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Resource theories of communication






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Abstract

A series of recent works has shown that placing communication channels in a coherent superposition of alternative configurations can boost their ability to transmit information. Instances of this phenomenon are the advantages arising from the use of communication devices in a superposition of alternative causal orders, and those arising from the transmission of information along a superposition of alternative trajectories. The relation among these advantages has been the subject of recent debate, with some authors claiming that the advantages of the superposition of orders could be reproduced, and even surpassed, by other forms of superpositions. To shed light on this debate, we develop a general framework of resource theories of communication. In this framework, the resources are communication devices, and the allowed operations are (a) the placement of communication devices between the communicating parties, and (b) the connection of communication devices with local devices in the parties' laboratories. The allowed operations are required to satisfy the minimal condition that they do not enable communication independently of the devices representing the initial resources. The resource-theoretic analysis reveals that the aforementioned criticisms on the superposition of causal orders were based on an uneven comparison between different types of quantum superpositions, exhibiting different operational features.

1. Introduction

A fundamental task in information theory is to quantify the amount of information that a given set of communication devices can transmit. Claude E Shannon addressed this question for devices operating according to the laws of classical physics [1], laying down the foundations of our current communication technology. At the fundamental level, however, the classical laws are just an approximation of the laws of quantum physics. The ability to transmit quantum data [2–5] was shown to offer remarkable advantages, such as the possibility of secure quantum key distribution [6, 7]. Over time, the study of communication protocols involving the exchange of quantum data led to the establishment of the field of *quantum Shannon theory* [8].

In a series of recent works, a further generalisation of quantum Shannon theory has been proposed where not only the transmitted data, but also the configuration of the communication devices can be

quantum [9–20]. This introduces a second level of quantisation of Shannon theory, generalising standard quantum Shannon theory where the transmitted data are quantum but the configuration of the communication channels is classical. In one of the new frameworks, the available communication channels are combined in a superposition of different causal orders [9–16, 21], using an operation known as the quantum SWITCH [22, 23]. In another framework, information can be sent along a superposition of trajectories [17–20], leading to superpositions of alternative quantum evolutions [19, 24, 25]. In both frameworks, the superposition is generated by letting a quantum system control the configuration of the communication channels, determining either their order, or which of them is used to transmit information. Coherent control over the channels' configuration has been shown to yield advantages in a wide range of communication scenarios, achieving rates beyond those that are possible in standard quantum Shannon theory. Some of these advantages stimulated experiments in quantum optics, both on the control of orders [12, 26] and on the control of trajectories [27].

Recently, the works on the coherent control of causal orders, in particular references [9–11], have been criticised on the grounds that similar advantages could be obtained with coherent control of the choice of communication devices [18], or coherent control over different choices of encoding and decoding operations [28]. Here we respond to these criticisms, by setting up a resource-theoretic framework that sheds light on the comparison between different extensions of quantum Shannon theory.

First, we point out that references [9–11] only claimed that the superposition of causal orders offers an advantage with respect to *standard* quantum Shannon theory, where communication devices are composed in a definite order and no coherent control over their configuration is allowed. The converse claim that every advantage over standard quantum Shannon theory must be due to control over the causal order was not made in [9–11], and, in fact, was known to be false, since Gisin *et al* had previously shown that control over the choice of channels offers advantages over the standard model of quantum communication [17].

Second, while it is clear that coherent control over devices generically leads to communication advantages, it is important to distinguish between different types of control. Three distinct types of control have been considered so far:

- (i) Control over the causal order of communication channels [9–11]
- (ii) Control over the choice of communication channels [17–19]
- (iii) Control over choices of encoding and decoding operations [28].

These three types of control are conceptually distinct and, as we will see, have different operational features.

In this paper, we construct a general framework for resource theories of communication, and use it to shed light on the different extensions of quantum Shannon theory that have been proposed so far. We formulate a minimal requirement of a resource theory of communication, namely that no allowed operation on the communication devices should bypass them, enabling communication independently of the communication devices available to the sender and receiver. Our framework captures the differences between the different types of control (i)–(iii), and helps clarify various comparisons that have been made across protocols using them.

Applying our resource-theoretic framework, we argue that (a) the comparison between control of causal orders and control of communication channels proposed in reference [18] is uneven, because the control of communication channels requires (in principle) stronger initial resources than the control of causal orders, and (b) the examples of communication with control over encoding and decoding proposed in reference [28] do not satisfy the minimal requirement of a resource-theory of communication.

In section 2 of this work, we formalise standard quantum Shannon theory as a resource theory. In section 3 we extend our framework to general resource theories of communication and formulate the minimal requirement that such a theory should satisfy. Section 4 presents the frameworks of superposition of causal orders [9–11] and superposition of trajectories [17–19], showing that both are consistent with a resource-theoretic description. Section 5 comments on the comparisons made in reference [18] between the two frameworks, while section 6 argues that the communication protocols put forward in reference [28] do not admit a resource-theoretic formulation.

2. Standard quantum Shannon theory as a resource theory of communication

We begin by reformulating standard quantum Shannon theory as a resource theory, setting the scene for its extension to more general resource theories of communication.

2.1. Quantum Shannon theory as a theory of resources

A central task in information theory is to quantify the amount of information that a given communication device can transmit. In general, the amount of information can be classical or quantum, or of other types. In this paper, we will focus on classical and quantum information. To make the quantification unambiguous, it is essential to specify how the given device can be used. The device represents a resource, and the rules on the possible uses of this resource can be formulated as a *resource theory* [29, 30].

A resource-theoretic approach to standard quantum Shannon theory was initiated by Devetak, Harrow, and Winter [31]. Further resource-theoretic formalisations have been put forward in references [9, 32–34] in a variety of communication scenarios. Related resource theories of quantum devices have been recently formulated in references [35–37] for purposes other than the theory of communication.

In this paper we will adopt the general framework for resource theories proposed by Coecke, Fritz, and Spekkens [29]. In this framework, the set of all possible resources is described by a set of objects, equipped with a set of operations acting on them. The set of operations is closed under sequential and parallel composition. For example, the set of operations, hereafter denoted by \mathbf{M} , could be the set of all quantum channels (completely positive trace-preserving maps) acting on finite-dimensional quantum systems (the objects). The central idea of the resource-theoretic framework is to define a subset of operations $\mathbf{M}_{\text{free}} \subseteq \mathbf{M}$, which are regarded as *free*. The notion of resource is then defined relative to the set of free operations: a state or an operation is a non-trivial resource if and only if it is not free, and a resource is more valuable than another if the former can be converted into the latter by means of free operations.

Different choices of free operations generally define different resources. Intuitively, the set of free operations is meant to capture some operational restriction, which makes some operations ‘easy to implement’. In principle, however, \mathbf{M}_{free} could be any subset of operations, as long as it is closed under sequential and parallel composition. In this respect, the resource-theoretic approach is a conceptual tool to understand the power of the set \mathbf{M}_{free} , irrespectively of whether implementing the operations in it is easy or not.

In quantum Shannon theory, the input resources are communication channels, or, more precisely, *uses* of communication channels. For example, the ability to transfer a single qubit from a sender to a receiver is modelled as a single use of a single-qubit identity channel.

To cast quantum Shannon theory in the resource-theoretic framework of reference [29], one has to regard the various types of quantum channels as objects, and to define the allowed operations that transform input channels into output channels. These operations are known as *quantum supermaps* [23, 38, 39]. In the following, we will define the sets of free supermaps \mathbf{M}_{free} for some of the basic scenarios in quantum Shannon theory, setting the scene for the generalisations studied in the rest of the paper.

2.2. Notation

We will denote by \mathcal{H}_A the Hilbert space associated to a given quantum system A , and by $L(\mathcal{H}_A)$ the space of all linear operators on \mathcal{H}_A . The set of all quantum states (positive semidefinite operators with unit trace) on \mathcal{H}_A will be denoted by $\text{St}(A) \subset L(\mathcal{H}_A)$. For simplicity, we will restrict our attention to finite-dimensional systems, although this is not essential for our framework.

The set of all linear maps from $L(\mathcal{H}_A)$ to $L(\mathcal{H}_B)$ will be denoted by $\text{Map}(A, B)$. The set of quantum channels (completely positive trace-preserving maps) will be denoted by $\text{Chan}(A, B) \subset \text{Map}(A, B)$. We will also use the shorthand $\text{Chan}(A) := \text{Chan}(A, A)$. When the input and output are arbitrary, we will simply write Chan . We will sometimes use the fact that the action of a generic quantum channel $\mathcal{N} \in \text{Chan}(A, B)$ on an input state $\rho \in \text{St}(A)$ can be written in the Kraus representation, as $\mathcal{N}(\rho) = \sum_i N_i \rho N_i^\dagger$, where $\{N_i\}$ is a set of linear operators satisfying the normalisation condition $\sum_i N_i^\dagger N_i = I$ [8].

We will denote by $A \otimes B$ the composite system consisting of subsystems A and B . We recall that every map $\mathcal{M} \in \text{Map}(A_1 \otimes A_2, B_1 \otimes B_2)$ can be decomposed into a sum of product maps, namely $\mathcal{M} = \sum_{j=1}^L \mathcal{M}_{1,j} \otimes \mathcal{M}_{2,j}$, with $\mathcal{M}_{1,j} \in \text{Map}(A_1, B_1)$ and $\mathcal{M}_{2,j} \in \text{Map}(A_2, B_2)$ for every $j \in \{1, \dots, L\}$.

A supermap is a linear transformation from $\text{Map}(A, B)$ to $\text{Map}(A', B')$, where A, A', B, B' are generic systems. The tensor product of two supermaps $\mathcal{S} : \text{Map}(A_1, B_1) \rightarrow \text{Map}(A'_1, B'_1)$ and $\mathcal{T} : \text{Map}(A_2, B_2) \rightarrow \text{Map}(A'_2, B'_2)$ is the supermap $\mathcal{S} \otimes \mathcal{T} : \text{Map}(A_1 \otimes A_2, B_1 \otimes B_2) \rightarrow \text{Map}(A'_1 \otimes A'_2, B'_1 \otimes B'_2)$ defined by the condition $(\mathcal{S} \otimes \mathcal{T})(\mathcal{M}_1 \otimes \mathcal{M}_2) := \mathcal{S}(\mathcal{M}_1) \otimes \mathcal{T}(\mathcal{M}_2)$ for every $\mathcal{M}_1 \in \text{Map}(A_1, B_1)$ and $\mathcal{M}_2 \in \text{Map}(A_2, B_2)$. Since all the maps in $\text{Map}(A_1 \otimes A_2, B_1 \otimes B_2)$ are linear combinations of product maps, this condition uniquely defines the supermap $\mathcal{S} \otimes \mathcal{T}$.

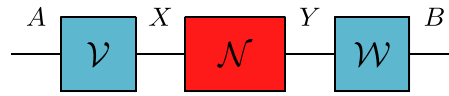


Figure 1. Basic placement supermap $\mathcal{S}_{\text{place}}^{A,B}(\mathcal{N}) := \mathcal{W}^B \circ \mathcal{N} \circ \mathcal{V}^A$. In this paper, the unplaced communication channels \mathcal{N} are drawn in red, while the placement supermaps (i.e. supermaps from unplaced channels to placed channels) are drawn in blue.

2.3. Direct communication from a sender to a receiver through a single channel

Consider the basic communication scenario where a sender (Alice) communicates directly to a receiver (Bob). At the fundamental level, the possibility of communication consists of two ingredients: the availability of a piece of hardware that serves as a communication device, and the placement of that piece of hardware between the sender and the receiver. For example, the piece of hardware could be an optical fibre, and the placement could be provided by a communication company that laid the fibre between the sender's and the receiver's locations. In some situations, the placement is implicit: for example, the sender and receiver could be communicating through a medium, such as the air between them, which has been placed there, as it were, by nature itself.

