On the Peak of the Impulse Response of Polytopic LTV Systems*

Graziano Chesi[1] and Tiantian Shen[2]

Abstract— **This paper addresses the problem of determining the peak of the impulse response of polytopic linear time-varying (LTV) systems. In particular, linear systems affected linearly by time-varying structured uncertainty constrained into a polytope are considered. A novel condition for establishing upper bounds of the sought peak is proposed in terms of feasibility of a system of linear matrix inequalities (LMIs). The proposed condition is sufficient for any degree of a polynomial introduced to describe the impulse response, moreover, its conservatism can be decreased by increasing this degree. Some examples illustrate the use of the proposed condition and show the application to a DC motor with time-varying uncertain parameters.**

I. INTRODUCTION

Input-output relationships dynamical systems can be characterized by various indexes. For instance, in the case of linear time-invariant (LTI) systems, common indexes are the H-infinity norm (i.e., the maximum amplitude gain of the frequency response) and the H-2 norm (i.e., the square root of the sum of the energies of the impulse responses). It is well-known that these indexes can play key roles for analysis and synthesis, therefore, methods for determining and controlling these indexes have been studied and developed since long time. See for instance [1], [9].

Another important index is the peak of the impulse response. This index provides the maximum amplitude of the output of a system in response to an impulse applied to one of its input channels. Hence, the knowledge of this index may be required in order to verify whether amplitude constraints are satisfied, and the manipulation of this index may be required in order to ensure the satisfaction of such constraints.

Unfortunately, the determination of this index is a challenging problem. Indeed, classical methods based on set invariance of quadratic Lyapunov functions are generally conservative for determining the peak of the impulse response of an LTI system, see for instance [3], [10]. This is sharply in contrast with the fact that the methodology of these methods is, however, nonconservative for determining other indexes such as the H-infinity norm and the H-2 norm of LTI systems.

Clearly, the problem is even more challenging if one leaves the realm of the LTI systems. In particular, when considering uncertain LTV systems, one has to face additional issues generated by the fact that, not only there exist a family of models for the system (instead of one model only), but also the model that describes the system changes in this family with the time. As it is well-known, this dependence on the time may generate instability even if all models in the family are stable, see for instance [2], [6].

This paper addresses the problem of determining the peak of the impulse response of polytopic LTV systems. In particular, linear systems affected linearly by time-varying structured uncertainty constrained into a polytope are considered. A novel condition for establishing upper bounds of the sought peak is proposed in terms of feasibility of a system of LMIs. This condition is obtained by describing the impulse response through the level set of a polynomial, by using projection techniques for establishing the extension of the level set, and by introducing polynomial transformations for establishing robustness. The proposed condition is sufficient for any degree of the polynomial, moreover, its conservatism can be decreased by increasing this degree. Some examples illustrate the use of the proposed condition and show the application to a DC motor with time-varying uncertain parameters.

The paper is organized as follows. Section II provides the problem formulation. Section III describes the proposed condition. Section IV presents the examples. Lastly, Section V reports the conclusions and future works. This paper extends our previous work [7] which only considers LTI systems not affected by uncertainties.

II. PROBLEM FORMULATION

The notation adopted in the paper is as follows. The set of natural numbers (including zero) and the set of real numbers set are denoted, respectively, by $\mathbb{N}$ and $\mathbb{R}$. The null matrix with size specified by the context is denoted by 0. The notation I_n denotes the $n \times n$ identity matrix. The Euclidean norm and the infinity norm of a matrix A are denoted, respectively, by $\|A\|_2$ and $\|A\|_\infty$. The notation $A \otimes B$ denotes the Kronecker product between matrices A and B. The transpose of a matrix A is denoted by A'. The notation $A > 0$ (respectively, $A \geq 0$) denotes a symmetric positive definite (respectively, semidefinite) matrix A. The convex hull of matrices $A_1, \ldots, A_r$ is denoted by $\mathrm{conv}\{\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{r}\}$. For a matrix A, the notation $\mathrm{sq}(\mathrm{A})$ denotes the matrix with the same dimension of A whose (i, j)-th entry is the square of the (i, j)-th entry of A. The degree of a polynomial $a(b)$ in the variable $b \in \mathbb{R}^n$ is denoted by $\deg(a(b))$. The symbol Σ denotes the set of polynomials that are sums of squares of polynomials.

