start with, we first consider the Lévy Process with $u = 0$. As mentioned in the introduction, we try to relate the $T$-deferred Gerber-Shiu function with the infinite-time Gerber-Shiu function as follows, by recalling (1.2),

$$\varphi(u,T) = \varphi(u) - \overline{\varphi}(u,T). \tag{2.1}$$

We here study how to compute the $T$-deferred Gerber-Shiu function, and then substitute it back to (2.1) to get the approximation of the finite-time Gerber-Shiu function. Conditioning on the values of $L_T$ and by the law of total expectation, we have,

$$\begin{aligned}\overline{\varphi}(0,T) &= \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-},|R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = 0]\\ &= \mathbb{E}[\mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-},|R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = 0, L_T, \tau]]. \end{aligned}\tag{2.2}$$

To calculate the corresponding inner conditional expectation in (2.2), we can simply shift the time parameter to commence at 0. Define $\tilde{R}_t := R_{t+T}$, $\tilde{\tau} := \tau - T$. Clearly, when $\tau \leq T$, we have $\mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-},|R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = 0, L_T, \tau \leq T] = 0$. Since $L_T > T$ implies $\tau \leq T$, so we only have to consider the remaining possibility of $L_T = x \in [0,T]$, then $\tilde{R}_0 = R_T = 0 + T - L_T = T - x$ and we have

$$\begin{aligned}&\mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-},|R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = 0, L_T = x, \tau > T]\\ &= \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-},|R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_T = T - x, \tau > T]\\ &= \mathbb{E}[e^{-\delta\tau}\kappa(\tilde{R}_{(\tau-T)^-},|\tilde{R}_{\tau-T}|)\mathbb{1}_{[0,\infty)}(\tau - T)|\tilde{R}_0 = T - x, \tau - T > 0]\\ &= \mathbb{E}[e^{-\delta(\tilde{\tau}+T)}\kappa(\tilde{R}_{\tilde{\tau}-},|\tilde{R}_{\tilde{\tau}}|)\mathbb{1}_{[0,\infty)}(\tilde{\tau})|\tilde{R}_0 = T - x, \tilde{\tau} > 0]\\ &= e^{-\delta T}\mathbb{E}[e^{-\delta\tilde{\tau}}\kappa(R_{\tilde{\tau}-},|R_{\tilde{\tau}}|)\mathbb{1}_{[0,\infty)}(\tilde{\tau})|\tilde{R}_0 = T - x]\\ &= e^{-\delta T}\varphi(T - x). \end{aligned}\tag{2.3}$$

Substitute this result into equation (2.2), we have

$$\begin{aligned}\overline{\varphi}(0,T) =&\mathbb{E}[\mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-},|R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = 0, L_T, \tau]]\\ =&e^{-\delta T}\varphi(T)\mathbb{P}(L_T = 0) + e^{-\delta T}\int_0^T \varphi(T - x)\mathbb{P}(L_T \in (x + \mathrm{d}x), \tau > T). \end{aligned}\tag{2.4}$$

Define the probability density $g_T(x)$ as

$$g_T(x)\mathrm{d}x := \mathbb{P}(L_T \in (x, x + \mathrm{d}x), \tau > T), \quad 0 < x < T, \tag{2.5}$$

and hence

$$\overline{\varphi}(0,T) = e^{-\delta T}\left[\varphi(T)\mathbb{P}(L_T = 0) + \int_0^T \varphi(T - x)g_T(x)\mathrm{d}x\right]. \tag{2.6}$$

By a continuous analogue of Ballot Theorem (also see the Generalized Ballot Theorem and its proof in Section SM1), we have

$$g_T(x) = \frac{T - x}{T}f_T(x), \quad 0 < x < T.$$

For the sake of computation of the Fourier transform of $g_T$, we propose to extend the defective domain $(0,T)$ of the density $g_T$ to the whole positive real line $(0,\infty)$, yet still denote the extended function by $g_T$:

$$g_T(x) = \frac{T-x}{T} f_T(x), \quad x > 0, \tag{2.7}$$

which is well-defined since $f_T(x)$ is defined for all $x > 0$.

The first term in the bracket of (2.6) involving the infinite-time Gerber-Shiu function can be easily calculated by various methods, for instance, those developed by [47], [8] and [48]. We here choose the method developed by [8] and represent the infinite-time Gerber-Shiu function by

$$\varphi(T) = h_1(0) + \int_0^T V(x)\mathrm{d}x, \tag{2.8}$$

where the definitions of the functions $h_1$ and $V$ together with further properties are included in Appendix A, in addition, we assume that $V \in \mathcal{L}^1(\mathbb{R}^+) \cap \mathcal{L}^2(\mathbb{R}^+)$ in the rest of this paper as we also adopted in [8] before.

For the second term of (2.6), by a simple calculation (see the derivation of (SM1.5) in Section SM1 for details), we can obtain that

$$\int_0^T \varphi(T-x) g_T(x)\mathrm{d}x = h_1(0)\left[1 - \mathbb{P}(L_T=0) - \frac{1}{T}\int_0^T S_T(x)\mathrm{d}x\right] + \int_0^T [V * g_T](x)\mathrm{d}x, \tag{2.9}$$

where $[V * g_T](x) := \int_0^x V(x-z)g_T(z)\mathrm{d}z$ is the convolution, since both the supports of $V$ and $g_T$ contain only non-negative numbers. Combining (2.8) and (2.9), we have:

$$\begin{aligned} &\varphi(T)\mathbb{P}(L_T = 0) + \int_0^T \varphi(T-x)g_T(x)\mathrm{d}x \\ &= h_1(0) + \int_0^T \left(\mathbb{P}(L_T = 0)V(x) - \frac{h_1(0)}{T} S_T(x) + [g_T * V](x)\right)\mathrm{d}x. \end{aligned} \tag{2.10}$$

Substituting (2.10) back into (2.6) and together with (2.1), we obtain a crucial formula for the finite-time Gerber-Shiu function,

$$\varphi(0,T) = h_1(0)\left(1 - e^{-\delta T}\right) - e^{-\delta T}\int_0^T \left(\mathbb{P}(L_T=0)V(x) - \frac{h_1(0)}{T} S_T(x) + [g_T * V](x)\right)\mathrm{d}x. \tag{2.11}$$

In the rest of this section, we shall propose an approximation based on (2.11) to which we apply the Fourier-cosine method. There is a common point in the three terms in the integrand in (2.11), which in turn connects with the effectiveness of the Fourier-cosine method, namely the Fourier transforms of each term can be readily obtained (to be discussed in Subsection 2.2).

**2.1. Fourier-cosine numerical scheme.** In this subsection, we sketch out the main idea behind the numerical approximation method for the integral in the following form:

$$\int_0^\gamma g(x)\mathrm{d}x = \int_0^a \mathbb{1}_{\{x \le \gamma\}} g(x)\mathrm{d}x =: J_\gamma. \tag{2.12}$$

To start with, for an arbitrary function $g$ defined on $[0,\pi]$, there is a natural symmetric extension of $g$ into an even function on $[-\pi,\pi]$ by defining $\breve{g}$ as

$$\breve{g}(x) = \begin{cases} g(x), & x \geq 0; \\ g(-x), & x < 0. \end{cases}$$

Clearly, every even function can be expressed as a Fourier-cosine series (see [16]) as follows:

$$\breve{g}(x) = {\sum_{k=0}^{\infty}}' \cos(kx)\frac{1}{\pi}\int_{-\pi}^{\pi}\breve{g}(x)\cos(kx)\mathrm{d}x = {\sum_{k=0}^{\infty}}' \cos(kx)\frac{2}{\pi}\int_{0}^{\pi} g(x)\cos(kx)\mathrm{d}x,$$

where the notation $\sum'$ denotes a summation with its first term weighted by a half. Since $g$ is a part of $\breve{g}$, the expansion is also valid for $g$ itself. For any general function with the support on $[0,a]$, its Fourier-cosine series expansion can be obtained through a simple change of variable $y := \frac{x}{a}\pi$.

Motivated by the above argument, we write

$$g(x) = {\sum_{k=0}^{\infty}}' A_k \cos\left(\frac{k\pi}{a}x\right), \quad \text{for } 0 \leq x \leq a,$$

where $a$ is a positive constant, to be determined, greater than $\gamma$, and

$$A_k = \frac{2}{a}\int_0^a g(s)\cos\left(\frac{k\pi}{a}s\right)\mathrm{d}s.$$

Since ${\sum_{k=0}^{n}}' A_k \cos\left(\frac{k\pi}{a}x\right)$ converges to ${\sum_{k=0}^{\infty}}' A_k \cos\left(\frac{k\pi}{a}x\right)$ in $\mathbb{L}^2$, by Fubini's theorem, we have

$$J_\gamma = \int_0^a \mathbb{1}_{\{x\leq\gamma\}} {\sum_{k=0}^{\infty}}' A_k \cos\left(\frac{k\pi}{a}x\right)\mathrm{d}x = {\sum_{k=0}^{\infty}}' A_k \chi_k(0,\gamma),$$

where

$$\chi_k(0,\gamma) = \int_0^\gamma \cos\left(\frac{k\pi}{a}x\right)\mathrm{d}x = \begin{cases} \frac{a}{k\pi}\sin\left(\frac{k\pi\gamma}{a}\right), & k \neq 0; \\ \gamma, & k = 0. \end{cases} \tag{2.13}$$

The Fourier-cosine method suggests that, if $a$ is large enough, it is tempting to replace the coefficient $A_k$ by the real part of the Fourier transform of $g(x)$, as shown below:

$$A_k = \frac{2}{a}\int_0^a g(s)\cos\left(\frac{k\pi}{a}s\right)\mathrm{d}s = \frac{2}{a}\Re\left\{\int_0^a g(s)e^{i\frac{k\pi}{a}s}\mathrm{d}s\right\} = F_k - \frac{2}{a}\Re\left\{\int_a^\infty g(s)e^{i\frac{k\pi}{a}s}\mathrm{d}s\right\}, \tag{2.14}$$

where $\Re\{x\}$ represents the real part of the complex number $x$, and

$$F_k = \frac{2}{a}\Re\left\{\hat{g}\left(\frac{k\pi}{a}\right)\right\}, \quad k = 0,1,2,... \tag{2.15}$$

In summary, $J_\gamma$ can be expressed as

$$J_\gamma = {\sum_{k=0}^{\infty}}' A_k\chi_k(0,\gamma) = {\sum_{k=0}^{K}}' F_k\chi_k(0,\gamma) + \eta_1 + \eta_2, \tag{2.16}$$

where the error terms $\eta_1$ and $\eta_2$ relating to $g$ are:

$$\eta_1 := -\sum_{k=0}^{K}{}' \frac{2}{a}\chi_k(0,\gamma)\Re\left\{\int_a^\infty g(s)e^{i\frac{k\pi}{a}s}\mathrm{d}s\right\}, \quad \eta_2 := \sum_{k=K+1}^{\infty} A_k\chi_k(0,\gamma). \tag{2.17}$$

Here $\eta_1$ quantifies the error arisen from replacing $A_k$ by $F_k$, which involves the Fourier transform of $g$; while $\eta_2$ quantifies the error arisen from approximating the infinite series by a truncated partial sum.

