



PAPER

Resource conversion between operational coherence and multipartite entanglement in many-body systems

OPEN ACCESS

RECEIVED

30 October 2020

REVISED


30 December 2020

ACCEPTED FOR PUBLICATION

8 January 2021

PUBLISHED

29 April 2021

Li-Hang Ren¹, Meng Gao¹, Jun Ren¹, Z D Wang^{2,*} and Yan-Kui Bai^{1,2,*} ¹ College of Physics and Hebei Key Laboratory of Photophysics Research and Application, Hebei Normal University, Shijiazhuang, Hebei 050024, People's Republic of China² Department of Physics and HKU-UCAS Joint Institute for Theoretical and Computational Physics at Hong Kong, The University of Hong Kong, Pokfulam Road, Hong Kong, People's Republic of China

* Authors to whom any correspondence should be addressed.

E-mail: zwang@hku.hk and ykbai@semi.ac.cn**Keywords:** quantum coherence, multipartite entanglement, operational resource conversion

Original content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.



Abstract

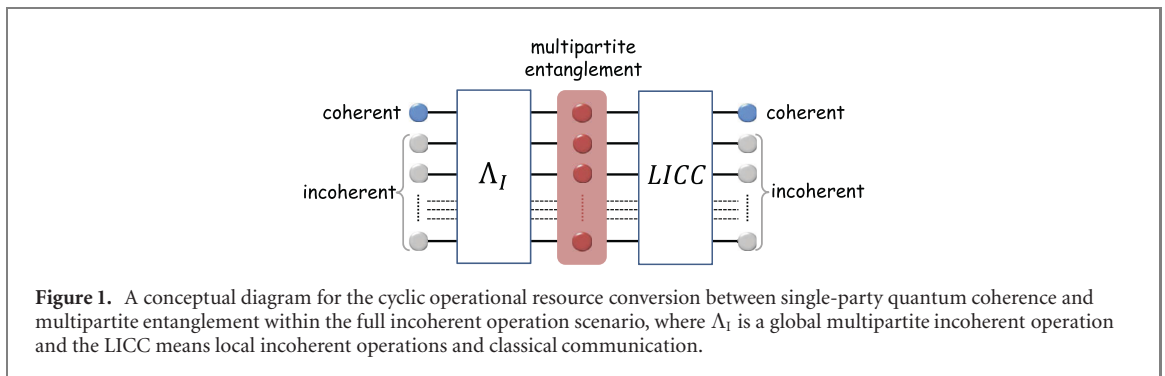
We establish a set of resource conversion relationships between quantum coherence and multipartite entanglement in many-body systems, where the operational measures of resource cost and distillation are focused. Under the multipartite incoherent operation, the initial coherence of single-party system bounds the maximal amounts of corresponding operational entanglement in an arbitrary bipartite partition as well as the genuine multipartite entanglement (GME) in many-body systems. Furthermore, the converted multipartite entanglement can be transferred to its subsystems and restored to coherence of a single party by means of local incoherent operations and classical communication, which constitutes a protocol of resource interconversion within the full incoherent operation scenario. As an example, we present a scheme for cyclic interconversion between coherence and GME in three-qubit systems without loss. Moreover, we analyze the property of bipartite and multipartite genuine multi-level entanglement by the initial coherence and investigate multipartite resource dynamics in the conversion.

1. Introduction

Quantum entanglement [1, 2], as an important physical resource, has been widely applied to quantum communication and quantum computation, such as quantum teleportation, quantum key distribution, one-way quantum computation and so on (see a review paper [3]). In recent years, quantum coherence [4, 5] has also been formulated as a physical resource in quantum information processing and becomes a vivid research topic [6, 7]. Both kinds of the resources can be considered as the particular manifestation of the superposition principle in quantum mechanics, and their interplay has attracted a lot of attention. For example, the resource consumption in the tasks of quantum state preparation and quantum state merging [8] can be quantified by pairs of quantum coherence and entanglement [9, 10]. Moreover, unified characterization and operational relations between the two kinds of resource measures were analyzed [11–15].

It is a practical problem to investigate the resource conversion between quantum coherence and entanglement under certain operation constraints. Streltsov *et al* first showed that single-party quantum coherence with respect to some fixed basis can be converted to entanglement via bipartite incoherent operations [16]. On the other hand, Chitambar *et al* provided the upper bound of assisted coherence distillation (ACD) from bipartite systems to one of the subsystems by local quantum incoherent operations and classical communication (LQICC) [17]. Other effective resource conversion methods for nonclassicality, quantum correlations and nonlocality were also presented [18–20], and some key experimental progress have been made in the optics and superconducting systems [21–24].

In the quantum resource theory (QRT) [25], there are two basic operational processes: one is the so-called resource distillation, and the other is resource formation. Motivated by these processes, Winter



and Yang established operational resource theory of coherence by focusing on the distillable coherence and coherence cost [26], which correspond to the distillable entanglement [27] and entanglement cost [28] in the QRT of entanglement. It is desirable and necessary to study the operational resource conversion between quantum coherence and entanglement, especially in multipartite systems. Moreover, the ACD in the LQICC scenario involves two parties (Alice and Bob) and their goal is to maximize the quantum coherence of Alice's subsystem by Bob performing arbitrary local quantum operations on his subsystem, while Alice is restricted to local incoherent operations assisted by classical communication between them [17]. It is noted that the LQICC can generate coherence in Bob's subsystem since local quantum operations are not incoherent operations in general [29]. In a cyclic resource conversion of coherence-entanglement-coherence, the operations should be considered within the full incoherent scenario, i.e., both Alice and Bob are restricted to local incoherent operations which is referred to as local incoherent operations and classical communication (LICC). The definitions can be further generalized to multipartite systems and we have the relation $LICC \subset LQICC$ [29]. In addition, since entanglement in many-body systems is a complex problem, it is meaningful to characterize the entanglement property via quantum coherence in the resource conversion. Meanwhile, the resource dynamics of quantum coherence and multipartite entanglement in the conversion needs to be addressed, because quantum systems interact unavoidably with the environment in realistic quantum information processing.

In this paper, focused on resource distillation and resource formation, we study the conversion between quantum coherence and multipartite entanglement in many-body systems, which is different from the previous results in bipartite quantum systems [12–16]. As shown in figure 1, we give a conceptual diagram for the cyclic resource conversion within the full incoherent operation scenario. In the conversion from coherence to entanglement, we restrict the operation to multipartite incoherent operations which are chosen to be a kind of global operations [11]. Moreover, in the conversion from multipartite entanglement to single-party coherence, we utilize the LICC [29] which does not generate coherence in the assisted subsystems and can overcome the flaw of LQICC in the cyclic resource conversion. It is found that, via multipartite incoherent operations, the initial operational coherence of single party bounds not only the generated bipartite operational entanglement but also the genuine multipartite entanglement (GME) in the composite systems. The converted multipartite entanglement can be further transferred to its subsystems and restored to the coherence of a single-party subsystem by the LICC, where we prove that the optimal resource conversion can make the relations saturated. As a typical application, we present a cyclic interconversion protocol between quantum coherence and multipartite entanglement in three-qubit systems without loss. In addition, we show that the coefficients of initial single-party coherent state can determine whether the converted quantum state is genuine multi-level entangled. Finally, under the depolarization environment, different resource dynamical properties are investigated in multipartite resource conversion.

2. Operational resource conversion in multipartite systems

In the QRT, a free state is the one without possessing a defined resource, and free operation cannot generate the resource and maps the set of free states onto itself. In regard to the operational resource characterization, resource distillation is a transformation from a mixed state to the unit resource, and resource formation is the reverse transformation from the unit resource to a mixed state, where both of the transformations are restricted to free operations. For entanglement theory, two motivated measures arising from the two operational tasks are distillable entanglement and entanglement cost [3],

$$E_d(\rho) = \sup\{r : \lim_{n \rightarrow \infty} [\inf_{\Gamma} \|\Gamma(\rho^{\otimes n}) - \Phi_{2+}^{\otimes n}\|_1] = 0\}, \quad (1)$$

$$E_c(\rho) = \inf\{r : \lim_{n \rightarrow \infty} [\inf_{\Gamma} \|\rho^{\otimes n} - \Gamma(\Phi_{2+}^{\otimes n})\|_1] = 0\}, \quad (2)$$

where r is the optimal rate in the tasks, $\|\cdot\|_1$ trace norm, $\Phi_{2+} = |\Phi_2^+\rangle\langle\Phi_2^+|$ the two-qubit Bell state $|\Phi_2^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (unit entanglement), and $\Gamma(\cdot)$ the free operation in resource theory of entanglement (i.e., local operations and classical communication (LOCC) [30]). It is noted that entanglement cost is not equal to the well-known entanglement of formation $E_f(\rho)$ in general [31, 32].

In the coherence resource theory formulated by Baumgratz *et al* [4], the free state has the form $\sigma = \sum_i p_i |i\rangle\langle i|$ with $\{|i\rangle\}$ being a fixed reference basis in finite dimensions, and the set of incoherent states is denoted by \mathcal{I} . The incoherent operation is the free operation, which is specified by a set of Kraus operators $\{K_l\}$ satisfying $\sum_l K_l^\dagger K_l = I$ and $K_l \mathcal{I} K_l^\dagger \subset \mathcal{I}$ for all l . The unit coherence resource is the maximal coherent single-qubit state $\Psi_2 = \frac{1}{2} \sum_{i,j=0}^1 |i\rangle\langle j|$. The framework can be further generalized to multipartite scenario, in which the N -partite incoherent state has the form [11] $\sigma = \sum_{\vec{i}} p_{\vec{i}} |\vec{i}\rangle\langle \vec{i}|$ where $|\vec{i}\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle$ with $|i_k\rangle$ being a pre-fixed local basis of the k th subsystem. The N -partite incoherent operation can also be expressed by a set of Kraus operators, where each incoherent operator maps the set of N -partite incoherent state onto itself (see the details in appendix A).

The coherence distillation is the process that extracts unit coherence from a mixed state by incoherent operations, and the dual coherence formation is a process that prepares a mixed state by consuming unit coherent states under incoherent operations. In the asymptotic limit of many copies of a state, it is shown that distillable coherence and coherence cost can be quantified by simple single-letter formulas [26]

$$C_d(\rho) = C_r(\rho) = S(\Delta(\rho)) - S(\rho), \quad (3)$$

$$C_c(\rho) = C_f(\rho) = \min_i \sum_i p_i S(\Delta(\psi_i)), \quad (4)$$

where $C_r(\rho)$ is the relative entropy of coherence [4] with $S(\rho) = -\text{tr} \rho \log_2 \rho$ being the von Neumann entropy and $\Delta(\rho) = \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$, and $C_f(\rho)$ is the coherence of formation with the minimum running over all the pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$.