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[1]Department of Electrical and Electronic Engineering, The University of Hong Kong, chesi@eee.hku.hk

[2]Department of Electronic Information Engineering, Hunan Normal University, titiansaint@foxmail.com

Let us consider the system

$$\begin{cases} \dot{x}(t) &=& A(\theta(t))x(t)+B(\theta(t))u(t) \\ y(t) &=& C(\theta(t))x(t) \\ \theta(\cdot) &\in& \Theta \end{cases} \quad (1)$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $\theta(t) \in \mathbb{R}^q$ is the time-varying uncertainty, Θ is the set of admissible time-varying uncertainties

$$\Theta = \{\theta : \mathbb{R} \to \mathcal{P}\} \quad (2)$$

where $\mathcal{P}$ is the convex bounded polytope

$$\mathcal{P} = \text{conv}\{\text{P}_1, \ldots, \text{P}_\text{r}\} \quad (3)$$

for given vectors $P_1, \ldots, P_r \in \mathbb{R}^q$, and $A(\theta(t)), B(\theta(t)), C(\theta(t))$ are affine linear matrix functions of suitable sizes, i.e.,

$$A(\theta(t)) = A_0 + \sum_{i=1}^{q} \theta_i(t) A_i \quad (4)$$

where $A_0, \ldots, A_q \in \mathbb{R}^{n\times n}$ are given matrices, and similarly for $B(\theta(t))$ and $C(\theta(t))$. This is a standard model adopted in the literature of uncertain systems, see for instance [6] and references therein. In spite of its simplicity, this model allows to consider various sets of admissible uncertainties through the choice of the vectors $P_1, \ldots, P_r$, moreover, the problem addressed in this paper is still unsolved for this model. Let us introduce the following definition.

Definition 1: The impulse response of the system (1) with respect to the i-th input channel, denoted as $Y_i(t)$, is the solution $y(t)$ of the system (1) for $x(0^-) = 0$ and $u(t) = \delta(t) I_m^{(i)}$, where $\delta : \mathbb{R} \to \mathbb{R}$ is the Dirac distribution and $I_m^{(i)}$ is the i-th column of I_m. □

The problem addressed in this paper is as follows.

Problem 1: Given $c \in (0, \infty)$, establish whether c is an upper bound of the peak of the impulse response of the system (1) with respect to all input channels for all admissible uncertainties, i.e.,

$$\|Y_i(t)\|_\infty < c \quad \forall t \geq 0 \ \forall \theta(\cdot) \in \Theta \ \forall i = 1, \ldots, m. \quad (5)$$

□

The dependence on the time will be omitted in the sequel for ease of notation unless specified otherwise.

III. Proposed Condition

Let $f : \mathbb{R}^s \to \mathbb{R}$ be a polynomial of degree not greater than $d \in \mathbb{N}$, and let us express $f(a)$, $a \in \mathbb{R}^s$, as

$$f(a) = \sum_{\substack{k \in \mathbb{N}^s \\ k_1+\ldots+k_s \leq d}} c_k \prod_{i=1}^{s} a_i^{k_i} \quad (6)$$

where $c_k \in \mathbb{R}$. For $g \in \mathbb{R}^s$, let us define

$$h(a) = \sum_{\substack{k \in \mathbb{N}^s \\ k_1+\ldots+k_s \leq d}} c_k (g'a)^{d-k_1-\ldots-k_s} \prod_{i=1}^{s} a_i^{k_i}. \quad (7)$$

For $g \neq 0$, it follows that $h(a)$ is a homogeneous polynomial in a of degree d. Moreover, the coefficients of $h(a)$ are linear functions of the coefficients of $f(a)$. Let us denote the definition of $h(a)$ from $f(a)$ and g as

$$h(a) = \Phi(f(a), a, g). \quad (8)$$

Next, let $s \in \mathbb{R}^r$ and define $\zeta : \mathbb{R}^r \to \mathbb{R}$ as

$$\zeta(s) = \sum_{i=1}^{r} s_i. \quad (9)$$

Also, let us define $Q : \mathbb{R}^r \to \mathbb{R}^q$ as

$$Q(s) = \sum_{i=1}^{r} s_i P_i. \quad (10)$$

Let us express the matrices $B(\theta)$ and $C(\theta)$ as

$$\begin{cases} B(\theta) &=& \left(\begin{array}{ccc} B^{(1)}(\theta) & \ldots & B^{(m)}(\theta) \end{array} \right) \\ C(\theta) &=& \left(\begin{array}{ccc} C^{(1)}(\theta) & \ldots & C^{(p)}(\theta) \end{array} \right)' \end{cases} \quad (11)$$

where $B^{(1)}, \ldots, B^{(m)}, C^{(1)}, \ldots, C^{(p)} : \mathbb{R}^q \to \mathbb{R}^n$.