*Remark* 2.1. Note that our Fourier-cosine scheme is slightly different from the original COS method first introduced in [16]. In [16], in order to compute the expectation

$$\mathbb{E}^{\mathbb{Q}}[v(y,T)|x] = \int_{\mathbb{R}} v(y,T)f_{Y|X}(y|x)\mathrm{d}y,$$

they first chopped off the integration range $\mathbb{R}$ to $[a,b]$, and then they used the Fourier-cosine expansion to approximate the truncated integral

$$\int_a^b v(y,T)f_{Y|X}(y|x)\mathrm{d}y,$$

and this in turn introduces an additional integration truncation error

$$\int_{\mathbb{R}\setminus[a,b]} v(y,T)f_{Y|X}(y|x)\mathrm{d}y.$$

As explained in [5] and [11], this truncation error is sensitive to the choice of $a$ and $b$ and may be problematic especially when the payoff function $v(y,T)$ grows rapidly with $y$ approaching infinity and the transition probability density $f_{Y|X}(y|x)$ has a fat right-tail. In contrast, we use the Fourier-cosine scheme directly on the integral

$$J_\gamma = \int_0^\gamma g(x)\mathrm{d}x,$$

which already has a finite integration range, and hence does not involve any integration range truncation error so that our present method is much more stable with the choice of $a$ as will be seen in the demonstration of our theory and the simulations. Our replacement error $\eta_1$ (see (2.17)) related to the parameter $a$ can be well-controlled simply by choosing a suitably large enough $a$, and then for a fixed $a$, we set another large enough $K$ so as to make $\eta_2$ sufficiently small. As for the recommended numerical choices of $a$ and $K$ in our scheme, we put them in Subsection 2.3 and Subsection 3.2.

**2.2. Approximation for finite-time Gerber-Shiu functions.** Apply the Fourier-cosine method summarised in Subsection 2.1 to the integral term in equation (2.11) and define $F_k^{(1)}, F_k^{(2)}, F_k^{(3)}$, recalling that $g_T(x) = \frac{T-x}{T}f_T(x)$, as

$$F_k^{(1)} := \frac{2}{a}\Re\left\{\widehat{V}\left(\frac{k\pi}{a}\right)\right\}, F_k^{(2)} := \frac{2}{a}\Re\left\{\widehat{S_T}\left(\frac{k\pi}{a}\right)\right\},\ F_k^{(3)} := \frac{2}{a}\Re\left\{[\widehat{V * g_T}]\left(\frac{k\pi}{a}\right)\right\}, \tag{2.18}$$

for $k = 0, 1, 2, 3, \cdots$. Then we can use (2.16) to replace the integral in (2.11) and obtain the following expression (also see (1.7)):

$$\varphi(0,T) = h_1(0)\left(1-e^{-\delta T}\right) - e^{-\delta T}\sum_{k=0}^{K}{}'\left[\mathbb{P}(L_T=0)F_k^{(1)} - \frac{h_1(0)}{T}F_k^{(2)} + F_k^{(3)}\right]\chi_k(0,T) + \eta, \tag{2.19}$$

where the total error $\eta$ is given by:

$$\eta := -e^{-\delta T}\left[\mathbb{P}(L_T=0)(\eta_1^{(1)}+\eta_2^{(1)}) - \frac{h_1(0)}{T}(\eta_1^{(2)}+\eta_2^{(2)}) + (\eta_1^{(3)}+\eta_2^{(3)})\right],$$

where the error terms $\eta_1^{(i)}$ and $\eta_2^{(i)}$ are the corresponding $\eta_1$ and $\eta_2$ in (2.16) for $i=1,2,3$, namely by applying the Fourier-cosine method to the three terms in the integrand in (2.11) respectively. The details of the error analysis will be shown in Section SM3 of the Supplementary Materials. And we have the following result:

THEOREM 2.2. *The total error $\eta$ in (2.19) is bounded by:*

$$|\eta| \leq 3\max\left\{\frac{h_1(0)}{T},1\right\}\left[\int_a^\infty |V(s)|+|S_T(s)|+|[V*g_T](s)|\mathrm{d}s + \frac{C_a}{K^{\min\{r^{(1)},r^{(2)},r^{(3)}\}}}\right],$$

*provided that the functions $V, S_T, V*g_T \in \mathcal{L}^2(\mathbb{R}^+)$, and when $k>\frac{1}{2}$ and $g=V, S_T$ or $V*g_T$ so that they all fulfill the condition*

$$\int_{-\infty}^{+\infty}(\Re(\hat{g}(s)))^2(1+s^2)^k\mathrm{d}s<\infty, \tag{2.20}$$

*where the constant $C_a$ depends only on $a$, and $r^{(1)}, r^{(2)}, r^{(3)}$ are the corresponding parameters in relation to the functions $V$, $S_T$ and $V*g_T$ in Proposition SM3.4 of Section SM3.*

Hence we can set the total error $\eta$ to be arbitrarily small by taking a large enough $a$. We can further improve our error bound by assuming additional decaying structures on the Fourier transforms of the functions $V$, $S_T$ and $V*g_T$, namely the algebraic index of convergence and the monotonicity. To this end, we define the algebraic index of convergence of a generic sequence $\{A_k, k=0,1,2,\cdots\}$ as follows.

DEFINITION 2.3. *A sequence $\{A_k, k=0,1,2,\cdots\}$ has an algebraic index of convergence of $s$ if it is the greatest possible real number such that $\limsup_{k\to\infty}|A_k|k^s<\infty$.*

THEOREM 2.4. *For $T\in[\epsilon, a-\epsilon]$ for an $\epsilon>0$, suppose that the sequences $\{F_k^{(i)}\}, i=1,2,3$ satisfy that:*

1. *For any $i=1,2,3$, the sequence $\{F_k^{(i)}\}$ has an algebraic index of convergence $\beta_i$, and so that $F_k^{(i)}\to 0$ as $k\to\infty$;*
2. *There exists a large enough $N'$ such that for all $i=1,2,3$, $\Delta F_k^{(i)} := F_{k+1}^{(i)} - F_k^{(i)}$ are of the same sign for all $k\geq N'$.*

*Then the total error $\eta$ can be bounded tighter in the sense that for any $K\geq N'$,*

$$|\eta| \leq 3\max\left\{\frac{h_1(0)}{T},1\right\}\left(\int_a^\infty |V(s)|+|S_T(s)|+|[V*g_T](s)|\mathrm{d}s + \frac{C_{a,\epsilon}}{K^{1+\min\{\beta_1,\beta_2,\beta_3\}}}\right),$$

*where the constant $C_{a,\epsilon}$ depends only on $a$ and $\epsilon$.*

*Proof.* The proof for bounding the error part caused by replacement in this enhanced theorem is identical to that for Theorem 2.2; while the error part caused by truncation can be shown in a similar manner by following the arguments in the proof of Theorem 4.5 in [9]. ∎

*Remark* 2.5. The condition in Theorem 2.4 is stronger than that in Theorem 2.2 in the sense that it requires the Fourier transforms of the three functions converge to 0 at a certain rate, while we only assume their overall integrability in Theorem 2.2. However, [8] showed that the conditions in Theorem 2.4 are fulfilled for the Fourier-cosine coefficients $\{F_k^{(1)}\}$ in most common models, such as the Compound Poisson-Exponential claim process (see Example 5.1 in [8]), the Lévy-Gamma Process (see Example 5.4 in [8]), for more legitimate examples one can refer to [8]. Besides, for the coefficients $\{F_k^{(2)}\}$ and $\{F_k^{(3)}\}$, the conditions on them can be also witnessed numerically and graphically.

