DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2017-0395
Math. Slovaca **70** (2020), No. 4, 821–828

ON REVERSE HÖLDER AND MINKOWSKI INEQUALITIES

Chang-Jian Zhao* — Wing Sum Cheung**

(*Communicated by Tomasz Natkaniec*)

ABSTRACT. In the paper, we give new improvements of the reverse Hölder and Minkowski integral inequalities. These new results in special case yield the Pólya-Szegö's inequality and reverse Minkowski's inequality, respectively.

1. Introduction

In [7], Pólya and Szegö's established a reverse Hölder's inequality as follows (see also [4: p. 62]). If $0 < m_1 \le u_k \le M_1$ and $0 < m_2 \le v_k \le M_2$, where $k = 1, 2, \dots, n$, then

$$\left(\sum_{k=1}^{n} u_k^2\right)\left(\sum_{k=1}^{n} v_k^2\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\sum_{k=1}^{n} u_kv_k\right)^2.$$

An integral analogue of the Pólya-Szegö's inequality easy follows.

If (E, A, x) is a measure space and $f, g\colon E \to \mathbb{R}$ be positive measurable functions and $f^2(x)$, $g^2(x)$ are integrable on E. If $0 < m_1 \le f(x) \le M_1$ and $0 < m_2 \le g(x) \le M_2$, then

$$\left(\int_E f^2(x)\mathrm{d}x\right)\left(\int_E g^2(x)\mathrm{d}x\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\int_E f(x)g(x)\mathrm{d}x\right)^2. \tag{1.1}$$

It should be noted that we write dx as a short replacement for $d\mu(x)$ in all integrals here and in the sequel.

The Pólya-Szegö's inequality was studied extensively and numerous variants, generalizations, and extensions appeared in the literatures (see [9], [10], [11], [2], [3], [5], [12], [1], [6], [13], [8] and the references cited therein). The first aim of this paper is to give a new improvement of the Pólya-Szegö's inequality, which is generalization of the Pólya-Szegö's inequality.

$$\begin{aligned}&\int_E \left(\Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right) f^{1/p}(x)g^{1/q}(x) - u^{1/p}(x)v^{1/q}(x)\right)\mathrm{d}x\\ &\ge \left(\int_E (f(x)-u(x))\mathrm{d}x\right)^{1/p}\left(\int_E (g(x)-v(x))\mathrm{d}x\right)^{1/q},\end{aligned} \tag{1.2}$$

2010 Mathematics Subject Classification: Primary 26D15.

Keywords: Pólya-Szegö's inequality, Pólya-Szegö's integral inequality, Bellman's inequality.
This work was supported by National Natural Science Foundation of China Grant No. 10971205, 11371334.

where $f(x)$, $g(x)$ are positive, and $u(x)$ and $v(x)$ are non-negative measurable functions on the measure space (E, A, x) and such that $f(x) > u(x)$ and $g(x) > v(x)$, $p > 1$, $1/p + 1/q = 1$, $0 < m_1 \leq f(x) \leq M_1$ and $0 < m_2 \leq g(x) \leq M_2$, and

$$\Gamma_{p,q}(\xi) = p^{-\frac{1}{p}} \cdot q^{-\frac{1}{q}} \frac{1-\xi}{(1-\xi^{1/p})^{1/p}(1-\xi^{1/q})^{1/q}} \cdot \xi^{-1/pq}, \tag{1.3}$$

for all $0 < \xi \leq 1$. Here $\Gamma_{p,q}(\xi)$ is continuous at $\xi_0 = 1$, and defined

$$\Gamma_{p,q}(1) = \lim_{\xi \to 1} \Gamma_{p,q}(\xi) = 1.$$

Obviously, (1.1) is a special case of (1.2).