Theorem 1: Let $i \in \{1, \ldots, m\}$ and $c \in (0, \infty)$. Assume without loss of generality that

$$\|C(\theta(0)) B^{(i)}(\theta(0))\|_\infty < c \quad \forall \theta(0) \in \mathcal{P}. \quad (12)$$

For the system (1) one has

$$\|Y_i(t)\|_\infty < c \quad \forall t \geq 0 \ \forall \theta(\cdot) \in \Theta \quad (13)$$

if there exist a polynomial $v : \mathbb{R}^n \to \mathbb{R}$ and a scalar $\varepsilon \in \mathbb{R}$ such that

$$\begin{cases} v(0) &=& 0 \\ \nabla v(0) &=& 0 \\ \varepsilon &>& 0 \end{cases} \quad (14)$$

and

$$f_l(\cdot), g_2(\cdot), h_{3,j,k}(\cdot) \in \Sigma \quad \forall \begin{cases} j = 0, 1 \\ k = 1, \ldots, p \\ l = 1, \ldots, r \end{cases} \quad (15)$$

where

$$f_l(x) = -\nabla v(x) A(P_l) \quad (16)$$

and

$$\begin{cases} g_1(s) &=& \Phi\left(1 - v(B^{(i)}(Q(s))), s, \zeta(s)\right) \\ g_2(s) &=& g_1(\text{sq}(\text{s})) \end{cases} \quad (17)$$

and

$$\begin{cases} h_{1,j,k}(x,s) = \Phi\left(v(x) - 1, x, \dfrac{C^{(k)}(Q(s))}{(-1)^j c}\right) - \varepsilon \|x\|_2^{\deg(v(x))} \\ h_{2,j,k}(x,s) = \Phi\left(h_{1,j,k}(x,s), s, \zeta(s)\right) \\ h_{3,j,k}(x,s) = h_{2,j,k}(x, \text{sq}(\text{s})). \end{cases} \quad (18)$$

Proof. Suppose that there exist $v(x)$ and ε such that (14)–(15) hold. This implies that $f_l(x)$, $g_2(s)$ and $h_{3,j,k}(x,s)$ are nonnegative. Let us consider the unitary sublevel set of $v(x)$:

$$\mathcal{V} = \{x \in \mathbb{R}^n : \ v(x) \le 1\}.$$

From the nonnegativity of $f_l(x)$, one has that the time derivative of $v(x)$ is nonpositive for all uncertain vectors $\theta(t)$ fixed at the vertices of $\mathcal{P}$, i.e., $P_1, \ldots, P_l$. Since $A(\theta(t))$ is affine linear in $\theta(t)$, it follows that any trajectory of the system (1) (for null input) starting in $\mathcal{V}$ remains in $\mathcal{V}$ for all admissible uncertainties. Applying the Dirac distribution to the i-th input channel with initial condition $x(0^-) = 0$ has the effect to move the initial condition to $x(0) = B^{(i)}(\theta(0))$. Hence, there is not loss of generality in assuming that (12) holds because (12) is equivalent to

$$\|Y_i(0)\|_\infty < c \ \ \forall \theta(\cdot) \in \Theta.$$

From (17) and the definition (6)–(8), one has that

$$g_1(s) = 1 - v(B^{(i)}(Q(s))) \ \ \forall s \in \mathcal{S}$$

where $\mathcal{S}$ is the simplex

$$\mathcal{S} = \{s \in \mathbb{R}^r : \ \zeta(s) = 1, \ s_i \ge 0 \ \forall i = 1, \ldots, r\}.$$

Moreover, the nonnegativity of $g_2(s)$ is equivalent to

$$g_1(s) \ge 0 \ \ \forall s \in \mathcal{S}.$$

This implies that $x(0) \in \mathcal{V}$ for all admissible uncertainties. Similarly, one has that

$$h_{2,j,k}(x,s) = v(x) - 1 - \varepsilon \|x\|_2^{\deg(v(x))} \ \ \forall x \in \mathcal{T}_{j,k}(c,s) \ \forall s \in \mathcal{S}$$

where $\mathcal{T}_{j,k}(c,s)$ is the level set

$$\mathcal{T}_{j,k}(c,s) = \left\{x \in \mathbb{R}^n : \ x' C^{(k)}(Q(s)) = (-1)^j c\right\}.$$

