Structured Feedback Synthesis for Stability and Performance of Switched Systems

Graziano Chesi[1], *Fellow, IEEE,* and Patrizio Colaneri[2], *Fellow, IEEE*

Abstract

This paper addresses the synthesis of fixed-order output feedback controllers for stability and performance of continuous-time switched linear systems with dwell time constraints or arbitrary switching. Specifically, the paper starts by considering the stabilization problem, which is addressed by searching for a family of homogeneous polynomial Lyapunov functions (HPLFs) parameterized polynomially by the sought controller. In order to conduct this search, polynomials are introduced for approximating the matrix exponential and for quantifying the feasibility of the Lyapunov inequalities. It is shown that a stabilizing controller exists if and only if a condition built solving three convex optimization problems with linear matrix inequalities (LMIs) holds for polynomials of degree sufficiently large. Analogous conditions for the existence of a controller ensuring desired upper bounds on the $\mathcal{H}_2$ norm and on the RMS gain of the closed-loop system are derived by searching for a family of homogeneous rational Lyapunov functions (HRLFs) parameterized rationally by the sought controller.

Index Terms

Switched system, Stability, $\mathcal{H}_2$ norm, RMS gain, Feedback synthesis.

I. Introduction

Switched systems are dynamical systems allowed to change with the time in a finite family as effect of a signal called switching rule. Switched systems play a fundamental role in automatic control, and can be found in a number of fields, such as mechanics [36], power systems [41] and systems biology [22], [28]. In this paper, the switching rule is assumed to be an exogenous deterministic signal.

[1]Graziano Chesi is with the Department of Electrical and Electronic Engineering, The University of Hong Kong. Contact information: see http://www.eee.hku.hk/˜chesi

[2]Patrizio Colaneri is with the Dipartimento di Elettronica, Informazione e Bioingegneria (DEIB), Politecnico di Milano e IEIIT-CNR. Contact information: see http://home.dei.polimi.it/colaneri

Switched systems are generally classified into two main classes, depending on the admissible switching rules: switched systems with dwell time constraints and switched systems with arbitrary switching. In the former class, the changes among the mathematical models can occur only after a minimum time, called dwell time, which can be hard or average. In the latter class, the changes among the mathematical models can occur arbitrarily fast. In this paper, (hard) dwell time constraints and arbitrary switching are considered.

A fundamental problem in switched systems is stability analysis. This problem has been investigated in numerous works for continuous-time switched linear systems. See for instance the books [3], [8], [29], [44], the surveys [18], [31], [32], [42], and the papers [5], [24], [30], [34], [37], [45]. Linear matrix inequality (LMI) conditions have been proposed in [16], [17], [19], [27], [35] based on various types of Lyapunov functions, such as quadratic Lyapunov functions, piecewise quadratic Lyapunov functions, and homogeneous polynomial Lyapunov functions (HPLFs).

Another fundamental problem in switched systems is performance analysis, in particular, concerning the $\mathcal{H}_2$ norm and the root mean square (RMS) gain. These indexes have been studied for switched linear systems in [23], [25], [32], [33] through techniques such as variational principles and worst-case control. In [13], [20], LMI conditions have been proposed in order to determine upper bounds through convex optimization based on the use of quadratic Lyapunov functions and homogeneous rational Lyapunov functions (HRLFs).

Control synthesis directly follows the problems mentioned above. One way to deal with control synthesis consists of designing a stabilizing switching rule, see for instance [2], [19], [21], [26], [38], [46]. Another way consists of designing a stabilizing feedback controller, see for instance [1], [44]. Unfortunately, this problem is not easy to solve because, by letting the controller be a decision variable in the existing conditions for establishing stability, one obtains nonconvex optimization problems, in general due to the presence of products between the Lyapunov function and the controller.

This paper addresses the synthesis of fixed-order output feedback controllers for stability and performance of continuous-time switched linear systems with dwell time constraints or arbitrary switching. Specifically, the paper starts by considering the determination of a mode-independent static output feedback controller that ensures stability for a strictly proper switched system. This problem is addressed by searching for a family of HPLFs parameterized polynomially by the sought controller that prove stability for the considered set of switching rules. In order to conduct this search, polynomials are introduced for approximating the matrix exponential and for quantifying the feasibility of the Lyapunov inequalities. It is shown that a stabilizing controller exists if and only if a condition built solving three convex optimization problems with LMIs holds for polynomials of degree sufficiently large. Hence, the paper continues by considering the determination of a mode-independent static output feedback controller that ensures desired upper bounds on the $\mathcal{H}_2$ norm and on the RMS gain of the closed-loop system, which is addressed by searching for a family of HRLFs parameterized rationally by the sought controller. Analogous necessary and sufficient LMI conditions are derived for the $\mathcal{H}_2$ norm in both cases of dwell time constraints or arbitrary switching, and for the RMS gain in the case of arbitrary switching. Lastly, the paper describes several extensions of the proposed methodology, which consider the cases of mode-dependent controller, dynamic controller, and non-strictly proper switched system. Some numerical examples illustrate the proposed methodology.

The paper is organized as follows. Section II introduces the preliminaries. Section III describes the proposed methodology for the stabilization problem. Section IV addresses the $\mathcal{H}_2$ norm control problem. Section V considers the RMS gain control problem. Section VI reports some comments and extensions. Section VII presents the numerical examples. Lastly, Section VIII concludes the paper with some final remarks. This paper extends the preliminary conference versions [14], [15] which do not address the $\mathcal{H}_2$ norm control problem (i.e., Section IV) and do not present the extensions (i.e., Section VI).

II. Preliminaries

Let us start by introducing the notation adopted in the paper. 0, I: null matrix and identity matrix of size specified by the context. $\mathbb{N}_0$, $\mathbb{N}$: sets of non-negative and positive integers. $\mathbb{R}$: set of real numbers. $\mathbb{S}^n$: set of symmetric matrices in $\mathbb{R}^{n\times n}$. A': transpose of A. $\det(A)$: determinant of A. $\exp(A)$: exponential of A. $\mathrm{he}(A)$: $A+A'$. $A\otimes B$: Kronecker's product between A and B. $A^{\otimes n}$: n-th Kronecker power, i.e., $A\otimes\cdots\otimes A$ where the number of occurrences of A is n. $\|A\|_2$, $\|A\|_\infty$ and $\|A\|_{Fro}$: 2-norm, ∞-norm and Frobenius' norm of A. $\|a(\cdot)\|_{\mathcal{L}_2}$: $\mathcal{L}_2$-norm of $a(t)$, i.e., $\|a(\cdot)\|_{\mathcal{L}_2}=\sqrt{\int_0^\infty \|a(t)\|_2^2 dt}$. $A\geq 0$ (respectively, $A>0$): symmetric positive semidefinite (respectively, definite) matrix A. $\star$: corresponding block in a symmetric matrix or generic subscript. s.t.: subject to. w.r.t.: with respect to.

A. Basic Model

In this section we introduce the basic model that will be considered for describing the proposed methodology. More general models will be discussed in Section VI. Let us start by considering the switched system

$$\begin{cases} \dot{x}(t) &= A_{1,\sigma(t)}x(t) + B_{1,\sigma(t)}u(t) + B_{2,\sigma(t)}w(t), \\ y(t) &= C_{1,\sigma(t)}x(t) + D_{1,\sigma(t)}u(t) + D_{2,\sigma(t)}w(t), \\ z(t) &= C_{2,\sigma(t)}x(t) + D_{3,\sigma(t)}u(t) + D_{4,\sigma(t)}w(t), \\ \sigma(\cdot) &\in \mathcal{D}, \end{cases} \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{m_1}$ is the control input, $w(t) \in \mathbb{R}^{m_2}$ is the performance input, $y(t) \in \mathbb{R}^{p_1}$ is the control output, $z(t) \in \mathbb{R}^{p_2}$ is the performance output, $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ is the switching rule, N is the number of subsystems, $\mathcal{D}$ is the set of admissible switching rules, and (by replacing $\sigma(t)$ with i) $A_{1,i}$, ..., $D_{4,i}$, $i = 1, \ldots, N$, are real matrices of suitable sizes. The system obtained for $\sigma(t) = i$ is called the i-th subsystem of the switched system (1).

This paper considers two sets of admissible switching rules, namely, the set of switching rules with dwell time $T > 0$, i.e.,

$$\begin{aligned} \mathcal{D}_T &= \{\sigma : \mathbb{R} \to \{1, \ldots, N\},\ \sigma(t) = constant \\ &\quad \forall t \in [t_i, t_{i+1}),\ i \in \mathbb{N}_0,\ t_{i+1} - t_i \geq T\}, \end{aligned} \tag{2}$$

and the set of arbitrary switching rules, i.e.,

$$\mathcal{D}_{arb} = \{\sigma : \mathbb{R} \to \{1, \ldots, N\}\}. \tag{3}$$

The reader is referred to the well-known monographs [29], [44] for more information on switched systems and switching rules.

Three main problems are addressed in this paper as it will be explained in the following sections, namely, stabilization, $\mathcal{H}_2$ norm control, and RMS gain control. For clarity of description, these problems will be firstly considered for a basic model, where it is assumed that:

1) the map between $u(t)$ and $y(t)$ in the switched system (1) is strictly proper, i.e.,

$$D_{1,i} = 0; \tag{4}$$

2) a mode-independent static output feedback controller is connected between the control output and the control input of the switched system (1), i.e.,

$$u(t) = Ky(t) \tag{5}$$

where $K \in \mathbb{R}^{m_1 \times p_1}$ has to be determined in the set

$$\mathcal{K} = \left\{ K \in \mathbb{R}^{m_1 \times p_1} : \ \|K\|_\infty \le \rho \right\} \tag{6}$$

where $\rho > 0$ is a given quantity. Let us observe that $\mathcal{K}$ is a compact set, in particular, a hypercube.

Hence, the closed-loop switched system obtained for the basic model is

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_{\sigma(t)}(K)x(t) + B_{\sigma(t)}(K)w(t), \\ z(t) & = & C_{\sigma(t)}(K)x(t) + D_{\sigma(t)}(K)w(t), \\ \sigma(\cdot) & \in & \mathcal{D} \end{array} \right. \tag{7}$$

where the i-th subsystem is described by the matrices

$$\left\{ \begin{array}{rcl} A_i(K) & = & A_{1,i} + B_{1,i}KC_{1,i}, \\ B_i(K) & = & B_{2,i} + B_{1,i}KD_{2,i}, \\ C_i(K) & = & C_{2,i} + D_{3,i}KC_{1,i}, \\ D_i(K) & = & D_{4,i} + D_{3,i}KD_{2,i}. \end{array} \right. \tag{8}$$

The formulation presented in this section includes the case of state feedback control, which occurs when $y(t) = x(t)$. If $x(t)$ cannot be measured but it is estimated through an observer, feedback controllers exploiting the estimate of the state can be designed by replacing $y(t)$ in (5) with the estimate of the state and by including the dynamics of the observer in the switched system (1) through state augmentation.

The dependence on t of the various quantities will be omitted in the sequel of the paper for ease of notation unless specified otherwise.