Another aim of this paper is to give the following new improvement of the well-known Pólya and Szegö's inequality.

$$\begin{aligned} &\left(\int\limits_E \left(\frac{1}{\Gamma^p_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)}(f(x)+g(x))^p - (u(x)+v(x))^p\right)\mathrm{d}x\right)^{1/p} \\ &\geq \left(\int\limits_E [f^p(x) - u^p(x)]\mathrm{d}x\right)^{1/p} + \left(\int\limits_E [g^p(x) - v^p(x)]\mathrm{d}x\right)^{1/p}, \end{aligned} \tag{1.4}$$

where $f(x), g(x)$ be positive measurable functions on the measure space (E, A, x), and $p > 1$, $0 < m_1 \leq f(x)/(f(x)+g(x))^{p-1} \leq M_1$ and $0 < m_2 \leq g(x)/(f(x)+g(x))^{p-1} \leq M_2$. Here $u(x)$, $v(x)$ are non-negative measurable functions with $f(x) > u(x)$ and $g(x) > v(x)$, and

$$\Gamma_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2) = \min\left\{\frac{1}{\Gamma_{p,\frac{p}{p-1}}\left(\frac{m_1}{M_1}\right)}, \frac{1}{\Gamma_{p,\frac{p}{p-1}}\left(\frac{m_2}{M_2}\right)}\right\}. \tag{1.5}$$

In order to establish inequality (1.4), we establish the following Pólya and Szegö type inequality, which is also a reverse Minkowski's inequality.

$$\left(\int\limits_E f^p(x)\mathrm{d}x\right)^{1/p} + \left(\int\limits_E g^p(x)\mathrm{d}x\right)^{1/p} \leq \frac{1}{\Gamma_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)}\left(\int\limits_E (f(x)+g(x))^p\mathrm{d}x\right)^{1/p}. \tag{1.6}$$

where $f(x), g(x)$, m_1, m_2, M_1, M_2 and p are as in (1.4).

2. Results

We need the following Lemmas to prove our main results.

Lemma 2.1 ([5]). *Let (E, A, x) be a measure space and $f, g\colon E \to \mathbb{R}$ be positive measurable functions, and $f^{1/p}g^{1/q}$ is integrable on E. Let $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. If $0 < m \leq f(x)/g(x) \leq M$, then*

$$\left(\int\limits_E f(x)\mathrm{d}x\right)^{1/p}\left(\int\limits_E g(x)\mathrm{d}x\right)^{1/q} \leq \Gamma_{p,q}\left(\frac{m}{M}\right)\int\limits_E f^{1/p}(x)g^{1/q}(x)\mathrm{d}x, \tag{2.1}$$

where $\Gamma_{p,q}\left(\frac{m}{M}\right)$ is as in (1.3).

Lemma 2.2. *Let (E, A, x) be a measure space and $f, g\colon E \to \mathbb{R}$ be positive measurable functions, and $f^{1/p}g^{1/q}$ is integrable on E. Let $p, q > 0, \frac{1}{p} + \frac{1}{q} = 1$. If $0 < m_1 \le f(x) \le M_1$ and $0 < m_2 \le g(x) \le M_2$, then*

$$\left(\int_E f(x)\mathrm{d}x\right)^{1/p}\left(\int_E g(x)\mathrm{d}x\right)^{1/q} \le \Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)\int_E f^{1/p}(x)g^{1/q}(x)\mathrm{d}x, \tag{2.2}$$

where $\Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)$ is as in (1.3).

P r o o f. This follows immediately from Lemma 2.1 with $\frac{m_1}{M_2} \le \frac{f(x)}{g(x)} \le \frac{M_1}{m_2}$. □

A special case of inequality (2.2) can be found in the book [6: p. 64].

Lemma 2.3. *Let (E, A, x) be a measure space and $f, g\colon E \to \mathbb{R}$ be positive measurable functions, and f^p, $g^{1/q}$ and $(f+g)^p$ are integrable on E. If $p > 1$, $0 < m_1 \le f(x)/(f(x)+g(x))^{p-1} \le M_1$ and $0 < m_2 \le g(x)/(f(x)+g(x))^{p-1} \le M_2$, then*

$$\left(\int_E f^p(x)\mathrm{d}x\right)^{1/p} + \left(\int_E g^p(x)\mathrm{d}x\right)^{1/p} \le \frac{1}{\Gamma_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)}\left(\int_E (f(x)+g(x))^p\mathrm{d}x\right)^{1/p}, \tag{2.3}$$

where $\Gamma_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)$ is as in (1.5).