In (2.19), the values of $h_1(0)$ and $F_k^{(1)}$ can be obtained explicitly, see Appendix A. The Fourier transform of $S_T(x)$ can be found by:

$$\begin{aligned} \widehat{S_T}(0) &:= \int_0^\infty S_T(x)\mathrm{d}x = \mathbb{E}(L_T) = \mu_\nu T, \\ \widehat{S_T}(s) &:= \int_0^\infty S_T(x)e^{isx}\mathrm{d}x = \frac{e^{isx}}{is}S_T(x)\bigg|_0^\infty + \frac{1}{is}\int_0^\infty f_T(x)e^{isx}\mathrm{d}x \\ &= \frac{\widehat{f_T}(s) - (1-\mathbb{P}(L_T=0))}{is}, \quad s \neq 0, \end{aligned} \tag{2.21}$$

where the Fourier transform of $f_T$ can be obtained by $\widehat{f_T}(s) = \mathbb{E}[\exp(isL_T), \{L_T \neq 0\}] = \mathbb{E}[\exp(isL_T)] - \mathbb{P}(L_T = 0) = \exp(T\Lambda(s)) - \mathbb{P}(L_T = 0)$, since there is a point mass of $L_T$ at 0. The Fourier transform of $g_T$ can be found as:

$$\begin{aligned} \widehat{g_T}(s) &= \int_0^\infty \frac{T-x}{T} f_T(x)e^{isx}\mathrm{d}x = \int_0^\infty f_T(x)e^{isx}\mathrm{d}x - \frac{1}{T}\int_0^\infty x f_T(x)e^{isx}\mathrm{d}x \\ &= \widehat{f_T}(s) - \frac{1}{iT}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\left[f_T(x)e^{isx}\right]\mathrm{d}x = \widehat{f_T}(s) + \frac{i}{T}\frac{\mathrm{d}}{\mathrm{d}s}\widehat{f_T}(s), \end{aligned} \tag{2.22}$$

where the last equation follows from Leibniz's rule. Finally, the Fourier transform of $[V * g_T]$ can be derived from the convolution rule and we get $\widehat{V * g_T}(x) = \hat{V}(x)\Big[\widehat{f_T}(s) + \frac{i}{T}\frac{\mathrm{d}}{\mathrm{d}s}\widehat{f_T}(s)\Big]$, where $\hat{V}$ is given by (A.4). Hence, we can calculate all the coefficients in (2.18) precisely:

$$\begin{aligned} F_k^{(1)} &= \frac{2}{a}\Re\left\{\widehat{V}\left(\frac{k\pi}{a}\right)\right\}, \qquad k \geq 0; \\ F_k^{(2)} &= \frac{2}{a}\Re\left\{\frac{\exp(T\Lambda(\frac{k\pi}{a}))-1}{i\frac{k\pi}{a}}\right\}, \quad k \geq 1; \quad F_0^{(2)} = \frac{2\mu_\nu T}{a}; \\ F_k^{(3)} &= \frac{2}{a}\Re\left\{\hat{V}\left(\frac{k\pi}{a}\right)\left[\exp\left(T\Lambda\left(\frac{k\pi}{a}\right)\right)\left(1+i\frac{\mathrm{d}\Lambda(s)}{\mathrm{d}s}\bigg|_{s=\frac{k\pi}{a}}\right) - \mathbb{P}(L_T=0)\right]\right\}, k \geq 0. \end{aligned} \tag{2.23}$$

**2.3. Numerical illustrations.** Throughout this paper, all the computer programs for numerical illustrations were written in Python 3, and they were run on a standard Macbook Pro(3.1 GHz Intel Core i5 processor and 8 GB RAM).

In the following numerical illustrations, we shall take the numerical choice of the parameter $a$ as:

$$a = u + T + \frac{L}{\eta} + L\sqrt{c_2}, \tag{2.24}$$

where $\eta = (1-\mu_\nu)/\mu_\nu$ is the safety loading factor by recalling $c = 1$, and $c_2$ is the second cumulant of $L_T$ (i.e. the variance of the random variable $L_T$) being given by:

$$c_2 = \widehat{[x^2 f_T]}(0) - \left[\widehat{[x f_T]}(0)\right]^2 = -T\Lambda''(0),$$

which can be derived by the formula (3.7). This choice for $a$ is similar to the formulae suggested for the computational purpose in the work of [16], though our definition of $a$ is different from the computational domain in their COS method. The term $(u+T)$ stems from the condition $a > u+T$ as demanded in Claim SM4.3, Claim SM4.4, Lemma SM4.5 and Lemma SM4.6. Note that $\Lambda''(0) = -\int_{(0,\infty)} x^2\nu(x)\mathrm{d}x$ is a negative real number, thus if we set a proper positive $L$ here, $L\sqrt{c_2} > 0$, such that the condition $a > u+T$ is fulfilled. We stress that our approximation is quite robust to the choice of $L$ as can be seen in the plots against $L$ in Figures 4 and 7 to 9, but throughout this paper, for the sake of convenience, we pick $L = 7$, which is certainly not the only suitable choice of $L$.

*Example* 2.6 (Finite-time ruin probability for a Compound Poisson Process). We consider the case when $\delta = 0$ and $\kappa(x,y) \equiv 1$, in which the finite-time Gerber-Shiu function becomes the finite-time ruin probability, i.e. $\varphi(u,T) = \mathbb{P}(\tau \le T)$. For $L_T$, we assume the Poisson rate to be $\lambda$ and exponentially distributed claim sizes with the mean of $1/\mu$. The Lévy measure for the process is $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x) = \lambda\mu e^{-\mu x}\mathrm{d}x$. $\Lambda(s)$ and the Fourier transform of $V$ are given by:

$$\Lambda(s) = \lambda\left(\frac{\mu}{\mu - is} - 1\right), \quad \hat{V}(s) = \frac{\lambda}{\mu}\cdot\frac{\lambda-\mu}{(\mu-\lambda)-is}.$$

Thus, the corresponding coefficients in (2.18) can be obtained by (2.23). Here, the numerical experiment is conducted with the following parameters: $\lambda = 0.87$, $\mu = 1$, $T = 60$ and $L = 7$. This corresponds to a loading factor $\eta$ of about 15% (usually 10% − 20%, see P3 of [3] for more interpretations) as the expected claim per unit time $\mu_\nu$ is 0.87. Figure 1 illustrates the ruin probability obtained by the approximation formula (2.19). The reference horizontal line is the true value of ruin probability based on the numerical integration formula (Proposition 1.3, Chapter V, [3]). As expected, the approximated probability tends to the true value as $K$ increases. Table 1 displays the absolute errors between the approximated and true ruin probabilities, and the convergence is clear. To check the rate of convergence, we obtain the approximations with a grid of larger values of $K$, and plot the negative logarithm of the absolute difference between the approximated and true values against $\log K$. Figure 2 plots the result, from which we observe that the error decays with an exponential rate and reaches the smallest possible error limit soon (an accuracy of around $10^{-14}$ for $K > 55$), which is determined by the parameter $a$. The time required for plotting Figure 1 (including 50 points) is only 0.039s.

| $K$ | 4 | 8 | 16 | 32 | 64 | 128 |
|---|---|---|---|---|---|---|
| Error | $3.0\cdot10^{-2}$ | $2.9\cdot10^{-3}$ | $3.6\cdot10^{-5}$ | $8.2\cdot10^{-10}$ | $8.3\cdot10^{-15}$ | $8.3\cdot10^{-15}$ |

TABLE 1

*Error magnitude for the Fourier-cosine approximation of the ruin probability with parameters* $\lambda = 0.87$, $\mu = 1$, $T = 60$ *and* $L = 7$.

*Example* 2.7 (Value-at-Risk for a Gamma Process). We consider the joint distribution of the deficit at ruin and the time of ruin $F(T,p) := \mathbb{P}(|R_\tau| \le p, \tau \le T)$. Let

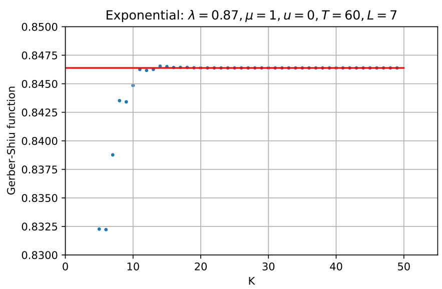


FIG. 1. *Approximation for the ruin probability, where the horizontal line (at 0.8464) is the true value.*

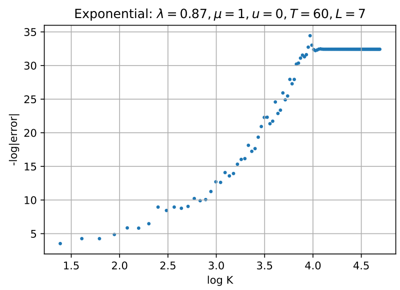


FIG. 2. *Plot of* $-\log|\text{error}|$ *against* $\log K$ *for the ruin probability approximation.*

$\delta = 0$ and the penalty function $\kappa(x,y) = \mathbb{1}_{[0,p]}(y)$, then the finite-time Gerber-Shiu function (1.2) becomes

$$\varphi(u,T) = \mathbb{E}[\mathbb{1}_{[0,p]}(y)\mathbb{1}_{[0,T]}(\tau)|R_0 = u] = \mathbb{P}(|R_\tau| \le p, \tau \le T).$$

If we further define

$$F_T(p) := \mathbb{P}(|R_\tau| \le p | \tau \le T) = F(T,p)/F(T,\infty),$$

then at the confidence level $\alpha$, $\mathrm{VaR}_\alpha$ satisfies $F_T(\mathrm{VaR}_\alpha) = \alpha$. Similar calculations can also be found in [28]. We consider a Gamma Process $L_t$ with parameters $\alpha = 0.4$ and $\beta = 0.5$, which has been used to evaluate infinite-time Gerber-Shiu functions in [8] and [53] as the underlying process for approximating ruin probabilities. Its Lévy measure is given by $\nu(\mathrm{d}x) = \left(\alpha e^{-\beta x}/x\right)\mathrm{d}x$, with $\mathbb{E}(L_t) = \alpha t/\beta$. In Figure 3, we plot the finite-time Gerber-Shiu function $\varphi(u,T) = F(T,p)$ with respect to the truncation number of terms $K$ in the left subfigure for the parameters $u = 0, p = 2, T = 24, L = 7$, and we also plot $\varphi(u,T)$ with respect to the parameter $p$ but for a fixed value of $K = 64$ in the right subfigure. We also provide a Monte Carlo simulation benchmark. For every path, since the Gamma Process contains infinitely many jumps and the jumping times are dense on any nontrivial compact time interval, which is different from a Compound Poisson Process, we partition the time interval $[0,T]$ uniformly with a mesh size of $1/2^{13}$, and simulate $L_t$ on the corresponding time grid by following the Monte Carlo procedure in [12], then use this step-wise path to evaluate the value of the corresponding finite-time Gerber-Shiu function. Note that there is always a negative bias of the Monte Carlo simulation compared with the true value since the possibility of ruin has always been underestimated due to the discretization of the path. We simulate 50,000 paths in each Monte Carlo simulation, and run the simulation for 50 times (50 data points) to calculate the mean and the standard deviation. The one standard deviation range (i.e. the range centered at the mean and with a radius of one standard deviation) is $0.7068 \pm 0.0025$ and is shown in Figure 3. We can see the approximation falls into the one standard deviation range very fast as $K$ increases. The time required to run the Monte Carlo simulation for 50 times (50 data points) is 85 minutes, while it only needs 9.4s to generate 500 points (the number of points in the second plot of Figure 3) by our Fourier-cosine method with $K = 64$, which is significantly faster and more effective than the Monte Carlo simulation; and in the insurance industry, most practitioners are still predominantly using the plain Monte Carlo simulation, indeed!