B. Gram Matrix Method

Here we report some basic information about the Gram matrix method, see for instance [9] and references therein for more details. Any homogeneous polynomial $v : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$, $d \in \mathbb{N}_0$, can be expressed through the Gram matrix method, also known as square matrix representation (SMR), as

$$v(x) = b(x,d)'Vb(x,d) \tag{9}$$

where $x \in \mathbb{R}^n$, $b(x,d) \in \mathbb{R}^{\varsigma_d}$ is a base vector for the homogeneous polynomials in x of degree d, and $V \in \mathbb{S}^{\varsigma_d}$. The matrix V, called Gram matrix of $v(x)$ w.r.t. $b(x,d)$, depends linearly on the coefficients of $v(x)$. The length of $b(x,d)$ is given by

$$\varsigma_d = \frac{(n+d-1)!}{(n-1)!d!}. \tag{10}$$

A typical way of choosing $b(x,d)$ is through the recursive rule

$$b(x,d) = \begin{cases} 1 & \text{if } d = 0, \\ \begin{pmatrix} x_1 b(X_1(x), d-1) \\ \vdots \\ x_n b(X_n(x), d-1) \end{pmatrix} & \text{if } d > 0, \end{cases} \tag{11}$$

where $X_i(x) = (x_i, \ldots, x_n)'$. The following are examples of vectors built with this rule:

$$\begin{aligned} n = 2,\ d = 3: \quad & b(x,d) = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)', \\ n = 3,\ d = 2: \quad & b(x,d) = (x_1^2, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2)'. \end{aligned}$$

The representation (9) is useful to establish if $v(x)$ is a sum of squares of polynomials (SOS), i.e., $v(x) = \sum_{i=1}^k v_i(x)^2$ for some polynomials $v_1(x), \ldots, v_k(x) \in \mathbb{R}$. Indeed, let us define the linear set

$$\mathcal{L}_d = \left\{ \tilde{L} \in \mathbb{S}^{\varsigma_d} : \ b(x,d)' \tilde{L} b(x,d) = 0 \right\} \tag{12}$$

whose dimension is

$$\tau_d = \frac{1}{2} \varsigma_d \left(\varsigma_d + 1 \right) - \varsigma_{2d}. \tag{13}$$

Then, $v(x)$ is SOS if and only if there exists $\tilde{L} \in \mathcal{L}_d$ satisfying the LMI $V + \tilde{L} \geq 0$.

The Gram matrix method can also be used in more general contexts, in particular, for matrix polynomials (i.e., matrices whose entries are polynomials in the entries of the variable). Specifically, any matrix polynomial $M : \mathbb{R}^n \to \mathbb{S}^m$ of degree not greater than $2d$, $d \in \mathbb{N}_0$, can be expressed as

$$M(x) = \left(\bar{b}(x,d) \otimes I \right)' \bar{M} \left(\bar{b}(x,d) \otimes I \right) \tag{14}$$

where $\bar{b}(x,d) \in \mathbb{R}^{\bar{\varsigma}_d}$ is base vector for the polynomials in x of degree not greater than d, with $\bar{\varsigma}_d = (n+d)!/(n!d!)$, and $\bar{M} \in \mathbb{S}^{m\bar{\varsigma}_d}$ is a matrix that depends linearly on the coefficients of $M(x)$. This representation is useful to establish if $M(x)$ is SOS, i.e., $M(x) = \sum_{i=1}^k M_i(x)' M_i(x)$ for some matrix polynomials $M_1(x), \ldots, M_k(x) \in \mathbb{R}^{m \times m}$. Indeed, let us define the linear set

$$\bar{\mathcal{L}}_{d,m} = \left\{ \bar{L} \in \mathbb{S}^{m\bar{\varsigma}_d} : \ \left(\bar{b}(x,d) \otimes I \right)' \bar{L} \left(\bar{b}(x,d) \otimes I \right) = 0 \right\}. \tag{15}$$

Then, $M(x)$ is SOS if and only if there exists $\bar{L} \in \bar{\mathcal{L}}_{d,m}$ satisfying the LMI $\bar{M} + \bar{L} \geq 0$.

III. Stabilization Problem

Let us start by introducing the following definition of stability for the switched system (7).

Definition 1: The switched system (7), in particular its autonomous part, is said to be *globally asymptotically stable (GAS)* if

$$\begin{cases} \forall \varepsilon > 0 \ \exists \delta > 0 : \ \|x(0)\|_2 \leq \delta \ \Rightarrow \ \|x(t)\|_2 \leq \varepsilon \ \forall t \geq 0, \\ \lim_{t\to\infty} x(t) = 0 \ \forall x(0) \end{cases} \tag{16}$$

for all $\sigma(\cdot) \in \mathcal{D}$ and for $w(t) = 0$. □

The first problem considered in this paper is as follows.

Problem 1: Find $K \in \mathcal{K}$ such that the switched system (7) is GAS in the cases $\mathcal{D} = \mathcal{D}_T$ and $\mathcal{D} = \mathcal{D}_{arb}$. □

The approach proposed in this paper for solving Problem 1 is based on the use of Lyapunov functions in the class of the homogeneous polynomials, i.e., HPLFs, which have been exploited in the literature to derive stability conditions for switched systems. Let us start by recalling the necessary and sufficient LMI condition proposed in [16] for establishing stability of the switched system (7) in the case of dwell time constraints through HPLFs. To this end, let $d \in \mathbb{N}$, and let $\Lambda_i(K) \in \mathbb{R}^{\varsigma_d \times \varsigma_d}$ be the matrix function that satisfies

$$\frac{db(x,d)}{dx} A_i(K)Tx = \Lambda_i(K)b(x,d). \tag{17}$$

The matrix function $\Lambda_i(K)$ can be built as explained in [16]. Let us observe that $\Lambda_i(K)$ is linear in $A_i(K)T$, and, from (8), affine linear in K.

Theorem 1 (see [16]): Consider K fixed and $\mathcal{D} = \mathcal{D}_T$, $T > 0$. The switched system (7) is GAS if and only if, for some finite $d \in \mathbb{N}$, there exist $V_i \in \mathbb{S}^{\varsigma_d}$ and $\Theta_i, \Omega_{i,j} \in \mathbb{R}^{\tau_d}$, $i, j = 1, \ldots, N$, $i \neq j$, satisfying the LMIs

$$\begin{cases} 0 & < & V_i, \\ 0 & > & \mathrm{he}(V_i \Lambda_i(K)) + L(\Theta_i), \\ 0 & < & V_i - \left(e^{\Lambda_i(K)}\right)' V_j e^{\Lambda_i(K)} + L(\Omega_{i,j}) \end{cases} \tag{18}$$

where $L(\cdot)$ is a linear parametrization of $\mathcal{L}_d$ in (12). □

The specialization of Theorem 1 to the case of arbitrary switching was proposed in [17] in the context of robust stability analysis of uncertain systems with polytopic time-varying uncertainty, and can be recovered from Theorem 1 by imposing $V_1 = \ldots = V_N$ and eliminating the third inequality in (18) (observe, in fact, that $T = 0$ cannot be used in Theorem 1). It is worth mentioning that there exist other conditions in the literature for establishing stability of switched systems in the case of arbitrary switching, see for instance [3], [5], [19], [27], [29], [35]. Also, let us observe that the value of d required in Theorem 1 depends on the system under consideration.

Unfortunately, the condition provided by Theorem 1 cannot be used directly for solving Problem 1 using the LMI machinery because, by letting K be free, the second and third inequalities of (18) would be nonlinear in the decision variables.

The first idea for coping with this problem is to adopt HPLFs depending polynomially on the controller as proposed in [11]. Specifically, these HPLFs can be expressed as, for $i = 1, \ldots, N$,

$$v_i(x, K) = b(x, d)' V_i(K) b(x, d) \tag{19}$$

where $d \in \mathbb{N}$ defines the degree in x (equal to $2d$), and $V_i(K) \in \mathbb{S}^{\varsigma_d}$ are matrix polynomials of degree not greater than r to be determined. This implies that V_i is replaced by $V_i(K)$ in (18). Similarly, let us replace in (18) matrices $\Theta_i, \Omega_{i,j}$ with matrix polynomials $\Theta_i(K), \Omega_{i,j}(K) \in \mathbb{R}^{\tau_d}$ to be determined (whose degree will be specified in the sequel).

Since $\Lambda_i(K)$ is affine linear in K, it follows that the first two inequalities obtained in the new condition (18) are polynomial in the entries of K. However, the third inequality obtained in this new condition is non-polynomial in K due to the presence of $e^{\Lambda_i(K)}$. In order to cope with this problem, we replace e^A, $A \in \mathbb{R}^{n \times n}$, with a polynomial approximation of it, denoted by $\Upsilon(A) \in \mathbb{R}^{n \times n}$, of chosen degree $q \in \mathbb{N}$. Such an approximation has to verify the condition

$$\lim_{q \to \infty} \Upsilon(A) = e^A, \quad \forall A \in \mathcal{A} \tag{20}$$

where $\mathcal{A} \subset \mathbb{R}^{n \times n}$ is any bounded set. While several polynomial approximations can be chosen, here we simply consider the Taylor expansion:

$$\Upsilon(A) = \sum_{i=0}^{q} \frac{A^i}{i!}. \tag{21}$$

Before proceeding, it is useful to observe that other strategies have been proposed in the literature for dealing with the matrix exponential, see for instance [1], where a bound is exploited to remove the matrix exponential, [7], where auxiliary matrix functions and their time derivatives are exploited, and [6] where the exponential matrix is avoided by proposing a solution based on differential linear matrix inequalities along with boundary constraints.

Based on these comments, let us define, for all $i,j=1,\ldots,N$, $i\neq j$, the matrix polynomials

$$\begin{cases} M_{i,1}(K) &= V_i(K)-I, \\ M_{i,2}(K) &= -\mathrm{he}(V_i(K)\Lambda_i(K))-L(\Theta_i(K))-\xi(K)I, \\ M_{i,j,3}(K) &= V_i(K)-\Upsilon(\Lambda_i(K))'V_j(K)\Upsilon(\Lambda_i(K)) \\ & \quad +L(\Omega_{i,j}(K))-\xi(K)I \end{cases} \tag{22}$$

where $L(\cdot)$ is a linear parametrization of $\mathcal{L}_d$ in (12), and $\xi(K)\in\mathbb{R}$ is an auxiliary polynomial to be determined. As it will become clear in the sequel, this auxiliary polynomial is introduced in order to quantify the positive definiteness of the matrix polynomials $V_i(K)$, $-\mathrm{he}(V_i(K)\Lambda_i(K))-L(\Theta_i(K))$ and $V_i(K)-\Upsilon(\Lambda_i(K))'V_j(K)\Upsilon(\Lambda_i(K))+L(\Omega_{i,j}(K))$, which implies that the considered value of K makes the switched system (7) GAS when the degree of the Taylor expansion (21) is large enough. Let us observe that $\xi(K)$ is not introduced in $M_{i,1}(K)$ (in order to eliminate an unnecessary degree of freedom in the definition of the variables), and that one could use different $\xi(K)$ for different matrix polynomials (this choice is not adopted for simplicity).

In order to determine matrix polynomials $V_i(K)$, $\Theta_i(K)$ and $\Omega_{i,j}(K)$ that make the matrix polynomials $V_i(K)$, $-\mathrm{he}(V_i(K)\Lambda_i(K))-L(\Theta_i(K))$ and $V_i(K)-\Upsilon(\Lambda_i(K))'V_j(K)\Upsilon(\Lambda_i(K))+L(\Omega_{i,j}(K))$ positive definite for some values of K over $\mathcal{K}$, we define the optimization problem

$$\sup_{\substack{\xi(K),V_i(K)\\ \Theta_i(K),\Omega_{i,j}(K)}} h \quad \text{s.t.} \quad \begin{cases} \text{all } M_\star(K) \text{ in (22) are SOS,} \\ h\leq 1 \end{cases} \tag{23}$$

where

$$h=\int_{\mathcal{K}}\xi(K)dK \tag{24}$$

and $\mathcal{K}$ is given by (6). The optimization problem (23) is a semidefinite program (SDP) because the cost function of (23) is linear in the decision variables (in particular, in the coefficients of $\xi(K)$), and because the constraints of (23) can be expressed as LMIs in the decision variables (in particular, in their coefficients) as explained in Section II-B[1]. SDPs belong to the class of convex

[1]The matrix polynomials $M_\star(K)$ in (22) are expressed in the variable K, which is a matrix. The case considered in Section II-B, where the matrix polynomials are expressed in a vector variable, can be recovered by grouping all the entries of K into a vector.

optimization problems since both cost function and feasible set are convex, see for instance [4]. Let us observe that only the matrix polynomials $M_\star(K)$ in (22) are required to be SOS in the SDP (23). Other matrix polynomials such as $\xi(K)$ are not required to be SOS in this SDP.