2.1. Converting operational coherence to multipartite entanglement via multipartite incoherent operations

Quantum coherence and entanglement are two kinds of fundamental nonclassical resources that can each be characterized within an operational resource theory. It is desirable to study the operational resource conversion between them. The conversion from quantum coherence to entanglement via bipartite incoherent operations was first studied by Streltsov *et al*, where two resources are quantified by distance-based measures such as relative entropy and Uhlmann fidelity [16]. Based on bipartite incoherent operations, Zhu *et al* further gave the conversion relationship between coherence and entanglement quantified by measures of convex-roof extension, l_1 -norm of coherence and negativity [13, 14].

It should be noted that the multipartite incoherent operation is not equivalent to that of the bipartite case in general [11, 16]. For example, the bipartite incoherent operation can convert a single maximal coherent qubit plus an incoherent ancilla to the Bell state, but the tripartite incoherent operation makes the coherent qubit plus two incoherent qubits into the GHZ state, which means that the multipartite operation has the ability to generate multipartite entangled states (see the details in the last paragraph of appendix A). Here, motivated by resource distillation and recourse cost, we study the operational resource conversion between coherence and entanglement in multipartite systems, and can obtain the following theorem.

Theorem 1. *Given a multipartite incoherent operation Λ_I applied to a single-party coherent state ρ_A and an ancillary N -partite incoherent state $\sigma_{B_1 B_2 \dots B_n}$, the generated operational entanglements are upper bounded by the operational coherences of single-party system*

$$C_d(\rho_A) \geq E_d[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \quad (5)$$

$$C_c(\rho_A) = C_f(\rho_A) \geq \max\{E_c[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], E_f[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})]\}, \quad (6)$$

where the coherent state has the form $\rho_A = \sum_{mn} \rho_{mn} |m\rangle\langle n|$ and the operational entanglements are in an arbitrary bipartite partition $\alpha|\bar{\alpha}$ for the multipartite systems with $\alpha \cup \bar{\alpha} = \{A B_1 B_2 \dots B_n\}$.

Proof. We first prove the relation in equation (5). Since the distillable coherence is equal to the relative entropy of coherence [26], we have

$$\begin{aligned} C_d(\rho_A) &= S(\rho_A \| \sigma_A) = S(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n} \| \sigma_A \otimes \sigma_{B_1 B_2 \dots B_n}) \\ &\geq S[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n}) \| \Lambda_I(\sigma_A \otimes \sigma_{B_1 B_2 \dots B_n})] \\ &\geq E_r(\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})) \geq E_d(\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})), \end{aligned} \tag{7}$$

where σ_A is the closest incoherent state to ρ_A in the first equation, the additive and contractive properties of relative entropy [33, 34] are used in the second equation and the first inequality, the result of the second inequality comes from the definition of the relative entropy of entanglement, and the last inequality is due to the fact that the relative entropy of entanglement is the upper bound of distillable entanglement [35, 36]. In the first inequality of the proof, the quantum state $\Lambda_I(\sigma_A \otimes \sigma_{B_1 B_2 \dots B_n})$ is multipartite incoherent which is also $(N + 1)$ -partite separable. Therefore, the distance of relative entropy in the first inequality is not less than the bipartite relative entropy of entanglement E_r in the second inequality where the separable state can be chosen in an arbitrary bipartite partition $\alpha|\bar{\alpha}$ such as $A|B_1 B_2 \dots B_n$, $AB_1|B_2 \dots B_n$, $AB_1 B_2|\dots B_n$ and so on, which further results in the last inequality in equation (7) being satisfied for the corresponding bipartite partition. Then, we prove the second relation shown in equation (6). Because the equality $C_c(\rho_A) = C_f(\rho_A)$ and the additivity of coherence of formation, we can obtain

$$\begin{aligned} C_c(\rho_A) &= C_f(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n}) \geq C_f[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} C_f[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})^{\otimes n}] \geq \lim_{n \rightarrow \infty} \frac{1}{n} E_f[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})^{\otimes n}] \\ &= E_c[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \end{aligned} \tag{8}$$

where the monotone property of C_f under incoherent operations is used in the first inequality, the second inequality holds due to the property $C_f(\rho) \geq E_f(\rho)$ [13] with the entanglement quantified in an arbitrary bipartite partition $\alpha|\bar{\alpha}$ for composite systems $AB_1 B_2 \dots B_n$, and the last equation comes from the relation that entanglement cost is equal to the regularized entanglement of formation [28] with the bipartite partition for E_c being arbitrary. Moreover, for the case of a single copy of coherent state ρ_A , we can derive that $C_f(\rho_A) \geq E_f[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})]$ by a similar analysis and the bipartite partition for E_f is also arbitrary. Although the two operational entanglements $E_c(\rho) \neq E_f(\rho)$ in general [32], we can get that, given a multipartite incoherent operation Λ_I , the coherence of formation $C_f(\rho_A)$ is not less than the maximum of operational entanglements for any bipartite partition in the multipartite system $AB_1 B_2 \dots B_n$ and then equation (6) is satisfied, such that we complete the proof of the theorem. \square

Compared with bipartite entanglement, the characterization of multipartite entanglement is much more complicated. In the pure state case, a multipartite quantum state is genuine multipartite entangled if it cannot be written as a bipartite product state under any bipartite partitions [37–39]. For the quantification of entanglement in many-body systems, there exist a kind of GME measures generalized by bipartite entanglement measures [40, 41],

$$E^{\text{GME}}(|\psi\rangle) = \min_{\{\alpha\}} E_{\alpha}(|\psi\rangle), \tag{9}$$

where $|\psi\rangle$ is an N -partite pure state and α represents all possible bipartite partitions $\alpha|\bar{\alpha}$ in the composite systems. When the bipartite measure E is chosen to be the operational entanglements E_d and E_f , we can obtain the GME measures E_d^{GME} and E_f^{GME} . In theorem 1, when the initial coherent state and final output state after the operation Λ_I are pure states, we can get the following conversion relations.

Corollary 1. *In the operational resource conversion between single-party coherence and multipartite entanglement, the GME measures quantified via E_d and E_f are upper bounded by the operational coherences,*

$$C_d(|\phi\rangle_A) \geq E_d^{\text{GME}}(|\psi\rangle_{AB_1 B_2 \dots B_n}), \tag{10}$$

$$C_f(|\phi\rangle_A) \geq E_f^{\text{GME}}(|\psi\rangle_{AB_1 B_2 \dots B_n}), \tag{11}$$

where $|\phi\rangle_A$ is the initial coherent pure state and $|\psi\rangle_{AB_1 B_2 \dots B_n}$ is the output state under the multipartite incoherent operation $\Lambda_I(|\phi\rangle \langle\phi|_A \otimes \sigma_{B_1 B_2 \dots B_n})$ with an N -partite incoherent state being ancillary.

Proof. According to theorem 1, we know that the single-party distillable coherence $C_d(|\phi\rangle_A)$ is the upper bound on the distillable entanglement $E_d(|\psi\rangle_{AB_1 B_2 \dots B_n})$ in an arbitrary bipartite partition of the multipartite systems. Therefore, $C_d(|\phi\rangle_A)$ is not less than $E_d(|\psi\rangle_{\alpha|\bar{\alpha}})$ which is the minimal bipartite distillable entanglement, and then we can obtain the inequality in equation (10) based on the definition of E_d^{GME} .

The situation for coherence of formation $C_f(|\phi\rangle_A)$ is similar and we can derive the inequality in equation (11) after analyzing the relation between E_f^{GME} and E_f in multipartite systems. □

The GME measure in equation (9) can be further generalized to mixed states by the convex roof extension [40]. A multipartite mixed state is said to be genuine multipartite entangled if, in any pure state decomposition of the mixed state, there exists at least one pure state component which cannot be written as bipartite product state with respect to any bipartite partition. For example, the GME based on entanglement of formation E_f can be quantified by

$$E_f^{\text{GME}}(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_f^{\text{GME}}(|\psi_i\rangle), \tag{12}$$

where the minimum runs over all the pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Although the GME measure for mixed states can be constructed by distillable entanglement, we do not utilize $E_d^{\text{GME}}(\rho)$ to characterize the multipartite entanglement in the operational resource conversions since E_d is not a measure based on the convex roof construction of pure state measure. Here we consider the multipartite relative entropy of entanglement which has the form [42]

$$E_r^{\text{M}}(\rho) = \min_{\sigma_s \in \mathcal{D}} S(\rho||\sigma_s), \tag{13}$$

where $S(x||y) \equiv -\text{tr}(x \log_2 y) + \text{tr}(x \log_2 x)$ is the relative entropy with $\sigma_s = \sum_i p_i |\varphi_1^i\rangle\langle\varphi_1^i| \otimes |\varphi_2^i\rangle\langle\varphi_2^i| \otimes \dots \otimes |\varphi_n^i\rangle\langle\varphi_n^i|$ being the element of the N -partite separable state set \mathcal{D} . In the resource conversion of mixed states, we have the following theorem.

Theorem 2. *Given a multipartite incoherent operation Λ_I applied to a single-party coherent state ρ_A and an ancillary N -partite incoherent state $\sigma_{B_1 B_2 \dots B_n}$, the generated multipartite entanglements are upper bounded by the operational coherences of initial quantum state*

$$C_d(\rho_A) \geq E_r^{\text{M}}[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \tag{14}$$

$$C_f(\rho_A) \geq E_f^{\text{GME}}[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \tag{15}$$

where E_r is $(N + 1)$ -partite relative entropy of entanglement and E_f^{GME} is the GME measure based on entanglement of formation.