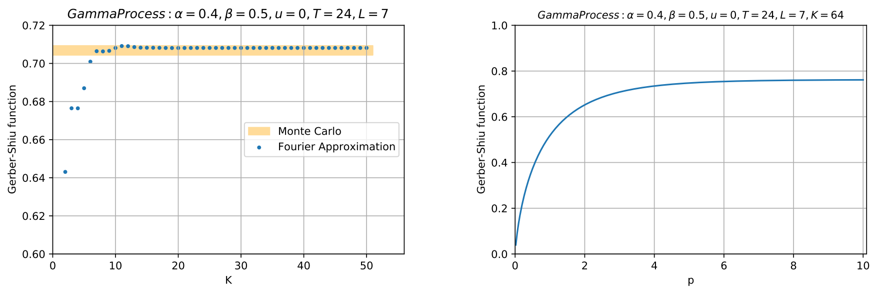


FIG. 3. *Gerber-Shiu function based on the Fourier-cosine approximation for the Generalized Stable Process.*

**3. With a positive initial surplus.** We next extend the Fourier-cosine method to the case $u > 0$. Recall that for Lévy subordinators with $\nu(\mathbb{R}^+) = \infty$, there are almost surely countably many jumping times which are dense in $[0, \infty)$, while for $0 < \nu(\mathbb{R}^+) < \infty$, there are infinitely countably many isolated jumping times which can be counted in an increasing order (but only finitely many in any finite interval), and the interarrival time has an exponential distribution with mean $1/\nu(\mathbb{R}^+)$. For more details about Lévy Processes, readers can refer to Theorem 21.3 of [45].

In the following subsection, we shall introduce the approximation for $T$-deferred Gerber-Shiu functions. To do so, we need to first deal with the probability $\mathbb{P}(L_T \in (x, x + \mathrm{d}x), \tau > T)$, which appears in our proposed expression (3.3) of the $T$-deferred Gerber-Shiu functions. Similar to the case $u = 0$, we use the idea of Generalized Ballot Theorem to determine that probability, see Section SM2 for details.

**3.1. Approximation for *T*-deferred Gerber-Shiu functions.** We now discuss the approximation for the $T$-deferred Gerber-Shiu function $\overline{\varphi}(u, T)$ for a given time $T > 0$ with an initial surplus $u > 0$. Similar to the case of $u = 0$, by the tower property, we have

$$\begin{aligned}\overline{\varphi}(u, T) &= \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = u]\\ &= \mathbb{E}[\mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = u, L_T, \tau > T]|R_0 = u].\end{aligned} \tag{3.1}$$

As in Section 2, we define $\tilde{R}_t := R_{t+T}$, $\tilde{\tau} := \tau - T$. Since $L_T > u + T$ implies $\tau \leq T$, so we only need to consider the possibility of $L_T = x \in [0, u + T]$, then $\tilde{R}_0 = R_T = u + T - L_T = u + T - x \geq 0$ and by the derivation for (2.3), the inner conditional expectation in (3.1) becomes

$$\begin{aligned}&\mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{(T,\infty)}(\tau)|R_0 = u, L_T = x, \tau > T]\\ =&\mathbb{E}[e^{-\delta(\tilde{\tau}+T)}\kappa(\tilde{R}_{\tilde{\tau}-}, |\tilde{R}_{\tilde{\tau}}|)\mathbb{1}_{[0,\infty)}(\tilde{\tau})|\tilde{R}_0 = u + T - x, \tilde{\tau} > 0]\\ =&e^{-\delta T}\varphi(u + T - x).\end{aligned} \tag{3.2}$$

Substitute (3.2) into Equation (3.1), and rewrite it in integral form, we have

$$\overline{\varphi}(u, T) = e^{-\delta T}\Bigg[\varphi(u + T)\mathbb{P}(L_T = 0, \tau > T) + \int_0^{u+T}\varphi(u + T - x)\mathbb{P}(L_T \in (x, x + \mathrm{d}x), \tau > T)\Bigg]. \tag{3.3}$$

For any $x \in (0, u)$, there will be no bankruptcy, and so $\mathbb{P}(L_T \in (x, x + \mathrm{d}x), \tau > T) = \mathbb{P}(L_T \in (x, x + \mathrm{d}x)) = f_T(x)$; while for any $x \in (u, u + T)$, we define and derive in Theorem SM2.1 that $h_T(x, u)\mathrm{d}x := \mathbb{P}(L_T \in (x, x + \mathrm{d}x), \tau > T) = f_T(x) - \mathbb{P}(L_{u+T-x} = 0) f_{x-u}(x) - \int_u^x g_{u+T-z}(x - z) f_{z-u}(z)\mathrm{d}z$. Hence, we get that

$$\overline{\varphi}(u,T) = e^{-\delta T}\Bigg[\mathbb{P}(L_T = 0)\varphi(u + T) + \int_0^u f_T(x)\varphi(u + T - x)\mathrm{d}x + \int_u^{u+T} h_T(x, u)\varphi(u + T - x)\mathrm{d}x\Bigg]; \tag{3.4}$$

by a simple calculation (see the derivation of (SM2.5) in Section SM2 for details), we can obtain that

$$\overline{\varphi}(u,T) = e^{-\delta T}\Bigg[\mathbb{P}(L_T = 0)\varphi(u + T) + [f_T * \varphi](u + T) - \int_0^T f_{T-z}(u + T - z)\Big(\mathbb{P}(L_z = 0)\varphi(z) + [g_z * \varphi](z)\Big)\mathrm{d}z\Bigg]. \tag{3.5}$$

Since all the Fourier transforms of the terms in (3.5) are already known, we can apply the Fourier-cosine method on (3.5) directly to obtain the approximation for $\overline{\varphi}(u, T)$. However, in order to enhance the convergence rate of the approximation error, we substitute (2.8) and (2.10) into (3.5), to finally achieve:

$$\overline{\varphi}(u,T) = e^{-\delta T}\Bigg\{\mathbb{P}(L_T = 0)\left[h_1(0) + \int_0^{u+T} V\mathrm{d}x\right] + \int_0^{u+T} [h_1(0)f_T(x) + [f_T * V](x)]\,\mathrm{d}x - \int_0^T f_{T-z}(u+T-z)\cdot\left[h_1(0) + \int_0^z\Big(\mathbb{P}(L_z = 0)V(x) - \frac{h_1(0)}{z}S_z(x) + [g_z * V](x)\Big)\mathrm{d}x\right]\mathrm{d}z\Bigg\}. \tag{3.6}$$

To this end, we shall apply the Fourier-cosine technique to calibrate term by term of (3.6) in order to obtain the approximation formula for $\overline{\varphi}(u, T)$. However, the error for approximating the term $f_t(u + t)$ directly by the Fourier-cosine method may be divergent when $t = T - z \to 0$. However, if it is valid that for any $t \in (0, T]$, the real part of the Fourier coefficient $\widehat{f}_t\left(\frac{k\pi}{a}\right)$ is monotone with respect to $k \geq K$, for a sufficiently large $K$, this error converges to zero and the proof can be done in parallel with the argument leading to Lemma SM4.5. Nevertheless this condition is hard to verify since it demands that $\widehat{f}_t\left(\frac{k\pi}{a}\right)$ is monotone with respect to $k$ for all $t \in (0, T]$ and it does not hold under some representative examples such as a Compound Poisson Process with beta distributed claims. So instead of tackling $f_t(u+t)$ directly, we can first approximate $(u + t)^n \cdot f_t(u + t)$ by the Fourier-cosine method, and then divide it by $(u + t)^n$ in order to get the approximation of the function $f_t(u + t)$. Note that the Fourier transform of $x^n f_t(x)$ is simply:

$$\widehat{[x^n f_t]}(s) = (-i)^n \cdot \frac{\mathrm{d}^n \widehat{f}_t(s)}{\mathrm{d}s^n} = (-i)^n \cdot \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(e^{t\Lambda(s)} - \mathbb{P}(L_t = 0)\right), \tag{3.7}$$

which is readily accessible and here $\Lambda(s) := \int_{(0,\infty)}(e^{isx} - 1)\zeta(x)\mathrm{d}x$ (also refer to (1.9)), where $\zeta$ denotes the density of the Lévy measure, i.e., $\nu(\mathrm{d}x) = \zeta(x)\mathrm{d}x$. In order to warrant the error $\eta^{(6)}(t)$ to decay uniformly (in $t$) and monotonically to zero when we attempt to approximate $x^n \cdot f_t(x)$ by the Fourier-cosine method for $t \in (0,T]$, we impose the following assumptions in the rest of Section 3:

**Assumption A.** There exists a positive integer $n_0$ such that

i) the magnitude of the first and $n_0$-th order derivatives of $\Lambda(s)$ possesses the same order of $|\Lambda^{(1)}(s)|^{n_0} = O\left(s^{-(1+\theta)}\right)$ and $|\Lambda^{(n_0)}(s)| = O\left(s^{-(1+\theta)}\right)$ for some $\theta > 0$;

ii) furthermore for $n_0 \geq 2$ and any integer $m = 2, \cdots, n_0$, $x^m\zeta(x)$ and $x^m\zeta'(x) \in \mathcal{L}^1(\mathbb{R}^+)$; otherwise if $n_0 = 1$, no additional condition is required.