The next step is to determine a candidate for the sought controller based on the solution of the SDP (23). To this end, let $\xi^*(K)$ be $\xi(K)$ evaluated for the optimal values of the decision variables found in the SDP (23). Let us define the polynomial

$$p(K) = \mu - \xi^*(K) - \sum_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,p_1}} (\rho^2 - K_{i,j}^2) s_{i,j}(K) \tag{25}$$

where the auxiliary scalar $\mu \in \mathbb{R}$ and polynomials $s_{i,j}(K) \in \mathbb{R}$ have to be determined, $K_{i,j}$ is the (i,j)-th entry of K, and ρ is used in the definition of $\mathcal{K}$ in (6). The second optimization problem that we define is the SDP

$$\inf_{\mu, s_{i,j}(K)} \mu \quad \text{s.t.} \quad p(K), s_{i,j}(K) \text{ are SOS}. \tag{26}$$

Let us observe that, in order to build the SDPs (23) and (26), one has to choose the degrees of polynomials involved. A guideline for choosing these degrees is as follows. First, choose d (which defines the degree of the HPLF candidates in (19) w.r.t. x), q (the degree of the Taylor expansion (21)) and r (the upper bound on the degree of $V_i(K)$). Second, set the degrees of $\Theta_i(K)$, $\Omega_{i,j}(K)$ and $\xi(K)$ not greater than the maximum degree of the matrix polynomials defined in (22) (evaluated for null $\Theta_i(K)$, $\Omega_{i,j}(K)$ and $\xi(K)$). Third, set the degrees of $s_{i,j}(K)$ not greater than the degree of $p(K)$ (evaluated for null $s_{i,j}(K)$). Summarizing, one chooses d, q and r, and the other degrees are automatically selected. This guideline will be adopted throughout the paper unless specified otherwise.

It is interesting to observe that both SDPs do not contain K as decision variable. Indeed, in the SDP (23), the decision variables are the coefficients of $\xi(K)$, $V_i(K)$, $\Theta_i(K)$ and $\Omega_{i,j}(K)$, and, in the SDP (26), the decision variables are μ and the coefficients of $s_{i,j}(K)$. Candidates for the sought controller will be determined based on the solutions of these SDPs. Specifically, let $p^*(K)$, μ^* and $s_{i,j}^*(K)$ be $p(K)$, μ and $s_{i,j}(K)$ evaluated for the optimal values of the decision variables found in the SDP (26). From $p^*(K)$, μ^* and $\xi^*(K)$, we define the set of controller candidates[2]

$$\mathcal{Z}(d,q,r) = \left\{ K \in \mathcal{K} : \; p^*(K) = 0, \; \xi^*(K) = \mu^* \right\}. \tag{27}$$

[2]We observe that the system of equations $p^*(K) = 0$ and $\xi^*(K) = \mu^*$ may admit also non-real solutions for K, however, they are not relevant for this paper.

The following result provides a necessary and sufficient condition for solving Problem 1 in the case of dwell time constraints.

Theorem 2: Consider $\mathcal{D} = \mathcal{D}_T$, $T > 0$. There exists $K \in \mathcal{K}$ such that the switched system (7) is GAS if and only if, for some finite $d, q \in \mathbb{N}$ and $r \in \mathbb{N}_0$, there exists $K \in \mathcal{Z}(d, q, r)$ such that the switched system (7) is GAS.

Proof. "Sufficiency". Suppose there exists $K \in \mathcal{Z}(d, q, r)$ such that the switched system (7) is GAS. Since $\mathcal{Z}(d, q, r)$ is a subset of $\mathcal{K}$, it follows that there exists $K \in \mathcal{K}$ such that the switched system (7) is GAS.

"Necessity". Suppose there exists $K \in \mathcal{K}$ such that the switched system (7) is GAS, and let us indicate such a value as $K^{\#}$. Let us replace K with $K^{\#}$ in the LMIs in (18), and let d be such that these LMIs are feasible (observe that such a d does exist finite from Theorem 1). Also, let us replace $e^{\Lambda_i(K^{\#})}$ with $\Upsilon(\Lambda_i(K^{\#}))$ in these LMIs, and let q be such that these LMIs are feasible (observe that such a q does exist finite since $\mathcal{K}$ is bounded, $\Upsilon(A)$ approximates arbitrarily well e^A for sufficiently large values of q, and the inequalities in (18) are strict). Let $V_i^{\#}$, $\Theta_i^{\#}$ and $\Omega_{i,j}^{\#}$ be values of V_i, Θ_i and $\Omega_{i,j}$ for which the obtained LMIs hold. Since these LMIs are homogeneous in the decision variables, one can assume without loss of generality that

$$V_i^{\#} > I$$

(indeed, if $V_i^{\#} \not> I$, then $V_i^{\#}$, $\Theta_i^{\#}$ and $\Omega_{i,j}^{\#}$ can be multiplied by a positive real number in order to ensure $V_i^{\#} > I$ and satisfaction of these LMIs). Moreover, there exists $\xi^{\#} > 0$ such that, for all $i, j = 1, \ldots, N$, $i \neq j$,

$$\begin{cases} 0 < V_i^{\#} - I, \\ 0 < -\mathrm{he}(V_i^{\#}\Lambda_i(K^{\#})) - L(\Theta_i^{\#}) - \xi^{\#} I, \\ 0 < V_i^{\#} - \Upsilon(\Lambda_i(K^{\#}))' V_j^{\#} \Upsilon(\Lambda_i(K^{\#})) + L(\Omega_{i,j}^{\#}) - \xi^{\#} I. \end{cases}$$

Given the continuity of these inequalities in $\xi^{\#}$, $V_i^{\#}$, $\Theta_i^{\#}$ and $\Omega_{i,j}^{\#}$, it follows that there exist matrix polynomials $\hat{\xi}(K)$, $\hat{V}_i(K)$, $\hat{\Theta}_i(K)$ and $\hat{\Omega}_{i,j}(K)$ such that

$$\hat{\xi}(K^{\#}) > 0$$

and, for all $i, j = 1, \ldots, N$, $i \neq j$, for all $l = 1, 2$, and for all $K \in \mathbb{R}^{m_1 \times p_1}$,

$$\begin{cases} 0 < \hat{M}_{i,l}(K), \\ 0 < \hat{M}_{i,j,3}(K) \end{cases}$$

where

$$\begin{cases} \hat{M}_{i,1}(K) = \hat{V}_i(K) - I, \\ \hat{M}_{i,2}(K) = -\mathrm{he}(\hat{V}_i(K)\Lambda_i(K)) - L(\hat{\Theta}_i(K)) - \hat{\xi}(K)I, \\ \hat{M}_{i,j,3}(K) = \hat{V}_i(K) - \Upsilon(\Lambda_i(K))'\hat{V}_j(K)\Upsilon(\Lambda_i(K)) \\ \qquad\qquad +L(\hat{\Omega}_{i,j}(K)) - \hat{\xi}(K)I. \end{cases}$$

Let us observe that the degrees of $\hat{\xi}(K)$, $\hat{V}_i(K)$, $\hat{\Theta}_i(K)$ and $\hat{\Omega}_{i,j}(K)$ can be always increased in order to satisfy the constraints of the SDP (23). Indeed, for $\eta \in \mathbb{N}_0$, let us define the polynomial

$$f(K) = \left(1 + \|K\|_{Fro}^2\right)^\eta .$$

From [40] it follows that there exists η such that the polynomials $f(K)\hat{M}_{i,l}(K)$ and $f(K)\hat{M}_{i,j,3}(K)$ are SOS. Since $f(K) - 1$ is SOS (being sum of powers of $\|K\|_{Fro}^2$ multiplied by positive coefficients), one has that the constraints of the SDP (23) can be satisfied by choosing

$$\begin{cases} \xi(K) &= f(K)\hat{\xi}(K), \\ V_i(K) &= f(K)\hat{V}_i(K), \\ \Theta_i(K) &= f(K)\hat{\Theta}_i(K), \\ \Omega_{i,j}(K) &= f(K)\hat{\Omega}_{i,j}(K) \end{cases}$$

which also satisfy

$$\xi(K^\#) > 0.$$

Let us observe that such matrix polynomials can be obtained by maximizing h in the SDP (23), and that the constraint $h \leq 1$ can be introduced without loss of generality. Indeed, if $h > 1$ for the $\xi(K)$ obtained so far, then $\xi(K)$ can be redefined as

$$\xi(K) \ \rightarrow \ \xi(K) - \delta$$

where

$$\delta = \left(\int_{\mathcal{K}} dK\right)^{-1} \left(\int_{\mathcal{K}} \xi(K) dK - 1\right),$$

and this ensures that the redefined $\xi(K)$ is still positive for some $K \in \mathcal{K}$ (since the integral of the redefined $\xi(K)$ over $\mathcal{K}$ is 1) and that the constraints of the SDP (23) hold (since $\delta > 0$).

Next, let us define

$$\bar{\mu} = \sup_{K \in \mathcal{K}} \xi^*(K)$$

where $\xi^*(K)$ is $\xi(K)$ evaluated for the optimal values of the decision variables found in the SDP (23). From the above discussion, it follows that $\bar{\mu} > 0$. Let us observe that $\rho^2 - K_{i,j}^2 \geq 0$ for all $i = 1, \ldots, m_1$ and $j = 1, \ldots, p_1$ if and only if $K \in \mathcal{K}$. This implies that

$$\mu^* \geq \bar{\mu}$$

for any chosen degrees of the polynomials $s_{i,j}(K)$, where μ^* is μ evaluated for the optimal values of the decision variables found in the SDP (26). Moreover, the polynomials $\rho^2 - K_{i,j}^2$ have even degree and the highest degree forms are zero if and only if $K = 0$. Hence, from Putinar's Positivstellensatz [39], it follows that

$$\mu^* = \bar{\mu}$$

for polynomials $s_{i,j}(K)$ with sufficiently large degrees. Let $K^* \in \mathcal{K}$ be such that $\xi^*(K^*) = \bar{\mu}$. Since $p^*(K)$ is SOS, it follows that

$$\begin{array}{rcl} 0 & \leq & p^*(K^*) \\ & = & \mu^* - \xi^*(K^*) - \sum_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,p_1}} (\rho^2 - (K_{i,j}^*)^2) s_{i,j}^*(K^*) \\ & \leq & 0. \end{array}$$

Hence, $p^*(K^*) = 0$ and, therefore, $K^* \in \mathcal{Z}(d,q,r)$ where r is any integer greater than or equal to the maximum among the degrees of the $V_i(K)$ for which the degrees of $\Theta_i(K)$, $\Omega_{i,j}(K)$ and $\xi(K)$ can be chosen with the guideline reported under (26). Moreover, $\xi^*(K^*) > 0$, which ensures that K^* solves the problem for sufficiently large values of q. $\square$

Theorem 2 provides a strategy for solving Problem 1 in the case of dwell time constraints based on the SDPs (23) and (26). This strategy consists of narrowing the search space for the sought controller from the original set, i.e., $\mathcal{K}$, to a subset of it, i.e., $\mathcal{Z}(d,q,r)$. Once $\mathcal{Z}(d,q,r)$ is found, one checks if any of the controllers included in such a set solves Problem 1, for instance by using Theorem 1. Let us observe that $\mathcal{K}$ affects $\mathcal{Z}(d,q,r)$ (and, hence, Theorem 2) through the cost function of the SDP (23) (which is the function h in (24)) and the first constraint of the SDP (26) (which is the condition that the polynomial $p(K)$ has to be SOS).

In order to understand the advantage of Theorem 2, let us observe that $\mathcal{Z}(d,q,r)$ typically contains one element only. Indeed, as seen in the proof of Theorem 2, μ^* is an upper bound of the maximum of $\xi^*(K)$ over $\mathcal{K}$. This means that $\mathcal{Z}(d,q,r)$ is either empty or its elements are global maximizers of $\xi^*(K)$ over $\mathcal{K}$ (i.e., points in $\mathcal{K}$ where $\xi^*(K)$ achieves its maximum over $\mathcal{K}$). Now, the set of polynomials of a specified degree that have more than one global maximizer over a specified compact set has a smaller dimension than the set of polynomials of such a degree (where the dimension is evaluated, for instance, in the space of the coefficients of the polynomials w.r.t. a specified basis). This explains why $\mathcal{Z}(d,q,r)$ typically contains one element only, and why this is indeed the case for all numerical examples in Section VII.