Mathematically, the communication device is described by a quantum channel $\mathcal{N} \in \text{Chan}(X, Y)$, which transforms systems of type X into systems of type Y . For example, the systems could be single qubits, encoded in the polarisation of single photons. At this level, the systems are not assigned a specific location in spacetime. Accordingly, we will call the systems X and Y *unplaced systems*, and the channel $\mathcal{N} \in \text{Chan}(X, Y)$ an *unplaced channel*.

The placement of the device can be described by introducing a *placement operation*, which corresponds to putting the input (output) system at the sender's (receiver's) location. Mathematically, a placement operation is a supermap that transforms channels in $\text{Chan}(X, Y)$ into channels in $\text{Chan}(A, B)$, where system A (B) is of the same type as system X (Y), denoted as $A \simeq X$ ($B \simeq Y$), and is placed at the sender's (receiver's) end, as illustrated in figure 1. Explicitly, we define the *basic placement supermap* as:

$$\mathcal{S}_{\text{place}}^{A,B}(\mathcal{N}) := \mathcal{W}^B \circ \mathcal{N} \circ \mathcal{V}^A, \quad (1)$$

where $\mathcal{V}^A \in \text{Chan}(A, X)$ and $\mathcal{W}^B \in \text{Chan}(Y, B)$ are unitary channels implementing the isomorphisms $A \simeq X$ and $Y \simeq B$, respectively.

We will call the systems A and B *placed systems*, and the channel $\mathcal{C} := \mathcal{S}_{\text{place}}^{A,B}(\mathcal{N})$ a *placed channel*. In the following, we will use the letters \mathcal{N} and \mathcal{C} for unplaced and placed channels, respectively. In figures, we will represent unplaced channels as red boxes, and placed channels as green boxes. This choice of colours reflects the fact that the placed channels are ready to be used by the communicating parties, while the unplaced channels have yet to be made available to them.

Once a device is in place, the sender and receiver can use it to communicate to one another. Typically, the communication is achieved by connecting the communication device with other devices present at the sender's and receiver's locations. For example, one end of an optical fibre could be connected to a computer, used by the sender to type an email, and the other end of the fibre could be connected to another computer, used by the receiver to read the email. The operations performed by the sender and receiver can be described by a supermap [23, 38, 39] transforming placed channels in $\text{Chan}(A, B)$ into placed channels in $\text{Chan}(A', B')$, where A' and B' are two new input and output systems, also placed in the sender's and receiver's locations, respectively.

Reference [38] showed that the most general supermap \mathcal{S} transforming a generic input channel $\mathcal{C} \in \text{Chan}(A, B)$ into an output channel $\mathcal{S}(\mathcal{C}) \in \text{Chan}(A', B')$ has the form

$$\mathcal{S}(\mathcal{C}) = \mathcal{D} \circ (\mathcal{C} \otimes \mathcal{I}_{\text{Aux}}) \circ \mathcal{E}, \quad (2)$$

where Aux is an auxiliary quantum system, and $\mathcal{E} \in \text{Chan}(A', A \otimes \text{Aux})$ and $\mathcal{D} \in \text{Chan}(B \otimes \text{Aux}, B')$ are quantum channels. These supermaps define the set of all possible operations on input channels, and play the role of the set \mathbf{M} in the general resource-theoretic framework described in section 2.1.

To specify the set of free operations, one has to specify a *subset* of the set of all supermaps. The standard choice (in the absence of additional resources such as shared entanglement or shared randomness) is to require the free operations to have the form

$$\mathcal{S}_{\mathcal{E},\mathcal{D}}(\mathcal{C}) := \mathcal{D} \circ \mathcal{C} \circ \mathcal{E} \quad (3)$$

(see figure 2 for an illustration). Operationally, this choice of \mathbf{M}_{free} is justified by the fact that the supermaps (3) can be achieved by performing a local encoding operation \mathcal{E} at the sender's side and a local decoding

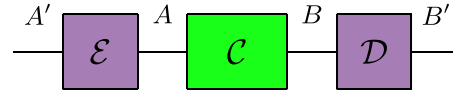


Figure 2. Encoding–decoding supermap $\mathcal{S}_{\mathcal{E},\mathcal{D}}(\mathcal{C}) := \mathcal{D} \circ \mathcal{C} \circ \mathcal{E}$. In this paper, the placed quantum channels are drawn in green, while the encoding–decoding supermaps are drawn in violet.

operation \mathcal{D} at the receiver’s side, without requiring the transmission of any system other than the system sent through the channel \mathcal{C} .

Note that while the supermaps (3) are the standard choice, other choices could be made. For example, one could consider quantum communication with the assistance of classical communication [40], or classical communication with the assistance of shared entanglement [41]. In these scenarios, the set of free supermaps is larger than the set of supermaps of the form (3), and contains supermaps that can be achieved with the additional resources under consideration. The characterisation of such supermaps is provided in appendix A. In the following, however, we will stick to the simplest choice of free supermaps, namely the choice in (3).

In general, we will refer to supermaps from unplaced channels to placed channels as *placement supermaps*, and we will interpret them as being performed either by a communication provider, or by nature itself. We will refer to supermaps from placed channels to placed channels as *party supermaps*, and will interpret them as being performed by the communicating parties.

2.4. Direct communication from a sender to a receiver through multiple channels

So far, we have considered operations on a single quantum channel. We now extend the resource-theoretic formulation to scenarios where multiple communication channels (or multiple uses of the same communication channel) are available.

Consider a communication protocol that uses k communication devices, described by k unplaced channels $\mathcal{N}_1, \dots, \mathcal{N}_k$, with $\mathcal{N}_i \in \mathbf{Chan}(X_i, Y_i)$ for $i \in \{1, \dots, k\}$. We denote by $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ the resource corresponding to a single use of each device. Again, the list $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ is interpreted as a description of the hardware before it is placed between the sender and receiver. For example, the hardware could be a list of optical fibres with some given specifications, viz attenuation coefficient, bandwidth, and length.

In appendix B.1 we show that the list $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ can be interpreted as an equivalent notation for the product channel $\mathcal{N}_1 \otimes \dots \otimes \mathcal{N}_k$, viewed as an element of a suitable set of channels (namely, k -partite no-signalling channels). In the following, we will use the list notation $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ as a visual reminder that the channels $\mathcal{N}_1, \dots, \mathcal{N}_k$ are unplaced.

In the direct communication scenario, it is understood that all the input systems are placed in Alice’s laboratory, and all the output systems are placed in Bob’s laboratory. Equivalently, this means that the communication devices are placed in parallel between the sender and the receiver. The operation of placing the devices in parallel is described by the *parallel placement supermap* $\mathcal{S}_{\text{par}}^{\mathbf{A},\mathbf{B}}$ defined by

$$\mathcal{S}_{\text{par}}^{\mathbf{A},\mathbf{B}}(\mathcal{N}_1, \dots, \mathcal{N}_k) := \mathcal{S}_{\text{place}}^{A_1, B_1}(\mathcal{N}_1) \otimes \dots \otimes \mathcal{S}_{\text{place}}^{A_k, B_k}(\mathcal{N}_k) \quad (4)$$

where $\mathbf{A} := (A_1, \dots, A_k)$ [$\mathbf{B} := (B_1, \dots, B_k)$] is a list of quantum systems placed in Alice’s (Bob’s) laboratory, with $A_i \simeq X_i$ and $B_i \simeq Y_i$ for every $i \in \{1, \dots, k\}$. The result of the supermap is a placed quantum channel in $\mathbf{Chan}(A_1 \otimes \dots \otimes A_k, B_1 \otimes \dots \otimes B_k)$.

A large body of results in standard quantum Shannon theory refers to channels combined in parallel as in equation (4). For example, Smith and Yard [42] showed that, surprisingly, the parallel composition of two channels with zero quantum capacity can give rise to a channel with non-zero quantum capacity. This phenomenon became known as *activation of the quantum capacity*.

2.5. Network communication from a sender to a receiver

Let us now consider a communication scenario where the sender (Alice) and receiver (Bob) communicate through a network of communication devices. To begin with, we focus on the simple case where Alice and Bob communicate through two devices, which are connected by an intermediate party (Ray), who serves as a ‘repeater’ passing to Bob the information received from Alice.

The initial resource is described by a pair of unplaced channels $(\mathcal{N}_1, \mathcal{N}_2) \in \mathbf{Chan}(X_1, Y_1) \times \mathbf{Chan}(X_2, Y_2)$. The operation of placing channel \mathcal{N}_1 between Alice and Ray, and channel \mathcal{N}_2 between Ray and Bob is described by the *sequential placement supermap* $\mathcal{S}_{\text{seq}}^{A, R, R', B}$ defined by

$$\mathcal{S}_{\text{seq}}^{A, R, R', B}(\mathcal{N}_1, \mathcal{N}_2) := \mathcal{S}_{\text{place}}^{A, R}(\mathcal{N}_1) \otimes \mathcal{S}_{\text{place}}^{R', B}(\mathcal{N}_2), \quad (5)$$

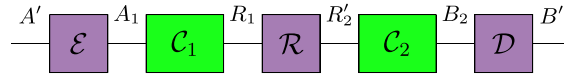


Figure 3. Encoding–repeater–decoding supermap $\mathcal{S}_{\mathcal{E},\mathcal{R},\mathcal{D}}(\mathcal{C}_1 \otimes \mathcal{C}_2) := \mathcal{D} \circ \mathcal{C}_2 \circ \mathcal{R} \circ \mathcal{C}_1 \circ \mathcal{E}$. Party supermaps (i.e. supermaps from placed channels to placed channels) are drawn in violet.

where system $A \simeq X_1$ is placed in Alice’s laboratory, systems $R \simeq Y_1$ and $R' \simeq X_2$ are placed in Ray’s laboratory, and system $B \simeq Y_2$ is placed in Bob’s laboratory.

Note that the sequential placement (5) is formally identical to the parallel placement (4): in both cases, the placement of multiple channels is the tensor product of the placement of individual channels. The difference between parallel and sequential placement arises from the different spacetime locations in which the inputs and outputs of the channels are placed. In the parallel placement, all the input systems \mathbf{A} are at the sender’s location, and all the output systems \mathbf{B} are at the receiver’s location. In the sequential placement, the systems A, R, R', B appear in a strict sequential order: A before R , R before R' , R' before B . This difference is crucial when it comes to specifying how the output of the placement supermap is to be used: in the case of parallel placement, the output of the supermap can be connected with local operations at the sender’s and receiver’s ends. In the case of sequential placement, intermediate operations are possible.

The difference between sequential and parallel placements is reflected by the different type of channels they generate. The output of the sequential placement supermap (5) is a *two-step quantum process*, where the first step represents the transfer of information from A to R , and the second step corresponds to the transfer of information from R' to B . Mathematically, a two-step process \mathcal{C} transforming system S_1 into system S'_1 in the first step, and system S_2 into system S'_2 in the second step, is described by a quantum channel $\mathcal{C} \in \mathbf{Chan}(S_1 \otimes S_2, S'_1 \otimes S'_2)$ satisfying the condition [43–45]

$$\mathrm{Tr}_{S'_2}[\mathcal{C}(\rho)] = \mathrm{Tr}_{S'_2} \left[\mathcal{C} \left(\mathrm{Tr}_{S_2}[\rho] \otimes \frac{I_{S_2}}{d_{S_2}} \right) \right] \forall \rho \in \mathbf{St}(S_1 \otimes S_2), \quad (6)$$

where Tr_S , I_S , and d_S denote the trace over \mathcal{H}_S , the identity on \mathcal{H}_S , and the dimension of \mathcal{H}_S , respectively. The difference between a two-step process and a generic bipartite channel is that the two-step process has to satisfy the additional condition (6), which ensures compatibility with the causal ordering of the systems S_1, S'_1, S_2 , and S'_2 .

Two-step quantum processes are known in the literature as quantum combs [39, 45], quantum memory channels [46, 47], and non-Markovian quantum processes [48]. Following references [39, 45], we will refer to two-step quantum processes as *quantum 2-combs*, and we will denote the corresponding set as $\mathbf{Comb}[(S_1, S'_1), (S_2, S'_2)]$.