*Remark* 3.1. The forms of $\Lambda(s)$ are simple for most common models, for instance, the Lévy measure of a Gamma Process with parameters $\alpha > 0$ and $\beta > 0$ is $\nu(\mathrm{d}x) = \left(\alpha e^{-\beta x}/x\right)\mathrm{d}x$, thus $\Lambda(s) = -\alpha\log(1 - \frac{is}{\beta})$, then we can simply choose $n_0 = 2$ so that Assumption A holds. More examples will be given in the following numerical illustrations in Subsection 3.2.

*Remark* 3.2. There is a supplement on choosing a suitable $n_0$ in Assumption A. To determine the order of $\Lambda^{(m)}(s) = i^m \int_0^\infty x^m\zeta(x)e^{isx}\mathrm{d}x$, that is to investigate the asymptotic behavior of the Fourier integral $\int_0^\infty h(x)e^{isx}\mathrm{d}x$ as $s \to \infty$, which is related to the well known Erdélyi lemma (see [14, 15]) and is entirely determined by the behavior of $h(x)$ in the neighborhood of the end points 0 and $\infty$ of the integration domain and the points at which $h(x)$ or some of its derivatives are discontinuous. If the Lévy density $\zeta(x) \in C^\infty(\mathbb{R}^+)$ with an exponential decay tail and has only one singularity at the origin of the type $x^{-\iota}$, i.e. $\zeta(x) \sim x^{-\iota}$ as $x \to 0^+$, where $\iota < 2$ due to the safety loading condition $\int_0^\infty x\zeta(x)\mathrm{d}x < \infty$, then by the Theorem 2 in [50] we have $|\Lambda^{(m)}(s)| = O\left(s^{-(m+1-\iota)}\right)$ for all integers $m$, and particularly, $|\Lambda^{(1)}(s)| = O\left(s^{-(2-\iota)}\right)$. For the finite Lévy measure case, we have $\int_0^\infty \zeta(x)\mathrm{d}x < \infty$, thus $\iota < 1$ and $n_0 = 1$, and they fulfill Assumption A, more illustrations can be found in our numerical examples in Subsection 3.2.1; while for the infinite Lévy measure case, $\iota \in (1,2)$, we can choose $n_0$ to be the smallest integer which is strictly larger than $\frac{1}{2-\iota}$, more illustrations can be found in our numerical examples in Subsection 3.2.2.

**Assumption B.** $f_t(x)$ is jointly continuous for $(x,t) \in \mathbb{R}^+ \times (0,T]$, and there is an $x_0 > 0$, such that $f_t'(x) < 0$ in $(x,t) \in (x_0,\infty) \times (0,T]$.

**Assumption C.** The algebraic index of convergence of $\hat{V}$, $\beta_V > 0$.

To write down the approximation formula for $\overline{\varphi}(u,T)$, we first introduce some notations. Define, for $k = 0, 1, 2, \ldots$, and $t > 0$,

(3.8)

$$F_k^{(1)} := \frac{2}{a}\Re\left\{\widehat{V}\left(\frac{k\pi}{a}\right)\right\}, F_k^{(2)}(t) := \frac{2}{a}\Re\left\{\frac{1}{t}\widehat{S}_t\left(\frac{k\pi}{a}\right)\right\}, F_k^{(3)}(t) := \frac{2}{a}\Re\left\{\widehat{[g_t * V]}\left(\frac{k\pi}{a}\right)\right\},$$
$$F_k^{(4)}(t) := \frac{2}{a}\Re\left\{\widehat{f}_t\left(\frac{k\pi}{a}\right)\right\}, F_k^{(5)}(t) := \frac{2}{a}\Re\left\{\widehat{[f_t * V]}\left(\frac{k\pi}{a}\right)\right\}, F_k^{(6)}(t) := \frac{2}{a}\Re\left\{\widehat{[x^{n_0} f_t]}\left(\frac{k\pi}{a}\right)\right\}.$$

Hence, by applying the Fourier-cosine approximation on each term of $\overline{\varphi}(u,T)$ in (3.6), we can then conclude with the following theorem and its complete proof is put in Section SM4 of the Supplementary Materials.

THEOREM 3.3. *Assume that the insurer has an initial surplus $u > 0$, and suppose that Assumptions A, B and C also hold. For a given $T > 0$ and any $\varepsilon > 0$, there exists a $K \in \mathbb{Z}^+$ and $a > 0$ such that, the $T$-deferred Gerber-Shiu function (also see*

(1.7)*)*

$$\overline{\varphi}(u,T)=e^{-\delta T}\Bigg\{\mathbb{P}\left(L_T=0\right)\left[h_1(0)+\sum_{k=0}^{K}{}'F_k^{(1)}\chi_k(0,u+T)\right]+\sum_{k=0}^{K}{}'\Big[h_1(0)F_k^{(4)}(T)$$

$$+F_k^{(5)}(T)\Big]\cdot\chi_k(0,u+T)-\int_0^T\frac{\left[\sum_{k=0}^{K}{}'F_k^{(6)}(T-z)\cos\frac{k\pi(u+T-z)}{a}\right]}{(u+T-z)^{n_0}}$$

$$\cdot\left[h_1(0)+\sum_{k=0}^{K}{}'\Big(\mathbb{P}\left(L_z=0\right)F_k^{(1)}-h_1(0)F_k^{(2)}(z)+F_k^{(3)}(z)\Big)\chi_k(0,z)\right]\mathrm{d}z\Bigg\}+\varepsilon_3, \tag{3.9}$$

*where the explicit formula for the approximation error* $\varepsilon_3$ *is given in* (SM4.11) *and* $|\varepsilon_3|<\varepsilon$.

This theorem presents only the approximation formula for the $T$-deferred Gerber-Shiu function $\overline{\varphi}(u,T)$. To get the final approximation for the finite-time Gerber-Shiu function $\varphi(u,T)$, we simply apply the formula $\varphi(u,T)=\varphi(u)-\overline{\varphi}(u,T)$ in (1.3) to get the desired approximation (1.7), and the approximation error in (1.7) is $\varepsilon_3' := \eta^{(1)}-\varepsilon_3$, where the definition and the bound of $\eta^{(1)}$ can be found in (SM3.1). And for the last integral term in the Equation (3.9), since it can be shown that the Fourier-cosine approximation converges uniformly in $z$ to the original integrand on the integration domain, where the proof is put in Section SM4, we can utilize a suitable numerical integration method to approximate it. In this paper, we choose Simpson's rule with a suitable partition size (say the number of partition points on $[0,T]$ is $N=200$) to calculate the corresponding integral numerically.

**3.2. Numerical illustrations.** In this subsection, we provide various numerical illustrations for the approximation of finite-time Gerber-Shiu functions under different processes with the initial surplus $u>0$. For the choice of $a$ when applying the formula $\varphi(u,T)=\varphi(u)-\overline{\varphi}(u,T)$ in (1.3), since the $T$-deferred Gerber-Shiu function $\overline{\varphi}(u,T)$ depends on $T$, but the infinite-time Gerber-Shiu function $\varphi(u)$ does not involve the parameter $T$, it is more reasonable to choose different $a$ for the two terms. More precisely, for the $T$-deferred Gerber-Shiu function $\overline{\varphi}(u,T)$, we still choose $a$ by (2.24), while for the infinite-time Gerber-Shiu function $\varphi(u)$, we suggest to use $a'$ as

$$a'=u+\frac{L}{\eta},$$

yet with the same $L=7$ as proposed before. As can be seen in our simulation study on the robustness against $L$ (see Figures 4 and 7 to 9), the approximations are stable for a sufficiently long range of values of $L$ as expected. Nevertheless, an exaggeratedly large value of $L$ may still cause a large approximation error; If $L$ is too small, the corresponding replacement error $\eta_1$ (see (2.17)) may be large; while if $L$ is too large, we need to increase $K$ accordingly to make the truncation error $\eta_2$ small enough, which would consume more time in computation.

We shall split our illustrations for each type of Lévy Processes, i.e. the Lévy Processes with a finite Lévy measure and those with an infinite Lévy measure.

**3.2.1. Finite Lévy measure case.** In this subsection, we shall first consider the case when $\nu(\mathbb{R}^+)<\infty$. Under this assumption, the Lévy Process $L_t$ is actually a Compound Poisson Process with Poisson intensity parameter $\lambda=\nu(\mathbb{R}^+)$. Moreover,

$\mathbb{P}(L_t = 0) = e^{-\lambda t}$ for all $t \geq 0$. For the claim distribution, we choose Exponential, Gamma and Beta distribution families to characterize the claim size random variables. These three distribution families are commonly used to fit insurance data with both flexibility and good performance. The gamma and the exponential distributions are positively skewed distributions over the positive real half line. Actuaries can use gamma distributions to easily control the tail behaviour of risks, especially to render the heavy-tail nature for insurance risks by taking appropriate parameters. While, having a bounded support, beta distributions are well suited for modeling insurance claims with ceilings. We also provide the corresponding Monte Carlo benchmark values for each example. For each Monte Carlo simulation we simulate 50,000 paths, and then produce 50 Monte Carlo results to calculate the mean and the standard deviation. As we can see in the numerical examples, our method is much faster and more accurate than the Monte Carlo one; and in each example, we shall count in the comparisons between time costs incurred.