The computation of $\mathcal{Z}(d,q,r)$ can be addressed using the Gram matrix of $p^*(K)$ found in the SDP (26), either in the typical case where $\mathcal{Z}(d,q,r)$ contains one element only or in other cases where this set may contain multiple elements. Indeed, since $p^*(K)$ is SOS, its zeros can be determined from the null space of the found positive semidefinite Gram matrix of $p^*(K)$. More specifically, this amounts to finding the values of K for which the vector of monomials in K used to define this Gram matrix belongs to the null space of this Gram matrix. This step can be addressed in various ways. For instance, a procedure reported in [10] and references therein involves pivoting operations and the computation of the roots of a polynomial in one variable. Once the zeros of $p^*(K)$ are found, one singles out those that belong to $\mathcal{K}$ and satisfy $\xi^*(K)=\mu^*$. For all numerical examples in Section VII, the computation of $\mathcal{Z}(d,q,r)$ is trivial because the null space of the found positive semidefinte Gram matrix of $p^*(K)$ has dimension one, and, hence, the zeros of $p^*(K)$ are determined by just scaling a vector.

The second constraint in the SDP (23) is introduced in order to ensure that the solution of this SDP is bounded. The constant 1 on the right hand side of this constraint can be replaced with any other positive number.

It is useful to observe that Theorem 2 not only extends to switched systems the idea of HPLFs depending polynomially on the controller introduced in [11] for uncertain systems, but also improves it. Indeed, the determination of these functions in the SDP (23) is achieved without introducing multipliers for imposing positive semidefiniteness of the matrix polynomials over $\mathcal{K}$.

Theorem 2 can be modified in order to solve Problem 1 in the case of arbitrary switching. This is explained in the following result.

Corollary 1: Consider $\mathcal{D}=\mathcal{D}_{arb}$. Modify the SDP (23) by imposing $V_1(K)=\ldots=V_N(K)$ and removing $M_{i,j,3}(K)$ from the constraints. There exists $K\in\mathcal{K}$ such that the switched system (7) is GAS if and only if, for some finite $d\in\mathbb{N}$ and $r\in\mathbb{N}_0$, there exists $K\in\mathcal{Z}(d,0,r)$ such that the switched system (7) is GAS.

Proof. Analogous to the proof of Theorem 2 by observing that, in the case $\mathcal{D} = \mathcal{D}_{arb}$, all the subsystems of the switched system (7) must share the same Lyapunov function. $\square$

IV. $\mathcal{H}_2$ Norm Control Problem

Let us start by providing the definition of $\mathcal{H}_2$ norm of the switched system (7).

Definition 2: The $\mathcal{H}_2$ norm of the switched system (7) is

$$\gamma_{\mathcal{H}_2} = \sqrt{\sup_{\sigma(\cdot)\in\mathcal{D}} \sum_{j=1}^{m_2} \|z^{(j)}(\cdot)\|_{\mathcal{L}_2}^2} \tag{28}$$

where $z^{(j)}(t)$ is the solution $z(t)$ for $x(0^-) = 0$ due to an impulse applied to the j-th entry of $w(t)$ (each impulse is applied independently on the others). $\square$

The second problem addressed in this paper is as follows.

Problem 2: Find $K \in \mathcal{K}$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than a desired value γ in the cases $\mathcal{D} = \mathcal{D}_T$ and $\mathcal{D} = \mathcal{D}_{arb}$. $\square$

Let us observe that, in order for the $\mathcal{H}_2$ norm of the switched system (7) to be bounded, the map between $w(t)$ and $z(t)$ has to be strictly proper. Hence, in this section we assume, without loss of generality,

$$D_i(K) = 0, \tag{29}$$

which holds, for instance, when $D_{4,i} = 0$, and $D_{3,i} = 0$ or $D_{2,i} = 0$.

The approach proposed in this paper for solving Problem 2 is based on the use of Lyapunov functions in the class of the homogeneous rational functions, i.e., functions that can be expressed as the ratio of homogeneous polynomials. Specifically, a function $v : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous rational function of total degree $\delta_1 \in \mathbb{N}_0$ and relative degree $\delta_2 \in \mathbb{N}_0$, with $\delta_1 \geq \delta_2$, if

$$v(x) = \frac{\phi(x)}{\psi(x)} \tag{30}$$

where $\phi(x)$ and $\psi(x)$ are homogeneous polynomials of degree δ_1 and $\delta_1 - \delta_2$, respectively. Lyapunov functions in the class of the homogeneous rational functions, i.e., HRLFs, have been introduced in [13] to derive upper bounds of the $\mathcal{H}_2$ norm of switched systems. In particular, HRLF candidates of total degree $2d$ and relative degree $2d-2$, $d \in \mathbb{N}$, are searched for. These candidates can be expressed as, for $i = 1, \ldots, N$,

$$\begin{cases} v_i(x) &= \dfrac{\phi_i(x)}{\psi(x)}, \\ \phi_i(x) &= b(x,d)'\Phi_i b(x,d), \\ \psi(x) &= b(x,d-1)'\Psi b(x,d-1) \end{cases} \tag{31}$$

where $\Phi_i \in \mathbb{S}^{\varsigma_d}$ and $\Psi \in \mathbb{S}^{\varsigma_{d-1}}$. Throughout the paper it is assumed that Ψ satisfies

$$\begin{cases} \Psi &> 0, \\ \psi(x) &= \|x\|_2^{2d-2}. \end{cases} \tag{32}$$

The following theorem, proposed in [13], provides a necessary and sufficient LMI condition for establishing upper bounds of the $\mathcal{H}_2$ norm of the switched system (7) in the case of dwell time constraints.

Theorem 3 (see [13]): Consider K fixed and $\mathcal{D} = \mathcal{D}_T$, $T > 0$. The $\mathcal{H}_2$ norm of the switched system (7) satisfies

$$\gamma_{\mathcal{H}_2} < \gamma \tag{33}$$

if and only if, for some finite $d \in \mathbb{N}$, there exist $\Phi_i \in \mathbb{S}^{\varsigma_d}$ and $\Theta_i, \Omega_{i,j} \in \mathbb{R}^{\tau_{2d-1}}$, $i, j = 1, \ldots, N$, $i \neq j$, satisfying the LMIs

$$\begin{cases} 0 &< \Phi_i, \\ 0 &> E_i(K,\Phi_i) + L(\Theta_i), \\ 0 &> Q_i(K,\Phi_i,\Phi_j,\exp(\cdot)) + L(\Omega_{i,j}), \\ 0 &< \gamma^2 - g_i(K,\Phi_i) \end{cases} \tag{34}$$

where $\exp(\cdot)$ is the exponential function, $L(\cdot)$ is a linear parametrization of $\mathcal{L}_{2d-1}$ in (12), and $E_i(K,\Phi_i)$, $Q_i(K,\Phi_i,\Phi_j,\exp(\cdot))$ and $g_i(K,\Phi_i)$ are the affine linear matrix functions in Φ_i and Φ_j defined in Appendix A. □

The counterpart of Theorem 3 for the case of arbitrary switching can be recovered by eliminating the third inequality in (34) and imposing $\Phi_1 = \ldots = \Phi_N$. As explained in [13], the choice for Ψ in (32) is introduced in order to guarantee that the LMI condition in Theorem 3 is not only sufficient but also necessary (indeed, sufficiency is achieved for all $\Psi > 0$). Let us observe that the HRLF candidates in (31) could be defined with different denominators, however, in [13] and in this paper these candidates share a common denominator for simplicity.

As in the case of Theorem 1, the condition provided by Theorem 3 cannot be used directly for solving Problem 2 because, by letting K be free, the second, third and fourth inequalities of (34) would be nonlinear in the decision variables.

The first idea for coping with this problem is to introduce HRLFs depending rationally on the controller. These functions can be expressed as, for $i = 1, \dots, N$,

$$v_i(x, K) = \frac{b(x, d)'\bar{\Phi}_i(K)b(x, d)}{b(x, d-1)'\Psi b(x, d-1)} \tag{35}$$

where

$$\bar{\Phi}_i(K) = \frac{\Phi_i(K)}{\zeta(K)} \tag{36}$$

and the auxiliary matrix polynomials $\Phi_i(K) \in \mathbb{S}^{\varsigma_d}$ and $\zeta(K) \in \mathbb{R}$ of degree not greater than r have to be determined. In order to use these functions, let us define the matrix polynomials

$$\begin{cases} \bar{E}_i(K, \Phi_i(K), \zeta(K)) = E_i(K, \bar{\Phi}_i(K))\zeta(K), \\ \bar{Q}_i(K, \Phi_i(K), \Phi_j(K), \zeta(K)) = Q_i(K, \bar{\Phi}_i(K), \bar{\Phi}_j(K), \Upsilon(\cdot)) \\ \qquad \cdot\zeta(K), \\ \bar{g}_i(K, \Phi_i(K), \zeta(K)) = \left(\prod_{j=1}^{m_2} \psi(B_i^{(j)}(K))\right) \\ \qquad \cdot\left(\gamma^2 - g_i(K, \bar{\Phi}_i(K))\right)\zeta(K) \end{cases} \tag{37}$$

where $B_i^{(j)}(K)$ is the j-th column of $B_i(K)$. Hence, for all $i, j = 1, \dots, N$, $i \neq j$, let us introduce the matrix polynomials

$$\begin{cases} M_{i,1}(K) = \Phi_i(K) - \xi(K)I, \\ M_{i,2}(K) = -\bar{E}_i(K, \Phi_i(K), \zeta(K)) - L(\Theta_i(K)) - \xi(K)I, \\ M_{i,j,3}(K) = \bar{Q}_i(K, \Phi_i(K), \Phi_j(K), \zeta(K)) + L(\Omega_{i,j}(K)) \\ \qquad -\xi(K)I, \\ M_{i,4}(K) = \bar{g}_i(K, \Phi_i(K), \zeta(K)) - \xi(K), \\ M_5(K) = \zeta(K) - 1 \end{cases} \tag{38}$$

where $L(\cdot)$ is a linear parametrization of $\mathcal{L}_{2d-1}$ defined by (12), and the auxiliary matrix polynomials $\xi(K) \in \mathbb{R}$ and $\Theta_i, \Omega_{i,j}(K) \in \mathbb{R}^{\tau_{2d-1}}$ have to be determined. Let us define the SDP

$$\sup_{\substack{\xi(K), \Phi_i(K) \\ \Theta_i(K), \Omega_{i,j}(K), \zeta(K)}} h \ \text{ s.t. } \begin{cases} \text{all } M_\star(K) \text{ in (38) are SOS}, \\ h \leq 1 \end{cases} \tag{39}$$

where h is defined in (24). Let us select the degrees of $\xi(K)$, $\Theta_i(K)$, $\Omega_{i,j}(K)$ automatically from d, q and r analogously to what done for the stabilization problem in Section III. The following result provides a necessary and sufficient condition for solving Problem 2 in the case of dwell time constraints.

Theorem 4: Consider $\mathcal{D} = \mathcal{D}_T$, $T > 0$. Let $\xi^*(K)$ in (25) be $\xi(K)$ evaluated for the optimal values of the decision variables found in the SDP (39). There exists $K \in \mathcal{K}$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ if and only if, for some finite $d, q \in \mathbb{N}$ and $r \in \mathbb{N}_0$, there exists $K \in \mathcal{Z}(d, q, r)$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ.

Proof. See Appendix C. □

Theorem 4 provides a strategy for solving Problem 2 in the case of dwell time constraints based on the SDPs (39) and (26). Analogously to the stabilization problem considered in the previous section, this strategy consists of narrowing the search space for the sought controller from the original set $\mathcal{K}$ to its subset $\mathcal{Z}(d, q, r)$, which typically contains one element only. Once $\mathcal{Z}(d, q, r)$ is found, one checks if any of the controllers included in such a set solves Problem 2, for instance by using Theorem 3.

Theorem 4 can be modified in order to solve Problem 2 in the case of arbitrary switching. This is explained in the following result.

Corollary 2: Consider $\mathcal{D} = \mathcal{D}_{arb}$. Modify the SDP (39) by imposing $\Phi_1(K) = \ldots = \Phi_N(K)$ and removing $M_{i,j,3}(K)$ from the constraints. Let $\xi^*(K)$ in (25) be $\xi(K)$ evaluated for the optimal values of the decision variables found in the SDP (39). There exists $K \in \mathcal{K}$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ if and only if, for some finite $d \in \mathbb{N}$ and $r \in \mathbb{N}_0$, there exists $K \in \mathcal{Z}(d, 0, r)$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ.