P r o o f. Noting that

$$\int_E (f(x)+g(x))^p\mathrm{d}x = \int_E f(x)(f(x)+g(x))^{p-1}\mathrm{d}x + \int_E g(x)(f(x)+g(x))^{p-1}\mathrm{d}x,$$

and let $q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, and by using Lemma 2.1, we obtain

$$\Gamma_{p,q}\left(\frac{m_1}{M_1}\right)\int_E f(x)(f(x)+g(x))^{p-1}\mathrm{d}x \ge \left(\int_E f(x)^p\mathrm{d}x\right)^{1/p}\left(\int_E (f(x)+g(x))^{(p-1)q}\mathrm{d}x\right)^{1/q},$$

and

$$\Gamma_{p,q}\left(\frac{m_2}{M_2}\right)\int_E g(x)(f(x)+g(x))^{p-1}\mathrm{d}x \ge \left(\int_E g(x)^p\mathrm{d}x\right)^{1/p}\left(\int_E (f(x)+g(x))^{(p-1)q}\mathrm{d}x\right)^{1/q}.$$

Hence

$$\begin{aligned}\int_E (f(x)+g(x))^p\mathrm{d}x \ge &\left(\frac{1}{\Gamma_{p,q}\left(\frac{m_1}{M_1}\right)}\left(\int_E f(x)^p\mathrm{d}x\right)^{1/p} + \frac{1}{\Gamma_{p,q}\left(\frac{m_2}{M_2}\right)}\left(\int_E g(x)^p\mathrm{d}x\right)^{1/p}\right)\\ &\times\left(\int_E (f(x)+g(x))^p\mathrm{d}x\right)^{1/q}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\int_E (f(x)+g(x))^p\mathrm{d}x\right)^{1/p}\\ &\ge \min\left\{\frac{1}{\Gamma_{p,q}\left(\frac{m_1}{M_1}\right)}, \frac{1}{\Gamma_{p,q}\left(\frac{m_2}{M_2}\right)}\right\}\left(\left(\int_E f(x)^p\mathrm{d}x\right)^{1/p} + \left(\int_E g(x)^p\mathrm{d}x\right)^{1/p}\right)\end{aligned}$$

$$= \min\left\{\frac{1}{\Gamma_{p,\frac{p}{p-1}}\left(\frac{m_1}{M_1}\right)}, \frac{1}{\Gamma_{p,\frac{p}{p-1}}\left(\frac{m_2}{M_2}\right)}\right\}\left(\left(\int_E f(x)^p \mathrm{d}x\right)^{1/p} + \left(\int_E g(x)^p \mathrm{d}x\right)^{1/p}\right)$$

$$= \Gamma_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)\left(\left(\int_E f(x)^p \mathrm{d}x\right)^{1/p} + \left(\int_E g(x)^p \mathrm{d}x\right)^{1/p}\right).$$

This completes the proof. □

We denote the set of non-negative real numbers by $\mathbb{R}_+$ in the rest of the paper.

Lemma 2.4 ([1: p. 38] Bellman's inequality). *Let $p > 1$ and $n \in \mathbb{N}$. Moreover let*

$$D_n = \{x = (x_0, x_1, x_2, \dots, x_n) \in \mathbb{R}_+^{n+1} | x_0^p \ge x_1^p + x_2^p + \dots + x_n^p\}$$

and

$$\phi(x) = \left(x_0^p - \sum_{i=1}^n x_i^p\right)^{1/p} \qquad (x = (x_0, x_1, x_2, \dots, x_n) \in D_n).$$

Then $\phi\colon D_n \to \mathbb{R}$ is superadditive (i.e., $x, y \in D_n$ implies $x + y \in D_n$ and

$$\phi(x + y) \ge \phi(x) + \phi(y), \tag{2.4}$$

with equality if and only if $x = \mu y$ where μ is a constant).

