Quantum Metrology with Indefinite Causal Order

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We address the study of quantum metrology enhanced by indefinite causal order, demonstrating a quadratic advantage in the estimation of the product of two average displacements in a continuous variable system. We prove that no setup where the displacements are used in a fixed order can have root-mean-square error vanishing faster than the Heisenberg limit $1/N$, where N is the number of displacements contributing to the average. In stark contrast, we show that a setup that probes the displacements in a superposition of two alternative orders yields a root-mean-square error vanishing with super-Heisenberg scaling $1/N^2$, which we prove to be optimal among all superpositions of setups with definite causal order. Our result opens up the study of new measurement setups where quantum processes are probed in an indefinite order, and suggests enhanced tests of the canonical commutation relations, with potential applications to quantum gravity.

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The traditional formulation of quantum mechanics assumes that the order of physical processes is well defined. Recently, a number of works started exploring new scenarios where the causal order is indefinite [1–6]. This extension is motivated by ideas in quantum gravity, where the order of events could be subject to quantum indefiniteness [7,8], and has potential applications in quantum information, where advantages have been found in channel discrimination tasks [9,10], nonlocal games [2,5], and communication complexity [11].

A paradigmatic example of process with indefinite causal order is the quantum SWITCH [1,4], a higher-order operation that combines two input gates in a quantum superposition of two alternative orders. When applied to two unitary gates U_1 and U_2, the quantum SWITCH generates the controlled unitary gate

$$S(U_1, U_2) \coloneqq |0\rangle\langle 0| \otimes U_2 U_1 + |1\rangle\langle 1| \otimes U_1 U_2 \quad (1)$$

by querying each of the two gates $\{U_1, U_2\}$ only once. Here first register on the right-hand side of Eq. (1) serves as a control of the order. When put in a coherent superposition of the states $|0\rangle$ and $|1\rangle$, it induces a coherent superposition of the two alternative orders $U_1 U_2$ and $U_2 U_1$. The quantum SWITCH has been shown to offer a number of information-processing advantages [9–11] and has inspired experiments in quantum optics [12–16], where the superposition of orders is reproduced by sending photons on a superposition of alternative paths [17]. Recently, it has stimulated an extension of Shannon theory to scenarios where the order of the communication channels is in a quantum superposition [18–20].

In this work, we show that the quantum SWITCH can boost the precision of quantum metrology, beating the limits associated with conventional schemes where processes are probed in a definite order. To illustrate this phenomenon, we consider a situation where an experimenter has access to $2N$ black boxes, each acting on a harmonic oscillator, with the promise that the first N boxes perform displacements generated by a given quadrature X, and the second N boxes perform displacements in the conjugate quadrature P. Displacements performed by different boxes are independent, and the task is to measure the product of the average displacement in X and the average displacement in P.

When the black boxes are used in a fixed order, we prove that the root mean square error (RMSE) cannot vanish faster than $f(E)/N$, where $f(E)$ is a function of the energy of the input states used to probe the black boxes. The scaling $1/N$ is consistent with the Heisenberg limit of quantum metrology [21], applied to the estimation of the two average displacements in X and P. In stark contrast, we show that a setup using the quantum SWITCH can achieve an error vanishing with super-Heisenberg scaling $1/N^2$, independently of the energy of the input states. Our result demonstrates that a setup that probes a sequence of processes in a coherent superposition of alternative orders can extract more information than any setup where the order of the processes is fixed. Furthermore, we show that the scaling $1/N^2$, achieved by our concrete setup, is optimal among all setups obtained by superposing causally ordered processes with bounded energy.

Our scenario can be described as follows. An experimenter has access to $2N$ black boxes, each implementing either a position displacement $D_{x_j} = e^{-ix_jP}$ or a momentum displacement $D_{p_k} = e^{ip_kX}$ ($j, k = 1, \ldots, N$), where X and P are the conjugate variables $X \coloneqq (a + a^\dagger)/\sqrt{2}$ and $P \coloneqq i(a^\dagger - a)/\sqrt{2}$, and a and $a^\dagger$ satisfy the canonical commutation relation $[a, a^\dagger] = I$. The displacements $\{x_j\}$ and $\{p_k\}$ are unknown, and vary independently within the range $[x_{\min}, x_{\max}]$ and $[p_{\min}, p_{\max}]$, respectively. The task is to estimate the product $A \coloneqq \bar{x} \cdot \bar{p}$ between the average displacements $\bar{x} \coloneqq \sum_{j=1}^N x_j/N$ and $\bar{p} \coloneqq \sum_{j=1}^N p_j/N$, by querying each black box only once in every run of the experiment. For simplicity, we will assume that the average displacements $\bar{x}$ and $\bar{p}$ are nonzero and converge to nonzero values in the large N limit.