*Example* 3.4 (Finite-time ruin probability for a Compound Poisson Process with exponential claims). Take $\delta = 0$ and $\kappa(x, y) \equiv 1$ like the zero initial surplus case in Example 2.6. The density function of an exponential claim is $\mu e^{-\mu x}$ and one can derive that $|\Lambda^{(1)}(s)| = \lambda|i\mu(\mu - is)^{-2}| = O(s^{-2})$, thus we can choose $n_0 = 1$ in Assumption A. The approximation formula (3.9) is applied to the same setting as in Example 2.6 but with a larger value of $u$, i.e., with $\lambda = 0.87$, $\mu = 1$, $T = 60$ and $L = 7$; also $u = 20$. Figure 4 plots the ruin probability as a function of $K$ (the number of terms used in each partial sum) and $L$ respectively. The highlighted band $(0.0174 \pm 0.0005)$ is the Monte Carlo simulation one standard deviation range centered at the mean. The total running time of the Monte Carlo simulation to generate 50 results (equivalently, 50 data points) is about 12.6 mins, while for the Fourier-cosine method it requires only 3.8s to generate even up to 100 points with parameter $K = 512$ in the second plot of Figure 4. Upon comparison with the true ruin probability based on the integration formula in [3], as we expected the approximation converges to the true value as $K$ increases. The absolute errors between the approximated and true ruin probabilities for several values of $K$ are given in Table 2, while a plot on the rate of convergence is in Figure 5. We observe that the convergence rate is of an order close to $O(K^{-3.0})$. Figure 6 plots the results of the finite-time ruin probability with respect to the time $T$, from the plot we observe that the Fourier-cosine approximation coincides with the reference value and converges to the infinite-time ruin probability as $T$ tends to infinity, which verifies that the Fourier-cosine method is stable for any sufficiently large $T$.

| $K$ | 32 | 64 | 128 | 256 | 512 | 1024 |
|---|---|---|---|---|---|---|
| Error | $5.1 \cdot 10^{-4}$ | $1.3 \cdot 10^{-4}$ | $2.4 \cdot 1.0^{-5}$ | $2.7 \cdot 10^{-6}$ | $1.4 \cdot 10^{-7}$ | $1.7 \cdot 10^{-7}$ |

TABLE 2

*Error magnitude for the Fourier-cosine approximation of the finite-time ruin probability with parameters $\lambda = 0.87$, $\mu = 1$, $T = 60$, $L = 7$ and $u = 20$.*

*Example* 3.5 (Finite-time Gerber-Shiu function for a Compound Poisson Process with Gamma claims). Let the penalty function $\kappa(x, y) = x + y$ (see [48] for more examples of penalty functions) and the claim size $Y$ be distributed as Gamma$(\alpha, \beta)$ with a density function $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$. One can easily derive that $|\Lambda^{(1)}(s)| = \lambda|\frac{\alpha i}{\beta}(1 - \frac{is}{\beta})^{-\alpha-1}| = O\left(s^{-1-\alpha}\right)$, so we can choose $n_0 = 1$ in Assumption A. In Figure 7, two illustrative plots of the finite-time Gerber-Shiu function against $K$ and $L$ with a parameter set $\{\delta = 0.05, \lambda = 2, \alpha = 0.5, \beta = 1.1, u = 3, T = 6, L = 7\}$ are given.

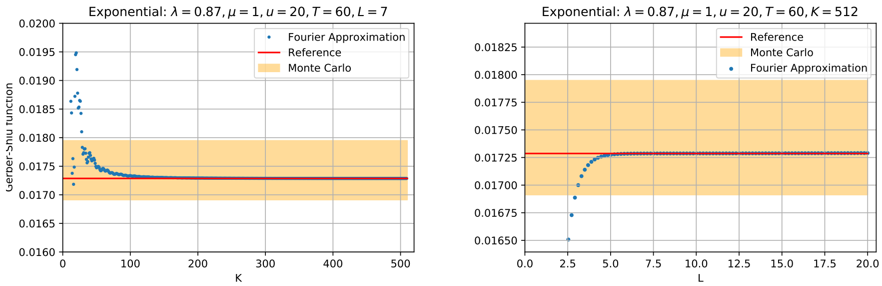


FIG. 4. *Ruin probability based on the Fourier-cosine approximation. The horizontal line (at 0.01729) is the true ruin probability.*

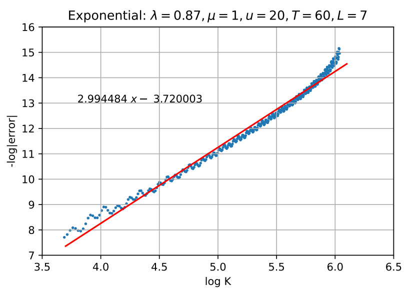


FIG. 5. *Plot of* $-\log|\text{error}|$ *against* $\log K$ *for the ruin probability approximation. The fitted straight line is also given.*

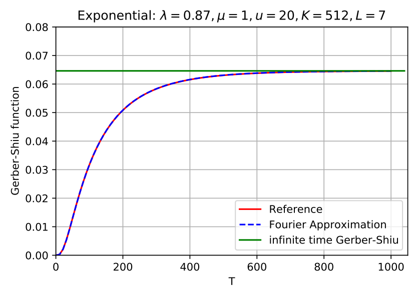


FIG. 6. *Finite-time ruin probability against* $T$*.*

The one standard deviation range of the Monte Carlo simulation is $(0.4010 \pm 0.0040)$, as shown in the highlighted region in the plots, the time required to run the Monte Carlo simulation 50 times (50 data points) is about 223s, while for the Fourier-cosine method it requires only 4.3s to generate 100 points with $K = 512$ in the second plot of Figure 7. The approximation of the Gerber-Shiu function stabilizes as $K$ increases, and for a large range of values of $L$, it falls into the one standard deviation range of the Monte Carlo simulation.

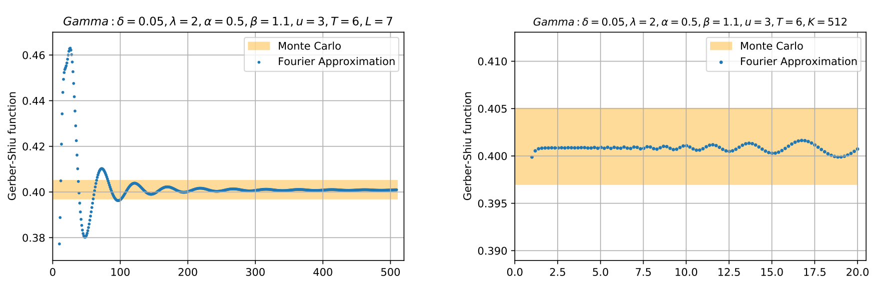


FIG. 7. *Gerber-Shiu functions based on the Fourier-cosine approximation for the compound Poisson model with gamma-distributed claims.*

*Example* 3.6 (Finite-time Gerber-Shiu function for a Compound Poisson Process with Beta claims). Let the penalty function $\kappa(x, y) = y$ and the claim size $Y$ be distributed as Beta$(\alpha, \beta)$ with a density function $\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$. Under this case, one can derive that

$$|\Lambda^{(1)}(s)| = \lambda \left| \frac{\alpha i}{\alpha+\beta} \, {}_1F_1(\alpha+1; \alpha+\beta+1; is) \right|$$

$$\sim \left| \Gamma(\alpha+\beta+1) \left( \frac{e^{is}(is)^{-\beta}}{\Gamma(\alpha+1)} + \frac{(-is)^{-\alpha-1}}{\Gamma(\beta)} \right) \right| = O\left(s^{-\max\{\beta,\alpha+1\}}\right),$$

$$|\Lambda^{(n_0)}(s)| \sim \lambda \left| {}_1F_1(\alpha+n_0; \alpha+\beta+n_0; is) \right| = O\left(s^{-\max\{\beta,\alpha+n_0\}}\right),$$

where ${}_1F_1(\cdot;\cdot;\cdot)$ is the hypergeometric function. Thus for $\beta > 1$ we can choose $n_0 = 1$ in Assumption A. In Figure 8, two more illustrations of the Gerber-Shiu function approximation as a function of $K$ and $L$ with a parameter set $\{\delta = 0.04, \lambda = 1.1, \alpha = 7, \beta = 2, u = 3, T = 10, L = 7\}$ are provided. The highlighted region $(0.0292 \pm 0.0004)$ is the one standard deviation range centered at the Monte Carlo simulation mean, the total running time of the Monte Carlo simulation for 50 times (50 data points) is about 111s, while for the Fourier-cosine method it requires only 3.8s to generate 100 points with $K = 512$ in the second plot of Figure 8. The approximation of the Gerber-Shiu function stabilizes as $K$ increases, and for a large range of values of $L$, it falls into the one standard deviation range of the Monte Carlo simulation.

*Remark* 3.7. When $\beta < 1$, Assumption A does not hold. However, the following numerical study (as shown in Figure 9) with parameters being set as $\delta = 0.06, \lambda = 1, \alpha = 9, \beta = 0.3, u = 5, T = 20, L = 7$ shows that our algorithm still converges even if $\beta < 1$ yet with a slower convergence rate (the Monte Carlo one standard deviation range is $(0.0359 \pm 0.0004)$). We leave the theoretical justification of this claim to future study by interested readers.

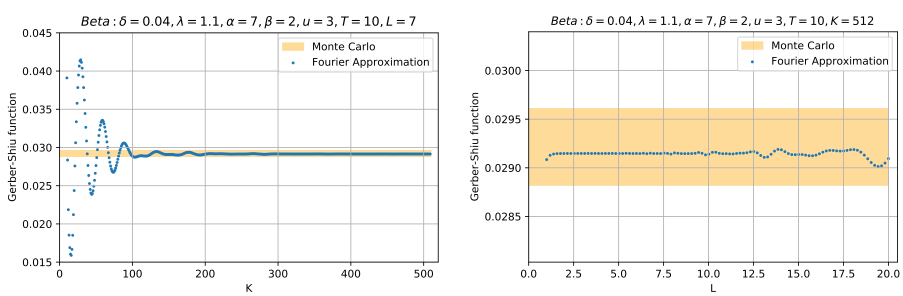


FIG. 8. *Gerber-Shiu functions based on the Fourier-cosine approximation for the compound Poisson model with beta-distributed claims.*

**3.2.2. Infinite Lévy measure case.** In this case, $L_t$ can be decomposed as a sum of a Compound Poisson Process and a pure jump process such that they are independent. On any nontrivial compact time interval with interior, the Lévy Process $L_t$ contains infinitely many jumps and the jumping times are dense in this arbitrary interval, see [45] for more discussion. In particular, for any time $t > 0$, $\mathbb{P}(L_t = 0) = 0$.