Proof. Analogous to the proof of Theorem 4 by observing that, in the case $\mathcal{D} = \mathcal{D}_{arb}$, all the subsystems of the switched system (7) must share the same Lyapunov function. □

V. RMS Gain Control Problem

Let us start by providing the definition of RMS gain of the switched system (7).

Definition 3: The RMS gain of the switched system (7) is

$$\gamma_{RMS} = \limsup_{\sigma(\cdot)\in\mathcal{D},w(\cdot)} \frac{\|z(\cdot)\|_{\mathcal{L}_2}}{\|w(\cdot)\|_{\mathcal{L}_2}} \tag{40}$$

where $z(t)$ is the solution for $x(0^-) = 0$. □

The third problem considered in this paper is as follows.

Problem 3: Find $K \in \mathcal{K}$ such that the RMS gain of the switched system (7) is smaller than a desired value γ in the cases $\mathcal{D} = \mathcal{D}_T$ and $\mathcal{D} = \mathcal{D}_{arb}$. □

Let us start by recalling the necessary and sufficient LMI condition proposed in [13] for establishing upper bounds of the RMS gain of the switched system (7) in the case of arbitrary switching through HRLFs. To this end, for $d \in \mathbb{N}$, let us define the vector

$$\tilde{b}(x,d,u) = \begin{pmatrix} b(x,2d-1) \\ b(x,2d-2)\otimes u \end{pmatrix}. \tag{41}$$

Also, let us define the linear space

$$\tilde{\mathcal{L}}_d = \left\{\tilde{L} = \tilde{L}' : \ \tilde{b}(x,d,u)'\tilde{L}\tilde{b}(x,d,u) = 0\right\} \tag{42}$$

whose dimension is denoted by $\tilde{\tau}_d$.

Theorem 5 (see [13]): Consider K fixed and $\mathcal{D} = \mathcal{D}_{arb}$. The RMS gain of the switched system (7) satisfies

$$\gamma_{RMS} < \gamma \tag{43}$$

if and only if, for some finite $d \in \mathbb{N}$, there exist $\Phi \in \mathbb{S}^{\varsigma_d}$ and $\Theta_i \in \mathbb{R}^{\tilde{\tau}_d}$, $i = 1,\ldots,N$, satisfying the LMIs

$$\begin{cases} 0 < \Phi \\ 0 > F_i(K,\Phi) + G_i(K) + L(\Theta_i) \end{cases} \tag{44}$$

where $L(\cdot)$ is a linear parametrization of $\tilde{\mathcal{L}}_d$ in (42), and $F_i(K,\Phi)$ and $G_i(K)$ are linear matrix functions in Φ whose definitions are reported in Appendix A. □

Theorem 5 can be exploited to solve Problem 3 by adopting the Lyapunov functions introduced in the previous section. Specifically, let us denote an HRLF depending rationally on the controller as

$$v(x,K) = \frac{b(x,d)'\bar{\Phi}(K)b(x,d)}{b(x,d)'\Psi b(x,d)} \tag{45}$$

where

$$\bar{\Phi}(K) = \frac{\Phi(K)}{\zeta(K)} \tag{46}$$

and the auxiliary matrix polynomials $\Phi(K) \in \mathbb{S}^{\varsigma_d}$ and $\zeta(K) \in \mathbb{R}$ of degree not greater than r have to be determined. Following the strategy proposed in Section IV, for all $i = 1, \ldots, N$, let us define the matrix polynomials

$$\begin{cases} M_1(K) &= \Phi(K) - \xi(K), \\ M_{i,2}(K) &= -F_i(K,\Phi(K)) - \zeta(K)G_i(K) - L(\Theta_i(K)) \\ & \quad -\xi(K)I, \\ M_3(K) &= \zeta(K) - 1 \end{cases} \tag{47}$$

where $L(\cdot)$ is a linear parametrization of $\tilde{\mathcal{L}}_d$ in (42), and the auxiliary matrix polynomials $\xi(K) \in \mathbb{R}$ and $\Theta_i(K) \in \mathbb{R}^{\tilde{\tau}_d}$ have to be determined. Let us define the SDP

$$\sup_{\substack{\xi(K),\Phi(K) \\ \Theta_i(K),\zeta(K)}} h \quad \text{s.t.} \quad \begin{cases} \text{all } M_\star(K) \text{ in (47) are SOS}, \\ h \leq 1 \end{cases} \tag{48}$$

where h is defined in (24). Let us select the degrees of $\xi(K)$ and $\Theta_i(K)$ automatically from d and r analogously to what done for the stabilization problem in Section III. The following result provides a necessary and sufficient condition for solving Problem 3 in the case of arbitrary switching.

Theorem 6: Consider $\mathcal{D} = \mathcal{D}_{arb}$. Let $\xi^*(K)$ in (25) be $\xi(K)$ evaluated for the optimal values of the decision variables found in the SDP (48). There exists $K \in \mathcal{K}$ such that the RMS gain of the switched system (7) is smaller than γ if and only if, for some finite $d \in \mathbb{N}$ and $r \in \mathbb{N}_0$, there exists $K \in \mathcal{Z}(d,0,r)$ such that the RMS gain of the switched system (7) is smaller than γ.

Proof. See Appendix C. □

Theorem 6 provides a strategy for solving Problem 3 in the case of arbitrary switching based on the SDPs (48) and (26). Analogously to the problems considered in the previous sections, this strategy consists of narrowing the search space for the sought controller from the original set $\mathcal{K}$ to its subset $\mathcal{Z}(d,0,r)$, which typically contains one element only. Once $\mathcal{Z}(d,0,r)$ is found, one checks if any of the controllers included in such a set solves Problem 3, for instance by using Theorem 5.

We conclude this section by mentioning that one can address Problem 3 in the case of dwell time constraints analogously, in particular, by combining the methodology proposed in Theorems 4 and 6 with the results proposed in [13].

VI. Comments and Extensions

In this section we report some comments on the methodology described in Sections III–V and some extensions of the basic model introduced in Section II.

A. *Comments*

Let us start by observing that Problems 1–3 do not always admit a solution. For instance, a necessary condition for the solvability of Problem 1 is that all the subsystems of the switched system (1) admit a common stabilizing static output feedback controller, and this necessary condition may not hold in some cases. Indeed, it is also possible that one or more subsystems of the switched system (1) do not even admit a stabilizing static output feedback controller.

Another comment concerns the set where the sought controllers are searched for, i.e., $\mathcal{K}$ defined in (6). The quantity ρ in (6), which defines the size of $\mathcal{K}$, is given. Using large values of ρ has the benefit of extending the space where the controller is searched for, however, this may increase the degrees d, q and r needed to determine a sought controller.

Let us also observe that, in general, $\mathcal{K}$ contains controllers that do not solve the problem under consideration. Hence, in order to possibly speed up the determination of a sought controller (i.e., reduce the degrees d, q and r needed), one could shrink $\mathcal{K}$ by removing some of such useless controllers. For instance, for Problem 1, one may remove some (and, if possible, all) K for which $A_i(K)$ is not Hurwitz for some $i=1,\ldots,N$. These values of K may be identified using standard methods such as the Routh-Hurwitz criterion. Hence, the model in Section II may be reformulated with a smaller $\mathcal{K}$. See Example 1 in Section VII for details.

The next comment concerns the degree q of the Taylor expansion (21). Let us observe that q is not used in Corollaries 1 and 2, and in Theorem 6, since the Taylor expansion (21) is not exploited in the case of arbitrary switching. Also, in these three results, if the quantity μ^* used to define $\mathcal{Z}(d,0,r)$ in (27) satisfies $\mu^* > 0$, then any controller in $\mathcal{Z}(d,0,r)$ solves the problem under consideration (i.e., one does not need to perform the final test on the controller candidates in $\mathcal{Z}(d,0,r)$).

Lastly, let us observe that Theorem 4 and Corollary 2 can be used to determine controllers that minimize the $\mathcal{H}_2$ norm of the switched system (7) through a bisection algorithm on γ. An analogous strategy can be used with Theorem 6 to determine controllers that minimize the RMS gain of the switched system (7).

B. *Extension: Mode-Dependent Controller*

The sought output feedback controller was assumed to be mode-independent in (5), being common to all the subsystems of the switched system (1). Whenever the switching rule is available in the controller, K can be replaced by $K_{\sigma(t)}$ in order to stabilize a larger class of switched systems. That is, (5) can be replaced with

$$u(t) = K_{\sigma(t)} y(t) \tag{49}$$

where $K_1, \ldots, K_N \in \mathbb{R}^{m_1 \times p_1}$ are matrices to be determined in $\mathcal{K}$. The design of these matrices can be readily addressed as described in Sections III–V by grouping $K_1, \ldots, K_N$ in a single matrix K.

C. *Extension: Dynamic Controller*

The sought output feedback controller was assumed to be static in (5). Here we suppose that a fixed-order dynamic output feedback controller is searched for, in particular of the form

$$\left\{ \begin{array}{rcl} \dot{x}_{con}(t) & = & A_{con} x_{con}(t) + B_{con} y(t), \\ u(t) & = & C_{con} x_{con}(t) + D_{con} y(t) \end{array} \right. \tag{50}$$

where $x_{con} \in \mathbb{R}^{n_{con}}$ is the state of chosen order, and A_{con}, B_{con}, C_{con} and D_{con} are matrices to be determined in a set analogous to $\mathcal{K}$. As it is well known, the design of a dynamic output controller can be reformulated as a static output feedback problem. As a matter of fact, the four unknown matrices in (50) can be grouped in a single matrix K that represents the gain of a static output feedback for an augmented system. Specifically, let us introduce an auxiliary input $\tilde{u}(t) \in \mathbb{R}^n$ and an auxiliary output $\tilde{y}(t) \in \mathbb{R}^{n_{con}}$. The static output feedback is

$$\begin{pmatrix} u(t) \\ \tilde{u}(t) \end{pmatrix} = K \begin{pmatrix} y(t) \\ \tilde{y}(t) \end{pmatrix}, \quad K = \begin{pmatrix} D_{con} & C_{con} \\ B_{con} & A_{con} \end{pmatrix} \tag{51}$$

and the augmented system is

$$\begin{cases} \begin{pmatrix} \dot{x}(t) \\ \dot{x}_{con}(t) \end{pmatrix} = \begin{pmatrix} A_{1,\sigma(t)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_{con}(t) \end{pmatrix} \\ \qquad + \begin{pmatrix} B_{1,\sigma(t)} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u(t) \\ \tilde{u}(t) \end{pmatrix} + \begin{pmatrix} B_{2,\sigma(t)} \\ 0 \end{pmatrix} w(t), \\ \begin{pmatrix} y(t) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} C_{1,\sigma(t)} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ x_{con}(t) \end{pmatrix} \\ \qquad + \begin{pmatrix} D_{1,\sigma(t)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ \tilde{u}(t) \end{pmatrix} + \begin{pmatrix} D_{2,\sigma(t)} \\ 0 \end{pmatrix} w(t), \\ z(t) = \begin{pmatrix} C_{2,\sigma(t)} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_{con}(t) \end{pmatrix} \\ \qquad + \begin{pmatrix} D_{3,\sigma(t)} & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ \tilde{u}(t) \end{pmatrix} + D_{4,\sigma(t)} w(t), \\ \sigma(\cdot) \in \mathcal{D}. \end{cases} \tag{52}$$

D. Extension: Non-Strictly Proper Switched System

The map between $u(t)$ and $y(t)$ was assumed to be strictly proper in (4). Here we want to allow one to consider the case where this map is proper only. Let us start by observing that, either (4) holds or not, the matrix functions $A_i(K)$, $B_i(K)$, $C_i(K)$ and $D_i(K)$ to be used in the switched system (7) are given by

$$\begin{cases} A_i(K) = A_{1,i} + B_{1,i} K \Gamma_i(K)^{-1} C_{1,i}, \\ B_i(K) = B_{2,i} + B_{1,i} K \Gamma_i(K)^{-1} D_{2,i}, \\ C_i(K) = C_{2,i} + D_{3,i} K \Gamma_i(K)^{-1} C_{1,i}, \\ D_i(K) = D_{4,i} + D_{3,i} K \Gamma_i(K)^{-1} D_{2,i} \end{cases} \tag{53}$$

where

$$\Gamma_i(K) = I - D_{1,i}K. \tag{54}$$

In such a case, the switched system (7) may be not well-posed since $\Gamma_i(K)$ may be singular. To cope with this issue, let us define the polynomials

$$M_{0,i}(K) = \det(\Gamma_i(K))^2 - \varepsilon \tag{55}$$

where $\varepsilon > 0$ is a chosen threshold. The non-negativity of these polynomials, in fact, ensures that all the subsystems of the switched system (7) are well-posed. Hence, these polynomials have to be included in the constraints of the SDPs (23), (39) and (48).