Lemma 2.5 ([6: p. 58] Popoviciu's inequality). *Let $p > 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $a = \{a_1, \dots, a_n\}$ and $b = \{b_1, \dots, b_n\}$ be two series of positive real numbers and such that $a_1^p - \sum\limits_{i=2}^n a_i^p > 0$ and $b_1^q - \sum\limits_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \le a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{2.5}$$

with equality if and only if $a = \mu b$, where μ is a constant.

Our main results are given in the following theorems.

Theorem 2.1. *Let (E, A, x) be a measure space and $f, g\colon E \to \mathbb{R}$ be positive measurable functions and $u_i(x)$ and $v_i(x)$ non-negative measurable functions such that $f(x) - \sum\limits_{i=1}^n u_i(x) > 0$ and $g(x) - \sum\limits_{i=1}^n v_i(x) > 0$, where $i = 1, 2, \dots, n$. If $0 < m_1 \le f(x) \le M_1$, $0 < m_2 \le g(x) \le M_2$, and $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. then*

$$\begin{aligned}\int_E &\left(\Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right) f^{1/p}(x) g^{1/q}(x) - \sum_{i=1}^n u_i^{1/p}(x) v_i^{1/q}(x)\right)\mathrm{d}x \\ &\ge \left(\int_E \left(f(x) - \sum_{i=1}^n u_i(x)\right)\mathrm{d}x\right)^{1/p}\left(\int_E \left(g(x) - \sum_{i=1}^n v_i(x)\right)\mathrm{d}x\right)^{1/q},\end{aligned} \tag{2.6}$$

where $\Gamma_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)$ is as in (1.3).

P r o o f. First, we prove the statement for $n=1$. From Hölder's inequality and Lemma 2.2, we obtain

$$\Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_E f^{1/p}(x)g^{1/q}(x)\mathrm{d}x \geq \left(\int_E f(x)\mathrm{d}x\right)^{1/p}\left(\int_E g(x)\mathrm{d}x\right)^{1/q}, \tag{2.7}$$

and

$$\int_E u_1^{1/p}(x)v_1^{1/q}(x)\mathrm{d}x \leq \left(\int_E u_1(x)\mathrm{d}x\right)^{1/p}\left(\int_E v_1(x)\mathrm{d}x\right)^{1/q}. \tag{2.8}$$

From (2.7) and (2.8) and by using Lemma 2.5, we have

$$\begin{aligned}
&\int_E \left(\Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right) f^{1/p}(x)g^{1/q}(x) - u_1^{1/p}(x)v_1^{1/q}(x)\right)\mathrm{d}x\\
&\geq \left(\int_E f(x)\mathrm{d}x\right)^{1/p}\left(\int_E g(x)\mathrm{d}x\right)^{1/q} - \left(\int_E u_1(x)\mathrm{d}x\right)^{1/p}\left(\int_E v_1(x)\mathrm{d}x\right)^{1/q}\\
&\geq \left(\int_E (f(x)-u_1(x))\mathrm{d}x\right)^{1/p}\left(\int_E (g(x)-v_1(x))\mathrm{d}x\right)^{1/q}.
\end{aligned}$$

This shows that (2.6) is true when $n=1$.