The simplest way to estimate A is to measure each displacement independently, as illustrated in Fig. 1(a). A bound on the RMSE follows immediately from the quantum Cramér-Rao bound [22–24], which can be applied to the estimation of a displacement z, yielding the lower bound $\Delta z \geq 1/\sqrt{8\nu E}$, where $E \coloneqq \langle\psi|(X^2 + P^2)|\psi\rangle/2$ is the average energy of the probe state, and ν is the number of repetitions of the experiment (see the Supplemental Material [25] for a derivation.) This bound implies that, once the energy E has been fixed, the error in the estimation of a single displacement is a constant. The error in the estimation of individual displacements then propagates to the estimation of the product, yielding an overall scaling $1/\sqrt{\nu N}$, corresponding to the standard quantum limit [21].

A better scaling can be obtained if, instead of measuring each displacement separately, one directly measures the two average displacements $\bar{x}$ and $\bar{p}$, by applying the total x displacement $D_{N\bar{x}} = D_{x_1}D_{x_2}\cdots D_{x_N}$ and the total p displacement $D_{N\bar{p}} = D_{p_1}D_{p_2}\cdots D_{p_N}$ to two independent probes, each of average energy E, as in Fig. 1(b). In this case, the Cramér-Rao bound implies that the RMSE for each average displacement is lower bounded by $1/(N\sqrt{8\nu E})$, and therefore error propagation gives the RMSE scaling as $1/(\sqrt{\nu}N)$ for the estimation of the product with any bounded energy E.

The $1/N$ scaling corresponds to the Heisenberg limit for the estimation of the average displacements $\bar{x}$ and $\bar{p}$ [21]. Later in the Letter we will prove that the scaling $1/N$ is optimal among all setups where the given black boxes are probed in a definite order, using a finite amount of energy.

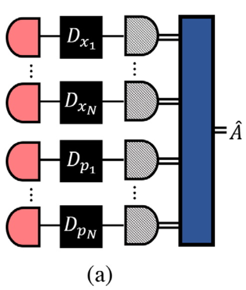

FIG. 1. Two causally ordered schemes. (a) Parallel scheme with measurements of individual displacements. $2N$ independent probes, each with average energy bounded by E, are used to estimate the $2N$ displacements $(x_i)_{i=1}^N$ and $(p_j)_{j=1}^N$. The average displacements $\bar{x} = \sum_i x_i/N$ and $\bar{p} = \sum_j p_j/N$, and their product $A = \bar{x}\,\bar{p}$ are then computed by classical postprocessing. The RMSE of the scheme has the standard quantum limit scaling $1/\sqrt{N}$. (b) Sequential scheme with independent x and p measurements. The average displacements $\bar{x}$ and $\bar{p}$ are measured directly by applying the total x displacement $D_{x_1}D_{x_2}\cdots D_{x_N}$ and the total p displacement $D_{p_1}D_{p_2}\cdots D_{p_N}$ to two independent probes, each with average energy bounded by E. The product $A = \bar{x}\,\bar{p}$ is then computed by classical postprocessing. The RMSE of this scheme has the Heisenberg scaling $1/N$.