Another point to observe is that, contrary to Section II where the matrix functions $A_i(K)$, $B_i(K)$, $C_i(K)$ and $D_i(K)$ were linear due to the assumption in (4), these matrix functions are now rational. As a consequence, the set of matrix polynomials $M_\star(K)$ considered in the SDPs (23), (39) and (48) now do not contain only matrix polynomials but also matrix rational functions. This issue can be dealt with by observing that these matrix rational functions can be expressed as ratios between matrix polynomials and even powers of $\det(\Gamma_i(K))$. Hence, one just uses the numerators of these ratios for the matrix polynomials $M_\star(K)$ in the SDPs (23), (39) and (48).

VII. Examples

In this section we present some illustrative examples of the proposed methodology. The SDPs are solved by using the toolbox SeDuMi [43] for Matlab on a personal computer with Windows 10, Intel Core i7, 3.4 GHz, 8 GB RAM. The computational time for solving each SDP is less than one second, and the number of LMI scalar variables in each SDP is reported (the reader is referred to [12] for more details about the numerical complexity of the LMIs obtained using the Gram matrix method described in Section II-B). The matrix polynomials $\Theta_i(K)$, $\Omega_{i,j}(K)$ and $\zeta(K)$ are chosen of degree 0. For brevity of description, it is assumed in the switched system (1) that $B_{1,i} = B_{2,i}$, $C_{1,i} = C_{2,i}$, and $D_{j,i} = 0$.

A. Example 1

Let us consider the switched system (1) with $N = 3$ and

$$\begin{cases} A_{1,1} = \begin{pmatrix} 0 & 1 \\ -5 & 2 \end{pmatrix}, & A_{1,2} = \begin{pmatrix} -1 & 0 \\ -5 & -2 \end{pmatrix}, \\ B_{1,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & B_{1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ C_{1,1} = \begin{pmatrix} 1 & -1 \end{pmatrix}, & C_{1,2} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ A_{1,3} = \begin{pmatrix} 0 & -3 \\ 0 & -5 \end{pmatrix}, & B_{1,3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ C_{1,3} = \begin{pmatrix} 1 & 1 \end{pmatrix}. \end{cases}$$

We want to solve Problem 1 with $\mathcal{D} = \mathcal{D}_T$, $T = 0.2$, and $\mathcal{K}$ given by (6) with $\rho = 10$. Let us observe that the switched system (7) is unstable for $K = 0$, for instance because the first subsystem is unstable.

Let us use the methodology proposed in Section III. First, we solve the SDP (23). We choose to search for HPLFs of degree $2d = 2$ constant in the controller (i.e., $r = 0$) using a first-order approximation of the matrix exponential (i.e., $q = 1$). We find

$$\xi^*(K) = -0.142K^2 + 1.035K - 6.54.$$

Second, we solve the SDP (26), finding the set

$$\mathcal{Z}(1, 1, 0) = \{3.657\}\,.$$

Third, we find that (18) is feasible for $K = K^*$ with $K^* = 3.657$, hence implying that K^* solves Problem 1. The number of LMI scalar variables in the three SDPs are 21, 2 and 10, respectively. Figure 1 shows the stabilizing and non-stabilizing controllers for this example found by brute force. Some numerical details about the construction of the SDPs (23) and (26) for this example are reported in Appendix B.

Lastly, let us observe that one could shrink $\mathcal{K}$ before applying the proposed methodology in order to possibly speed up the convergence as discussed in Section VI-A. Indeed, a necessary condition for the switched system (7) to be GAS is that $A_i(K)$ is Hurwitz for all $i = 1, 2, 3$, which is satisfied if and only if $K \in (2, 5)$. Hence, one could reformulate Problem 1 with a smaller ρ in (6), in particular $\rho = 1.5$ instead of $\rho = 10$, by introducing the changes

$$\begin{cases} A_i(K) \;\rightarrow\; A_i(K+3.5) \;\; \forall i = 1, 2, 3, \\ \mathcal{K} \;\rightarrow\; \{K \in \mathbb{R} : \; |K| \leq 1.5\} \, . \end{cases}$$

With these changes, by using the same d, q and r, we similarly obtain a controller that solves Problem 1, in particular $K^* = 3.724$ (expressed in the coordinates of the original system). For completeness, we report in Figure 2 the polynomial $\xi^*(K)$ found with these changes for some values of q and r. For all values considered, the obtained controller candidate is the maximizer of $\xi^*(K)$, and this candidate solves Problem 1.

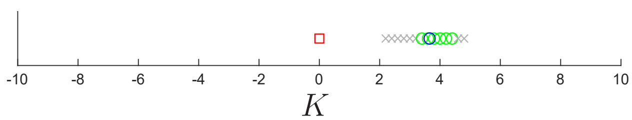

Fig. 1. Example 1. White area: controllers for which one or more subsystems are unstable; gray crosses: controllers for which stability cannot be established with quadratic Lyapunov functions (possibly unstable); green circles: controllers for which stability can be established with quadratic Lyapunov functions; red square: null controller (open loop system); blue circle: found controller.

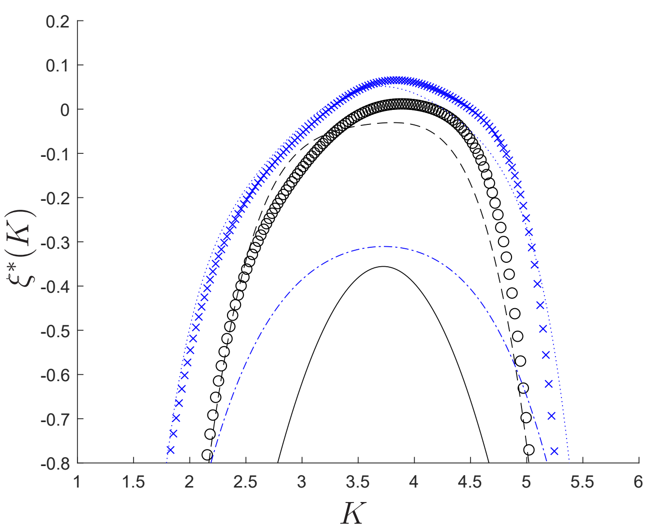

Fig. 2. Example 1. Polynomial $\xi^*(K)$ found for $d = 1$, $r = 0$ and $q = 1$ (black solid line), $q = 2$ (black dashed line), $q = 3$ (black circles). Also, polynomial $\xi^*(K)$ found for $d = 1$, $r = 2$ and $q = 1$ (blue dashdot line), $q = 2$ (blue dashed line), $q = 3$ (blue crosses).

B. Example 2

Let us consider the switched system (1) with $N = 2$ and

$$\begin{cases} A_{1,1} = \begin{pmatrix} 0 & -4 & -4 \\ 1 & 0 & 2 \\ 2 & 1 & -2 \end{pmatrix}, & A_{1,2} = \begin{pmatrix} -2 & -5 & -4 \\ 1 & 0 & 1 \\ 2 & 0 & -2 \end{pmatrix}, \\ B_{1,1} = \begin{pmatrix} 0.5 \\ -1 \\ 1.5 \end{pmatrix}, & B_{1,2} = \begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}, \\ C_{1,1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & C_{1,2} = \begin{pmatrix} 1 & 1 & 1 \\ 0.5 & 0 & 0.5 \end{pmatrix}. \end{cases}$$

We want to solve Problem 2 with $\mathcal{D} = \mathcal{D}_T$, $T = 0.5$, $\gamma = 3$, and $\mathcal{K}$ given by (6) with $\rho = 10$. Let us observe that the $\mathcal{H}_2$ norm of the switched system (7) is unbounded for $K = 0$, for instance because the first subsystem is unstable.

Let us use the methodology proposed in Section IV. First, we solve the SDP (39). We choose to search for HRLFs of total degree $2d = 2$ constant in the controller (i.e., $r = 0$) using a first-order approximation of the matrix exponential (i.e., $q = 1$). By using the notation $K = (k_1, k_2)'$, we find

$$\begin{aligned} \xi^*(K) &= -0.015k_1^2 - 0.001k_1k_2 - 0.179k_1 - 0.004k_2^2 \\ &\quad -0.093k_2 - 2.571. \end{aligned}$$

Second, we solve the SDP (26), finding the set

$$\mathcal{Z}(1,1,0) = \{(-5.681, -10)'\}.$$

Third, we find that (34) is infeasible for $K = (-5.681, -10)'$.

Hence, we repeat the above procedure using a second-order approximation of the matrix exponential (i.e., $q = 2$), finding the set

$$\mathcal{Z}(1,2,0) = \{(-5.037, 5.420)'\}$$

and that (34) is feasible for $K = K^*$ with $K^* = (-5.037, 5.420)'$. Hence, the found controller K^* solves the problem. In particular, this controller ensures that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than 2.354 (upper bound found with $2d = 2$). The number of LMI scalar variables in the three SDPs are 364, 19 and 13, respectively. Figure 3 shows the controllers that solve the problem considered in this example found by brute force.

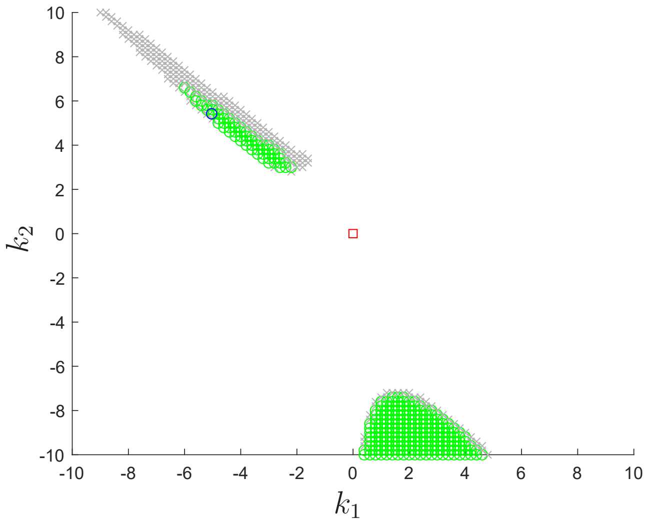

Fig. 3. Example 2. White area: controllers for which one or more subsystems are unstable; gray crosses: controllers for which the upper bound on the $\mathcal{H}_2$ norm cannot be established with quadratic Lyapunov functions (possibly upper bound does not hold); green circles: controllers for which the upper bound on the $\mathcal{H}_2$ norm can be established with quadratic Lyapunov functions; red square: null controller (open loop system); blue circle: found controller.