From the nonnegativity of $h_{3,j,k}(x,s)$, one has that

$$v(x) > 1 \ \ \forall x \in \mathcal{T}_{j,k}(c,s) \ \forall s \in \mathcal{S}.$$

Hence, we conclude that $\mathcal{V}$ does not intersect the set of states for which the output has infinity norm equal to c for all admissible uncertainties, and that the trajectory of the system (1) corresponding to the impulse response with respect to the i-th input channel remains in $\mathcal{V}$ for all admissible uncertainties. Since this trajectory starts from $x(0) = B^{(i)}(\theta(0))$ for which the output has infinity norm less than c due to (12), and since this trajectory is continuous, it follows that (13) holds. □

Let us observe that (14) is a set of linear equalities and inequalities on the coefficients of $v(x)$ and on ε. Also, let us observe that (15) is equivalent to a set of LMIs because the coefficients of the polynomials $f_l(x)$, $g_2(s)$ and $h_{3,j,k}(x,s)$ depend affine linearly on the coefficients of $v(x)$ and on ε, and because the condition that any of these polynomials is in Σ can be equivalently reformulated as an LMI feasibility test as explained in [4] and references therein. See also [3], [5] for formulas about the numerical complexity. In conclusion, the condition of Theorem 1 is equivalent to establish feasibility of a system of LMIs.

Problem 1 can be addressed by repeating the condition of Theorem 1 for all channels, i.e, for all $i = 1, \ldots, m$. That is, (5) holds if, for all $i = 1, \ldots, m$ there exist a polynomial $v_i : \mathbb{R}^n \to \mathbb{R}$ and a scalar $\varepsilon_i \in \mathbb{R}$ such that (14)–(15) hold with $v(x)$ and ε replaced by $v_i(x)$ and ε_i.

It is interesting to observe that the condition of Theorem 1 is obtained by evaluating the projection of $v(x) - 1$ onto the set of states for which the output has infinity norm equal to c. This is realized by the function in (6)–(8) and has the benefit of avoiding the introduction of multipliers.

IV. Examples

In this section we present some examples. The toolbox SeDuMi [11] for Matlab is adopted to test the mentioned LMI conditions.

A. Example 1

In this first example we consider for Problem 1 the polytopic LTV system

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \begin{pmatrix} 2\theta(t) - 2 & 1 \\ -3 & -\theta(t) \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) \\ \theta(t) & \in & [0,1]. \end{array}\right.$$

We want to determine the peak of the impulse response of the system for all admissible uncertainties. To this end, we exploit the condition proposed in Theorem 1. In particular, for $\deg(v(x)) = 2, 4, 6$, we obtain the upper bounds $c_2 = 2$, $c_4 = 1.729$ and $c_6 = 1.59$.

Figure 1 shows the trajectories of the system that correspond to the impulse response for some randomly generated admissible uncertainties (found by solving the differential equations for each generated uncertainty). As we can see, there exist trajectories that almost touches the level set $\|Cx\|_\infty = c_6$, which means that the upper bound c_6 is close to the sought value.

For comparison, we test the classical LMI condition based on quadratic Lyapunov function in [3], [10] by considering the matrix $A(\theta)$ at the vertices of the polytope, finding that the guaranteed upper bound is c_2.

B. Example 2

In this second example we consider for Problem 1 the model of a DC motor, specifically (see for instance [8])

$$\left\{\begin{array}{rcl} J_m \ddot{\psi}_m(t) + b_m \dot{\psi}_m(t) & = & K_t i_a(t) \\ L_a \dot{i}_a(t) + R_a i_a(t) & = & -K_e \dot{\psi}_m(t) + v_a(t) \end{array}\right.$$

where $\psi_m(t)$ is the angle, $i_a(t)$ is the current, $v_a(t)$ is the voltage, and J_m, b_m, K_t, L_a, R_a and K_e are parameters. Let us define

$$\left\{\begin{array}{rcl} x(t) & = & (\psi_m(t), \dot{\psi}_m(t), i_a(t))' \\ u(t) & = & v_a(t) \\ y(t) & = & \psi_m(t). \end{array}\right.$$