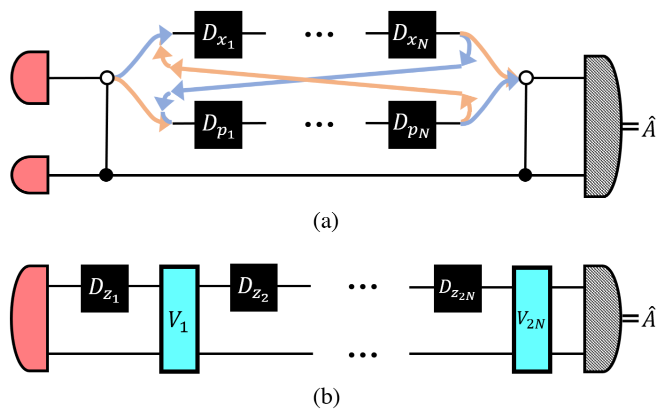

FIG. 2. Definite vs indefinite order in a quantum metrology setup. (a) Estimation scheme using the quantum SWITCH. The total x displacements $D_{x_1}D_{x_2}\cdots D_{x_N}$ and p displacements $D_{p_1}D_{p_2}\cdots D_{p_N}$ act in a coherent superposition of two alternative orders, controlled by the state of a control qubit. If the control is prepared in the state $|0\rangle$ ($|1\rangle$), the probe will experience the displacements in the order corresponding to the blue (orange) path. By preparing the probe in the minimum-energy state $|0\rangle$ and the control qubit in the state $|+\rangle$, this scheme achieves the super-Heisenberg scaling $1/N^2$ of the RMSE. (b) Generic causally ordered scheme. A probe and an auxiliary system are prepared in a generic state, with average energy of the probe bounded by E. Then, the probe undergoes a sequence of displacements, arranged in a fixed order $(z_1, \ldots, z_{2N})$, where $(z_1, \ldots, z_{2N})$ is an arbitrary permutation of the sequence $(x_1, \ldots, x_N, p_1, \ldots, p_N)$. Each displacement operation z_i is followed by a unitary gate V_i, acting jointly on the probe and the auxiliary system. Finally, a joint measurement is performed on the probe and the auxiliary system. Every estimation scheme of this form, including the schemes in Figs. 1(a) and 1(b), must have the RMSE vanishing no faster than $1/N$.

We now show that a setup using the quantum SWITCH can achieve the super-Heisenberg scaling $1/N^2$. The setup creates a coherent superposition of two configurations: one where all the x displacements are used first, and one where all the p displacements are used first, as in Fig. 2(a). The process experienced by the probe is a unitary with a qubit control

$$W = |0\rangle\langle 0| \otimes \prod_{j=1}^{N} D_{p_j} \prod_{j=1}^{N} D_{x_j} + |1\rangle\langle 1| \otimes \prod_{j=1}^{N} D_{x_j} \prod_{j=1}^{N} D_{p_j}. \tag{2}$$

Our scheme for estimating A is illustrated in Fig. 2(a). It consists of the following steps: (1) Prepare the control of the quantum SWITCH in the state $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$. (2) Prepare the probe in an arbitrary state $|\psi\rangle$, such as, e.g., the minimum-energy state $|0\rangle$. (3) Apply the gate W to the input state $|+\rangle \otimes |\psi\rangle$. (4) Measure the control using the projective measurement $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ with $|-\rangle := (|0\rangle - |1\rangle)/\sqrt{2}$. (5) Repeat the above procedure for ν rounds and output the maximum likelihood estimate $\hat{A} := \arg\max_A \log p(m_1, \ldots, m_\nu|A)$, where $m_j \in \{+,-\}$ is the jth measurement outcome, and $p(m_1, \ldots, m_\nu|A)$ is the probability of obtaining the measurement outcomes $\{m_1, \ldots, m_\nu\}$ conditioned on the parameter being A.

Using the Weyl relation $e^{ipX}e^{-ixP} = e^{ixpI}e^{-ixP}e^{ipX}$, the output unitary of the SWITCH [Eq. (2)] can be cast into the product form

$$W = (|0\rangle\langle 0| + e^{iN^2A}|1\rangle\langle 1|) \otimes \left(\prod_{j=1}^{N} D_{p_j} \prod_{j=1}^{N} D_{x_j}\right). \tag{3}$$

Then, one can immediately see that the final state of the control qubit is $(|0\rangle + e^{iN^2A}|1\rangle)/\sqrt{2}$, and the probability of getting the outcome $\pm$ is $p(\pm|A) = [1 \pm \cos(N^2A)]/2$.