C. *Example 3*

Let us consider the switched system (1) with $N = 2$ and

$$\begin{cases} A_{1,1} = \begin{pmatrix} 0 & 1 & 0 \\ -3 & 0 & -1 \\ -0.5 & 0.5 & -1 \end{pmatrix}, A_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ -10 & -0.5 & -1.5 \\ -1 & 0 & -4.5 \end{pmatrix}, \\ B_{1,1} = \begin{pmatrix} 0 \\ 1 \\ 0.5 \end{pmatrix}, \qquad B_{1,2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ C_{1,1} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}, \qquad C_{1,2} = \begin{pmatrix} -1 & 0.5 & -0.5 \end{pmatrix}. \end{cases}$$

We want to solve Problem 3 with $\mathcal{D} = \mathcal{D}_{arb}$, $\gamma = 5$, and $\mathcal{K}$ given by (6) with $\rho = 10$. It can be verified that the RMS gain of the switched system (7) is unbounded for $K = 0$ (though the subsystems are asymptotically stable). Indeed, the open loop switched system is unstable as shown in Figure 4, where the trajectory of the system is reported for $u(t) = 0$, $x(0) = (0, 1, 5)'$ and

$$\sigma(t) = \begin{cases} 1 & \text{if } t \in [1.5i, 1.5i + 1),\ i \in \mathbb{N}_0, \\ 2 & \text{otherwise.} \end{cases}$$

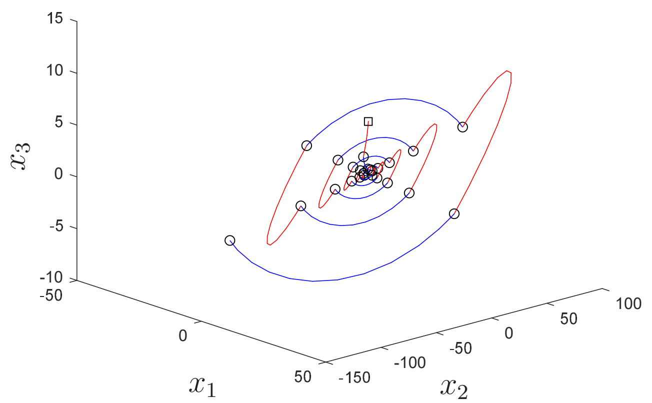

Fig. 4. Example 3. A trajectory of the open loop switched system in the absence of input. The square is the initial condition, and the circles are the points where the switches occur.

Let us use the methodology proposed in Section V. First, we solve the SDP (48). We choose to search for HRLFs of total degree $2d = 2$ constant in the controller (i.e., $r = 0$). We find

$$\xi^*(K) = -0.001K^2 - 0.006K - 1.059.$$

Second, we solve the SDP (26), finding the set

$$\mathcal{Z}(1, 0, 0) = \{-5.404\}.$$

Third, we find that (44) is infeasible for $K = -5.404$.

Hence, we repeat the above procedure by searching for HRLFs of total degree $2d = 4$, finding the set

$$\mathcal{Z}(2, 0, 0) = \{-2.474\}.$$

and (44) is feasible for $K = K^*$ with $K^* = -2.474$. Hence, the found controller K^* solves the problem. In particular, this controller ensures that the RMS gain of the switched system (7) is smaller than 3.884 (upper bound found with $2d = 4$). The number of LMI scalar variables in the three SDPs are 408, 2 and 166, respectively. Figure 5 shows the controllers that solve the problem considered in this example found by brute force.

VIII. Conclusions

This paper has addressed the synthesis of fixed-order output feedback controllers for stability and performance of continuous-time switched linear systems with dwell time constraints or

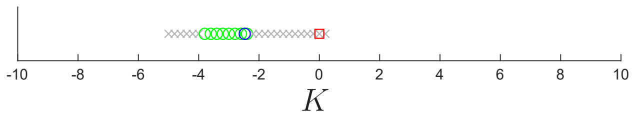

Fig. 5. Example 3. White area: controllers for which one or more subsystems are unstable; gray crosses: controllers for which the upper bound on the RMS gain cannot be established with HRLFs of total degree $2d = 4$ (possibly upper bound does not hold); green circles: controllers for which the upper bound on the RMS gain can be established with HRLFs of total degree $2d = 4$; red square: null controller (open loop system); blue circle: found controller.

arbitrary switching. Necessary and sufficient LMI conditions have been provided for determining stabilizing controllers and controllers ensuring a desired upper bound on the $\mathcal{H}_2$ norm and RMS gain. These conditions have been obtained through the use of HPLFs and HRLFs parameterized by the sought controller, and through the introduction of polynomials for approximating the matrix exponential and for quantifying the feasibility of the Lyapunov inequalities.

Several directions can be considered in future work. For instance, one could investigate the structural conditions under which the problems addressed in this paper admit solutions. Also, one could explore the possibility of reducing the numerical complexity, as it quickly grows with the size of the system and with the degree of the polynomials. Lastly, one could extend the proposed methodology to other classes of switched systems and switching rules.

Acknowledgements

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Appendix A

In this section we provide the definition of the quantities used in Theorems 3 and 5. Let us consider Theorem 3. The matrix $E_i(K, \Phi_i) \in \mathbb{S}^{\varsigma_{2d-1}}$ is defined as

$$E_i(K, \Phi_i) = E_{i,1}(K, \Phi_i) - E_{i,2}(K, \Phi_i) + E_{i,3}(K)$$

where

$$\left\{ \begin{array}{rcl} E_{i,1}(K, \Phi_i) & = & J_1' \left(\mathrm{he}(\Phi_i \Lambda_i(K)) \otimes \Psi \right) J_1, \\ E_{i,2}(K, \Phi_i) & = & J_1' \left(\Phi_i \otimes \mathrm{he}(\Psi \tilde{\Lambda}_i(K)) \right) J_1, \\ E_{i,3}(K) & = & J_2' \left(\Psi^{\otimes 2} \otimes C_i(K)' C_i(K) \right) J_2, \end{array} \right.$$

$J_1 \in \mathbb{R}^{\varsigma_d \varsigma_{d-1} \times \varsigma_{2d-1}}$ and $J_2 \in \mathbb{R}^{n \varsigma_{d-1}^2 \times \varsigma_{2d-1}}$ satisfy

$$\left\{ \begin{array}{rcl} b(x, d) \otimes b(x, d-1) & = & J_1 b(x, 2d-1), \\ b(x, d-1)^{\otimes 2} \otimes x & = & J_2 b(x, 2d-1), \end{array} \right.$$

and $\tilde{\Lambda}_i(K) \in \mathbb{R}^{\varsigma_d \times \varsigma_d}$ satisfies

$$\frac{db(x,d-1)}{dx} A_i(K)Tx = \tilde{\Lambda}_i(K) b(x,d-1).$$

Also, the matrix $Q_i(K,\Phi_i,\Phi_j,U(\cdot)) \in \mathbb{S}^{\varsigma_{2d-1}}$ is defined as

$$\begin{aligned} Q_i(K,\Phi_i,\Phi_j,U(\cdot)) &= Q_{i,1}(K,\Phi_j,U(\Lambda_i(K))) \\ &\quad -Q_{i,2}(K,\Phi_i,U(\tilde{\Lambda}_i(K))) \\ &\quad +Q_{i,3}(K,U(\tilde{\Lambda}_i(K)),U(A_i(K))) \end{aligned}$$

where

$$\begin{cases} Q_{i,1}(K,\Phi_j,X) = J_1'\left((X'\Phi_j X)\otimes\Psi\right)J_1, \\ Q_{i,2}(K,\Phi_i,X) = J_1'\left(\Phi_i\otimes(X'\Psi X)\right)J_1, \\ Q_{i,3}(K,X,Y) = J_2'\left(\Psi\otimes(X'\Psi X)\otimes\Delta_i(K,Y)\right)J_2 \end{cases}$$

and

$$\Delta_i(K,Y) = \int_0^T \left(C_i(K)Y^t\right)' C_i(K)Y^t dt.$$

Lastly, the scalar $g_i(K,\Phi_i) \in \mathbb{R}$ is defined as

$$g_i(K,\Phi_i) = \sum_{j=1}^{m_2} v_i(B_i^{(j)}(K))$$

where $B_i^{(j)}(K)$ is the j-th column of $B_i(K)$.

Next, let us consider Theorem 5. The matrices $F_i(K,\Phi), G_i(K) \in \mathbb{S}^{\varsigma_{2d-1}+m_1\varsigma_{2d-2}}$ are defined as

$$\begin{cases} F_i(K,\Phi) = \begin{pmatrix} E_{i,1}(K,\Phi)-E_{i,2}(K,\Phi) & \star \\ F_{i,1}(K,\Phi)'-F_{i,2}(K,\Phi)' & -\gamma^2 F_{i,3} \end{pmatrix}, \\ G_i(K) = \begin{pmatrix} G_{i,1}(K) & G_{i,2}(K) \\ \star & G_{i,3}(K) \end{pmatrix} \end{cases}$$

where

$$\begin{cases} F_{i,1}(K,\Phi) = J_1'\left(\Phi\tilde{B}_{i,0}(K)\otimes\Psi\right)J_3, \\ F_{i,2}(K,\Phi) = J_1'\left(\Phi\otimes\Psi\tilde{B}_{i,1}(K)\right)J_4, \\ F_{i,3} = J_5'\left(\Psi^{\otimes 2}\otimes I\right)J_5, \\ G_{i,1}(K) = J_2'\left(\Psi^{\otimes 2}\otimes C_i(K)'C_i(K)\right)J_2, \\ G_{i,2}(K) = J_2'\left(\Psi^{\otimes 2}\otimes C_i(K)'D_i(K)\right)J_5, \\ G_{i,3}(K) = J_5'\left(\Psi^{\otimes 2}\otimes D_i(K)'D_i(K)\right)J_5, \end{cases}$$

$J_3 \in \mathbb{R}^{m_1 \varsigma_{d-1}^2 \times m_1 \varsigma_{2d-2}}$, $J_4 \in \mathbb{R}^{m_1 \varsigma_d \varsigma_{d-2} \times m_1 \varsigma_{2d-2}}$ and $J_5 \in \mathbb{R}^{m_1 \varsigma_{d-1}^2 \times m_1 \varsigma_{2d-2}}$ satisfy

$$\begin{cases} b(x,d-1) \otimes u \otimes b(x,d-1) = J_3 \tilde{b}(x,2d-2,u), \\ b(x,d) \otimes b(x,d-2) \otimes u = J_4 \tilde{b}(x,2d-2,u), \\ b(x,d-1)^{\otimes 2} \otimes u = J_5 \tilde{b}(x,2d-2,u) \end{cases}$$

with

$$\tilde{b}(x,d,u) = b(x,d) \otimes u,$$

and $\tilde{B}_{i,s}(K) \in \mathbb{R}^{\varsigma_{d-s} \times m_1 \varsigma_{d-s-1}}$ satisfies, for $s = 0, 1$,

$$\frac{db(x,d-s)}{dx} B_i(K) u = \tilde{B}_{i,s}(K) \tilde{b}(x,d-s-1,u).$$

Appendix B

In this section we provide some details about the derivation of the solution found in Example 1. Let us start with the construction of the SDP (23). The matrices $A_i(K)$ of the switched system (7) are

$$A_1(K) = \begin{pmatrix} 0 & 1 \\ -5+k & 2-K \end{pmatrix}, \; A_2(K) = \begin{pmatrix} -1 & 0 \\ -5+K & -2 \end{pmatrix},$$

$$A_3(K) = \begin{pmatrix} 0 & -3 \\ k & -5+K \end{pmatrix}.$$

Since $d = 1$, $q = 1$ and $T = 0.2$, it follows that

$$b(x,d) = x, \; \Lambda_i(A) = 0.2 A_i(K), \; L(\cdot) = 0, \; \Upsilon(A) = I + A.$$

Since $r = 0$, $V_i(K)$ has degree 0. With the guideline reported after Theorem 2, $\xi(K)$ has degree 2. Let us parameterize $V_i(K)$ and $\xi(K)$ as

$$V_1(K) = \begin{pmatrix} a_1 & a_2 \\ \star & a_3 \end{pmatrix}, \; V_2(K) = \begin{pmatrix} a_4 & a_5 \\ \star & a_6 \end{pmatrix},$$

$$V_3(K) = \begin{pmatrix} a_7 & a_8 \\ \star & a_9 \end{pmatrix}, \; \xi(K) = \xi_1 + K\xi_2 + K^2 \xi_3$$

where $a_1, \ldots, a_9, \xi_1, \xi_2, \xi_3 \in \mathbb{R}$ are decision variables. It follows that there are 12 matrix polynomials in the constraints of the SDP (23), and 2 of them are

$$\begin{aligned} M_{1,1}(K) &= \begin{pmatrix} -1+a_1 & a_2 \\ \star & -1+a_3 \end{pmatrix}, \\ M_{1,2}(K) &= \begin{pmatrix} -K^2\xi_3 - 2Ka_2 - K\xi_2 + 10a_2 - \xi_1 & Ka_2 - Ka_3 - a_1 - 2a_2 + 5a_3 \\ \star & -K^2\xi_3 + 2Ka_3 - K\xi_2 - 2a_2 - 4a_3 - \xi_1 \end{pmatrix} \end{aligned}$$

(the other 10 matrix polynomials are omitted for brevity). The cost function of the SDP (23) is

$$h = 20\xi_1 + 666.667\xi_3.$$

The constraints of the SDP (23) are converted into LMIs by exploiting the Gram matrix method for matrix polynomials mentioned in Section II-B. For brevity, consider only $M_{1,2}(K)$. One has

$$M_{1,2}(K) = \begin{pmatrix} I & KI \end{pmatrix} \tilde{M} \begin{pmatrix} I & KI \end{pmatrix}'$$

where

$$\tilde{M} = \begin{pmatrix} 10a_2 - \xi_1 & -a_1 - 2a_2 + 5a_3 & -a_2 - 0.5\xi_2 & -\alpha + 0.5a_2 - 0.5a_3 \\ \star & -2a_2 - 4a_3 - \xi_1 & \alpha + 0.5a_2 - 0.5a_3 & a_3 - 0.5\xi_2 \\ \star & \star & -\xi_3 & 0 \\ \star & \star & \star & -\xi_3 \end{pmatrix}$$

and $\alpha \in \mathbb{R}$ is a decision variable. The condition for $M_{1,2}(K)$ to be SOS is hence replaced by the LMI $\tilde{M} \geq 0$.