The sequential placement supermap (5) transforms a pair of unplaced channels $(\mathcal{N}_1, \mathcal{N}_2)$ into a placed 2-comb $\mathcal{C}_1 \otimes \mathcal{C}_2$, with $\mathcal{C}_1 := \mathcal{S}_{\mathrm{place}}^{A,R}(\mathcal{N}_1)$ and $\mathcal{C}_2 := \mathcal{S}_{\mathrm{place}}^{R',B}(\mathcal{N}_2)$. Note that, in general, the set of 2-combs also contains maps that are not of the product form $\mathcal{C}_1 \otimes \mathcal{C}_2$. These maps correspond to two-step processes where a memory is passed from the first step to the second.

Once the devices have been placed, the sender, repeater, and receiver can connect them with their local devices, thus establishing a single channel that transfers information directly from the sender to the receiver. The most general supermaps from quantum combs to quantum channels have been characterised in reference [39]. Their action on product combs $\mathcal{C}_1 \otimes \mathcal{C}_2$ is given by

$$\mathcal{S}(\mathcal{C}_1 \otimes \mathcal{C}_2) = \mathcal{D} \circ (\mathcal{C}_2 \otimes \mathcal{I}_{\mathrm{Aux}_2}) \circ \mathcal{R} \circ (\mathcal{C}_1 \otimes \mathcal{I}_{\mathrm{Aux}_1}) \circ \mathcal{E}, \quad (7)$$

where Aux_1 and Aux_2 are auxiliary systems, and \mathcal{E} , \mathcal{R} , and \mathcal{D} are arbitrary channels in $\mathbf{Chan}(A', A \otimes \mathrm{Aux}_1)$, $\mathbf{Chan}(R \otimes \mathrm{Aux}_1, R' \otimes \mathrm{Aux}_2)$, and $\mathbf{Chan}(B \otimes \mathrm{Aux}_2, B')$, respectively.

The standard choice of free supermaps is the supermaps that are achievable without the auxiliary systems Aux_1 and Aux_2 , that is, the supermaps of the form

$$\mathcal{S}_{\mathcal{E},\mathcal{R},\mathcal{D}}(\mathcal{C}_1 \otimes \mathcal{C}_2) := \mathcal{D} \circ \mathcal{C}_2 \circ \mathcal{R} \circ \mathcal{C}_1 \circ \mathcal{E}, \quad (8)$$

illustrated in figure 3.

Communication through a network of $k \geq 2$ devices is described by a direct generalisation of the above example. Consider the situation where a sender communicates to a receiver with the assistance of $k - 1$ intermediate repeaters. The communication devices are described by a list of unplaced channels $(\mathcal{N}_1, \dots, \mathcal{N}_k) \in \mathbf{Chan}(X_1, Y_1) \times \mathbf{Chan}(X_2, Y_2) \times \dots \times \mathbf{Chan}(X_k, Y_k)$. The placement of the devices

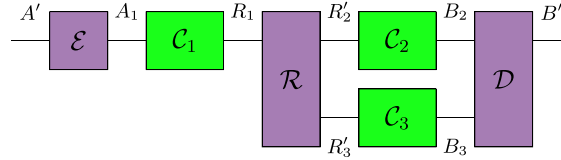


Figure 4. A composite supermap. The figure shows the placement of $k = 3$ channels between a sender, a single repeater ($r = 1$), and a receiver. The placement is then followed by an encoding–repeater–decoding supermap (in violet), representing the local operations performed by sender, repeater, and receiver.

between the sender, repeaters, and receiver is described by the supermap

$$\mathcal{S}_{\text{seq}}^{A, R_1, R'_1, \dots, R_{k-1}, R'_{k-1}, B}(\mathcal{N}_1, \dots, \mathcal{N}_k) := \mathcal{S}_{\text{place}}^{A, R_1}(\mathcal{N}_1) \otimes \mathcal{S}_{\text{place}}^{R'_1, R_2}(\mathcal{N}_2) \otimes \dots \otimes \mathcal{S}_{\text{place}}^{R'_{k-1}, B}(\mathcal{N}_k), \quad (9)$$

where system $A \simeq X_1$ is placed in the sender's laboratory, system $B \simeq Y_k$ is placed in the receiver's laboratory, and systems $R_i \simeq Y_i$ and $R'_i \simeq X_{i+1}$ are placed in the laboratory of the i th repeater, for $i \in \{1, \dots, k-1\}$.

The output of the supermap $\mathcal{S}_{\text{seq}}^{A, R_1, R'_1, \dots, R_{k-1}, R'_{k-1}, B}$ is a k -step quantum processes [46], also known as a *quantum k -comb* [39]. A quantum k -comb transforming system S_i into system S'_i at the i th step is a quantum channel $\mathcal{C} \in \mathbf{Chan}(S_1 \otimes \dots \otimes S_k, S'_1 \otimes \dots \otimes S'_k)$ satisfying a generalisation of condition (6) to k steps (see appendix B.1 for the precise definition). The set of quantum k -combs with the above input/output systems will be denoted by $\mathbf{Comb}[(S_1, S'_1), \dots, (S_k, S'_k)]$.

Once the available devices have been placed, the communicating parties can connect their local devices to the placed communication channels. The corresponding supermap has the form

$$\mathcal{S}_{\mathcal{E}, \mathcal{R}_1, \dots, \mathcal{R}_{k-1}, \mathcal{D}}(\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_k) := \mathcal{D} \circ \mathcal{C}_k \circ \mathcal{R}_{k-1} \circ \mathcal{C}_{k-1} \circ \dots \circ \mathcal{R}_1 \circ \mathcal{C}_1 \circ \mathcal{E}, \quad (10)$$

where $\mathcal{E} \in \mathbf{Chan}(A', A)$ is the encoding operation performed by the sender, $\mathcal{R}_i \in \mathbf{Chan}(R_i, R'_i)$ is the repeater operation performed by the i th intermediate party, and $\mathcal{D} \in \mathbf{Chan}(B, B')$ is the decoding operation performed by the receiver.

More generally, one can consider any placement of $k \geq 2$ devices with $r \leq k-1$ intermediate repeaters. This includes placing some channels in parallel between two subsequent parties, in which case the placed channel is a quantum $(r+1)$ -comb. An example of this situation is illustrated in figure 4. The most general placement supermaps corresponding to a definite causal structure of communicating parties are described in appendix C.

2.6. Terminology

In the rest of the paper, the study of communication protocols involving only parallel placement between a sender and a receiver will be called *standard quantum Shannon theory for direct communication*. The study of communication protocols involving both parallel and sequential placements between a sender, a receiver, and intermediate parties will be called *standard quantum Shannon theory for network communication*, or simply, *standard quantum Shannon theory*. We will not consider assisted scenarios, such as entanglement-assisted communication, which can nevertheless be incorporated in our framework as discussed in appendix A.

3. General resource theories of communication

Here we extend the framework of standard quantum Shannon theory to general resource theories of communication, arguing that any such theory must not include operations that enable communication independently of the communication devices initially available to the communicating parties.

3.1. Basic structure

The resource-theoretic formulation of standard quantum Shannon theory, discussed in the previous section, suggests a general scheme for constructing new resource theories of communication. The basic scheme is as follows:

- (i) One use of each communication device is described by an *unplaced quantum channel*, specifying how a given system type is transformed into another system type, but without assigning these system types to specific locations in spacetime.

- (ii) The (uses of the) available communication devices are described by a *list of unplaced quantum channels*.
- (iii) The sender, receiver, and possibly a set of intermediate parties are assigned spacetime regions, whose causal structure specifies who can send messages to whom. The physical systems accessed by the communicating parties are *placed systems*, that is, systems assigned specific locations in spacetime.
- (iv) The placement of the communication devices in between the communicating parties is described by a *placement supermap*, that is, a supermap transforming lists of unplaced quantum channels into placed quantum channels. A placed quantum channel has placed systems as inputs and outputs, and can in general be a multistep process, represented by a quantum comb.
- (v) The operations performed by the sender, receiver, and intermediate parties are described by a *party supermap*, that is, a supermap on the set of placed quantum channels.

In the above scheme, a resource theory of communication is formulated by specifying which operations are considered as ‘free’ in points (iv) and (v) above.

Free operations on placed channels (party supermaps) are interpreted as being implemented by the sender, the receiver, or intermediate parties. Free operations from unplaced to placed channels (placement supermaps) are interpreted as being performed by an external agent, e.g. a communication provider, or nature itself. This is consistent with the intuitive idea that a communication infrastructure has to be set up before communication takes place. Overall, a resource theory of communication describes the actions performed by the communicating parties and by an external agent that places the communication devices between them.

In principle, one could also consider a third type of operations, from unplaced channels to unplaced channels. These operations would be performed by the third party *before the channels are placed between the sender and receiver*. For example, the third party could decide to discard one of the devices in the list $(\mathcal{N}_1, \dots, \mathcal{N}_k)$, and use only the remaining devices. For completeness, we will include the possibility of these ‘pre-placement operations’ in our general scheme.

3.2. Resource theories of communication

For a resource theory of communication, the broader set of operations \mathbf{M} from which the free operations \mathbf{M}_{free} are chosen consists of (1) supermaps from unplaced channels to unplaced channels, (2) supermaps from unplaced channels to placed channels, and (3) supermaps from placed channels to placed channels. The mathematical classification of these three types of admissible supermaps is given in appendix B.2.

A resource theory of communication is then specified by fixing the set of free operations:

Definition 1 (Resource theory of communication). A resource theory of communication is specified by a set of free supermaps $\mathbf{M}_{\text{free}} \subset \mathbf{M}$, closed under sequential and parallel composition, containing (1) free supermaps from unplaced channels to unplaced channels, called *pre-placement supermaps*, (2) free supermaps from unplaced channels to placed channels, called *placement supermaps*, and (3) free supermaps from placed channels to placed channels, called *party supermaps*.

In pictures, we represent the placement supermaps by blue boxes, and the party supermaps by violet boxes.

Mathematically, the different channel types are objects in a symmetric monoidal category, and the free operations \mathbf{M}_{free} correspond to the morphisms between them. This scheme matches the general framework of Coecke, Fritz and Spekkens [29].

The set \mathbf{M}_{free} can be specified by a generating set of operations [29]. For example, standard quantum Shannon theory is the resource theory of communication where the free operations $\mathbf{M}_{\text{standard}}$ are generated from the following types of free operations:

- (i) *Basic placement*: for a single channel $\mathcal{N} \in \mathbf{Chan}(X, Y)$, the map

$$\mathcal{S}_{\text{place}}^{A,B}(\mathcal{N}) := \mathcal{W}^B \circ \mathcal{N} \circ \mathcal{V}^A,$$

where $\mathcal{V} \in \mathbf{Chan}(A, X)$ [$\mathcal{W} \in \mathbf{Chan}(Y, B)$] is the unitary channel implementing the isomorphism between the unplaced system X (Y) and the placed system A (B).

- (ii) *Insertion of local devices*: for l placed channels $\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_l \in \mathbf{Comb}[(A, R_1), (R'_1, R_2), \dots, (R'_{l-1}, B)]$, the *encoding map*

$$\mathcal{S}_{\mathcal{E}}(\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_l) := (\mathcal{C}_1 \circ \mathcal{E}) \otimes \mathcal{C}_2 \otimes \dots \otimes \mathcal{C}_l, \quad (11)$$

the *repeater map*

$$\mathcal{S}_{\mathcal{R}_m}(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_l) := \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_{m-1} \otimes (\mathcal{C}_{m+1} \circ \mathcal{R}_m \circ \mathcal{C}_m) \otimes \mathcal{C}_{m+2} \otimes \cdots \otimes \mathcal{C}_l, \quad (12)$$

and the *decoding map*

$$\mathcal{S}_{\mathcal{D}}(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_l) := \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_{l-1} \otimes (\mathcal{D} \circ \mathcal{C}_l), \quad (13)$$

where $\mathcal{E} \in \mathbf{Chan}(A', A)$, $\mathcal{R}_m \in \mathbf{Chan}(R_m, R'_m)$, and $\mathcal{D} \in \mathbf{Chan}(B, B')$ are quantum channels representing local devices at the sender's, m th repeater's, and receiver's end, respectively.