Proof. For the single-party quantum state ρ_A , its distillable coherence C_d is equal to the relative entropy of coherence C_r , and we have

$$\begin{aligned} C_d(\rho_A) &= C_r(\rho_A) = S(\rho_A||\sigma_A) \\ &\geq S[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})||\Lambda_I(\sigma_A \otimes \sigma_{B_1 B_2 \dots B_n})] \\ &\geq E_r^{\text{M}}[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \end{aligned} \tag{16}$$

where the additive and contractive properties of relative entropy are used in the first inequality, and the second inequality is satisfied due to $\Lambda_I(\sigma_A \otimes \sigma_{B_1 B_2 \dots B_n})$ being an $(N + 1)$ -partite incoherent state which is a multipartite separable state. For the coherence of formation, we can obtain

$$\begin{aligned} C_f(\rho_A) &\geq C_f[\Lambda_I(\rho_A \otimes \sigma_{B_1 \dots B_n})] = \sum_i p_i C_f(|\psi_i'\rangle_{AB_1 B_2 \dots B_n}) \\ &\geq \sum_i p_i E_f(|\psi_i'\rangle_{\alpha|\bar{\alpha}}) \geq \sum_i p_i E_f^{\text{GME}}(|\psi_i'\rangle_{AB_1 B_2 \dots B_n}) \geq E_f^{\text{GME}}[\Lambda_I(\rho_A \otimes \sigma_{B_1 \dots B_n})], \end{aligned} \tag{17}$$

where the additive and contractive properties are used in the first inequality, the optimal pure state decomposition $\Lambda_I(\rho_A \otimes \sigma_{B_1 \dots B_n}) = \sum_i p_i |\psi_i'\rangle\langle\psi_i'|_{AB_1 B_2 \dots B_n}$ for coherence of formation is utilized in the next equality, the second inequality holds due to the property $C_f(|\psi_i'\rangle) \geq E_f(|\psi_i'\rangle)$ with the entanglement in an arbitrary bipartite partition $\alpha|\bar{\alpha}$, the last two inequalities come from the definitions of multipartite entanglement E_f^{GME} for pure states and mixed states, and then we complete the proof. □

Entanglement monogamy is an important property in many-body quantum systems, which gives the trade-off relations on the distribution of entanglement among different subsystems [43–47]. As is known, the monogamy property can be used for constructing multipartite entanglement measures and indicators

[43, 47–51]. For the operational entanglement measures in an N -qubit quantum state $\rho_N = \rho_{A_1 A_2 \dots A_n}$, we can define two multipartite entanglement indicators

$$\tau_{\text{MED}}(\rho_N) = \left\{ \max \left[0, E_d^2(\rho_{A_1|\bar{A}_1}) - \sum_{i=2}^n E_d^2(\rho_{A_1 A_i}) \right] \right\}^{\frac{1}{2}}, \tag{18}$$

$$\tau_{\text{MEF}}(\rho_N) = \left[E_f^2(\rho_{A_1|\bar{A}_1}) - \sum_{i=2}^n E_f^2(\rho_{A_1 A_i}) \right]^{\frac{1}{2}}, \tag{19}$$

where $\bar{A}_1 = A_2 A_3 \dots A_n$ is the subsystem other than qubit A_1 . The indicators τ_{MED} and τ_{MEF} can indicate multipartite entanglement which cannot be stored in two-qubit subsystems. The residual entanglement in τ_{MEF} is always nonnegative according to the monogamy property of squared entanglement of formation [47, 51]. In appendix B, we show that the distillable entanglement is monogamous in N -qubit pure states and some kinds of mixed states, which can be used to indicate the existence of the genuine multi-qubit entanglement when τ_{MED} is positive. In the operational resource conversion between coherence and multi-qubit entanglement, we have the following theorem.

Theorem 3. For a single-qubit coherent state ρ_A accompanied by an N -qubit incoherent state $\sigma_{B_1 B_2 \dots B_n}$, the generated multi-qubit entanglements via multipartite incoherent operation Λ_I are upper bounded by operational coherences of initial quantum state

$$C_d(\rho_A) \geq \tau_{\text{MED}}[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \tag{20}$$

$$C_f(\rho_A) \geq \tau_{\text{MEF}}[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})], \tag{21}$$

where τ_{MED} and τ_{MEF} are multi-qubit entanglement indicators based on distillable entanglement and entanglement of formation.

Proof. We first consider the distillable coherence and multi-qubit entanglement indicator τ_{MED} and set $\rho_{AB_1 B_2 \dots B_n} = \Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})$. According to theorem 1, we have

$$C_d(\rho_A) \geq E_d(\rho_{A|B_1 B_2 \dots B_n}) \geq \left[\max \left\{ 0, E_d^2(\rho_{A|B_1 B_2 \dots B_n}) - \sum_{i=1}^n E_d^2(\rho_{AB_i}) \right\} \right]^{\frac{1}{2}} = \tau_{\text{MED}}(\rho_{AB_1 B_2 \dots B_n}), \tag{22}$$

where in the first inequality we choose the bipartite partition $A|B_1 B_2 \dots B_n$ for the distillable entanglement, and in the second inequality we use the nonnegative property of two-qubit distillable entanglements. For the coherence of formation, we can obtain

$$C_f(\rho_A) \geq E_f(\rho_{A|B_1 B_2 \dots B_n}) \geq \left[E_f^2(\rho_{A|B_1 B_2 \dots B_n}) - \sum_{i=1}^n E_f^2(\rho_{AB_i}) \right]^{\frac{1}{2}} = \tau_{\text{MEF}}(\rho_{AB_1 B_2 \dots B_n}), \tag{23}$$

where the first inequality holds due to equation (6) in theorem 1, and the second inequality comes from the monogamy property of squared entanglement of formation, which completes the proof of this theorem. □

Now we consider the optimal conversion from the coherence to quantum entanglement via multipartite incoherent operations. It was shown that the generalized controlled-not operation is the optimal bipartite incoherent operation in bipartite entanglement conversion [16], when the ancilla is in the zero state $|0\rangle$. The multipartite generalized controlled-not operation U_{mcn} can realize the transformation

$$|i\rangle |j_1\rangle \dots |j_n\rangle \rightarrow |i\rangle |(i + j_1) \bmod d\rangle \dots |(i + j_n) \bmod d\rangle, \tag{24}$$

which is the optimal multipartite incoherent operation Λ_I in the operational resource conversion described by the above theorems when the state of the ancilla is $\sigma_{B_1 \dots B_n} = |0\rangle \langle 0|^{\otimes n}$. In appendix A, we first show that the operation U_{mcn} is a multipartite incoherent operation different from that of the bipartite case, and then prove its optimality for the multipartite resource conversion in theorem 1 and its corollary. Under the optimal multipartite incoherent operation, the generated multipartite quantum state is $U_{\text{mcn}}[\rho_A \otimes |0\rangle \langle 0|^{\otimes n}_{B_1 \dots B_n}] U_{\text{mcn}}^\dagger$ and can be written as

$$\rho_{AB_1 B_2 \dots B_n}^{\text{mc}} = \sum_{m,n=0}^{d-1} \rho_{mn} |mm \dots m\rangle \langle nn \dots n|, \tag{25}$$

which is the so-called maximally correlated state (MCS) [52] and is multipartite entangled with all its reduced states being separable. For the case of MCS, we have the desired properties $E_d(\rho^{\text{mc}}) = E_r(\rho^{\text{mc}})$ [53] and $E_c(\rho^{\text{mc}}) = E_f(\rho^{\text{mc}})$ [54] in an arbitrary bipartite partition of multipartite systems $AB_1B_2 \dots B_n$, which saturate the inequalities in theorem 1 and make the resource conversions optimal. Since the saturated equalities in theorem 1 are satisfied in any bipartite partition, we can get that the conversions between coherence and multipartite entanglement in corollary 1 are also optimal according to the definition of E^{GME} . Moreover, the multipartite generalized controlled-not operation U_{mcn} is also optimal for multipartite resource conversions described by theorems 2 and 3, which makes the corresponding inequalities saturated and we give the detailed proofs in appendix C.

2.2. Resource conversion from multipartite entanglement to coherence via LICC

It is useful to establish the operational relations between entanglement and quantum coherence. Zhu *et al* proved a one-to-one mapping between the two kinds of resource measures based on convex roof extension, and showed the relation $E_f(\rho_{AB}) \leq C_f(\rho_{AB})$ in bipartite quantum states [13]. For multipartite systems, we can obtain the following theorem for operational resource measures.

Theorem 4. *In an N -partite quantum state $\rho_N = \rho_{A_1A_2 \dots A_n}$, the operational entanglements and quantum coherences are connected by the relations*

$$E_d(\rho_{\alpha|\bar{\alpha}}) \leq E_r^M(\rho_N) \leq C_d(\rho_N), \quad (26)$$

$$E_f^{\text{GME}}(\rho_N) \leq E_f(\rho_{\alpha|\bar{\alpha}}) \leq C_f(\rho_N), \quad (27)$$

where $\alpha|\bar{\alpha}$ is an arbitrary bipartite partition in the composite system, and the bounds are saturated by the MCSs.

Proof. For the N -partite quantum state, its distillable entanglement in the partition $\alpha|\bar{\alpha}$ is not greater than the relative entropy of entanglement $E_d(\rho_{\alpha|\bar{\alpha}}) \leq E_r(\rho_{\alpha|\bar{\alpha}})$ [35, 36]. According to the definitions of bipartite and multipartite relative entropy of entanglements, we have $E_r^M(\rho_N) = \min_{\{\sigma_s \in \mathcal{D}\}} S(\rho_N \| \sigma_s) = S(\rho_N \| \sigma_s^N) = S(\rho_{\alpha|\bar{\alpha}} \| \sigma_s^{\alpha|\bar{\alpha}}) \geq E_r(\rho_{\alpha|\bar{\alpha}})$, in which $\sigma_s^N = \sigma_s^{A_1A_2 \dots A_n}$ is the nearest N -partite separable state for E_r^M and $\sigma_s^{\alpha|\bar{\alpha}}$ is the bipartite partition expression of σ_s^N . Since the set of N -partite incoherent states is a subset of N -partite separable states, we can get $E_r^M(\rho_N) \leq C_r(\rho_N) = C_d(\rho_N)$, and then the inequality in equation (26) is satisfied. For the optimal pure state decomposition of $\rho_{\alpha|\bar{\alpha}}$, the bipartite entanglement of formation has the property $E_f(\rho_{\alpha|\bar{\alpha}}) = \sum_i p_i E_f(|\psi^i_{\alpha|\bar{\alpha}}\rangle) \geq \sum_i p_i E_f^{\text{GME}}(|\psi^i_{\alpha|\bar{\alpha}}\rangle) \geq E_f^{\text{GME}}(\rho_N)$. Combining this property with the relation $C_f(\rho_N) \geq E_f(\rho_{\alpha|\bar{\alpha}})$, we have the inequality in equation (27). Moreover, when the N -partite quantum state is the MCS $\rho_N^{\text{mc}} = \sum \rho_{mn} |mm \dots m\rangle \langle nn \dots n|$, we have $E_d(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = E_r^M(\rho_N^{\text{mc}}) = C_d(\rho_N^{\text{mc}})$ and $E_f^{\text{GME}}(\rho_N^{\text{mc}}) = E_f(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = C_f(\rho_N^{\text{mc}})$, which completes the proof. \square

Next, we consider the resource conversion from entanglement to quantum coherence in multipartite systems. In reference [17], the authors introduced a task of ACD in bipartite systems, where both parties work together to generate the maximal possible coherence on one of the subsystems and the operations are limited to the LQICC in which the target party performs local incoherent operations and the assisted party utilizes local quantum operations. It is noted that the task does not limit the output of the coherence in the assisted subsystem and local quantum operations can generate quantum coherence on the subsystem in general (for example, the projection measurement with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$).