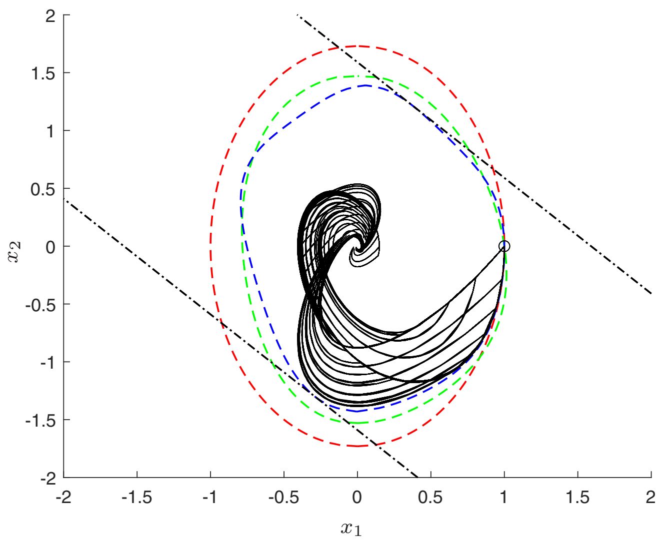

Fig. 1. Example 1: level set $\|Cx\|_\infty = c_6$ (black dash-dot line), impulse response for some admissible uncertainties randomly generated (black solid lines), and unitary level set of the found polynomial $v(x)$ for $\deg(v(x)) = 2, 4, 6$ (red, green and blue dashed lines).

It follows that the model can be rewritten as

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{b_m}{J_m} & \dfrac{K_t}{J_m} \\ 0 & -\dfrac{K_e}{L_a} & -\dfrac{R_a}{L_a} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{L_a} \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t). \end{array}\right.$$

Let us choose the plausible values

$$b_m = 0.2,\ K_t = 1,\ L_a = 0.5,\ R_a = 1,\ K_e = 0.5.$$

Moreover, we consider that the parameter J_m is a time-varying uncertainty in the interval $[1, 3]$. This situation can be considered by the polytopic LTV system (1) by defining

$$\left\{\begin{array}{rcl} \theta(t) & = & \dfrac{1}{J_m} \\ \mathcal{P} & = & [1/3, 1]. \end{array}\right.$$

We want to determine the peak of the impulse response of the system for all admissible uncertainties. To this end, we exploit the condition proposed in Theorem 1. In particular, for $\deg(v(x)) = 2, 4, 6$, we obtain the upper bounds $c_2 = \infty$, $c_4 = 13.349$ and $c_6 = 6.183$. Figure 2 shows the level set of the Lyapunov function found for $\deg(v(x)) = 6$.

Also in this example, we test for comparison the classical LMI condition based on quadratic Lyapunov function in [3], [10] by considering the matrix $A(\theta)$ at the vertices of the polytope, finding that the guaranteed upper bound is c_2.

V. Conclusions

This paper has proposed a novel condition for establishing upper bounds of the peak of the impulse response of polytopic LTV systems. The proposed condition requires to establish feasibility of a system of LMIs, it is sufficient for any degree of a polynomial introduced to describe the

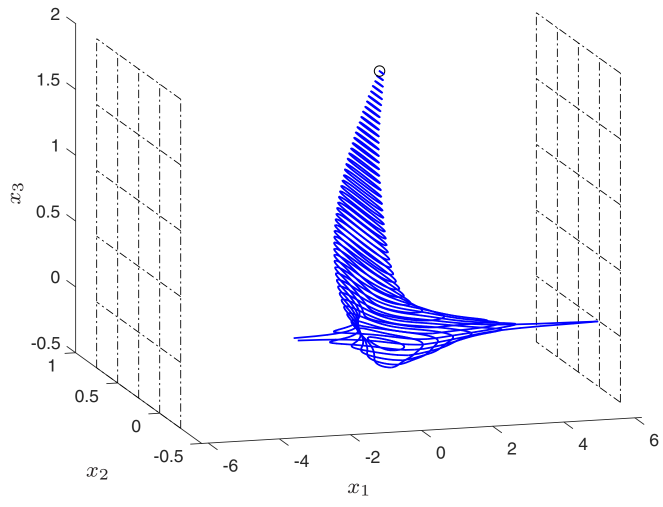

Fig. 2. Example 2: level set $\|Cx\|_\infty = c_6$ (black grid) and unitary level set of the found polynomial $v(x)$ for $\deg(v(x)) = 6$ (blue surface).

impulse response, and its conservatism can be decreased by increasing this degree. Some examples have shown the use of the proposed condition and its advantage with respect to classical methods.

Several directions can be considered in future work. In particular, it will be interesting to investigate the extension of the proposed condition to the design of feedback controllers for ensuring desired upper bounds on the peak of the impulse response. Another direction could consider the extension of the proposed conditions to nonlinear systems.

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