Next, let us consider the construction of the SDP (26). With the guideline reported after Theorem 2, $s_{1,1}(K)$ has degree 0. We parameterize $s_{1,1}(K)$ as $s_{1,1}(K) = s$ where $s \in \mathbb{R}$ is a decision variable. By using $\xi^*(K)$ found in the solution of the SDP (23) and reported in Example 1, it follows that

$$p(K) = \mu + K^2 s + 0.142K^2 - 1.035K - 100s + 6.54.$$

By converting the constraints of the SDP (26) into LMIs, we obtain

$$\begin{cases} 0 \leq \begin{pmatrix} \mu - 100s + 6.54 & -0.517 \\ \star & s + 0.142 \end{pmatrix}, \\ 0 \leq s. \end{cases}$$

Lastly, let us consider the determination of $\mathcal{Z}(1,1,0)$. From the solution of the SDP (26) we have

$$\begin{cases} p^*(K) & = & 0.142K^2 - 1.035K + 1.892 \\ \mu^* & = & -4.648. \end{cases}$$

Since $p^*(K)$ is quadratic, one finds trivially that $p^*(K) = 0$ if and only if $K = K^* = 3.657$. Moreover, $\xi^*(K^*) = \mu^*$, and, hence, we conclude that $\mathcal{Z}(1,1,0) = \{K^*\}$.

APPENDIX C

In this section we report the proofs of Theorems 4 and 6.

(Theorem 4) *Proof*. "Sufficiency". Suppose there exists $K \in \mathcal{Z}(d,q,r)$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ. Since $\mathcal{Z}(d,q,r)$ is a subset of $\mathcal{K}$, it follows that there exists $K \in \mathcal{K}$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ.

"Necessity". Suppose there exists $K \in \mathcal{K}$ such that the $\mathcal{H}_2$ norm of the switched system (7) is smaller than γ, and let us indicate such a value as $K^\#$. For the chosen $\Psi > 0$, let us replace K with $K^\#$ in the LMIs in (34), and let d be such that these LMIs are feasible (observe that such a d does exist finite from Theorem 3). Also, let us replace $\exp(\cdot)$ with $\Upsilon(\cdot)$ in these LMIs, and let q be such that these LMIs are feasible (observe that such a q does exist finite since $K^\#$ is bounded, $\Upsilon(A)$ approximates arbitrarily well e^A for sufficiently large values of q, and the inequalities in (34) are strict). Let $\Phi_i^\#$, $\Theta_i^\#$ and $\Omega_{i,j}^\#$ be values of Φ_i, Θ_i and $\Omega_{i,j}$ for which the obtained LMIs hold. It follows that there exists $\xi^\# > 0$ such that, for all $i,j = 1,\ldots,N$, $i \neq j$,

$$\begin{cases} 0 & < & \Phi_i^\# - \xi^\# I, \\ 0 & < & -\bar{E}_i(K^\#, \Phi_i^\#, 1) - L(\Theta_i^\#) - \xi^\# I, \\ 0 & < & \bar{Q}_i(K^\#, \Phi_i^\#, \Phi_j^\#, 1) + L(\Omega_{i,j}^\#) - \xi^\# I, \\ 0 & < & \bar{g}_i(K^\#, \Phi_i^\#, 1) - \xi^\#. \end{cases}$$

Given the continuity of these inequalities on $\xi^\#$, $\Phi_i^\#$, $\Theta_i^\#$ and $\Omega_{i,j}^\#$, it follows that there exist matrix polynomials $\hat{\xi}(K)$, $\hat{\Phi}_i(K)$, $\hat{\Theta}_i(K)$ and $\hat{\Omega}_{i,j}(K)$ such that $\hat{\xi}(K^\#) > 0$ and, for all $i,j = 1,\ldots,N$, $i \neq j$, for all $l = 1,2,4$, and for all $K \in \mathbb{R}^{m_1 \times p_1}$,

$$\begin{cases} 0 & < & \hat{M}_{i,l}(K), \\ 0 & < & \hat{M}_{i,j,3}(K) \end{cases}$$

where

$$\begin{cases} \hat{M}_{i,1}(K) &= \hat{\Phi}_i(K) - \hat{\xi}(K)I, \\ \hat{M}_{i,2}(K) &= -\bar{E}_i(K, \hat{\Phi}_i(K), 1) - L(\hat{\Theta}_i(K)) - \hat{\xi}(K)I, \\ \hat{M}_{i,j,3}(K) &= \bar{Q}_i(K, \hat{\Phi}_i(K), \hat{\Phi}_j(K), 1) + L(\hat{\Omega}_{i,j}(K)) \\ & \quad -\hat{\xi}(K)I, \\ \hat{M}_{i,4}(K) &= \bar{g}_i(K, \hat{\Phi}_i(K), 1) - \hat{\xi}(K). \end{cases}$$

The degrees of $\hat{\xi}(K)$, $\hat{\Phi}_i(K)$, $\hat{\Theta}_i(K)$ and $\hat{\Omega}_{i,j}(K)$ can be always increased in order to satisfy the constraints of the SDP (39). Indeed, from [40] one has that, by choosing $f(K) = (1 + \|K\|_{Fro}^2)^\eta$ for some $\eta \in \mathbb{N}_0$, the polynomials $f(K)\hat{M}_{i,l}(K)$ and $f(K)\hat{M}_{i,j,3}(K)$ are SOS. Since $f(K) - 1$ is SOS as well, one has that the constraints of the SDP (39) can be satisfied by choosing the matrix polynomials

$$\begin{cases} \xi(K) = f(K)\hat{\xi}(K), \\ \Phi_i(K) = f(K)\hat{\Phi}_i(K), \\ \Theta_i(K) = f(K)\hat{\Theta}_i(K), \\ \Omega_{i,j}(K) = f(K)\hat{\Omega}_{i,j}(K), \\ \zeta(K) = f(K) \end{cases}$$

which also satisfy $\xi(K^\#) > 0$. The proof is completed by proceeding analogously to the last part of the proof of Theorem 2. □

(Theorem 6) *Proof*. “Sufficiency”. Suppose there exists $K \in \mathcal{Z}(d, 0, r)$ such that the RMS gain of the switched system (7) is smaller than γ. Since $\mathcal{Z}(d, 0, r)$ is a subset of $\mathcal{K}$, it follows that there exists $K \in \mathcal{K}$ such that the RMS gain of the switched system (7) is smaller than γ.

“Necessity”. Suppose there exists $K \in \mathcal{K}$ such that the RMS gain of the switched system (7) is smaller than γ, and let us indicate such a value as $K^\#$. For the chosen $\Psi > 0$, let us replace K with $K^\#$ in the LMIs (44), and let d be such that these LMIs are feasible (observe that such a d does exist finite from Theorem 5). Let $\Phi^\#$ and $\Theta_i^\#$ be values of Φ and Θ_i for which the obtained LMIs hold. It follows that there exists $\xi^\# > 0$ such that, for all $i = 1, \ldots, N$,

$$\begin{cases} 0 &< \Phi^\# - \xi^\# I, \\ 0 &< -F_i(K^\#, \Phi^\#) - G_i(K^\#) - L(\Theta_i^\#) - \xi^\# I. \end{cases}$$

Given the continuity of these inequalities on $\xi^{\#}$, $\Phi^{\#}$ and $\Theta_i^{\#}$, it follows that there exist matrix polynomials $\hat{\xi}(K)$, $\hat{\Phi}(K)$ and $\hat{\Theta}_i(K)$ such that $\hat{\xi}(K^{\#}) > 0$ and, for all $i = 1, \ldots, N$ and for all $K \in \mathbb{R}^{m_1 \times p_1}$,

$$\begin{cases} 0 & < & \hat{\Phi}(K) - \hat{\xi}(K)I, \\ 0 & < & -F_i(K, \hat{\Phi}(K)) - G_i(K) - L(\hat{\Theta}_i(K)) - \hat{\xi}(K)I. \end{cases}$$

The proof is completed by proceeding analogously to the last part of the proof of Theorem 4. □

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Graziano Chesi is a Professor at the Department of Electrical and Electronic Engineering of the University of Hong Kong. He received the Laurea in Information Engineering from the University of Florence in 1997, and the PhD in Systems Engineering from the University of Bologna in 2001. He joined the University of Siena in 2000 and the University of Hong Kong in 2006. He served as Associate Editor for various journals, including Automatica, the European Journal of Control, the IEEE Control Systems Letters, the IEEE Transactions on Automatic Control, the IEEE Transactions on Computational Biology and Bioinformatics, and Systems and Control Letters. He also served as Guest Editor for the IEEE Transactions on Automatic Control, the International Journal of Robust and Nonlinear Control, and Mechatronics. He founded and served as chair of the Technical Committee on Systems with Uncertainty of the IEEE Control Systems Society. He also served as chair of the Best Student Paper Award Committees for the IEEE Conference on Decision and Control and the IEEE Multi-Conference on Systems and Control. He is author of the books ”Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems” (Springer 2009) and ”Domain of Attraction: Analysis and Control via SOS Programming” (Springer 2011). He is a IEEE Fellow of the IEEE.

Patrizio Colaneri was born in Palmoli, Italy, in 1956. He received the Laurea degree in Electrical Engineering in 1981 and the Ph.D. degree (Dottorato di Ricerca) in Automatic Control in 1987. After a few years in industry and at the National Research Council of Italy, he joined the Politecnico di Milano where he is full professor of Automatica and served as head of the Ph.D. school on ICT (2007-2009). He spent a semester at the Systems Research Center of the University of Maryland (1989) and at the Hamilton Institute of the National University of Ireland (2009). He also collaborates with the Institute for Design and Control of Mechatronical Systems of Johannes Kepler University in Linz (Austria) since 2000. Dr. Colaneri was the chair of the IFAC Coordinating Committee on Design Methods for six years, the chair of the Technical Committee on Control Design for six years, a member of the Council of EUCA (European Union Control Association) for three years, an Associate Editor of Automatica for six years (certificate of outstanding service). He is at present a member of the Technical Board of IFAC, a Subject Editor of the International Journal of Robust and Nonlinear Control, a senior Editor of the IFAC journal Nonlinear Analysis Hybrid Systems and a Senior Editor of the IEEE Transactions on Automatic Control. He was elevated to the degree of IEEE Fellow for contributions on periodic and switching control. He is also a Fellow of IFAC (International Federation of Automatic Control). His main interests are in the area of periodic systems and control, robust filtering and control, and switching control. He has authored/co-authored more than 220 papers and seven books, including the following four international monographies: “Control Theory and Design: an RH2 and RH∞ viewpoint”, Academic Press 1997, “Periodic Systems: Filtering and Control”, Springer Verlag, 2009, “Positive Markov Jump Linear Systems”, Now Publishing 2015, “Switched Positive Linear Systems”, Now Publishing 2015.