Here, we consider the multi-party LICC scenario [29] where all the parties are in the distance labs and each local one can perform only local incoherent operations assisted by classical communication among different parties. In this situation, all of the local parties cannot create quantum coherence and we have the relation $\text{LICC} \subset \text{LQICC}$ for the two classes of operations due to the local incoherent operation being a subset of local quantum operation. In the cyclic resource conversion, since the entanglement in multipartite systems comes from quantum coherence via multipartite incoherent operations $\Lambda_I(\rho_A \otimes \sigma_{B_1B_2 \dots B_n})$, the procedure of conversion from entanglement to quantum coherence should be restricted to incoherent operations. Therefore, in the distance lab paradigm, it is desirable to confine the operations to the LICC, which is the free operation for the resource conversion in the whole multipartite system.

In the previous stage from single-party coherence to multipartite entanglement, the generic output state is $\varrho_{AB_1B_2 \dots B_n} = \Lambda_I(\rho_A \otimes \sigma_{B_1B_2 \dots B_n})$ which is not the MCS form under a multipartite incoherent operation assisted by an N -partite incoherent ancilla. In this case, the relations on operational entanglement and coherence are characterized by the bounds in theorem 4. When the local parties want to restore the coherence to the initial party A , they can use the protocol of ACD via the LICC [29], and the corresponding distillable coherence $C_{\text{LICC}}^{B_1B_2 \dots B_n|A}$ is upper bounded by its operational coherence $C_d(\varrho_{AB_1B_2 \dots B_n})$ since quantum coherence is monotone under the LICC. Moreover, the ACD via LICC is also upper bounded by the

quantum-incoherent (QI) relative entropy [17, 29]

$$C_{\text{LICC}}^{B_1 B_2 \dots B_n | A}(\rho_{AB_1 B_2 \dots B_n}) \leq C_r^{B_1 B_2 \dots B_n | A}(\rho_{AB_1 B_2 \dots B_n}) = \min_{\chi \in \mathcal{QI}} S(\rho \| \chi), \quad (28)$$

with χ being the QI state in the bipartition $A|B_1 B_2 \dots B_n$ of multipartite systems.

In the following, we consider the optimal output state under the multipartite incoherent operation U_{mch} which has the MCS form. Due to the MCS making the inequalities in theorem 4 saturated, we obtain the following theorem in the resource conversion from multipartite entanglement to the operational coherence of single-party system.

Theorem 5. *For the optimal output state under multipartite incoherent operations, its operational entanglements are equal to corresponding multipartite entanglements, which are the upper bounds on the converted coherence via the LICC*

$$E_r^M(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = E_d(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) \geq C_d[\Lambda_{\text{LICC}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}})] \geq C_d(\rho_r^{\text{LICC}}), \quad (29)$$

$$E_f^{\text{GME}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = E_f(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) \geq C_f[\Lambda_{\text{LICC}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}})] \geq C_f(\rho_r^{\text{LICC}}), \quad (30)$$

where $\alpha|\bar{\alpha}$ is an arbitrary bipartite partition in the multipartite systems, Λ_{LICC} is the multipartite transformation under the LICC, and $\rho_r^{\text{LICC}} = \text{Tr}_{\bar{r}}[\Lambda_{\text{LICC}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}})]$ is the reduced state of the multipartite system with \bar{r} being the traced subsystem.

Proof. According to theorem 4, we have the relation $E_d(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = E_r^M(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = C_d(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}})$ for the MCS. The first inequality in equation (29) is satisfied because the distillable coherence C_d is monotone under the free operation Λ_{LICC} . With the relation $C_d(\rho) = C_r(\rho)$ and the property that the relative entropy of a state is not increasing after tracing a subsystem out [34]

$$S(\text{Tr}_p \rho \| \text{Tr}_p \sigma) \leq S(\rho \| \sigma), \quad (31)$$

where Tr_p is a partial trace, we can obtain the second inequality in equation (29). Next, we analyze the resource conversion relations in equation (30) for which we have $E_f^{\text{GME}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = E_f(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = C_f(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}})$. The first inequality is satisfied due to the coherence of formation being monotone under the LICC. Furthermore, via the concavity of von Neumann entropy, we can obtain the entropy of diagonal state $\Delta(\psi_i)$ is not increasing after a partial trace, i.e.,

$$S(\text{Tr}_p \Delta(\psi_i)) \leq S(\Delta(\psi_i)). \quad (32)$$

Combining the above property with the convexity of C_f , we have the second inequality in equation (30), and then the proof is completed.

According to theorem 5, we know that the converted coherence in multipartite systems under a general LICC is upper bounded by the multipartite entanglement in the MCS. Moreover, since the LICC is a subset of the LOCC, the operational quantum entanglement for any initial state in the resource conversion is monotone under the LICC.

When the multipartite system is composed of qubits, the multipartite entanglement indicators τ_{MED} and τ_{MEF} are the upper bounds on the converted coherence via the LICC in the arbitrary subsystems,

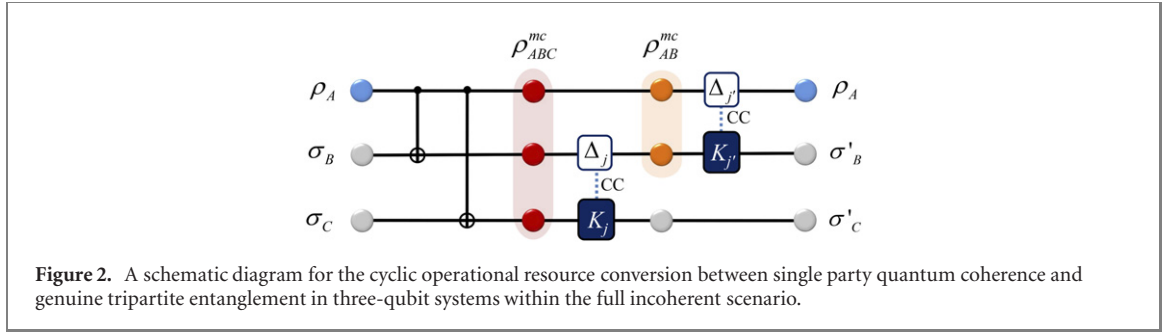
$$\tau_{\text{MED}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) \geq C_d(\rho_r^{\text{LICC}}), \quad (33)$$

$$\tau_{\text{MEF}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) \geq C_f(\rho_r^{\text{LICC}}), \quad (34)$$

where $\rho_r^{\text{LICC}} = \text{Tr}_{\bar{r}}[\Lambda_{\text{LICC}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}})]$ is the reduced state of the multipartite system with \bar{r} being the traced subsystem.

Next, we analyze the resource conversion from multipartite entanglement in the MCS to quantum coherence under the optimal LICC. In this case, the local incoherent operation can be chosen to be the set of Kraus operators $K_j = |j\rangle\langle\varphi_j|$ with $|\varphi_j\rangle_j = 1/\sqrt{d} \sum_k e^{i\phi_k^j} |k\rangle$ being the mutually orthogonal maximally coherent state and $j = 0, 1, \dots, d-1$ (in fact, all the mutually unbiased bases [55–57] except for the coherence-dependent basis can be used in the incoherent measurement). After the incoherent measurement K_j is performed on a subsystem, the quantum state of the remained subsystems can be transformed to the MCS form with the help of classical communication (measurement results on the j) and an incoherent unitary operation $U_j = \sum_k e^{i\phi_k^j} |k\rangle\langle k|$ [29].

Due to the state of remained subsystem being the MCS and having the same nonzero matrix elements, the operational entanglement and coherence are transferred to the subsystem and keep the same amount.



Repeating the incoherent measurement and the incoherent unitary operation on all the subsystems B_i , the quantum state of remained subsystem A becomes $\rho_A = \sum_{m,n=0}^{d-1} \rho_{mn} |m\rangle \langle n|$ and the coherence is restored to the single-party subsystem. In addition, it is a similar case to convert multipartite entanglement to the coherence of an arbitrary single-party subsystem B_i . Thus, we have completed the characterization on resource interconversion between quantum coherence and entanglement in multipartite systems within a full incoherent operation scenario.

2.3. Cyclic resource interconversion between operational coherence and multipartite entanglement within the full incoherent operation scenario

Recently, a cyclic interconversion between coherence and quantum correlation [42, 58, 59] has been investigated experimentally on bipartite optical systems [22], where the ACD via the LQICC is utilized. When the operations are restricted to the LICC, it is desirable to generalize the cyclic conversion to multipartite systems.

In this subsection, we show that the cyclic interconversion can be generalized to operational resource measures for multipartite systems within a full incoherent operation scenario. As shown in figure 2, a schematic diagram is given for the cyclic interconversion protocol of coherence-entanglement-coherence in three-qubit systems without loss. The initial coherent state is $\rho_A = \sum_{m,n=0}^1 \rho_{mn} |m\rangle \langle n|$ and the states of ancillas are $\sigma_B = \sigma_C = |0\rangle \langle 0|$. In the conversion from quantum coherence to entanglement, the optimal multipartite incoherent operation is the generalized tripartite controlled-not gate, which can be realized by two controlled-not gates. Then the output state of the tripartite system $U_{\text{mcn}}(\rho_A \otimes \sigma_B \otimes \sigma_C)U_{\text{mcn}}^\dagger$ is the MCS and can be expressed as

$$\rho_{ABC}^{\text{mc}} = \sum_{m,n=0}^1 \rho_{mn} |mmm\rangle \langle nnn|, \quad (35)$$

which is a genuine three-qubit entangled state belonging to the GHZ class under the classification of stochastic local operations and classical communication [60, 61]. The multipartite entanglements of the MCS are

$$E_r^M(\rho_{ABC}^{\text{mc}}) = \tau_{\text{MED}}(\rho_{ABC}^{\text{mc}}) = C_d(\rho_A), \quad (36)$$

$$E_f^{\text{GME}}(\rho_{ABC}^{\text{mc}}) = \tau_{\text{MEF}}(\rho_{ABC}^{\text{mc}}) = C_f(\rho_A). \quad (37)$$

Thus we realize the optimal conversion from single-qubit coherence to genuine three-qubit entanglements via multipartite incoherent operation. It is noted that, manipulating the structure of incoherent unitary operation U , one can not change the entangled state from GHZ class to W class. The incoherent unitary operation is restricted to $U = \sum_i |f(i)\rangle \langle i|$ with $f(i)$ being the one-to-one mapping from the fixed incoherent basis to itself. When the initial state is a coherent qubit plus two incoherent ancilla, the output state under the multipartite incoherent U has at most two coherent terms in the fixed basis, while the entangled state of W class has three coherent terms $a|100\rangle + b|010\rangle + c|001\rangle$ [61].