Note that we omitted pre-placement supermaps, because the set of such supermaps is trivial in standard quantum Shannon theory.

The other supermaps shown earlier in section 2 can be decomposed into the basic supermaps (i), (ii). For example, the parallel placement (4) and sequential placement (9) are just the product of basic placement supermaps (i), which place individual channels in the appropriate configuration. Similarly, the encoding–decoding supermap (3) and the encoding–repeater–decoding–supermap (8) are just the result of multiple insertions of local devices (ii).

3.3. Generalised channel capacities

In standard quantum Shannon theory, the classical (quantum) capacity of a quantum channel \mathcal{N} is defined as the maximum number bits (qubits) that can be transmitted over n parallel uses of \mathcal{N} , per channel use and with vanishing error in the asymptotic limit $n \rightarrow \infty$. This is equivalent to the maximum number of classical (quantum) identity channels $\mathcal{I}_{\text{clas}}$ (\mathcal{I}) that the n parallel uses of \mathcal{N} can simulate, per channel use and with vanishing error in the asymptotic limit $n \rightarrow \infty$, using arbitrary encoding/decoding channels [8] (the classical identity channel $\mathcal{I}_{\text{clas}}$ being defined as the perfect dephasing channel with respect to a given orthonormal basis).

The standard definition of classical (quantum) capacity is appropriate for placed channels, which have already been arranged in between the sender and receiver, and therefore can only be used in parallel. However, unplaced channels could be arranged in more general configurations, generating a broader class of communication protocols.

In a general resource theory of communication, we define the *generalised classical (quantum) capacity* of \mathcal{N} as the maximum number of classical (quantum) identity channels $\mathcal{I}_{\text{clas}}$ (\mathcal{I}) that can be generated by performing free operations of \mathbf{M}_{free} , per channel use and with vanishing error in the asymptotic limit of $n \rightarrow \infty$. Other types of generalised capacities can be defined similarly, with respect to some given ideal reference channel.

The generalised capacity is (trivially) a resource monotone [29, 30], meaning that it cannot be increased by applying free operations. Moreover, the generalised capacity increases (or stays the same) whenever the set of free operations is enlarged. Examples of this situation are the capacity enhancements observed in the presence of quantum control over the causal orders [9–14]: in these protocols, the set of placements of standard quantum Shannon theory is enlarged to include placements in a superposition of alternative orders, and consequently various channel capacities have been shown to increase.

3.4. A minimal requirement for any resource theory of communication

Formally, every set of free supermaps defines a resource theory of communication. However, such a resource theory may not be a meaningful one. We argue that every meaningful resource theory of communication should at least satisfy a minimal requirement: the free operations should not allow the sender and receiver to communicate independently of the communication devices from which their communication protocol is built.

To illustrate this idea, consider the situation where two parties, Alice and Bob, communicate through a noisy telephone line. In the standard theory of communication, the key question is how to use this communication resource to transmit information reliably. Now, if Alice were to walk into Bob's room, he would clearly be able to hear her through the air, but this would not be a new way to use the telephone line. Rather, it would be a way to bypass it. The air would act as a side-channel, allowing Alice and Bob to communicate to each other independently of how good or how bad their telephone line is.

The telephone line example has the following structure. Initially, Alice and Bob have access to a noisy communication channel $\mathcal{N} \in \mathbf{Chan}(A, B)$. The operation of Alice moving into Bob's lab can be modelled

as a *side-channel supermap*

$$\begin{aligned} \mathcal{S}_{\text{side}}^{(E,E')}: \text{Chan}(A, B) &\rightarrow \text{Chan}(A \otimes E, B \otimes E') \\ \mathcal{S}(\mathcal{N}) : \mathcal{N} &\mapsto \mathcal{N} \otimes \mathcal{I}_{E,E'}, \end{aligned} \quad (14)$$

which juxtaposes the noisy channel \mathcal{N} with a side-channel $\mathcal{I}_{E,E'} \in \text{Chan}(E, E')$ acting on some additional systems E and E' (the air in the proximity of Alice and Bob, respectively). If the channel $\mathcal{I}_{E,E'}$ is ideal, then the supermap $\mathcal{S}_{\text{side}}^{(E,E')}$ would let Alice communicate perfectly to Bob. This communication ‘enhancement’, however, is independent of the original channel \mathcal{N} . Every operation of the form (14) trivialises the notion of communication enhancement, and therefore should not be allowed in a resource theory of communication.

Building on the above example, we now propose a general notion of a side-channel generating operation:

Definition 2 (Side-channel generating operations). A supermap $\mathcal{S} \in \mathbf{M}$ generates a classical (quantum) side-channel if there exist two free supermaps $\mathcal{S}_1 \in \mathbf{M}_{\text{free}}$ and $\mathcal{S}_2 \in \mathbf{M}_{\text{free}}$ such that, for all choices of input channels $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ for supermap \mathcal{S}_1 , one has

$$(\mathcal{S}_2 \circ \mathcal{S} \circ \mathcal{S}_1)(\mathcal{N}_1, \dots, \mathcal{N}_k) = \mathcal{C}, \quad (15)$$

where \mathcal{C} is a placed quantum channel with non-zero classical (quantum) capacity.

The above definition captures the idea that the supermap \mathcal{S} can be used to construct a communication protocol that works independently of the communication devices originally available to the communicating parties. In the telephone line example, the channel \mathcal{C} is the ideal channel $\mathcal{I}_{E,E'}$ describing the transmission of a message through the air between Alice and Bob.

We demand that any sensible resource theory of communication should forbid side-channel generating operations:

Condition 1 (No side-channel generation). In a resource theory of classical (quantum) communication, no free operation $\mathcal{S} \in \mathbf{M}_{\text{free}}$ should generate a classical (quantum) side-channel.

We stress that condition 1 is a *minimal* requirement, and that, in particular cases, one may want to impose even stronger conditions on the allowed operations. In other words, we are not claiming that every resource theory of communication satisfying condition 1 is an interesting one. Rather, condition 1 is a bottom line that has to be satisfied when defining new resource theories of communication.

It is immediate to verify that standard quantum Shannon theory satisfies condition 1. In the following, we will show that:

- (i) Quantum Shannon theory with superpositions of causal orders satisfies condition 1
- (ii) Quantum Shannon theory with superpositions of trajectories satisfies condition 1
- (iii) Quantum Shannon theory with superpositions of encoding and decoding operations violates condition 1.

In appendix D, we comment on the difference between our framework and the frameworks of references [32–34], discussing an alternative to condition 1, where the free supermaps are required to transform constant channels into constant channels. In appendix E we discuss the difference between our definition of side-channels and another notion of side-channels proposed in reference [28], assessing some of the claims made therein.

4. Superposition of orders and superposition of trajectories

Here we formulate the resource theories of quantum Shannon theory with superpositions of causal orders and quantum Shannon theory with superpositions of trajectories, and we show that both theories satisfy the requirement of no side-channel generation.

4.1. Quantum Shannon theory with superpositions of causal orders

The information-theoretic advantages of indefinite causal order in quantum computation were envisaged by Hardy [49], and fleshed out a few years later with the introduction of the quantum SWITCH [22, 23], a higher-order operation that places two quantum devices in a superposition of two alternative causal orders. Since then, information-processing advantages of the quantum SWITCH have been found in a variety of contexts, including quantum query complexity [50, 51], quantum communication complexity [52], and

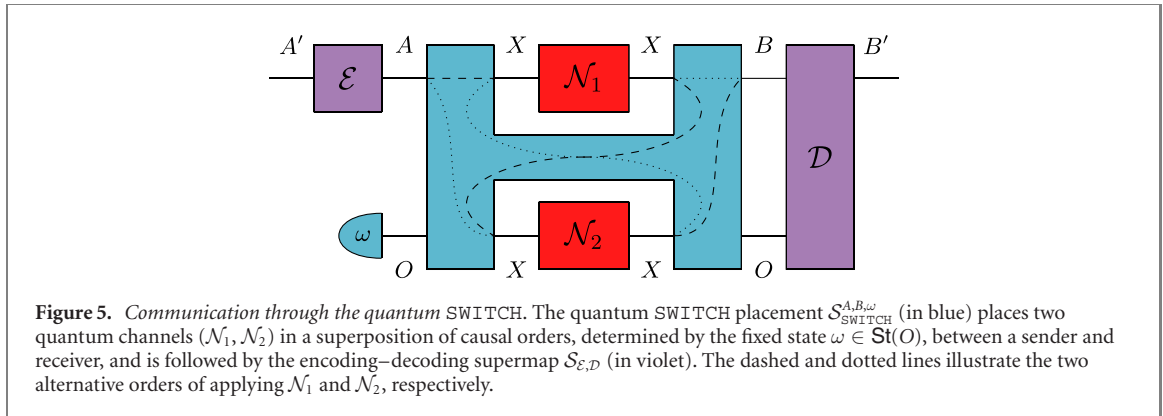


Figure 5. Communication through the quantum SWITCH. The quantum SWITCH placement $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}$ (in blue) places two quantum channels ($\mathcal{N}_1, \mathcal{N}_2$) in a superposition of causal orders, determined by the fixed state $\omega \in \text{St}(O)$, between a sender and receiver, and is followed by the encoding–decoding supermap $\mathcal{S}_{\mathcal{E},\mathcal{D}}$ (in violet). The dashed and dotted lines illustrate the two alternative orders of applying \mathcal{N}_1 and \mathcal{N}_2 , respectively.

quantum metrology [53]. Other forms of indefinite causal order, and their advantages in non-local games, have been demonstrated in reference [54]. In all the above works, the combination of quantum devices in an indefinite causal order was shown to offer performances that cannot be matched by any quantum protocol that uses the input devices in a definite order.

A different category of advantages arises in the context of quantum communication [9–14, 16, 21]. Here, protocols that combine communication channels through the quantum SWITCH have been shown to offer advantages with respect to the protocols allowed in standard quantum Shannon theory, as defined earlier in this paper. These advantages are *not* advantages with respect to all possible protocols with definite causal order. They cannot be so, because the set of all protocols with definite causal order includes also trivial protocols where the original communication channels are juxtaposed with noiseless channels, as in the telephone line example of equation (14).

The proper way to interpret the communication advantages shown in references [9–14, 16, 21] is to regard them as a comparison between two different resource theories of communication: standard quantum Shannon theory, and an extended resource theory that includes the quantum SWITCH among its placements.

Here we explicitly define such a resource theory, which we call quantum Shannon theory with superpositions of causal orders (SCO). The corresponding set of free operations will be denoted by \mathbf{M}_{SCO} . The generating free operations are operations (i), (ii) of standard quantum Shannon theory, plus an additional placement supermap, based on the quantum SWITCH:

- (iii) The quantum SWITCH placement $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}$ maps a pair of unplaced quantum channels $(\mathcal{N}_1, \mathcal{N}_2) \in \text{Chan}(X) \times \text{Chan}(X)$ into a placed quantum channel $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2) \in \text{Chan}(A, B \otimes O)$, where $A \simeq X$ ($B \simeq X$) is a quantum system placed at the sender’s (receiver’s) end, and O is a qubit system, called the *order qubit*, placed at the receiver’s end. Explicitly, the quantum channel $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)$ is defined as

$$\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)(\rho) = \sum_{ij} S_{ij}(\rho \otimes \omega) S_{ij}^\dagger, \quad (16)$$

where $\omega \in \text{St}(O)$ is a state of the order qubit, and

$$S_{ij} := N_i^{(2)} N_j^{(1)} \otimes |0\rangle\langle 0| + N_j^{(1)} N_i^{(2)} \otimes |1\rangle\langle 1|, \quad (17)$$

$\{|0\rangle, |1\rangle\}$ being an orthonormal basis for the order qubit. The quantum channel $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)$ is independent of the Kraus decomposition of the channels \mathcal{N}_1 and \mathcal{N}_2 .