Suppose that (2.6) holds when $n=k-1$, we have

$$\begin{aligned}
&\int_E \left(\Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right) f^{1/p}(x)g^{1/q}(x) - \sum_{i=1}^{k-1} u_i^{1/p}(x)v_i^{1/q}(x)\right)\mathrm{d}x\\
&\geq \left(\int_E (f(x)-\sum_{i=1}^{k-1}u_i(x))\mathrm{d}x\right)^{1/p}\left(\int_E (g(x)-\sum_{i=1}^{k-1}v_i(x))\mathrm{d}x\right)^{1/q},
\end{aligned} \tag{2.9}$$

From (2.8) and (2.9) and by using Lemma 2.5, we have

$$\begin{aligned}
&\int_E \left(\Gamma_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right) f^{1/p}(x)g^{1/q}(x) - \sum_{i=1}^{k} u_i^{1/p}(x)v_i^{1/q}(x)\right)\mathrm{d}x\\
&\geq \left(\int_E \left(f(x)-\sum_{i=1}^{k-1}u_i(x)\right)\mathrm{d}x\right)^{1/p}\left(\int_E \left(g(x)-\sum_{i=1}^{k-1}v_i(x)\right)\mathrm{d}x\right)^{1/q}\\
&\quad - \left(\int_E u_k(x)\mathrm{d}x\right)^{1/p}\left(\int_E v_k(x)\mathrm{d}x\right)^{1/q}\\
&\geq \left(\int_E \left(f(x)-\sum_{i=1}^{k}u_i(x)\right)\mathrm{d}x\right)^{1/p}\left(\int_E \left(g(x)-\sum_{i=1}^{k}v_i(x)\right)\mathrm{d}x\right)^{1/q}.
\end{aligned}$$

The completes the proof. □

Remark 1. Putting $n=1$, (2.6) becomes (1.2) stated in the introduction. For $u_i(x)=v_i(x)=0$, (2.6) reduces to (2.2).

If taking for $p = q = 2$ and $u_i(x) = v_i(x) = 0$ $(i = 1, 2, \ldots, n)$ in (2.6), and in view of

$$\Gamma_{2,2}\left(\frac{m_1m_2}{M_1M_2}\right) = \frac{1}{2}\left(\sqrt[4]{\frac{M_1M_2}{m_1m_2}} + \sqrt[4]{\frac{m_1m_2}{M_1M_2}}\right),$$

then (2.6) changes to the following result.

$$\left(\int_E f(x)\mathrm{d}x\right)^{1/2}\left(\int_E g(x)\mathrm{d}x\right)^{1/2} \le \frac{1}{2}\left(\sqrt[4]{\frac{M_1M_2}{m_1m_2}} + \sqrt[4]{\frac{m_1m_2}{M_1M_2}}\right)\int_E f^{1/2}(x)g^{1/2}(x)\mathrm{d}x, \tag{2.10}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional. Replace $f^{1/2}(x)$ and $f^{1/2}(x)$ by $f(x)$ and $g(x)$ in (2.10), respectively and hence in view of $m_i^{1/2}(x)$ and $M_i^{1/2}(x)$ are replaced by m_i and M_i $(i = 1, 2)$, respectively. Therefore

$$\left(\int_E f^2(x)\mathrm{d}x\right)^{1/2}\left(\int_E g^2(x)\mathrm{d}x\right)^{1/2} \le \frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)\int_E f(x)g(x)\mathrm{d}x.$$

This is just the Pólya-Szegö integral inequality (1.1).

Theorem 2.2. *Let (E, A, x) be a measure space and $f, g\colon E \to \mathbb{R}$ be positive measurable functions and $u_i(x)$ and $v_i(x)$ non-negative measurable functions such that $f^p(x) - \sum\limits_{i=1}^{n} u_i^p(x) > 0$ and $g^p(x) - \sum\limits_{i=1}^{n} v_i^p(x) > 0$, where $i = 1, 2, \ldots, n$. If $p > 1$, $0 < m_1 \le f(x)/(f(x) + g(x))^{p-1} \le M_1$ and $0 < m_2 \le g(x)/(f(x) + g(x))^{p-1} \le M_2$, then for $n \in \mathbb{N}$*

$$\begin{aligned}&\left(\int_E \left(\Gamma^{-p}_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)(f(x) + g(x))^p - \sum_{i=1}^{n}(u_i(x) + v_i(x))^p\right)\mathrm{d}x\right)^{1/p}\\ &\ge \left(\int_E \left(f^p(x) - \sum_{i=1}^{n} u_i^p(x)\right)\mathrm{d}x\right)^{1/p} + \left(\int_E \left(g^p(x) - \sum_{i=1}^{n} v_i^p(x)\right)\mathrm{d}x\right)^{1/p},\end{aligned} \tag{2.11}$$

where $\Gamma_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)$ is as in (1.5).