The tripartite entanglement can be transferred to bipartite systems by an incoherent measurement $\{K_j\}$ with the Kraus operators $K_0 = |0\rangle \langle +|$ and $K_1 = |1\rangle \langle -|$ in which $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Via the classical communication ($j = 0$ or 1) between subsystems B and C, the observer B performs the corresponding incoherent operation Δ_j on subsystem B where $\Delta_0 = I_2$ and $\Delta_1 = \sigma_z$. After the operation Δ_j , the genuine three-qubit entanglement is transferred to two-qubit subsystem and its quantum state is $\rho_{AB}^{\text{LICC}} = \sum_{m,n=0}^1 \rho_{mn} |mm\rangle \langle nn|$. In this case, the operational bipartite entanglement and genuine three-qubit entanglement are connected by equations $E_r^M(\rho_{ABC}) = E_d(\rho_{AB}^{\text{LICC}})$ and $E_f^{\text{GME}}(\rho_{ABC}) = E_f(\rho_{AB}^{\text{LICC}})$. Furthermore, the two-qubit entanglement can be converted to single-party quantum coherence of subsystem A by a set of similar local incoherent operations $\{K_j, \Delta_j\}$ assisted by the classical communication about the incoherent measurement

result $j' = 0$ or 1 . After these operations, the quantum state of local subsystem A has the form

$$\rho_A^{\text{LICC}} = \sum_{m,n=0}^1 \rho_{mn} |m\rangle \langle n|, \quad (38)$$

which is the same as that of initial single qubit coherent state ρ_A , and the operational bipartite entanglements are restored to single-qubit operational coherence, i.e., $E_d(\rho_{AB}^{\text{LICC}}) = C_d(\rho_A^{\text{LICC}})$ and $E_f(\rho_{AB}^{\text{LICC}}) = C_f(\rho_A^{\text{LICC}})$. In a similar way, we can choose to convert the entanglement to the single-party coherence in subsystem B when we perform the incoherent measurement on subsystem A.

We have shown that, within a full incoherent operation scenario, the single-qubit coherence and multipartite entanglement can be cyclically interconverted without loss

$$C_d(\rho_A) = E_r^M(\rho_{ABC}^{\text{mc}}) = C_d(\rho_A^{\text{LICC}}), \quad (39)$$

$$C_f(\rho_A) = E_f^{\text{GME}}(\rho_{ABC}^{\text{mc}}) = C_f(\rho_A^{\text{LICC}}), \quad (40)$$

where the multipartite relative entropy of entanglement and the GME measure based on entanglement of formation are not equal in general since the coherence resource theory is irreversible. The cyclic resource conversion can also be generalized to multipartite multi-level systems.

3. Genuine multi-level entanglement and resource dynamical properties

Entanglement of high-dimensional quantum systems is an important resource, which can enhance the capabilities of certain quantum communication protocols [62, 63]. There is a kind of high-dimensional bipartite entanglement that cannot be simulated with copies of low-dimensional bipartite systems, which is referred to as genuine multilevel bipartite entanglement [64]. For example, although two Bell pairs can mimic the test of nonlocality [65] in a $4 \otimes 4$ entangled state, there exist the genuine four-dimensional bipartite entangled states which cannot be decomposed into two Bell-like states [66, 67]. In the case that the multi-level bipartite entanglement can be simulated with the copies of low-dimensional systems, the entanglement is called as decomposable multi-level entanglement. Kraft *et al* presented a general theory to characterize the genuine multi-level entanglement [64]. In particular, for the two-ququart state after the Schmidt decomposition $|\psi\rangle_{AB} = s_0|00\rangle + s_1|11\rangle + s_2|22\rangle + s_3|33\rangle$ (with the assumption $s_0 \geq s_1 \geq s_2 \geq s_3$), they prove that the entangled state $|\psi\rangle_{AB}$ is decomposable if and only if the determinant of matrix $S = [s_0, s_1; s_2, s_3]$ is zero.

In the resource conversion from quantum coherence to entanglement, the generated quantum state is a multi-level entangled when the input single party state is multi-level coherent. It is an interesting problem that whether or not one can judge the multi-level entanglement property via the initial quantum coherent state. Here, we consider a four-level coherent state

$$|\varphi\rangle_A = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle, \quad (41)$$

where, without loss of generality, we assume that the moduli of amplitudes obey the relation $|\alpha_0| \geq |\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$. In the optimal bipartite resource conversion, the generated bipartite entangled state is a four-level MCS $\rho_{AB}^{\text{mc}} = |\varphi\rangle \langle\varphi|_{AB}$ with $|\varphi\rangle_{AB} = \alpha_0|00\rangle + \alpha_1|11\rangle + \alpha_2|22\rangle + \alpha_3|33\rangle$, and its multi-level entanglement property is related to the coefficients of initial coherent state. After analyzing the determinant of S matrix for the MCS, we can obtain the following observation.

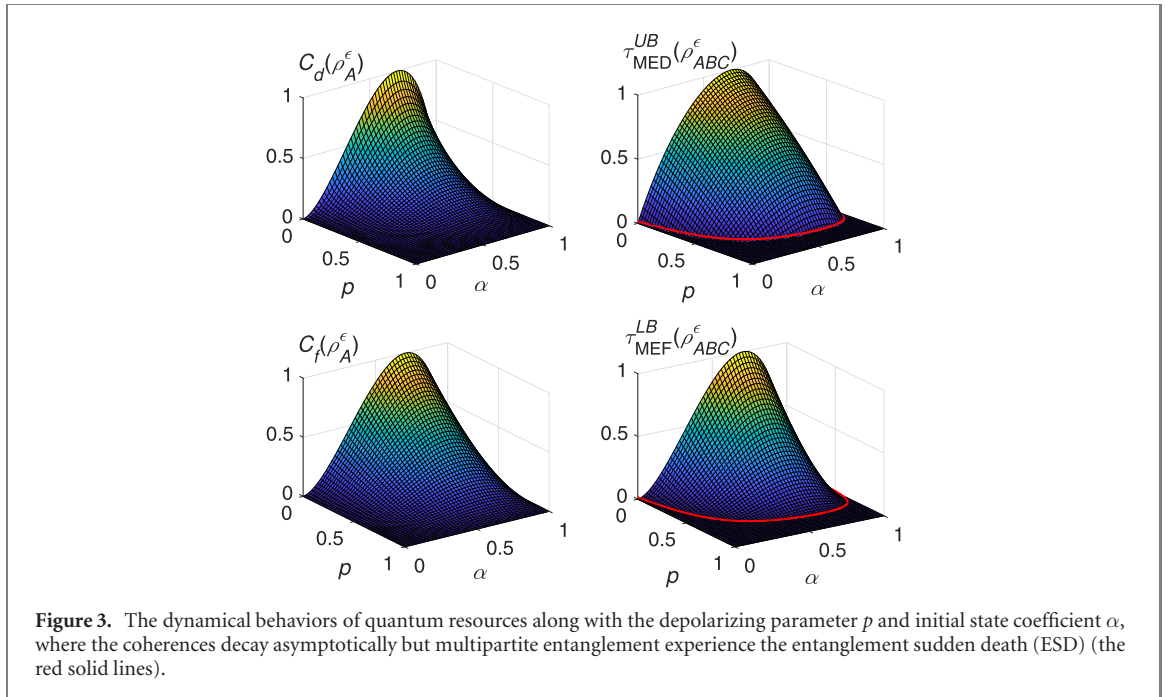
Observation 1. The generated two-ququart state $|\varphi\rangle_{AB}$ in the optimal resource conversion is decomposable if and only if the coefficients of initial coherent state satisfy the relation

$$|\alpha_0\alpha_3| = |\alpha_1\alpha_2|, \quad (42)$$

and the bipartite quantum state is the genuine multi-level entangled when the equality is violated.

We note that, when the initial coherent state can be decomposed into the tensor product state of two qubits $|\varphi\rangle_A = (a|0\rangle + b|1\rangle)_{A_1} \otimes (c|0\rangle + d|1\rangle)_{A_2}$, the equality in equation (42) is satisfied and then the generated entanglement in the conversion is decomposable. However, when the initial coherent state is not tensor product, for example, $|\varphi\rangle_{A_1A_2} = \alpha|+\rangle|0\rangle + \beta|-\rangle|1\rangle$, the output entangled state is also decomposable. Therefore, we conclude that the decomposable property of initial coherent state is only a sufficient but not necessary condition for the decomposable property of the generated entangled state in the resource conversion.

Resource dynamics is a fundamental problem in the practical quantum information processing, because quantum systems interact unavoidably with the environment and may lose their quantum coherence or



entanglement. In the operational resource conversion between quantum coherence and multipartite entanglement, it is worth analyzing the dynamical behaviors of two kinds of resources under the typical noise environment. Here, we investigate the case of the optimal conversion in three-qubit systems as shown in figure 2. The initial coherent state is chosen to be $|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$ with real coefficients and the output MCS has the form $|\psi\rangle_{ABC}^{mc} = \alpha|000\rangle + \beta|111\rangle$ after the operations of two controlled-not gates. The noise environment we consider is the depolarizing channel $\mathcal{E}(\rho) = pI/d + (1-p)\rho$ [68] with the parameter p being the depolarization probability, under which the coherent state and the converted entangled state are $\rho_A^\epsilon = \mathcal{E}(|\psi\rangle_A \langle\psi|)$ and $\rho_{ABC}^\epsilon = \mathcal{E}(|\psi\rangle_{ABC}^{mc} \langle\psi|)$, respectively. After some calculation, we can obtain the values of two operational coherence measures

$$C_d(\rho_A^\epsilon) = h(x_1) - h(p/2), \tag{43}$$

$$C_f(\rho_A^\epsilon) = h\left[(1 + \sqrt{1 - x_2^2})/2\right], \tag{44}$$

where $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy, and the parameters are $x_1 = (1-p)\alpha^2 + p/2$ and $x_2 = 2(1-p)\alpha\beta$, respectively.