A communication protocol using the quantum SWITCH placement is given in figure 5. Note that the initial state of the order qubit is fixed as part of the placement, and is thus inaccessible to the sender [9, 10].

We stress that the quantum SWITCH placement should be understood here as an abstract supermap from two quantum channels to a new quantum channel. Whether this supermap can be physically realised, and how it can be realised, is entirely another matter. Various ways to reproduce the action of the quantum SWITCH have been proposed, using conventional physics [12, 26, 55–58], closed timelike curves [23], or quantum gravity scenarios [59, 60]. However, the resource theory \mathbf{M}_{SCO} should be considered as the abstract resource theory associated with the quantum SWITCH transformation, without reference to a specific physical implementation.

The motivation for including the quantum SWITCH among the free operations is to understand how the world *could* be, if quantum devices could be combined in a superposition of alternative orders. The

study of quantum Shannon theory with the addition of the quantum SWITCH is similar in spirit the study of information tasks assisted by the Popescu–Rohrlich box [61], a fictional device that generates stronger than quantum correlations. Like the Popescu–Rohrlich box, the quantum SWITCH serves as a conceptual device, used to better understand standard quantum theory by comparing it to possible alternatives.

4.2. Quantum Shannon theory with superpositions of trajectories

The superposition of alternative evolutions was defined in references [24, 25], and applied to quantum communication in references [17, 27], where the ability to send quantum particles along a superposition of different trajectories provided the working principle for a new technique called error filtration. Shannon-theoretic advantages of the superposition of trajectories were demonstrated more recently in references [18–20].

Here, we formulate the resource theory of quantum Shannon theory with superpositions of trajectories (ST) [19]. The set of free operations in this resource theory, denoted by \mathbf{M}_{ST} , is generated by the standard free operations (i)–(ii), with the addition of a *superposition placement* (iii)*, which creates a superposition of two alternative communication channels.

In order to define the superposition placement, we need to revise the way in which the communication hardware is modelled. Normally, a quantum communication channel $\mathcal{N} \in \mathbf{Chan}(X)$ describes the action of a communication device *when a system is transmitted*. However, the communication device also exists when no system is sent through it. The action of the device in the lack of an input can be modelled by introducing a vacuum state, which can be sent to the device in alternative to states of system X . Hence, the overall action of the communication device is described not by the original channel $\mathcal{N} \in \mathbf{Chan}(X)$, but by another channel $\tilde{\mathcal{N}}$ that acts as \mathcal{N} when the input is restricted to X , and as the identity transformation \mathcal{I}_{vac} when the input is in the vacuum state.

The channel $\tilde{\mathcal{N}}$ is called a *vacuum extension* of the quantum channel \mathcal{N} [19]. Mathematically, $\tilde{\mathcal{N}}$ is an element of $\mathbf{Chan}(\tilde{X})$, where $\tilde{X} := X \oplus \text{vac}$ is the quantum system with Hilbert space $\mathcal{H}_{\tilde{X}} := \mathcal{H}_X \oplus \mathcal{H}_{\text{vac}}$, \mathcal{H}_{vac} being the Hilbert space of the vacuum. Note that, in general, a density matrix of system \tilde{X} can also have off-diagonal elements of the form $|\psi\rangle\langle\text{vac}|$, with $|\psi\rangle \in \mathcal{H}_X$ and $|\text{vac}\rangle \in \mathcal{H}_{\text{vac}}$, corresponding to the presence of quantum coherence between system X and the vacuum.

In the following, we will assume for simplicity that the vacuum Hilbert space is one-dimensional, meaning that there exists a unique vacuum state $|\text{vac}\rangle$, up to global phases. With this assumption, the conditions for a channel $\tilde{\mathcal{N}} \in \mathbf{Chan}(\tilde{X})$ to be a vacuum extension of channel $\mathcal{N} \in \mathbf{Chan}(X)$ are

$$\tilde{\mathcal{N}}(|\text{vac}\rangle\langle\text{vac}|) = |\text{vac}\rangle\langle\text{vac}|, \quad (18)$$

and

$$\tilde{\mathcal{N}}(P_X \rho P_X) = \mathcal{N}(P_X \rho P_X) \quad \forall \rho \in \mathbf{St}(\tilde{X}), \quad (19)$$

where $P_X := I - |\text{vac}\rangle\langle\text{vac}|$ is the projector on the subspace corresponding to system X .

Conditions (18) and (19) imply that the Kraus operators of channel $\tilde{\mathcal{N}}$ are of the form

$$\tilde{N}_i = N_i \oplus \nu_i |\text{vac}\rangle\langle\text{vac}|, \quad (20)$$

where $\{N_i\}$ are Kraus operators for \mathcal{N} , and $\{\nu_i\}$ are complex numbers satisfying the condition $\sum_i |\nu_i|^2 = 1$. In the following, the numbers $\{\nu_i\}$ will be called the *vacuum amplitudes* of channel $\tilde{\mathcal{N}}$.

The action of channel $\tilde{\mathcal{N}}$ on a generic quantum state $\rho \in \mathbf{St}(\tilde{X})$ is

$$\tilde{\mathcal{N}}(\rho) = \mathcal{N}(P_X \rho P_X) + \langle\text{vac}|\rho|\text{vac}\rangle |\text{vac}\rangle\langle\text{vac}| + F \rho |\text{vac}\rangle\langle\text{vac}| + |\text{vac}\rangle\langle\text{vac}| \rho F^\dagger, \quad (21)$$

where the operator

$$F := \sum_i \bar{\nu}_i N_i, \quad (22)$$

is called the *vacuum interference operator* [19]. Note that the operator F depends only on the channel $\tilde{\mathcal{N}}$, and not on the choice of Kraus operators, as one can see by comparing the two sides of equation (21).

If the vacuum interference operator is zero, then the output state (21) is an incoherent mixture of a state of system X and the vacuum.

Definition 3. For $F = 0$, we say that the vacuum extension $\tilde{\mathcal{N}}$ has *no coherence with the vacuum*, and we call it the *incoherent vacuum extension* of channel \mathcal{N} . For $F \neq 0$, we say that the vacuum extension $\tilde{\mathcal{N}}$ has *coherence with the vacuum*.

Mathematically, the vacuum extension of a given quantum channel is highly non-unique: every channel has infinitely many vacuum extensions [18, 19]. Physically, the choice of vacuum extension is part of the specification of the communication device, and can be determined through process tomography [25].

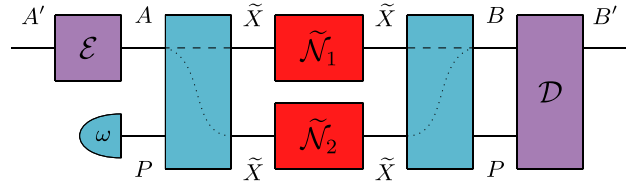


Figure 6. Communication through a superposition of quantum channels. The supermap $\mathcal{S}_{\text{sup}}^{A,B,\omega}$ (in blue) places two vacuum-extended channels $(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2)$ on two alternative paths, and lets the transmitted system travel along both paths (dashed line and dotted line, respectively) in a quantum superposition, determined by the state $\omega \in \text{St}(P)$. The resulting channel then undergoes the encoding–decoding supermap $\mathcal{E}_{\mathcal{E},\mathcal{D}}$ (in violet), describing the local operations performed at the sender’s and receiver’s ends.

Vacuum-extended channels represent communication devices that can act on the information carrier, or on the vacuum, or on any coherent superposition of the two. Using this feature, it is possible to coherently control the choice of channel through which the information carrier is sent. The result can be interpreted as a placement of the given different channels in a superposition of being on the path of the information carrier:

- (iii*) The *superposition placement* $\mathcal{S}_{\text{sup}}^{A,B,\omega}$ maps a pair of unplaced vacuum-extended channels $(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2) \in \text{Chan}(\tilde{X}) \times \text{Chan}(\tilde{X})$ into a placed quantum channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2) \in \text{Chan}(A, B \otimes P)$, where $A \simeq X$ ($B \simeq X$) is a quantum system placed at the sender’s (receiver’s) end, and P is a qubit system, called the *path qubit*, placed at the receiver’s end. Explicitly, the quantum channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2)$ is defined as

$$\begin{aligned} \mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2)(\rho) = & \langle 1|\omega\rangle\langle 1| \mathcal{N}_1(\rho) \otimes |1\rangle\langle 1| + \langle 2|\omega\rangle\langle 2| \mathcal{N}_2(\rho) \otimes |2\rangle\langle 2| \\ & + \langle 1|\omega\rangle\langle 2| F_1 \rho F_2^\dagger \otimes |1\rangle\langle 2| + \langle 2|\omega\rangle\langle 1| F_2 \rho F_1^\dagger \otimes |2\rangle\langle 1|, \end{aligned} \quad (23)$$

where $\omega \in \text{St}(P)$ is a state of the path qubit, $\{|1\rangle, |2\rangle\}$ is an orthonormal basis for the path qubit, and F_1 and F_2 are the vacuum interference operators associated to channels $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$, respectively.

An example of a communication protocol using the superposition placement is shown in figure 6. The superposition placement is physically implementable in photonic systems, making the resource theory \mathbf{M}_{ST} interesting both from a purely information-theoretic point of view as well as a practical point of view.

For simplicity of presentation, here we considered only superpositions of two channels, both for the superposition of trajectories and for the superposition of orders. Both the superposition placement (i*) and the quantum SWITCH placement (i) can be straightforwardly generalised to k channels. The corresponding definitions can be found in references [19, 62], respectively.

4.3. Superpositions of causal orders and superpositions of trajectories do not generate side-channels

We now show that the supermaps (iii) or (iii*), combined with (i) and (ii), do not generate side-channels.

Proposition 1. *No supermap composed from the quantum SWITCH placement (iii), basic placement (i), and insertion of local devices (ii) generates side-channels.*

Proof. The supermaps (i) and (ii) of standard quantum Shannon theory do not generate side channels. Hence, it is sufficient to prove that the quantum SWITCH does not generate side-channels.

This is done by finding a choice of adversarial channels \mathcal{N}_1 and \mathcal{N}_2 such that $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)$ is a channel with zero classical capacity. One such choice is to pick \mathcal{N}_1 to be the identity channel \mathcal{I} (with a single Kraus operator $N_i^{(1)} = I$, $i = 1$) and \mathcal{N}_2 to be the constant channel $\mathcal{N}_2(\rho) = |\psi_0\rangle\langle\psi_0|$ (with Kraus operators $N_j^{(2)} = |\psi_0\rangle\langle j|$, for some orthonormal basis $\{|j\rangle\}$). With this choice, the Kraus operators (17) in the definition of the channel $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)$ are

$$S_{ij} = |\psi_0\rangle\langle j| \otimes I, \quad (24)$$

and therefore one has

$$\begin{aligned} \mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)(\rho) &:= \sum_{ij} S_{ij}(\rho \otimes \omega) S_{ij}^\dagger \\ &= |\psi_0\rangle\langle\psi_0| \otimes \omega \quad \forall \rho \in \text{St}(A). \end{aligned} \quad (25)$$

Since the output of the channel $\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_1, \mathcal{N}_2)$ is independent of its input, the channel has zero capacity (both classical and quantum), and no combination of it with the other supermaps (i) and (ii) can generate a channel with non-zero capacity. \square

Proposition 2. *No supermap composed from the superposition placement (iii*), basic placement (i), and insertion of local devices (ii) generates side-channels.*

Proof. As in the proof of proposition 1, it is sufficient to prove that the superposition placement (a) does not generate side-channels. This is done by finding a choice of adversarial vacuum-extended channels $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$ such that $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2)$ is a channel with zero classical capacity. One such choice is to pick the vacuum-extended channels $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$ defined by

$$\tilde{\mathcal{N}}_1(\rho) = \tilde{\mathcal{N}}_2(\rho) = \rho_0 \text{Tr}[\rho(I - |\text{vac}\rangle\langle\text{vac}|)] + |\text{vac}\rangle\langle\text{vac}| \rho |\text{vac}\rangle\langle\text{vac}| \quad \forall \rho \in \text{St}(\tilde{X}). \quad (26)$$

In other words, $\tilde{\mathcal{N}}_1 = \tilde{\mathcal{N}}_2$ is the incoherent vacuum-extension of the constant channel that maps every state into the fixed state ρ_0 . For the vacuum-extended channels $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$, the vacuum interference operators are $F_1 = F_2 = 0$, and the superposition placement then yields the channel

$$\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2)(\rho) = \rho_0 \otimes \text{diag}(\omega), \quad (27)$$

with $\text{diag}(\omega) := \langle 1|\omega|1\rangle|1\rangle\langle 1| + \langle 2|\omega|2\rangle|2\rangle\langle 2|$, as one can verify from equation (23). Since the channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_1, \tilde{\mathcal{N}}_2)$ is constant, it has zero (classical and quantum) capacity.