P r o o f. First, we prove the statement for $n = 1$. From Minkowski's inequality and Lemma 2.3, it is easy to obtain

$$\Gamma^{-1}_{p,\frac{p}{p-1}}(m_1, m_2, M_1, M_2)\left(\int_E (f(x)+g(x))^p\mathrm{d}x\right)^{1/p} \ge \left(\int_E f^p(x)\mathrm{d}x\right)^{1/p} + \left(\int_E g^p(x)\mathrm{d}x\right)^{1/p}, \tag{2.12}$$

and

$$\left(\int_E (u_1(x) + v_1(x))^p\mathrm{d}x\right)^{1/p} \le \left(\int_E u_1^p(x)\mathrm{d}x\right)^{1/p} + \left(\int_E v_1^p(x)\mathrm{d}x\right)^{1/p}. \tag{2.13}$$

From (2.12), (2.13) and by using Lemma 2.4, we have

$$\begin{aligned}&\left(\int_E \left(\Gamma^{-p}_{p,\frac{p}{p-1}}(m_1,m_2,M_1,M_2)(f(x)+g(x))^p-(u_1(x)+v_1(x))^p\right)\mathrm{d}x\right)^{1/p}\\&\geq\left\{\left[\left(\int_E f^p(x)\mathrm{d}x\right)^{1/p}+\left(\int_E g^p(x)\mathrm{d}x\right)^{1/p}\right]^p-\left[\left(\int_E u_1^p(x)\mathrm{d}x\right)^{1/p}+\left(\int_E v_1^p(x)\mathrm{d}x\right)^{1/p}\right]^p\right\}^{1/p}\\&\geq\left(\int_E [f^p(x)-u_1^p(x)]\mathrm{d}x\right)^{1/p}+\left(\int_E [g^p(x)-v_1^p(x)]\mathrm{d}x\right)^{1/p}.\end{aligned}\tag{2.14}$$

This shows that (2.11) is true when $n=1$.

Suppose that (2.11) holds when $n=k-1$, we have

$$\begin{aligned}&\left(\int_E \left(\Gamma^{-p}_{p,\frac{p}{p-1}}(m_1,m_2,M_1,M_2)(f(x)+g(x))^p-\sum_{i=1}^{k-1}(u_i(x)+v_i(x))^p\right)\mathrm{d}x\right)^{1/p}\\&\geq\left(\int_E \left[f^p(x)-\sum_{i=1}^{k-1}u_i^p(x)\right]\mathrm{d}x\right)^{1/p}+\left(\int_E \left[g^p(x)-\sum_{i=1}^{k-1}v_i^p(x)\right]\mathrm{d}x\right)^{1/p},\end{aligned}\tag{2.15}$$

On the other hand

$$\left(\int_E (u_i(x)+v_i(x))^p\mathrm{d}x\right)^{1/p}\leq\left(\int_E u_i^p(x)\mathrm{d}x\right)^{1/p}+\left(\int_E v_i^p(x)\mathrm{d}x\right)^{1/p}.\tag{2.16}$$