Since our estimator is unbiased, its RMSE satisfies the Cramér-Rao bound [28–30]

$$\Delta A_{\text{SWITCH}} \geq \frac{1}{\sqrt{\nu F_A}} \tag{4}$$

where F_A is the Fisher information of the parameter A, given by

$$F_A := \sum_{m\in\{+,-\}} p(m|A)\left[\frac{\partial \ln p(m|A)}{\partial A}\right]^2 = N^4. \tag{5}$$

The Cramér-Rao bound [Eq. (4)] is achievable in the large ν limit, and we have the asymptotic equality

$$\Delta A_{\text{SWITCH}} = \frac{1}{\sqrt{\nu}N^2}. \tag{6}$$

Hence, the estimation scheme based on the quantum SWITCH achieves the super-Heisenberg scaling $1/N^2$ in terms of the number of displacements contributing to the average. Notice that the $1/N^2$ scaling is independent of the energy of the probe, meaning that the quantum SWITCH allows one to extract precise information even in the low-energy regime.

Our estimation scheme provides an accurate estimate for small values of the parameter A, i.e., values not exceeding the period of the functions $p(+|A)$ and $p(-|A)$. Alternatively, our estimation scheme can be seen as a way to estimate the total phase $\phi := \sum_{i,j} x_i p_j \mod 2\pi$ with RMSE $\Delta\phi_{\text{SWITCH}} = 1/\sqrt{\nu}$. This scaling cannot be achieved with the causally ordered estimation scheme of Fig. 1(b), because the total displacements in x and p grow as N, and therefore error propagation implies that the RMSE of their product grows as N, thus making the estimation of the phase ϕ unreliable whenever N is large compared to 2π. More generally, we will see that no causally ordered scheme can achieve the RMSE scaling $\Delta\phi = 1/\sqrt{\nu}$.

Note that our scheme does not involve any measurement on the probe. The scheme can be further improved by measuring the probe with a heterodyne measurement, whose measurement operators are projections on coherent states. When the probe is initialized in a coherent state, such as the minimum-energy state $|0\rangle$, we show that our scheme can achieve RMSE

$$\Delta A'_{\text{SWITCH}} = \frac{1}{\sqrt{\nu}N^2}\sqrt{\frac{\bar{x}^2 + \bar{p}^2}{\bar{x}^2 + \bar{p}^2 + 1/N^2}}. \tag{7}$$

The derivation of Eq. (7) can be found in the Supplemental Material [25].

We now show that the error scaling $1/N^2$ cannot be achieved if the unknown displacements are used in a definite order. Specifically, we will show that every estimation strategy with fixed order [see Fig. 2(b)] will have a RMSE vanishing no faster than $1/N$. Suppose that the first displacement operation in the sequence is D_{x_1}. In this case, every estimation scheme with fixed causal order can also be used to estimate A in the less challenging scenario where all the displacements except x_1 are known. In this scenario, the RMSE is simply $\Delta x_1/|\partial x_1/\partial A| = |\bar{p}|\Delta x_1/N$, where Δx_1 is the error in estimating x_1 from the displacement operation D_{x_1}. Similarly, if the first displacement operation is D_{p_1}, one obtains RMSE $\Delta p_1/|\partial p_1/\partial A| = |\bar{x}|\Delta p_1/N$, where Δp_1 is the error in estimating p_1 from the displacement operation D_{p_1}. In general, the RMSE for the estimation of A in any fixed causal order is lower bounded as

$$\Delta A_{\text{fixed}} \geq \frac{\min_j |c_j| \cdot \Delta z_j}{N}, \tag{8}$$