The distillable entanglement is very difficult to compute in a generic mixed state, but the measure E_d is upper bounded by the logarithmic negativity $E_N(\rho) = \log_2 \|\rho^{TA}\|_1$ [69, 70]. On the other hand, Chen *et al* gave a tight lower bound for entanglement of formation in an arbitrary bipartite mixed state, and for the qubit–qudit system, the lower bound can be expressed as a binary entropy function $E_f^{LB}(\rho) = h[(1 + \sqrt{1 - (\Lambda - 1)^2})/2]$ in which Λ can be the trace norm $\|\rho^{TA}\|_1$ (or the norm of realignment matrix) [71]. For the generated entangled state under the depolarizing channel, we can derive the multipartite entanglement indicators via the corresponding bounds of the operational entanglement measures,

$$\begin{aligned} \tau_{MED}^{UB}(\rho_{ABC}^\epsilon) &= \left[E_d^{UB}(\rho_{A|BC}^\epsilon)^2 - \sum_{k \in \{B,C\}} E_d^2(\rho_{Ak}^\epsilon) \right]^{1/2}, \\ &= \log_2(1 + \zeta/4), \end{aligned} \tag{45}$$

$$\tau_{MEF}^{LB}(\rho_{ABC}^\epsilon) = \left[E_f^{LB}(\rho_{A|BC}^\epsilon)^2 - \sum_{k \in \{B,C\}} E_f^2(\rho_{Ak}^\epsilon) \right]^{1/2} = h[(\sqrt{\omega} + \sqrt{2 - \omega})^2/4], \tag{46}$$

where we use the property that the operational entanglement E_d and E_f are zero in two-qubit subsystems, and the parameters are $\zeta = \max[0, 8(1-p)|\alpha\beta| - p]$, $\omega = \|(\rho_{A|BC}^\epsilon)^{TA}\|_1$, respectively.

In figure 3, we plot the resource measures along with the depolarization parameter p and the state coefficient α , where the coherences C_d and C_f decay in an asymptotic way but the multipartite

entanglements $\tau_{\text{MED}}^{\text{UB}}$ and $\tau_{\text{MEF}}^{\text{LB}}$ disappear in finite time (i.e., ESD). After some calculation, we have the ESD lines for the multipartite entanglement in the figure, which have the same expression,

$$p = \frac{8\alpha\sqrt{1-\alpha^2}}{1+8\alpha\sqrt{1-\alpha^2}}. \quad (47)$$

These behaviors coincide with the previous study on quantum correlation dynamics [72–75], where quantum systems experience disentanglement in a finite time under the noise environment, even if their coherence is lost asymptotically.

4. Discussion and conclusion

In the resource conversion from coherence to entanglement, we restrict the operation to multipartite incoherent operation [11] which is different from the bipartite case [16, 22]. For the generated multipartite entangled state, its distillable entanglement and entanglement of formation in an arbitrary bipartite partition $\alpha|\bar{\alpha}$ are upper bounded by the corresponding operational single-party coherences. Furthermore, the output state is multipartite entangled in general, and we have established a set of relationships between single-party coherence and multipartite entanglement. In particular, we have shown that the multipartite generalized controlled-not operation U_{mcn} is the optimal operation which makes all the resource conversion relationships saturated.

The operational framework between single-party coherence and multipartite entanglement can be further generalized. When the initial state is chosen to be a multipartite coherent state $\rho_{AB_1B_2\dots B_n}$, the similar relations like those in theorems 1–3 and corollary 1 are still satisfied where the distillable coherence and coherence of formation are replaced with $C_d(\rho_{AB_1B_2\dots B_n})$ and $C_f(\rho_{AB_1B_2\dots B_n})$, and the bipartite or multipartite entanglement $E[\Lambda_I(\rho_A \otimes \sigma_{B_1B_2\dots B_n})]$ is changed to $E[\Lambda_I(\rho_{AB_1B_2\dots B_n})]$. That is to say, the coherence of the whole initial multipartite systems will set the upper bound over the converted entanglements. In a specific case that the B subsystem starts off with some amount of coherence and the initial state is chosen to be $\rho_A \otimes \rho_{B_1B_2\dots B_n}$, the converted bipartite and multipartite entanglements via multipartite incoherent operations are bounded by $C(\rho_A) + C(\rho_{B_1B_2\dots B_n})$ due to the additive property of C_r and C_f [26]. However, in the above cases, the multipartite generalized controlled-not operation U_{mcn} is not the optimal multipartite incoherent operation, and the exploration for the optimal operation is still an open problem need to be studied in future.

In a generic N -partite quantum state $\rho_{A_1A_2\dots A_n}$, we prove that the multipartite entanglements E_f^{M} and E_f^{GME} are not larger than its operational coherences C_d and C_f , and the bounds are saturated for the MCS. Moreover, the conversion from multipartite entanglement of the MCS to single-party coherence is restricted to the LICC [29], which makes the cyclic resource conversion within a full incoherent operation scenario. In contrast to the LQICC protocol [17, 22], the LICC scheme keeps the coherence of whole multipartite system from increasing. In section 2.3, we presented the scheme of cyclic resource interconversion in three-qubit systems where the single party coherence, bipartite entanglement and genuine tripartite entanglement can be freely interconverted without loss.

The cyclic resource conversion without loss [22] can provide potential flexibilities on utilizing quantum coherence or multipartite entanglement to perform certain tasks in quantum information processing, where one may obtain the operational benefits from the resource conversion. Moreover, the cyclic conversion makes quantum coherence and multipartite entanglement be compared quantitatively under the full incoherent operation scenario, which is useful to give a unified characterization on the two kinds of resources in a certain operational framework.

High-dimensional entanglement can enhance the capabilities of quantum communication protocols [62, 63]. In the optimal resource conversion from coherence to entanglement, we have provided a method for detecting genuine bipartite multi-level entanglement [64] via the coefficients of initial coherent state. This method can be further generalized to multipartite case. For an N -partite output state $|\psi_N\rangle$ in the optimal conversion, we can judge whether it is genuine bipartite multi-level entangled by the initial coherent coefficients. When $|\psi_N\rangle$ is not decomposable in an arbitrary partition $\alpha|\bar{\alpha}$, we can obtain that the quantum state is genuine multipartite multi-level entangled [64]. It should be noted that the decomposability of initial coherent state is not equivalent to that of the generated multi-level entangled state.

In conclusion, we have established a set of resource conversion relationships between coherence and entanglements in multipartite systems within a full incoherent operation scenario, where the operational resource measures and related multipartite quantifiers are focused. Via the multipartite incoherent operation and the ACD by LICC where coherence of the global state is not a freely available resource, we

can realize the interconversion between single-party coherence and multipartite entanglement. Moreover, through the procedure of resource conversion, we have been able to bridge the coherent states and genuine multi-level entangled states by the initial coherent coefficients, and to analyze the asymptotical decay of coherence and ESD behavior of multipartite entanglement in a noise environment. By uncovering the operational connection between coherence and entanglement in multipartite systems, the present work provides a set of useful tools for QRT in many-body systems.

Acknowledgments

Y-KB would like to thank M-Y Ye and M-L Hu for many useful discussions. This work was supported by the URC fund of HKU, NSF-China (Grants Nos. 11575051, 11947090, and 11904078), Hebei NSF (Grants Nos. A2016205215, A2019205263 and A2019205266), and project of China Postdoctoral Science Foundation (Grant No. 2020M670683). JR and L-HR were also funded by Science and Technology Project of Hebei Education Department (Grant No. QN2019092) and Hebei Normal University (Grants Nos. L2018B02 and L2019B07).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. The optimal multipartite incoherent operation

The coherence resource theory formulated by Baumgratz *et al* [4] can be extended into multipartite scenario [11]. The N -partite incoherent state has the form

$$\sigma_n = \sum_{\vec{i}} p_{\vec{i}} |\vec{i}\rangle \langle \vec{i}|, \quad (\text{A.1})$$

where $p_{\vec{i}}$ are probabilities and $|\vec{i}\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_n\rangle$ with $|i_k\rangle$ being a pre-fixed local basis of the k th subsystem. The N -partite incoherent operation can be described by a completely positive trace preserving map Λ_I which has the form

$$\Lambda_I(\rho) = \sum_l K_l \rho K_l^\dagger, \quad (\text{A.2})$$

where the set of Kraus operators $\{K_l\}$ satisfies the properties $\sum_l K_l^\dagger K_l = I$ and $K_l \mathcal{I} K_l^\dagger \subset \mathcal{I}$ with \mathcal{I} being now the set of N -partite incoherent states.

Below equation (24) of the main text, we pointed out that the multipartite generalized controlled-not operation U_{mcn} is the optimal multipartite incoherent operation in the conversion from quantum coherence to entanglement in multipartite systems. Here, we first show that U_{mcn} is a multipartite incoherent operation. In this case, we consider that the incoherent state $\sigma_{AB_1 \dots B_n}$ is $(N+1)$ -partite, where the pre-fixed basis is $\{|i\rangle \otimes |j_1\rangle \otimes \dots \otimes |j_n\rangle\}$ with $|i\rangle$ being local basis of subsystem A and $|j_k\rangle$ local basis of the k th subsystem B_k . The multipartite generalized controlled-not operation is unitary and can be written as

$$\begin{aligned} U_{\text{mcn}} = & \sum_{i=0}^{d-1} \sum_{j_1=0}^{d-1} \dots \sum_{j_n=0}^{d-1} |i\rangle \langle i| \otimes |(i+j_1) \bmod d\rangle \langle j_1| \otimes \dots \\ & \times \otimes |(i+j_n) \bmod d\rangle \langle j_n|, \end{aligned} \quad (\text{A.3})$$

where the dimensions of all the local systems are equal and the operation can realize the transformation $|i\rangle |j_1\rangle \dots |j_n\rangle \rightarrow |i\rangle |(i+j_1) \bmod d\rangle \dots |(i+j_n) \bmod d\rangle$. Moreover, when the dimension d_k of subsystem B_k is larger than that of subsystem A , the transformation U_{mcn} does not change the prefixed basis for $j_k \geq d$. In the case that the dimension d_k of subsystem B_k is less than that of subsystem A , one can add an extra ancilla B'_k to the subsystem B_k , which can enlarge the Hilbert space of the new subsystem $\tilde{B}_k = B_k B'_k$. Therefore, the operation U_{mcn} maps the set of pre-fixed basis of multipartite systems into itself, and then it is a multipartite incoherent operation satisfying the property shown in equation (A.2).

Next, we analyze the optimality of the multipartite generalized controlled-not operation U_{mcn} in the operational resource conversion of multipartite systems. In theorem 1 of the main text, we prove that the operational coherences of single party A are not less than the generated entanglement of multipartite systems $AB_1 \dots B_n$ under a multipartite incoherent operation Λ_I . When the initial state $\rho_{AB_1 \dots B_n}$ is a

single-party coherent state $\rho_A = \sum_{mn} \rho_{mn} |m\rangle\langle n|$ accompanied by the ancillary state $\sigma_{B_1 B_2 \dots B_n} = |0\rangle\langle 0|^{\otimes n}$, we can choose the multipartite incoherent operation Λ_1 to be U_{mcn} . Then the output state is

$$\begin{aligned} \Lambda_1(\rho_{AB_1 \dots B_n}) &= U_{\text{mcn}} \left(\rho_A \otimes |0\rangle\langle 0|^{\otimes n}_{B_1 \dots B_n} \right) U_{\text{mcn}}^\dagger \\ &= \sum_{m,n=0}^{d-1} \rho_{mn} |mm \dots m\rangle\langle nn \dots n| \\ &= \rho_{AB_1 \dots B_n}^{\text{mc}}. \end{aligned} \tag{A.4}$$

where $\rho_{AB_1 \dots B_n}^{\text{mc}}$ is the MCS in the $(N + 1)$ -partite systems. Since the MCS has the properties $E_d(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = C_d(\rho_A)$ and $E_c(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = E_f(\rho_{\alpha|\bar{\alpha}}^{\text{mc}}) = C_f(\rho_A)$ with $\alpha|\bar{\alpha}$ being an arbitrary bipartite partition in the multipartite systems, we can obtain that the inequalities in theorem 1 and corollary 1 are saturated and the operation U_{mcn} is the optimal multipartite incoherent operation in the resource conversion.