Propositions 1 and 2 show that both quantum Shannon theory with superpositions of causal orders and quantum Shannon theory with superpositions of trajectories satisfy the requirement of no side-channel generation, as stated in condition 1. \square

5. Reply to Abbott *et al*

In reference [18], Abbott *et al* give an example of a communication protocol where two completely depolarising channels are coherently superposed. The authors quantify the transmission of information in terms of the Holevo information (a lower bound for the classical capacity [8]), and show that the Holevo information achievable by superposing the two channels is greater than the Holevo information achievable by putting them in the quantum SWITCH.

This observation is presented as a comparison between two alternative ways to turn two depolarising channels into a new quantum channel with non-zero capacity. Based on this comparison, the authors argue that the communication advantages of the quantum SWITCH ‘*should therefore rather be understood as resulting from coherent control of quantum communication channels*’, as opposed to being specifically due to indefinite causal order.

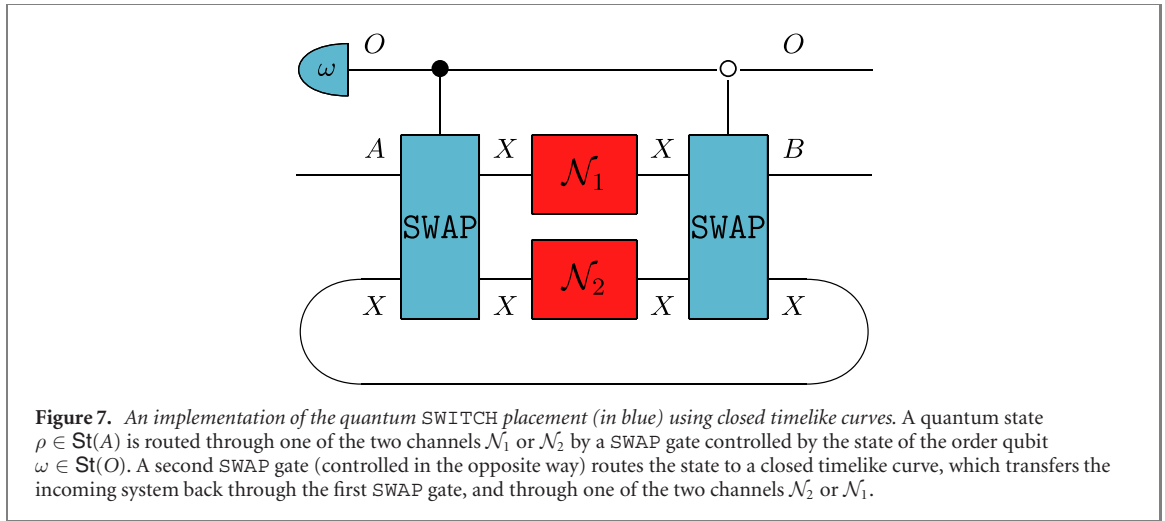
The logic of this conclusion, however, does not seem to pass a careful scrutiny. First, it is not clear how a comparison between the values of the Holevo information for the quantum SWITCH and for the superposition of channels could be used to make any deduction on the ‘true origin’ of the respective advantages. If anything, the comparison would show that the ability to control trajectories is more powerful than the ability to control causal orders. Second, the comparison made in [18] is uneven, because:

- (i) It does not compare supermaps acting on the same input channels, and
- (ii) It does not compare superpositions where the depolarising channels act the same number of times.

A detailed analysis of these two points is provided in the following.

(i) *Different input channels.* In quantum Shannon theory with superpositions of trajectories, the input resources are vacuum-extended channels, while in quantum Shannon theory with superpositions of causal orders the input resources are ordinary (non-vacuum-extended) channels.

A vacuum-extended channel is a stronger resource than the corresponding channel, because it can have coherence with the vacuum, in the sense of definition 3. We now argue that coherence with the vacuum is indeed the underlying resource implicit in the communication advantages of reference [18]. Suppose that a particle is sent in a superposition of two paths, going through two communication devices, each of which



acts as a completely depolarising channel on the internal degree of freedom of the particle. The two devices are described by vacuum extensions of the completely depolarising channel, and act as

$$\tilde{\mathcal{N}}_{\text{dep}}(\rho) = (1 - \langle \text{vac} | \rho | \text{vac} \rangle) \frac{I}{d} + \langle \text{vac} | \rho | \text{vac} \rangle |\text{vac}\rangle \langle \text{vac}| + F \rho |\text{vac}\rangle \langle \text{vac}| + |\text{vac}\rangle \langle \text{vac}| \rho F^\dagger, \quad (28)$$

where F is the vacuum interference operator defined in equation (22). Now, if the channels have no coherence with the vacuum (that is, if $F = 0$), then their superposition yields the constant channel

$$\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_{\text{dep}}, \tilde{\mathcal{N}}_{\text{dep}})(\rho) = \frac{I}{d} \otimes \text{diag}(\omega), \quad (29)$$

following from equation (27) with $\rho_0 = I/d$. Since the output is independent of the input, the channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}$ cannot be used to communicate.

The above analysis shows that the presence of coherence with the vacuum is necessary for the advantages observed by Abbott *et al* [18]. In contrast, the presence of coherence with the vacuum is, in principle, unnecessary for the advantages of the quantum SWITCH. For example, the implementation of the quantum SWITCH via closed timelike curves [22, 23], illustrated in figure 7, does not require any coherence with the vacuum.

In summary, the advantages of references [9, 18] arise from different input resources, with the resources used in [18] (vacuum-extended channels exhibiting coherence with the vacuum) being strictly stronger than the resources used in [9] (ordinary, non-vacuum-extended channels, possibly without coherence with the vacuum).

(ii) *Different numbers of uses of the depolarising channel.* References [9, 18] refer to two different communication scenarios:

- (a) In reference [18], the particle travels through *only one* depolarising channel (*either* \mathcal{N}_1 *or* \mathcal{N}_2),
- (b) In reference [9] the particle travels through *two* depolarising channels (*both* \mathcal{N}_1 *and* \mathcal{N}_2).

From this point of view, there is little surprise that scenario (a) allows more communication than scenario (b), given that in scenario (b) the particle is exposed twice to depolarising noise, as acknowledged also by the authors of reference [18].

One may argue that the difference between scenarios (a) and (b) is irrelevant, because the completely depolarising channel $\mathcal{N}_{\text{dep}}(\cdot) := I/d \text{Tr}[\cdot]$ satisfies the equality

$$\mathcal{N}_{\text{dep}} \circ \mathcal{N}_{\text{dep}} = \mathcal{N}_{\text{dep}}, \quad (30)$$

meaning that applying the channel twice in a row is the same as applying it once.

However, the input resource for the superposition of two channels is not two depolarising channels themselves, but rather their vacuum extensions. Crucially, algebraic identities like the one in equation (30) do not carry over to the vacuum extensions: in general, the relation $\mathcal{N}_1 \circ \mathcal{N}_2 = \mathcal{N}_3$ *does not* imply the relation $\tilde{\mathcal{N}}_1 \circ \tilde{\mathcal{N}}_2 = \tilde{\mathcal{N}}_3$. In the particular case of depolarising channels, we have the following result:

Proposition 3. *The condition $\tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}} = \tilde{\mathcal{N}}_{\text{dep}}$ is satisfied if and only if the vacuum extension $\tilde{\mathcal{N}}_{\text{dep}}$ has no coherence with the vacuum.*

The proof is given in appendix F.

In summary, the only case in which the equation $\tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}} = \tilde{\mathcal{N}}_{\text{dep}}$ would justify a comparison between the Holevo information with a single depolarising channel and the Holevo information with two depolarising channels is exactly the case in which the vacuum extension $\tilde{\mathcal{N}}_{\text{dep}}$ has no coherence with the vacuum, and therefore the protocol of reference [18] provides no advantage.

In order to make an even comparison with the quantum SWITCH, one should analyse the scenario where information is sent along a superposition of two paths, each visiting *two* depolarising channels. Mathematically, this superposition is described by the channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}}, \tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}})$, instead of the channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_{\text{dep}}, \tilde{\mathcal{N}}_{\text{dep}})$ considered in reference [18].

The Holevo information of the channel $\mathcal{S}_{\text{sup}}^{A,B,\omega}(\tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}}, \tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}})$ was recently evaluated in reference [20], for a set of vacuum extensions constructed from the representation of the completely depolarising channel as a uniform mixture of the four Pauli unitaries. For this set of vacuum extensions, the maximum Holevo information turned out to be 0.018, which is strictly less than the value 0.049 of the Holevo information for the quantum SWITCH. While this comparison is limited to a specific set of vacuum extensions, it already shows that bringing the comparison to an even ground may actually change the conclusions of reference [18].

For the above reasons, we argue that the comparison between the quantum SWITCH [9] and superposition of independent communication channels presented in reference [18] is uneven. We nevertheless acknowledge the importance of the initial question posed in reference [18], namely to what extent indefinite causal order per se, as opposed to the common element of coherent control, is responsible for the communication advantages of the quantum SWITCH. With respect to this open question, we point out that there exist several partial indications that indefinite causal order does indeed exhibit specific features that differentiate it from the coherent control of communication channels.

First, references [10, 11] showed that the quantum SWITCH enables noiseless quantum communication through two noisy channels, a phenomenon that is impossible through the coherent control of the same channels, *even if access to vacuum extensions is granted*. In other words, even if one overlooks the fact that the control over orders and the control over channels build on different initial resources, there still exist phenomena that are specific to the control over orders.

Second, reference [21] presented numerical evidence that the underlying mechanism for activation of communication capacity through the quantum SWITCH is different from the activation arising from the superposition of trajectories. Specifically, reference [21] observed that the superposition of channels generically increases their capacity, whereas the quantum SWITCH of channels can either increase or decrease their capacity, with this behaviour appearing to depend on the amount of non-commutativity of the input channels, as measured by a certain function of their Kraus operators. On average, the authors found that when the quantum SWITCH increases the Holevo information, it has higher probability to increase it by larger amounts compared to the superposition of channels.

These findings provide a numerical indication that the communication advantages of the quantum SWITCH arise from an interplay between the coherent control of channels *and* the non-commutativity of the Kraus operators, a feature not present in the superposition of independent channels.

6. Reply to Guérin *et al*

A recent paper by Guérin, Rubino, and Brukner [28] argues that, in order to claim meaningful communication advantages, the quantum SWITCH should be compared to a general class of operations termed ‘superpositions of direct pure processes’. In this section we analyse their arguments and examples, concluding that they rest on a communication model that violates the basic resource-theoretic framework. Before analysing that communication model, we also reply to two criticisms directed at the papers [9–11].

6.1. Reply to criticisms

References [9–11] proved that a quantum Shannon theory enriched with the quantum SWITCH offers advantages over standard quantum Shannon theory. Reference [28] criticises the fact that the comparison is restricted to standard quantum Shannon theory, writing ‘[...] it is also important to keep a relatively large class of causally ordered processes against which the process under consideration can be compared; otherwise any advantage would be empty of practical significance’.