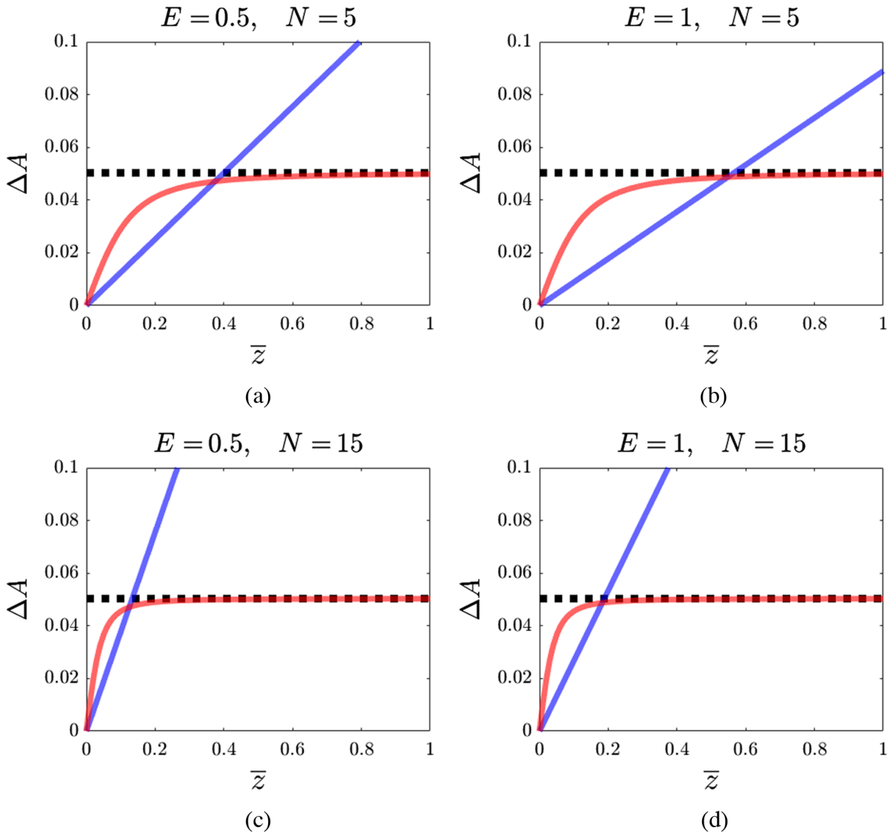

FIG. 3. Definite vs indefinite order in the nonasymptotic regime. The RMSE achievable with the quantum SWITCH is plotted against the lower bound to the RMSE for every estimation scheme with definite causal order. The four plots correspond to the parameter values $|\bar{x}| = |\bar{p}| = \bar{z} > 0$, $\nu = 10$, and (a) $E = 0.5$, $N = 5$; (b) $E = 1$, $N = 5$; (c) $E = 0.5$, $N = 15$; (d) $E = 1$, $N = 15$. The y axis shows the RMSE ΔA in units of $2\pi/N^2$. The solid red lines show the RMSE $\Delta A'_{\text{SWITCH}}$, achievable by measuring the probe and the control [Eq. (7)]. The dashed lines show the RMSE ΔA_{SWITCH}, achievable by measuring the control alone [Eq. (6)]. The blue lines show the lower bound of the RMSE ΔA_{fixed} [Eq. (9)].

where $\{z_j\}$ are the $2N$ displacements, and $c_j = \bar{p}$ ($\bar{x}$) if z_j is a position (momentum) displacement. Since the RMSE in estimating a displacement z_j is lower bounded by $1/\sqrt{8\nu E}$ with E being the initial energy of the probe, Eq. (8) yields the bound

$$\Delta A_{\text{fixed}} \geq \frac{\min\{|\bar{x}|, |\bar{p}|\}}{\sqrt{8\nu E}N}. \tag{9}$$

A more formal derivation of the bound Eq. (9) is provided in the Supplemental Material [25].

The advantage of indefinite causal order can immediately be identified when comparing the RMSEs Eqs. (7) and (9). Using a quantum SWITCH, the error vanishes as $1/N^2$ instead of $1/N$. In terms of the phase $\phi = N^2 A \bmod 2\pi$, the quantum SWITCH offers RMSE scaling as $1/\sqrt{\nu}$ with the number of repetitions of the experiment, while every scheme with definite causal order has RMSE scaling at best as $N/\sqrt{\nu}$ in the $\nu \gg N$ regime. In Fig. 3 we compare the RMSE Eq. (7) with the lower bound Eq. (9) for various values of N and E.

A natural question is whether more general forms of indefinite causal order, other than the quantum SWITCH, can beat the scaling $1/N^2$. As it turns out, the answer is negative for all superpositions of definite causal orders. The argument can be sketched as follows. The RMSE in the estimation of A is lower bounded by the RMSE in the situation where all displacements except one (say x_1) are known. In that case, we have $\Delta A = \bar{p}\Delta x_1/N$. We then show that no superposition of causal orders with bounded energy can achieve RMSE Δx_1 vanishing faster than $1/N$. Putting everything together, this means that the RMSE for the estimation of A cannot vanish faster than N^2 (see the Supplemental Material [25] for the full argument.)