It should be noted that the multipartite incoherent operation is different from that of the bipartite case. Therefore, the results in theorem 1 is not a trivial extension in which the subsystems $B_1 B_2 \dots B_n$ are regarded as a whole system B . For example, when the input state is a single-qubit maximally coherent state $|+\rangle_A = (|0\rangle + |1\rangle)/\sqrt{2}$ accompanied by two ancillary qubits $|00\rangle_{B_1 B_2}$, the output state under the optimal tripartite incoherent operation (tripartite generalized controlled-not gate) is a GHZ state $(|000\rangle_{AB_1 B_2} + |111\rangle_{AB_1 B_2})/\sqrt{2}$, which is a genuine tripartite entangled state and makes the inequalities in theorem 1 saturated in an arbitrary partition such as $AB_1|B_2$ and $AB_2|B_1$. Meantime, the multipartite entanglement inequalities in corollary 1 are saturated too. However, when the input state is $|+\rangle_A |0\rangle_B$ in $2 \otimes 4$ systems, the output state for the optimal bipartite incoherent operation (bipartite controlled-not gate) is a bipartite Bell state $(|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$.

Appendix B. Monogamy property of distillable entanglement in multi-qubit systems

In equation (18) of the main text, we define the multipartite entanglement indicator

$$\tau_{\text{MED}}(\rho_N) = \left[\max \left\{ 0, E_d^2(\rho_{A_1|\bar{A}_1}) - \sum_{i=2}^n E_d^2(\rho_{A_1 A_i}) \right\} \right]^{\frac{1}{2}}, \tag{B.1}$$

which is based on the distribution of squared distillable entanglement. Entanglement monogamy is an important property in many-body quantum systems and the residual entanglement can characterize the GME in the composite system.

We first analyze the monogamy relation of the squared distillable entanglement in an N -qubit pure state $|\psi\rangle_N = |\psi\rangle_{A_1 A_2 \dots A_n}$. In this case, the multipartite entanglement indicator $\tau_{\text{MED}}(\rho_N)$ is effective and we have the property

$$E_d^2(|\psi\rangle_{A_1|A_2 \dots A_n}) - \sum_{i=2}^n E_d^2(\rho_{A_1 A_i}) \geq E_f^2(|\psi\rangle_{A_1|A_2 \dots A_n}) - \sum_{i=2}^n E_f^2(\rho_{A_1 A_i}) \geq 0, \tag{B.2}$$

where $E_d(|\psi\rangle_{A_1|A_2 \dots A_n})$ quantifies bipartite distillable entanglement in the N -qubit system and $E_d(\rho_{A_1 A_i})$ two-qubit distillable entanglement, the relations $E_d(|\psi\rangle_{A_1|A_2 \dots A_n}) = E_f(|\psi\rangle_{A_1|A_2 \dots A_n})$ and $E_f(\rho_{A_1 A_i}) \geq E_d(\rho_{A_1 A_i})$ are used in the first inequality, and the monogamy property of E_f^2 [47] is utilized in the second inequality.

For the case of mixed states $\rho_{A_1 A_2 \dots A_n}$, it is still an open problem that whether the distillable entanglement is monogamous. However, when the residual entanglement is positive, namely, $E_d^2(|\psi\rangle_{A_1|A_2 \dots A_n}) - \sum_{i=2}^n E_d^2(\rho_{A_1 A_i}) > 0$, the indicator τ_{MED} can indicate that there exists GME which cannot be stored in two-qubit subsystems. For example, in the optimal resource conversion from coherence to entanglement, the output state has the form $\rho_{N+1}^{\text{mc}} = \rho_{AB_1 B_2 \dots B_n}^{\text{mc}} = \sum_{m,n=0}^{d-1} \rho_{mn} |mm \dots m\rangle\langle nn \dots n|$. Since all the two-qubit reduced states of ρ_{N+1}^{mc} are separable, we have

$$\begin{aligned} \tau_{\text{MED}}(\rho_{N+1}^{\text{mc}}) &= \left[E_d^2(\rho_{A|B_1 B_2 \dots B_n}^{\text{mc}}) - \sum_{i=1}^n E_d^2(\rho_{A_1 B_i}) \right]^{1/2} \\ &= E_d(\rho_{A|B_1 B_2 \dots B_n}^{\text{mc}}), \end{aligned} \tag{B.3}$$

which indicates the existence of the GME in multi-qubit systems. Moreover, it is a similar situation for the entanglement dynamics under depolarizing noise that the nonzero $\tau_{\text{MED}}^{\text{UB}}(\rho_{\text{ABC}}^\epsilon)$ indicates the genuine three-qubit entanglement as shown in figure 3 of the main text.

Appendix C. The optimality of U_{mcn} for multipartite entanglement conversion

In theorem 2 of the main text, when the initial single-party coherent state $\rho_A = \sum \rho_{mn} |m\rangle\langle n|$ is accompanied by ancilla state being in $|00 \dots 0\rangle_{B_1 B_2 \dots B_n}$, the output state under multipartite generalized controlled-not operation U_{mcn} is

$$\rho_{AB_1 B_2 \dots B_n}^{\text{mc}} = \sum_{m,n=0}^{d-1} \rho_{mn} |mm \dots m\rangle \langle nn \dots n|. \quad (\text{C.1})$$

For this output state, the multipartite relative entropy of entanglement has the property

$$\begin{aligned} E_r^M(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) &= S(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}} \| \sigma_s^{N+1}) = S(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}} \| \sigma_s^{A|B_1 B_2 \dots B_n}) \\ &\geq E_r(\rho_{A|B_1 B_2 \dots B_n}^{\text{mc}}) \geq S(\rho_A^d) - S(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = C_d(\rho_A), \end{aligned} \quad (\text{C.2})$$

where in the first equality we use the nearest $(N+1)$ -partite separable state σ_s^{N+1} for multipartite relative entropy of entanglement, in the second equality we cut the $(N+1)$ -partite separable state into bipartite partition $A|B_1 B_2 \dots B_n$ which results in the relative entropy being not less than the bipartite relative entropy of entanglement, in the second inequality we use the lower bound for $E_r(\rho)$ [35, 36], and in the last equality we use the relation $S(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = S(\rho_A)$ and the definition of the distillable coherence. Combining the relation $C_d(\rho_A) \geq E_r^M[\Lambda_I(\rho_A \otimes \sigma_{B_1 B_2 \dots B_n})]$ in theorem 2 with equation (C.2), we have

$$E_r^M(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = C_d(\rho_A), \quad (\text{C.3})$$

and then the multipartite generalized controlled-not operation U_{mcn} is the optimal multipartite incoherent operation. Next, we consider the GME based on entanglement of formation, for which the value of $\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}$ under optimal pure state decomposition $\{p_i, |\psi_i\rangle\}$ is

$$\begin{aligned} E_f^{\text{GME}}(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) &= \sum_i p_i E_f^{\text{GME}}(|\psi_i\rangle) = \sum_i p_i E_f(|\psi_i\rangle_{\alpha|\bar{\alpha}}) \\ &= \sum_i p_i E_f(|\psi_i\rangle_{A|B_1 B_2 \dots B_n}) = \sum_i p_i C_f(|\psi_i\rangle_{AB_1 B_2 \dots B_n}) \\ &= C_f(\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}) = C_f(\rho_A), \end{aligned} \quad (\text{C.4})$$

where in the second equality we use the minimal entanglement in bipartite partition $\alpha|\bar{\alpha}$, in the third equality we use the property that any $|\psi_i\rangle$ in the support of the MCS has the form $|\psi_i\rangle = \sum_j q_j |jj \dots j\rangle$ with $\sum_j |q_j|^2 = 1$ and the minimal entanglement can choose the partition $A|B_1 B_2 \dots B_n$, in the fourth equality the property $E_f(|\psi_i\rangle) = C_f(|\psi_i\rangle)$, and the last equality holds since $\rho_{AB_1 B_2 \dots B_n}^{\text{mc}}$ and ρ_A have the same nonzero matrix elements. Therefore, we obtain that the multipartite generalized controlled-not operation U_{mcn} is optimal in the resource conversion from single-party coherence of formation to the GME based on entanglement of formation.

In theorem 3 of the main text, we prove that, in the resource conversion via multipartite incoherent operation, the operational coherences of single-party system are not less than the multipartite entanglement indicators based on operational entanglements. When we choose the operation U_{mcn} , the output state is the MCS state in equation (C.1) for which its multipartite entanglement indicators are

$$\tau_{\text{MED}} = E_d(\rho_{A|B_1 B_2 \dots B_n}^{\text{mc}}) = C_d(\rho_A), \quad (\text{C.5})$$

$$\tau_{\text{MEF}} = E_f(\rho_{A|B_1 B_2 \dots B_n}^{\text{mc}}) = C_f(\rho_A), \quad (\text{C.6})$$

where we use the property of MCS that its two-qubit reduced state ρ_{AB_i} is separable, and the operational entanglements of MCS are equal to the operational coherences. Such that the U_{mcn} is optimal.