The work [28] built an implementable numerical scheme to approximate the finite-time Gerber-Shiu functions when the risk processes are meromorphic ones belonging to Beta and Theta families of Lévy Processes, which were first introduced in [26] and [27]; their method relies on inverting the Laplace transform of $\varphi(u, T)$ with respect to the $T$-variable which can be given by a closed form expression in terms of the corresponding infinite-time Gerber-Shiu counterpart. Their work was a breakthrough in

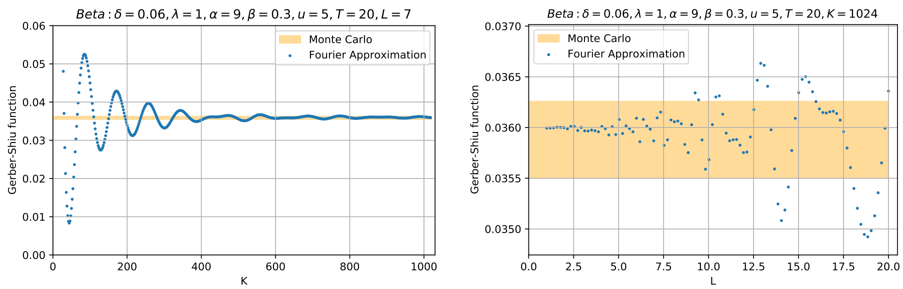


FIG. 9. *Gerber-Shiu functions based on the Fourier-cosine approximation for the compound Poisson model with beta-distributed claims.*

the contemporary literature and they also provided a comprehensive numerical study and demonstrated the efficiency of their method, for instance, computing the Value-at-Risk of the deficit at the ruin, conditional on the event that the ruin happens before the deterministic time $T$, particularly under the Theta families of risk processes with the density of the corresponding Lévy measure having a singularity at zero of order 3/2. Nevertheless, their approach is apparently workable only for a restricted class of Lévy Processes under which the infinite-time Gerber-Shiu functions acquire a closed form. Now, under our proposed Fourier-cosine method, in addition to demonstrating the approximation of the conditional VaR under the special case of Theta families as considered in [28] as discussed above; we further extend our numerical scheme to the more general classes of risk processes under which the infinite-time Gerber-Shiu functions fail to have a closed form. For instance, Gamma Processes and Generalized Stable Processes will be considered, and again they are beyond the scope of [28]. To this end, we first compute the conditional distribution function

$$F_T(p) := \mathbb{P}(|R_\tau| \le p | \tau \le T) = \frac{\mathbb{P}(|R_\tau| \le p, \tau \le T)}{\mathbb{P}(\tau \le T)}, \tag{3.10}$$

then by defining at the confidence level $\alpha$, $\mathrm{VaR}_\alpha$ satisfies $F_T(\mathrm{VaR}_\alpha) = \alpha$. To find the value $\mathrm{VaR}_\alpha$, we compute the finite-time ruin probability $\mathbb{P}(\tau \le T)$ first in each example, and then compute the finite-time Gerber-Shiu function $F(T,p) := \mathbb{P}(|R_\tau| \le p, \tau \le T)$ as in Example 2.7 and use (3.10) to get the desired value.

*Example* 3.8 (Value-at-Risk for Theta families with a singularity of order 3/2). The Lévy measure of Theta families is $\nu(\mathrm{d}x) = \frac{c\beta}{\mu\pi} e^{-\alpha\beta x}\Theta_1(\beta x)$, where $\Theta_1(y) = 2\sum_{n=1}^{\infty} n^2 e^{-n^2 y}$ †, with $\mathbb{E}(L_t) = \frac{ct}{2\mu\beta}(\alpha^{-1/2}\coth(\pi\sqrt{\alpha}) - \pi\sinh^{-2}(\pi\sqrt{\alpha}))$. The first order derivative of $\Lambda(s)$ is

$$\Lambda^{(1)}(s) = \frac{ci}{2\mu\beta}\left[(\alpha - \frac{is}{\beta})^{-1/2}\coth(\pi\sqrt{\alpha - \frac{is}{\beta}}) - \pi\sinh^{-2}(\pi\sqrt{\alpha - \frac{is}{\beta}})\right],$$

and one can check that $|\Lambda^{(1)}(s)| = O(s^{-1/2})$ as $s \to \infty$, as a result we can choose $n_0 = 3$ in Assumption A. The illustration is conducted with parameters $\mu = 20, c = 5.4, \alpha = 0.5, \beta = 0.35, u = 2, T = 20$. The finite-time ruin probability $\mathbb{P}(\tau \le T)$ is shown in Figure 10, and the finite-time Gerber-Shiu function $F(T,p) := \mathbb{P}(|R_\tau| \le$

†Note that the function $\Theta_1(y)$ is just the first order derivative of the Theta function $\theta_3(0, e^{-y})$, see [27].

$p, \tau \le T)$ is shown in Figure 11. We can see that the approximations appear to stabilize as $K$ increases. Then we use the formula (3.10) to compute the conditional distribution $F_T(p)$, and the plot of $F_T(p)$ against $p$ is given in Figure 12. From the plot, we can see the $\mathrm{VaR}_{0.95}$ for Theta families with a singularity at zero of order 3/2 is 5.47, while the benchmark value in [28] is 5.472856602. For reference, we also provide a table of more accurate values of $\mathrm{VaR}_\alpha$ for $\alpha = \{0.95, 0.96, 0.97, 0.98, 0.99\}$ in Table 3 by setting a denser grid and using the fourth order Lagrange polynomial interpolation to improve the precision (see [28] for details). The time needed for generating Figure 10, Figure 11 and Figure 12 (each includes 500 points) are 16s, 25s and 29s, respectively. We also provide a plot of $F(T, p)$ against $T$ in Figure 13, from the plot, we observe that as $T$ tends to infinity, the approximation approaches the value of the corresponding infinite-time Gerber-Shiu function, which justifies that our method is stable with the time parameter $T$ and is consistent with the corresponding Fourier-cosine approximation of the infinite-time Gerber-Shiu function $\varphi(u)$.

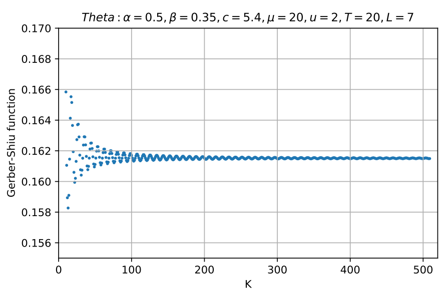


FIG. 10. *Approximation of ruin probability for Theta families.*

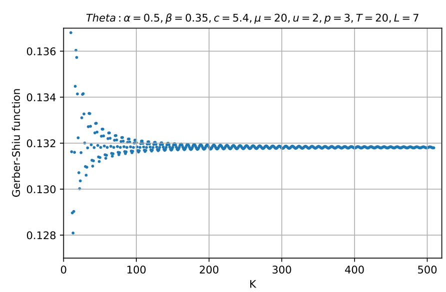


FIG. 11. *Approximation of $F(T, p)$ for Theta families.*

| $\alpha$ | 0.95 | 0.96 | 0.97 | 0.98 | 0.99 |
|---|---|---|---|---|---|
| $\mathrm{VaR}_\alpha$ | 5.472452523 | 5.897355828 | 6.445308494 | 7.217829473 | 8.539040167 |

TABLE 3
*$VaR_\alpha$ of different $\alpha$ for Theta families.*

*Example* 3.9 (Value-at-Risk for a Gamma Process). The Lévy measure of a Gamma Process with parameters $\alpha > 0$ and $\beta > 0$ is $\nu(\mathrm{d}x) = \left(\alpha e^{-\beta x}/x\right)\mathrm{d}x$, with $\mathbb{E}(L_t) = \alpha t/\beta$ and $\Lambda^{(1)}(s) = \frac{\alpha i}{\beta}(1 - \frac{is}{\beta})^{-1}$. In this case we can choose $n_0 = 2$ in Assumption A since one can check that $|\Lambda^{(1)}(s)| = O(s^{-1})$. The illustration is conducted with the same parameters in Example 2.7 but with a positive $u = 4$. The finite-time ruin probability $\mathbb{P}(\tau \le T)$ is shown in Figure 14, and the finite-time Gerber-Shiu function $F(T, p) := \mathbb{P}(|R_\tau| \le p, \tau \le T)$ is shown in Figure 15. The setting of Monte Carlo simulation is the same with the one in Example 2.7 and the corresponding one standard deviation range for $\mathbb{P}(\tau \le T)$ and $F(T, p)$ are $(0.2575 \pm 0.0019)$ and $(0.2271 \pm 0.0019)$, respectively. Again, we can see that both approximations fall rapidly into the one standard deviation limited range as $K$ increases. Then we use the formula (3.10) to compute the conditional distribution $F_T(p)$, and the plot of $F_T(p)$ against $p$ is given in Figure 16. From the graph, we can see the $\mathrm{VaR}_{0.95}$ for this Gamma Process is 4.38. We also provide a table of more accurate values of $\mathrm{VaR}_\alpha$ for $\alpha = \{0.95, 0.96, 0.97, 0.98, 0.99\}$ in Table 4 for readers' references. The time for running the Monte Carlo simulation 50 times (50 data points) is around 85 minutes, while the time for generating Figure 14, Figure 15 and Figure 16 (each includes 500 points) are 13s, 2 mins and 3.5 mins, respectively.