From (2.15), (2.16) and by using Lemma 2.4 again, we have

$$\begin{aligned}&\left(\int_E \left(\Gamma^{-p}_{p,\frac{p}{p-1}}(m_1,m_2,M_1,M_2)(f(x)+g(x))^p-\sum_{i=1}^{k}(u_i(x)+v_i(x))^p\right)\mathrm{d}x\right)^{1/p}\\&\geq\left\{\left[\left(\int_E [f^p(x)-\sum_{i=1}^{k-1}u_i^p(x)]\mathrm{d}x\right)^{1/p}+\left(\int_E [g^p(x)-\sum_{i=1}^{k-1}v_i^p(x)]\mathrm{d}x\right)^{1/p}\right]^p\right.\\&\quad\left.-\left[\left(\int_E u_i^p(x)\mathrm{d}x\right)^{1/p}+\left(\int_E v_i^p(x)\mathrm{d}x\right)^{1/p}\right]^p\right\}^{1/p}\\&\geq\left(\int_E \left[f^p(x)-\sum_{i=1}^{k}u_i^p(x)\right]\mathrm{d}x\right)^{1/p}+\left(\int_E \left[g^p(x)-\sum_{i=1}^{k}v_i^p(x)\right]\mathrm{d}x\right)^{1/p}.\end{aligned}$$

This completes the proof. □

Remark 2. Putting $n=1$, (2.11) becomes (1.6) stated in the introduction. For $u_i(x)=v_i(x)\equiv 0$, (2.11) reduces to (2.3).

Taking for $p=q=2$ and $u_i(x)=v_i(x)\equiv 0$ in (2.11), (2.11) changes to the following result.

$$\left(\int_E f^2(x)\mathrm{d}x\right)^{1/2}+\left(\int_E g^2(x)\mathrm{d}x\right)^{1/2}\leq\Gamma^{-1}_{2,2}(m_1,m_2,M_1,M_2)\cdot\left(\int_E (f(x)+g(x))^2\,\mathrm{d}x\right)^{1/2}.$$

REFERENCES

[1] BECHENBACH, E. F.—BELLMAN, R.: *Inequalities*, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1961.
[2] DRAGOMIR, S. S.: *Asupra unor inegalităţi, Caiete Metodico-Şiinţifice*, Matematica **13**, Univ. Timisoara, 1984.
[3] DRAGOMIR, S. S.—KHAN, L.: *Two discrete inequalities of Grüss type via Pólya-Szegö and Shisha-Mond results for real numbers*, Tamkang J. Math. **35** (2004), 117–128.
[4] HARDY, G. H.—LITTLEWOOD, J. E.—PÓLYA, G.: *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.
[5] LIU, X. H.: *On reverse Hölder inequality*, Math. Pract. Theory **1990** (1990), 84–88.
[6] MITRINOVIĆ, D. S.: *Analytic Inequalities*, Springer-Verlag Berlin, Heidelberg, New York, 1970.
[7] PÓLYA, G.—SZEGÖ, G.: *Aufgaben und Lehrsätze aus der Analysis I*, Berlin, 1925, p. 57 and 213–214.
[8] WANG, C.-L.: *On development of inverses of Cauchy and Hölder inequalities*, SIAM Review. **21** (1979), 550–557.
[9] WU, S. H.: *Generalization of a sharp Hölder's inequality and its application*, J. Math. Anal. Appl. **332** (2007), 741–750.
[10] WU, S. H.: *A new sharpened and generalized version of Hölder's inequality and its applications*, Appl. Math. Comput. **197** (2008), 708–714.
[11] WU, S. H.: *Some improvements of Aczél's inequality and Popoviciu's inequality*, Computers. Math. Appl. **56** (2008), 1196–1205.
[12] YANG, S. G.: *Reverse Minkowski inequality and its applications*, J. Tongling College **13** (2002), 71–76.
[13] ZHAO, C.-J.—CHEUNG, W. S.: *On Pólya-Szegö's inequality*, J. Inequal. Appl. **2013** (2013), #591.

Received 17. 5. 2019
Accepted 31. 1. 2020

* *Department of Mathematics*
China Jiliang University
Hangzhou 310018
Zhejiang
P. R. CHINA
E-mail: chjzhao@163.com; chjzhao@cjlu.edu.cn

** *Department of Mathematics*
The University of Hong Kong
Pokfulam Road
HONG KONG
E-mail: wscheung@hku.hk