ORCID iDs

Yan-Kui Bai  <https://orcid.org/0000-0002-6750-6044>

References

- [1] Einstein A, Podolsky B and Rosen N 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47** 777

- [2] Bohr N 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **48** 696
- [3] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Quantum entanglement *Rev. Mod. Phys.* **81** 865
- [4] Baumgratz T, Cramer M and Plenio M B 2014 Quantifying coherence *Phys. Rev. Lett.* **113** 140401
- [5] Aberg J 2006 Quantifying superposition (arXiv:quant-ph/0612146)
- [6] Streltsov A, Adesso G and Plenio M B 2017 Colloquium: quantum coherence as a resource *Rev. Mod. Phys.* **89** 041003
- [7] Hu M-L, Hu X, Wang J, Peng Y, Zhang Y-R and Fan H 2018 Quantum coherence and geometric quantum discord *Phys. Rep.* **762–764** 1–100
- [8] Horodecki M, Oppenheim J and Winter A 2005 Partial quantum information *Nature* **436** 673–6
- [9] Chitambar E and Hsieh M-H 2016 Relating the resource theories of entanglement and quantum coherence *Phys. Rev. Lett.* **117** 020402
- [10] Streltsov A, Chitambar E, Rana S, Bera M N, Winter A and Lewenstein M 2016 Entanglement and coherence in quantum state merging *Phys. Rev. Lett.* **116** 240405
- [11] Yao Y, Xiao X, Ge L and Sun C P 2015 Quantum coherence in multipartite systems *Phys. Rev. A* **92** 022112
- [12] Tan K C, Kwon H, Park C-Y and Jeong H 2016 Unified view of quantum correlations and quantum coherence *Phys. Rev. A* **94** 022329
- [13] Zhu H, Ma Z, Cao Z, Fei S-M and Vedral V 2017 Operational one-to-one mapping between coherence and entanglement measures *Phys. Rev. A* **96** 032316
- [14] Zhu H, Hayashi M and Chen L 2018 Axiomatic and operational connections between the l_1 -norm of coherence and negativity *Phys. Rev. A* **97** 022342
- [15] Zhou H, Yuan X and Ma X 2019 Unification of quantum resources in distributed scenarios *Phys. Rev. A* **99** 022326
- [16] Streltsov A, Singh U, Dhar H S, Bera M N and Adesso G 2015 Measuring quantum coherence with entanglement *Phys. Rev. Lett.* **115** 020403
- [17] Chitambar E, Streltsov A, Rana S, Bera M N, Adesso G and Lewenstein M 2016 Assisted distillation of quantum coherence *Phys. Rev. Lett.* **116** 070402
- [18] Killoran N, Steinhoff F E S and Plenio M B 2016 Converting nonclassicality into entanglement *Phys. Rev. Lett.* **116** 080402
- [19] Ma J, Yadin B, Girolami D, Vedral V and Gu M 2016 Converting coherence to quantum correlations *Phys. Rev. Lett.* **116** 160407
- [20] Xi Y, Zhang T, Zheng Z-J, Li-Jost X and Fei S-M 2019 Converting quantum coherence to genuine multipartite entanglement and nonlocality *Phys. Rev. A* **100** 022310
- [21] Wu K-D, Hou Z, Zhong H-S, Yuan Y, Xiang G-Y, Li C-F and Guo G-C 2017 Experimentally obtaining maximal coherence via assisted distillation process *Optica* **4** 454
- [22] Wu K-D et al 2018 Experimental cyclic interconversion between coherence and quantum correlations *Phys. Rev. Lett.* **121** 050401
- [23] Qiao L-F et al 2018 Entanglement activation from quantum coherence and superposition *Phys. Rev. A* **98** 052351
- [24] Wang W et al 2019 Witnessing quantum resource conversion within deterministic quantum computation using one pure superconducting qubit *Phys. Rev. Lett.* **123** 220501
- [25] Chitambar E and Gour G 2019 Quantum resource theories *Rev. Mod. Phys.* **91** 025001
- [26] Winter A and Yang D 2016 Operational resource theory of coherence *Phys. Rev. Lett.* **116** 120404
- [27] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Mixed-state entanglement and quantum error correction *Phys. Rev. A* **54** 3824
- [28] Hayden P M, Horodecki M and Terhal B M 2001 The asymptotic entanglement cost of preparing a quantum state *J. Phys. A: Math. Gen.* **34** 6891–8
- [29] Streltsov A, Rana S, Bera M N and Lewenstein M 2017 Towards resource theory of coherence in distributed scenarios *Phys. Rev. X* **7** 011024
- [30] Two-way LOCC operations can be written as $\Gamma_1(\rho_{AB}) = \sum_i (A_i \otimes I_{\rho_{AB}} A_i^\dagger \otimes I) \otimes |i\rangle\langle i|_B$ and $\Gamma_2(\rho_{AB}) = \sum_i (I \otimes B_i \rho_{AB} I \otimes B_i^\dagger) \otimes |i\rangle\langle i|_{A'}$ in which $\sum_i A_i^\dagger A_i = I_A$ and $\sum_i B_i^\dagger B_i = I_B$, and the form of other class of LOCC can see section 11.2 in reference [3].
- [31] Hastings M B 2009 Superadditivity of communication capacity using entangled inputs *Nat. Phys.* **5** 255
- [32] Fukuda M and Wolf M M 2007 Simplifying additivity problems using direct sum constructions *J. Math. Phys.* **48** 072101
- [33] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 Quantifying entanglement *Phys. Rev. Lett.* **78** 2275
- [34] Vedral V 2002 The role of relative entropy in quantum information theory *Rev. Mod. Phys.* **74** 197
- [35] Horodecki M, Horodecki P and Horodecki R 2000 Limits for entanglement measures *Phys. Rev. Lett.* **84** 2014–7
- [36] Devetak I and Winter A 2005 Distillation of secret key and entanglement from quantum states *Proc. R. Soc. A* **461** 207–35
- [37] Dür W, Cirac J I and Tarrach R 1999 Separability and distillability of multiparticle quantum systems *Phys. Rev. Lett.* **83** 3562
- [38] de Vicente J I and Huber M 2011 Multipartite entanglement detection from correlation tensors *Phys. Rev. A* **84** 062306
- [39] Huber M and Sengupta R 2014 Witnessing genuine multipartite entanglement with positive maps *Phys. Rev. Lett.* **113** 100501
- [40] Dai Y, Dong Y, Xu Z, You W, Zhang C and Gühne O 2020 Experimentally accessible lower bounds for genuine multipartite entanglement and coherence measures *Phys. Rev. Appl.* **13** 054022
- [41] Szalay S 2015 Multipartite entanglement measures *Phys. Rev. A* **92** 042329
- [42] Modi K, Paterek T, Son W, Vedral V and Williamson M 2010 Unified view of quantum and classical correlations *Phys. Rev. Lett.* **104** 080501
- [43] Coffman V, Kundu J and Wootters W K 2000 Distributed entanglement *Phys. Rev. A* **61** 052306
- [44] Koashi M and Winter A 2004 Monogamy of quantum entanglement and other correlations *Phys. Rev. A* **69** 022309
- [45] Osborne T J and Verstraete F 2006 General monogamy inequality for bipartite qubit entanglement *Phys. Rev. Lett.* **96** 220503
- [46] Kim J S and Sanders B C 2010 Monogamy of multi-qubit entanglement using Rényi entropy *J. Phys. A: Math. Theor.* **43** 445305
- [47] Bai Y-K, Xu Y-F and Wang Z D 2014 General monogamy relation for the entanglement of formation in multiqubit systems *Phys. Rev. Lett.* **113** 100503
- [48] Ou Y-C and Fan H 2007 Monogamy inequality in terms of negativity for three-qubit states *Phys. Rev. A* **75** 062308
- [49] Bai Y-K, Yang D and Wang Z D 2007 Multipartite quantum correlation and entanglement in four-qubit pure states *Phys. Rev. A* **76** 022336
- [50] Bai Y-K, Ye M-Y and Wang Z D 2009 Entanglement monogamy and entanglement evolution in multipartite systems *Phys. Rev. A* **80** 044301
- [51] Bai Y-K, Xu Y-F and Wang Z D 2014 Hierarchical monogamy relations for the squared entanglement of formation in multipartite systems *Phys. Rev. A* **90** 062343
- [52] Rains E M 1999 Bound on distillable entanglement *Phys. Rev. A* **60** 179
- [53] Rains E M 2001 A semidefinite program for distillable entanglement *IEEE Trans. Inf. Theory* **47** 2921

- [54] Vidal G, Dür W and Cirac J I 2002 Entanglement cost of bipartite mixed states *Phys. Rev. Lett.* **89** 027901
- [55] Bandyopadhyay S, Boykin P O, Roychowdhury V and Vatan F 2002 A new proof for the existence of mutually unbiased bases *Algorithmica* **34** 512
- [56] Durt T, Englert B-G, Bengtsson I and Życzkowski K 2010 On mutually unbiased bases *Int. J. Quantum Inf.* **08** 535
- [57] Kalev A, Mann A and Revzen M 2013 Choice of measurement as the signal *Phys. Rev. Lett.* **110** 260502
- [58] Henderson L and Vedral V 2001 Classical quantum and total correlations *J. Phys. A: Math. Gen.* **34** 6899
- [59] Ollivier H and Zurek W H 2001 Quantum discord: a measure of the quantumness of correlations *Phys. Rev. Lett.* **88** 017901
- [60] Dür W, Vidal G and Cirac J I 2000 Three qubits can be entangled in two inequivalent ways *Phys. Rev. A* **62** 062314
- [61] Acín A, Bruß D, Lewenstein M and Sanpera A 2001 Classification of mixed three-qubit states *Phys. Rev. Lett.* **87** 040401
- [62] Bechmann-Pasquinucci H and Tittel W 2000 Quantum cryptography using larger alphabets *Phys. Rev. A* **61** 062308
- [63] Cerf N J, Bourennane M, Karlsson A and Gisin N 2002 Security of quantum key distribution using d-level systems *Phys. Rev. Lett.* **88** 127902
- [64] Kraft T, Ritz C, Brunner N, Huber M and Gühne O 2018 Characterizing genuine multi-level entanglement *Phys. Rev. Lett.* **120** 060502
- [65] Brunner N, Pironio S, Acin A, Gisin N, Méthot A A and Scarani V 2008 Testing the dimension of Hilbert spaces *Phys. Rev. Lett.* **100** 210503
- [66] Cong W, Cai Y, Bancal J-D and Scarani V 2017 Witnessing irreducible dimension *Phys. Rev. Lett.* **119** 080401
- [67] Guo Y, Hu X-M, Liu B-H, Huang Y-F, Li C-F and Guo G-C 2018 Experimental witness of genuine high-dimensional entanglement *Phys. Rev. A* **97** 062309
- [68] Nielsen M A and Chuang I L 2014 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [69] Plenio M B 2005 Logarithmic negativity: a full entanglement monotone that is not convex *Phys. Rev. Lett.* **95** 090503
- [70] Vidal G and Werner R F 2002 Computable measure of entanglement *Phys. Rev. A* **65** 032314
- [71] Chen K, Albeverio S and Fei S-M 2005 Entanglement of formation of bipartite quantum states *Phys. Rev. Lett.* **95** 210501
- [72] Życzkowski K, Horodecki P, Horodecki M and Horodecki R 2001 Dynamics of quantum entanglement *Phys. Rev. A* **65** 012101
- [73] Scheel S, Eisert J, Knight P L and Plenio M B 2003 Hot entanglement in a simple dynamical model *J. Mod. Opt.* **50** 881
- [74] Yu T and Eberly J H 2006 Quantum open system theory: bipartite aspects *Phys. Rev. Lett.* **97** 140403
- [75] Yu T and Eberly J H 2009 Sudden death of entanglement *Science* **323** 598