The last comment on the ‘practical significance’ appears to be misplaced, given that standard quantum Shannon theory (and not other models of communication with causally ordered processes) underpins all

implementations of quantum communication currently considered in practice. This said, we stress that the motivation for studying the communication advantages of the quantum SWITCH is not directly a practical one: the motivation is to explore how the theory of quantum communication as we currently know it would be affected by the possibility to combine quantum channels in a superposition of orders. As mentioned in section 4.1, the interest in quantum communication assisted by the quantum SWITCH is similar to the interest in communication and computation assisted by Popescu–Rohrlich boxes, namely to better understand standard quantum theory by comparing it to possible alternatives.

We agree with the authors of reference [28] that it may be interesting to contrast the superposition of causal orders with other extensions of quantum Shannon theory. However, the particular extension proposed in reference [28] appears to be problematic, in that it does not satisfy the minimal requirement for a resource theory of communication: as we will see in the following subsections, the operations proposed in reference [28] generally create side-channels.

Reference [28] also criticises the use of the term *causal activation* to describe the phenomenon in references [9–11] of achieving a non-zero capacity when combining two zero-capacity channels in a superposition of alternative orders. The reason for the criticism is that ‘[...] *there are causally ordered processes that offer the same advantages and can be considered as equivalent resources*’.

Again, the criticism appears to be misplaced. The term ‘causal activation’ was not meant to be a statement about the origin of the advantage. Instead, it was meant to be a way to distinguish the new type of activation from the already known ‘activation of the quantum capacity’, introduced in the seminal work of Smith and Yard [42]. Since the activation phenomenon observed in references [9–11] was radically different from the standard activation of the quantum capacity, the authors added the attribute ‘causal’ to stress the different context of the activation observed in their work.

Besides the choice of terminology, the claim that ‘there are causally ordered processes that offer the same advantages and can be considered as equivalent resources’ appears to be unclear, as the authors of reference [28] did not provide a resource-theoretic analysis. In the following, we will assess their claim within the resource-theoretic framework developed in this paper.

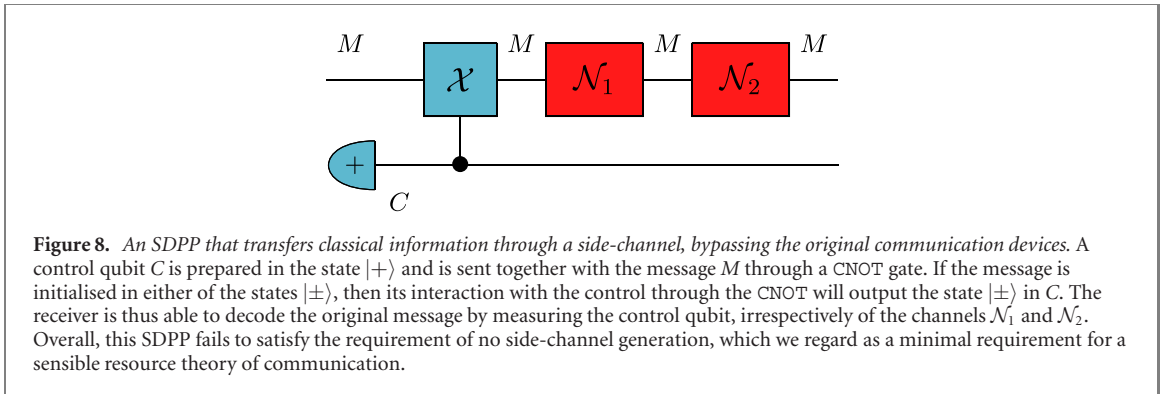
6.2. The framework of SDPPs

The authors of reference [28] argue that quantum Shannon theory with superpositions of causal orders should be considered within a general framework of ‘superpositions of direct pure processes’ (SDPPs), which includes the quantum SWITCH: ‘*it seems that any reasonable resource theory that contains the quantum switch—a superposition of direct pure processes with different causal orders—should also allow superpositions of direct pure processes with the same causal order*’ [28]. It is claimed, therefore, that the advantages of the quantum SWITCH should be compared to SDPPs with a definite causal order. In the following, we analyse the above claim, showing that, while the quantum SWITCH and the SDPPs considered in reference [28] share a similar mathematical structure, they have different operational features: in particular, the specific SDPPs compared with the quantum SWITCH in reference [28] generate side-channels, making the proposed advantages trivial from the resource-theoretic point of view.

In the language of this paper, the SDPPs of reference [28] are supermaps that take two channels $(\mathcal{N}_1, \mathcal{N}_2)$, and return a superposition of k channels which are individually of the form $\mathcal{D}_j \circ \mathcal{N}_2 \circ \mathcal{R}_j \circ \mathcal{N}_1 \circ \mathcal{E}_j$ or $\mathcal{D}'_j \circ \mathcal{N}_1 \circ \mathcal{R}'_j \circ \mathcal{N}_2 \circ \mathcal{E}'_j$, $j \in \{1, 2, \dots, k\}$, for some encoding, repeater and decoding operations $\mathcal{E}_j, \mathcal{E}'_j, \mathcal{R}_j, \mathcal{R}'_j, \mathcal{D}_j$ and \mathcal{D}'_j . Here, a superposition of N channels $\mathcal{A}_j \in \text{Chan}(A_j)$, for $j \in \{1, \dots, N\}$, is defined in the most general way as any channel $\mathcal{S} \in \text{Chan}(A_1 \oplus \dots \oplus A_N)$ which acts as \mathcal{A}_j when the input is restricted to state in sector A_j [19]. SDPPs with a definite causal order are defined as SDPPs which are superpositions of terms where the input channels \mathcal{N}_1 and \mathcal{N}_2 occur in the same, fixed order.

In the resource-theoretic scheme of our paper, the SDPPs should be regarded as a set of free operations. One could, for example, consider them as a broader set of party supermaps, or alternatively, as a set of placement supermaps with internal encoding, decoding, and repeater operations, which are in a quantum superposition controlled by some quantum degree of freedom that is part of the placement.

The resource theory based on SDPPs is different from the resource theory of quantum Shannon theory with superpositions of trajectories. An important difference is that SDPPs are supermaps acting directly on the original channels, rather than their vacuum extensions. In this respect, SDPPs and the quantum SWITCH operate on the same kind of input resource, and a comparison between them would indeed be even. However, in the following we will show that building a resource theory of communication where all SDPPs are taken as free operations is problematic, because it violates the requirement of no side-channel generation. This fact will be illustrated by analysing the specific examples of SDPPs proposed in reference [28].



6.3. Some SDPPs generate classical side-channels

One of the SDPPs proposed in reference [28] is the supermap depicted in figure 8. This supermap corresponds to a protocol where two qubit channels \mathcal{N}_1 and \mathcal{N}_2 are applied after a CNOT gate, acting on the message qubit, and on an additional control qubit C . The supermap, here denoted as $\mathcal{F} : (\mathcal{N}_1, \mathcal{N}_2) \mapsto \mathcal{F}(\mathcal{N}_1, \mathcal{N}_2)$, produces the output channel defined by

$$\mathcal{F}(\mathcal{N}_1, \mathcal{N}_2)(\rho) = [(\mathcal{N}_2 \circ \mathcal{N}_1) \otimes \mathcal{I}_C] \circ \mathcal{U}^{\text{CNOT}}(\rho \otimes |+\rangle\langle +|), \quad (31)$$

where $\mathcal{U}^{\text{CNOT}} := U^{\text{CNOT}\dagger}(\cdot)U^{\text{CNOT}}$ is the unitary channel corresponding to the CNOT gate

$$U^{\text{CNOT}} := I_M \otimes |0\rangle\langle 0|_C + X_M \otimes |1\rangle\langle 1|_C, \quad (32)$$

X being the NOT gate. To avoid overloading the notation, here we have omitted the isomorphisms between unplaced and placed systems, and simply denoted the (placed and unplaced) message system by M .

Now, the map \mathcal{F} enables perfect classical communication of one bit independently of the communication channels \mathcal{N}_1 and \mathcal{N}_2 [19, 28]. Using the phase kickback mechanism of the CNOT gate [63], information encoded in the states $|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$ is transferred from the message M to the control C before the noisy channels \mathcal{N}_1 and \mathcal{N}_2 are applied. Then, the information is safely carried by the control system to the receiver, completely bypassing the communication channels \mathcal{N}_1 and \mathcal{N}_2 , and avoiding the resulting noise. In other words, this example of an SDPP is analogous to the example of the noisy telephone line discussed in section 3.4: it achieves communication by completely bypassing the original channels.

More formally, one can see that the operation \mathcal{F} generates a classical side-channel in the sense of definition 2. Indeed, one can consider the party supermap corresponding to the encoding channel $\mathcal{E} = \mathcal{I}_M$ and the decoding channel $\mathcal{D} = \text{Tr}_M$, which discards the message qubit. The result is the channel

$$\begin{aligned} \mathcal{C}(\cdot) &= \mathcal{D} \circ \mathcal{F}(\mathcal{N}_1, \mathcal{N}_2) \circ \mathcal{E}(\cdot) \\ &= |+\rangle\langle +| \cdot |+\rangle\langle +| + |-\rangle\langle -| \cdot |-\rangle\langle -|, \end{aligned} \quad (33)$$

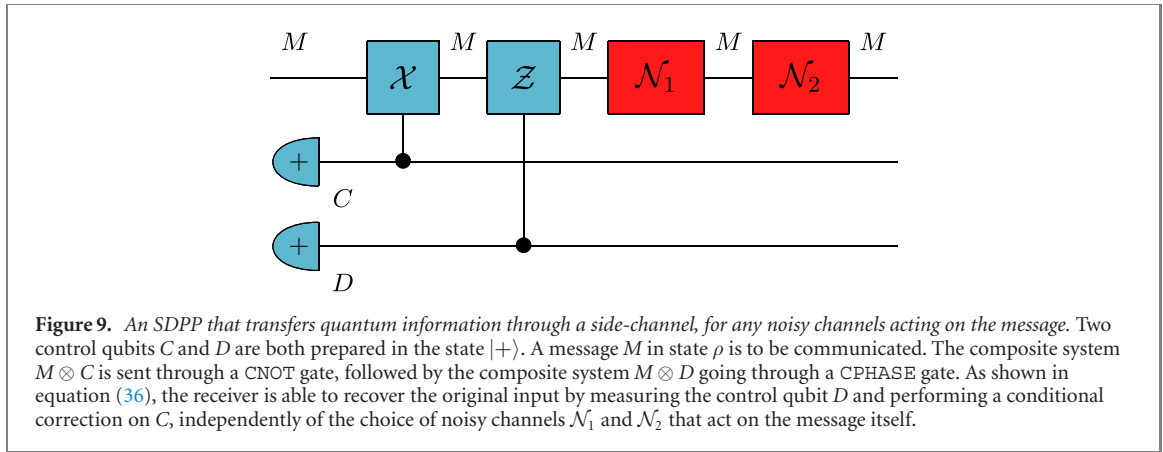
which is independent of \mathcal{N}_1 and \mathcal{N}_2 and provides a perfect transmission line for classical communication. In conclusion, the ‘communication enhancement’ of the SDPP (31) arises from a classical side-channel, which completely bypasses the original communication devices.

The authors of reference [28] also consider a specific interferometric implementation of the operation \mathcal{F} , which they claim avoids the criticism above. In appendix G, we analyse the arguments provided in reference [28], and conclude that, in fact, the above criticism still applies.

6.4. Some SDPPs generate quantum side-channels

In appendix B of reference [28], the authors present an SDPP, stating that it ‘allows us to perfectly transmit one qubit of quantum information, for all channels [...]’. This statement is an explicit acknowledgement that the SDPP model permits the strongest possible kind of side-channels: perfect side-channels for quantum communication.