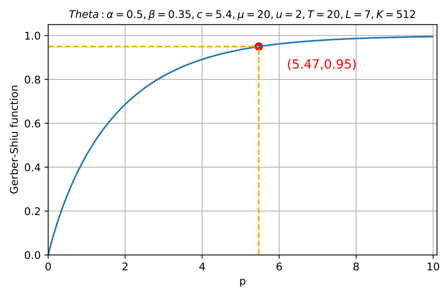


FIG. 12. *Approximation of $F_T(p)$ for Theta families.*

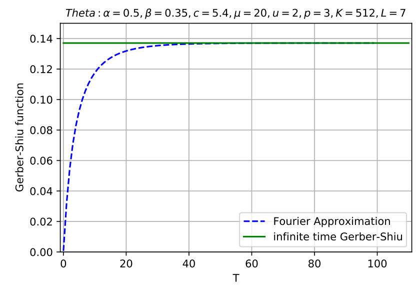


FIG. 13. *Approximation of $F(T,p)$ against $T$ for Theta families.*

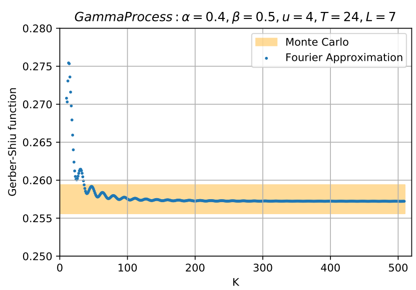


FIG. 14. *Approximation of ruin probability for the Gamma Process.*

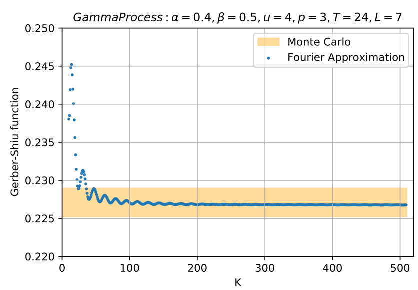


FIG. 15. *Approximation of $F(T,p)$ for the Gamma Process.*

| $\alpha$ | 0.95 | 0.96 | 0.97 | 0.98 | 0.99 |
|---|---|---|---|---|---|
| $\mathrm{VaR}_\alpha$ | 4.378096648 | 4.743214622 | 5.218267029 | 5.89531416 | 7.070394497 |

TABLE 4
*$\mathrm{VaR}_\alpha$ of different $\alpha$ for the Gamma Process.*

*Example* 3.10 (Value-at-Risk for a Generalized Stable Process). The Lévy measure of a Generalized Stable Process with parameters $\beta \in (0,1)$ and $\lambda > 0$ is $\nu(\mathrm{d}x) = \frac{\beta e^{-\lambda x}}{\Gamma(1-\beta)x^{\beta+1}}\mathrm{d}x$, with $\mathbb{E}(L_t) = \beta t \lambda^{\beta-1}$ and $\Lambda^{(1)}(s) = \beta i(\lambda - is)^{\beta-1}$. One can check that $|\Lambda^{(1)}(s)| = O(s^{-(1-\beta)})$, thus we can choose $n_0$ as the smallest integer larger than $\frac{1}{1-\beta}$ in Assumption A. The illustration is conducted with parameters $\lambda = 0.3, \beta = 0.45, u = 24, T = 120$. The finite-time ruin probability $\mathbb{P}(\tau \le T)$ is shown in Figure 17, and the finite-time Gerber-Shiu function $F(T,p) := \mathbb{P}(|R_\tau| \le p, \tau \le T)$ is shown in Figure 18. We can see that both approximations appear to stabilize as $K$ increases. Then we use the formula (3.10) to compute the conditional distribution $F_T(p)$, and the plot of $F_t(p)$ against $p$ is given in Figure 19. From the plot, we can see the $\mathrm{VaR}_{0.95}$ for this Generalized Stable Process is 6.19. We also provide a table of more accurate values of $\mathrm{VaR}_\alpha$ for $\alpha = \{0.95, 0.96, 0.97, 0.98, 0.99\}$ in Table 5 for reference. The time needed for generating Figure 17, Figure 18 and Figure 19 (each includes 500 points) are 15s, 3.6 mins and 9.6 mins, respectively. Note that the calculations for the Gamma Process and the Generalized Stable Process are much slower than for the Theta families, the reason is that for the Gamma Process and the Generalized Stable Process, the corresponding Fourier transforms of $V$ involve the computation of incomplete Gamma functions which we cannot use vectorization in Python to compute for approximating $F_t(p)$.

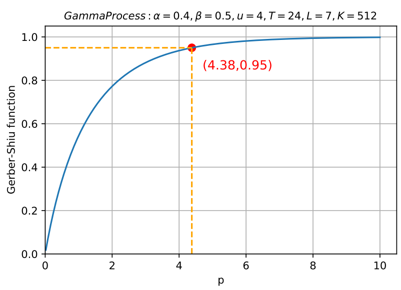


FIG. 16. *Approximation of $F_T(p)$ for the Gamma Process.*

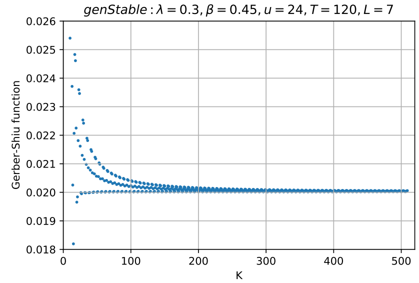


FIG. 17. *Approximation of ruin probability for the Generalized Stable Process.*

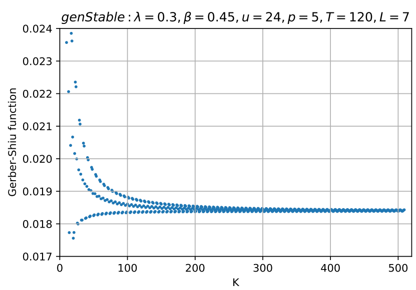


FIG. 18. *Approximation of $F(T,p)$ for the Generalized Stable Process.*

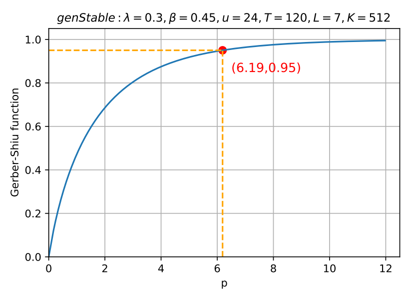


FIG. 19. *Approximation of $F_T(p)$ for the Generalized Stable Process.*

| $\alpha$ | 0.95 | 0.96 | 0.97 | 0.98 | 0.99 |
|---|---|---|---|---|---|
| $\text{VaR}_\alpha$ | 6.185790705 | 6.737914506 | 7.459858661 | 8.494905258 | 10.305576768 |

TABLE 5
*$VaR_\alpha$ of different $\alpha$ for the Generalized Stable Process.*

**Supplementary Materials.** All the proofs and the detailed error analyses are presented in the Supplementary Materials. The proofs for the Generalized Ballot Theorem and the non-crossing probability as well as the derivations of Equation (2.9) and Equation (3.5) are put in Section SM1 and Section SM2; the detailed error analyses for Theorem 2.2 and Theorem 3.3 are shown in Section SM3 and Section SM4, respectively.

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**Appendix A. Overview on Gerber-Shiu functions.** [22] first introduced the function $\varphi$ named after them, its effectiveness was demonstrated by the systematic characterization of important financial quantities in actuarial science. In their first work, the classical risk model was used, and they showed that $\varphi$ satisfies a defective renewal equation, to which the solution can be expressed as an infinite sum of the order of convolution products. This result has been generalized to the model (1.8) in [20], with the following representation:

$$\varphi(u) = \sum_{k=0}^{\infty} h_1 * h_2^{*k}(u), \tag{A.1}$$

where $v^{*k}$ denotes the $k$-th order convolution for a function $v$ such that the custom of $v^{*1} = v$ and $v^{*k} = v^{*(k-1)} * v$ is adopted; and we denote $f * v^{*0} = f$. The functions $h_1$ and $h_2$ are given by

$$h_1(x) := \int_x^{\infty} \int_0^{\infty} e^{-\rho(z-x)} \kappa(z,y) \zeta(z+y) \mathrm{d}y \mathrm{d}z, \quad h_2(x) := \int_x^{\infty} e^{-\rho(y-x)} \zeta(y) \mathrm{d}y, x \geq 0,$$

where $\zeta$ denotes the density of the Lévy measure, i.e., $\nu(\mathrm{d}y) = \zeta(y)\mathrm{d}y$, and the constant $\rho$ is the unique non-negative solution of the following equation in $\lambda$,

$$\delta - \lambda - \Lambda(i\lambda) = 0.$$

It has been shown in [8] that the Gerber-Shiu function has the following representation:

$$\varphi(u) = h_1(0) + \int_0^u V(x) \mathrm{d}x, \tag{A.2}$$

where

$$V(x) := h_1(0) \sum_{k=1}^{\infty} h_2^{*k}(x) + \rho \sum_{k=0}^{\infty} h_1 * h_2^{*k}(x) - \sum_{k=0}^{\infty} h_3 * h_2^{*k}(x), \tag{A.3}$$

and

$$h_3(x) := \rho h_1(x) - h_1'(x) = \int_0^\infty \kappa(x,y)\zeta(x+y)\mathrm{d}y.$$

Notice that from (A.2) we have $\varphi(0) = h_1(0)$. In our work, we demonstrated that the Fourier transform of functions $h_1$, $h_2$ and $h_3$ are easy to calculate, and it can be shown that $|\hat{h}_2(s)| < 1$ for all $s \in \mathbb{R}$ under the safety loading condition (see [8] for details). Thus from (A.3), the Fourier transform of $V$ can be calculated by

$$\hat{V} = h_1(0)\sum_{k=1}^{\infty} \hat{h}_2^k + \sum_{k=0}^{\infty} \hat{h}_1\hat{h}_2^k - \sum_{k=0}^{\infty} \hat{h}_3\hat{h}_2^k = \frac{h_1(0)\hat{h}_2 + \rho\hat{h}_1 - \hat{h}_3}{1-\hat{h}_2}. \tag{A.4}$$

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