Our protocol suggests a way to test modifications of the canonical commutation relations, such as those envisaged in certain theories of quantum gravity [31–34]. For example, Ref. [34] argues that the commutation relation should be replaced by $[X, P] = i(I + \beta P^2)$, where $\beta \ll 1$ is a suitable coefficient. Using the quantum SWITCH setup one can in principle create the superposition

$$|\Psi\rangle = \frac{(I \otimes D_p D_x)(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle)}{\sqrt{2}} \tag{10}$$

where U is the unitary operator

$$U = D_{-x}D_{-p}D_xD_p = e^{-ixp}e^{-i\beta x(pP^2+p^2P+\frac{1}{3}p^3)} + O(\beta^2). \quad (11)$$

Choosing the state $|\psi\rangle$ to be close to an eigenstate of the momentum operator, we then obtain the state $|\Psi\rangle \approx D_xD_p|\psi\rangle \otimes (|0\rangle + e^{-ixp[1+(7/3)\beta p^2]}|1\rangle)/\sqrt{2}$. If the size of the displacements grows linearly, namely, $x = N\bar{x}$ and $p = N\bar{p}$ for two fixed values $\bar{x}$ and $\bar{p}$, then the constant β can be measured with RMSE scaling as $1/N^4$. In other words, our scheme offers a favorable scaling with the size of the displacements.

Other theories of quantum gravity [32] exhibit noncommutativity of the position operators associated with different Cartesian coordinates. For example, the position operators X and Y can become conjugate variables, satisfying the canonical commutation relation $[X, Y] = ic_{xy}I$ where c_{xy} is a small constant. Therefore, in this scenario protocol could in principle offer a way to measure the constant and to discover small amounts of noncommutativity of the two coordinates X and Y.

These potential applications motivate the search for experimental implementations of our setup. For discrete variables, the quantum SWITCH can be reproduced on photonic systems using superpositions of paths [12,13,15]. For continuous variables, Ref. [35] suggests that a quantum SWITCH could be implemented in new experiments with Gaussian quantum optics. However, no photonic realization of the continuous-variable quantum SWITCH has been proposed to date. Alternatively, we suggest that the continuous-variable quantum SWITCH could be implemented with massive particles with a continuous-variable internal degree of freedom, using the path of the particle to control the order of different displacement operations. For example, the internal degrees of freedom could be the vibrational modes of a molecule or the internal states of a Bose-Einstein condensate. Another alternative is to reproduce our setup in ion trap systems, where the spin and the axial mode of motion of an ion can be coupled together in a way that implements the control-unitary gates $U_j = |0\rangle\langle 0| \otimes D_{x_j} + |1\rangle\langle 1| \otimes D_{x_j}^\dagger$ and $V_j = |0\rangle\langle 0| \otimes D_{x_j}^\dagger + |1\rangle\langle 1| \otimes D_{x_j}$ [36,37]. In this scenario, the quantum SWITCH can be simulated by first applying all the gates U_j (with j running from 1 to n), then all the displacements D_{p_j}, and finally all the gates V_j. Overall, this sequence of gates results in the gate $(|0\rangle\langle 0| + e^{2N^2Ai}|1\rangle\langle 1|) \otimes D_{N\bar{x}}^\dagger D_{N\bar{p}} D_{N\bar{x}}$, from which the parameter A can be estimated with RMSE $\Delta A = 1/(2\sqrt{\nu}N^2)$.

In summary, we showed the quantum metrology schemes using indefinite causal orders can sometime outperform the standard schemes where quantum processes are probed in a definite order. Specifically, we showed that every estimation scheme that probes N pairs of displacements in a definite order has an error vanishing no faster than $1/N$ for the estimation of the product of the average displacements. Instead, we showed that an estimation scheme using the quantum SWITCH achieves the enhanced scaling $1/N^2$. Our result opens up a new area of research on the study of quantum metrology schemes powered by indefinite causal order.

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Note added.—Recently, we found two studies on the application of the quantum SWITCH in quantum thermometry [38] and channel identification [39]. These works showed an increase of the quantum Fisher information by a constant amount when the order of two channels is put in a coherent superposition, but did not address the comparison with the performances of arbitrary schemes with definite causal order.

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[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.190503, which contains Refs. [26] and [27], as well as a proof of the error bound for the estimation of a single displacement, proofs of Eqs. (7) and (9), and a proof that the scaling $1/N^2$ is optimal among all superpositions of causally ordered circuits.
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