The example in appendix B of reference [28] is presented as one that ‘generalises, and improves upon, the observations made in the main text’, the improvement being that only four control qubits are used instead of eight, which is the number of qubits used by the protocol in the main text. Here we review the example, showing that, in fact, one can improve it even further: the same perfect qubit side-channel can be generated by an SDPP that uses only two control qubits, instead of four.



Our improved version of the SDPP in reference [28] is depicted in figure 9. It uses two control qubits C and D , in addition to the message qubit M . The corresponding supermap is given by

$$\mathcal{G}^{\omega, \xi} : (\mathcal{N}_1, \mathcal{N}_2) \mapsto \mathcal{G}^{\omega, \xi}(\mathcal{N}_1, \mathcal{N}_2) \quad (34)$$

$$\mathcal{G}^{\omega, \xi}(\mathcal{N}_1, \mathcal{N}_2)(\rho_M) = [(\mathcal{N}_2 \circ \mathcal{N}_1) \otimes \mathcal{I}_C \otimes \mathcal{I}_D] \circ (\mathcal{U}_{MD}^{\text{CPHASE}} \otimes \mathcal{I}_C) \circ (\mathcal{U}_{MC}^{\text{CNOT}} \otimes \mathcal{I}_D)(\rho_M \otimes \omega_C \otimes \xi_D),$$

where ω and ξ are quantum states of the control qubits C and D , respectively, and $\mathcal{U}_{MC}^{\text{CNOT}}$ and $\mathcal{U}_{MD}^{\text{CPHASE}}$ are the CNOT and CPHASE gates, respectively, defined by

$$\begin{aligned} \mathcal{U}_{MC}^{\text{CNOT}} &:= I_M \otimes |0\rangle\langle 0|_C + X_M \otimes |1\rangle\langle 1|_C \\ \mathcal{U}_{MD}^{\text{CPHASE}} &:= I_M \otimes |0\rangle\langle 0|_D + Z_M \otimes |1\rangle\langle 1|_D, \end{aligned} \quad (35)$$

X and Z being Pauli gates.

Explicit calculation reveals that for both control qubits initialised in the $|+\rangle$ state, one obtains

$$\text{Tr}_M \left[\mathcal{G}^{|\+\rangle\langle +|, |\+\rangle\langle +|}(\mathcal{N}_1, \mathcal{N}_2)(\rho_M) \right] = \rho_C \otimes |+\rangle\langle +|_D + X \rho_X C \otimes |-\rangle\langle -|_D. \quad (36)$$

Therefore, the initial state ρ can be perfectly recovered independently of the noisy channels \mathcal{N}_1 and \mathcal{N}_2 , by measuring D in the Fourier basis and then applying a NOT gate on C if the outcome is $|-\rangle$. The supermap defined by equation (34) is an example of an SDPP that generates a perfect quantum side-channel, as it can perfectly transmit one qubit of quantum information for any choice of noisy channels.

The authors of reference [28] conclude with regard to their protocol: ‘*this example shows that SDPPs [...] can be used to perfectly send one qubit of information, essentially trivialising the problem of enhancing quantum and classical channel capacity if one were to take the set of all SDPPs as a resource*’. We agree, and argue that this is the reason why the set of all SDPPs does not define a sensible resource theory of communication.

A possible direction of future research would be to compare the quantum SWITCH with the subset of SDPPs that have definite causal order and do not generate side-channels. This may shed light on the mechanism that leads to enhancements in the quantum SWITCH, and on whether or not the characteristics of this mechanism can be reproduced by SDPPs with definite causal order.

More interestingly, it would be important to compare the side-channel non-generating SDPPs with definite causal order with *all* the side-channel non-generating SDPPs with indefinite causal order, rather than just restricting the comparison to the quantum SWITCH. For a given pair of channels, the maximum communication capacity achievable with indefinite causal order is—by definition—always larger than or equal to the maximum communication capacity achievable with definite causal order. The interesting question is whether there is a gap between the two, meaning that there exist communication advantages that can be achieved *only* with indefinite causal order.

7. Summary and outlook

We established a general framework of resource theories of communication. In our framework, the input resources are communication devices, which can be placed between the communicating parties, and combined with local operations performed by the communicating parties. A resource theory is specified by

a choice of placement operations, describing how the communication devices are arranged, and by a choice of party operations, describing the action of the communicating parties.

We formulated a minimal requirement that every resource theory of communication should satisfy: no combination of the allowed operations should be able to bypass the communication devices initially available to the communicating parties. We have shown that quantum Shannon theory with superpositions of causal order of communication channels [9] and quantum Shannon theory with superpositions of trajectories of information carriers [19] satisfy this requirement, while quantum Shannon theory with superpositions of encoding and decoding operations [28] does not.

We pointed out the importance of distinguishing between different forms of coherent control, rather than conflating them into a generic label. Specifically, we distinguished between three different types of superpositions: the superposition of causal orders of communication channels, the superposition of trajectories through independent communication channels, and the superposition of encoding/decoding operations. We observed that the superposition of causal orders is in principle different from the superposition of trajectories, because these two different superpositions use different input resources (the original channels in the former case, and an extension of the original channels in the latter case). In turn, the superposition of orders and the superposition of trajectories are strikingly different from the superposition of encoding/decoding operations, in that the former do not generate side-channels, while the latter does.

Our definition of resource theories of communication can be extended straightforwardly to allow correlated quantum channels as input resources, where the noisy processes occurring in the application of a device at time $t + 1$ may be affected by the application of the same device at time t . Such correlated channels are known as quantum memory channels [46, 64], quantum combs [38, 39] and non-Markovian quantum processes [48, 65], and have been shown to provide interesting communication advantages over uncorrelated channels [19, 20, 64].

Overall, the resource-theoretic framework proposed in this paper allows for rigorous comparisons between different resource theories of communication, and can be used for the exploration of new models of quantum communication, with both foundational and practical implications.

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Appendix A. Free supermaps in assisted communication scenarios

Here we provide examples of supermaps that arise in the presence of assistance from classical communication and entanglement.

Let us consider first the assistance of free classical communication [40], as illustrated in figure A.1. In this case, the free supermaps on placed channels have the form

$$\mathcal{S}_{\mathcal{E}, \mathcal{D}, \text{clas}}(\mathcal{C}) := \mathcal{D} \circ (\mathcal{C} \otimes \mathcal{I}_{\text{Aux}}^{\text{clas}}) \circ \mathcal{E}, \quad (\text{A1})$$

where $\mathcal{I}_{\text{Aux}}^{\text{clas}}$ is the classical identity channel, defined as $\mathcal{I}_{\text{Aux}}^{\text{clas}}(\rho) = \sum_j |j\rangle\langle j| \rho |j\rangle\langle j|$ for some orthonormal basis $\{|j\rangle\}$, and $\mathcal{E} \in \mathbf{Chan}(A', A \otimes \text{Aux})$ and $\mathcal{D} \in \mathbf{Chan}(B \otimes \text{Aux}, B')$ are quantum channels.

Let us consider now classical communication with the assistance of shared entanglement [41]. In this case, the free operations on placed channels are those that can be achieved by performing encoding and decoding operations that act on a shared entangled state, as shown in figure A.2. Mathematically, these

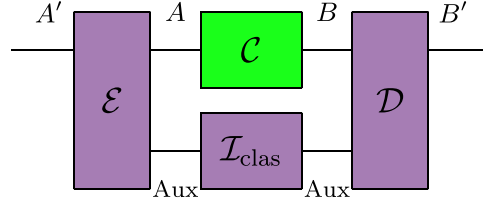


Figure A.1. Supermap describing encoding and decoding operations assisted by free classical communication.

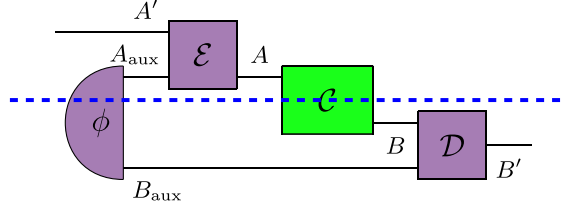


Figure A.2. Supermap describing encoding and decoding operations assisted by shared entanglement. The blue dashed line denotes the partition between Alice (top) and Bob (bottom).

operations correspond to free supermaps of the form

$$\mathcal{S}_{\mathcal{E}, \mathcal{D}, \text{ent}}(\mathcal{C}) := \mathcal{D}_{BB_{\text{aux}}} \circ (\mathcal{C}_A \circ \mathcal{E}_{A'A_{\text{aux}}} \otimes \mathcal{I}_{B_{\text{aux}}}) \circ (\mathcal{I}_{A'} \otimes \phi_{A_{\text{aux}}B_{\text{aux}}}), \quad (\text{A2})$$

where $\phi_{A_{\text{aux}}B_{\text{aux}}}$ is an entangled state on system $A_{\text{aux}} \otimes B_{\text{aux}}$, $\mathcal{E} \in \text{Chan}(A' \otimes A_{\text{aux}}, A)$ and $\mathcal{D} \in \text{Chan}(B \otimes B_{\text{aux}}, B')$ are encoding and decoding channels, respectively, and the subscripts indicate the input systems of all channels.

Appendix B. Mathematical structure of unplaced channels, placed channels and admissible supermaps

B.1. Placed and unplaced channels

Here we specify the mathematical structure of placed and unplaced channels. In the following, the subset of completely positive (CP) maps will be denoted by $\text{CP}(A, B) \subset \text{Map}(A, B)$, and the subset of trace-preserving (TP) maps will be denoted by $\text{TP}(A, B) \subset \text{Map}(A, B)$.

For a single use of a single device, the distinction between placed and unplaced channels concerns only the type of inputs and outputs. Mathematically, placed and unplaced channels are both described by completely positive trace-preserving maps.

When multiple devices are involved, the distinction is more substantial. As described in sections 2.4 and 2.5, k devices can be placed in parallel, giving rise to a multipartite quantum channel, or in sequence, giving rise to a quantum k -comb, or, more generally, in any combination of parallel and sequence, giving rise to a quantum l -comb, with any l from 1 to k .

Mathematically, a quantum k -comb transforming system S_i into system S'_i at the i th step is a completely positive map $\mathcal{C} \in \text{CP}(S_1 \otimes \dots \otimes S_k, S'_1 \otimes \dots \otimes S'_k)$ satisfying the conditions

$$\begin{aligned} \text{Tr}_{S'_k}[\mathcal{C}(\rho)] &= \text{Tr}_{S'_k} \left[\mathcal{C} \left(\text{Tr}_{S_k}[\rho] \otimes \frac{I_{S_k}}{d_k} \right) \right] \\ \text{Tr}_{S'_{k-1}S'_k}[\mathcal{C}(\rho)] &= \text{Tr}_{S'_{k-1}S'_k} \left[\mathcal{C} \left(\text{Tr}_{S_{k-1}S_k}[\rho] \otimes \frac{I_{S_{k-1}}}{d_{k-1}} \otimes \frac{I_{S_k}}{d_k} \right) \right] \\ &\vdots \\ \text{Tr}_{S'_1 \dots S'_k}[\mathcal{C}(\rho)] &= 1 \quad \forall \rho \in \text{St}(S_1 \otimes \dots \otimes S_k). \end{aligned} \quad (\text{B1})$$

The set of quantum k -combs with the input/output systems above is denoted by $\text{Comb}[(S_1, S'_1), \dots, (S_k, S'_k)]$.

In contrast, unplaced channels belong to a different set of completely positive maps. We will now specify this set explicitly. So far, we represented unplaced channels by lists, such as $(\mathcal{N}_1, \dots, \mathcal{N}_k)$. However, the set

of lists is not closed under probabilistic mixtures, which arise quite naturally when some of the parameters of the communication devices are subject to random fluctuations.

Probabilistic mixtures can be described by convex combinations of the form $\sum_{i=1}^L p_i (\mathcal{N}_{1,i}, \dots, \mathcal{N}_{k,i})$, where $(p_i)_{i=1}^L$ is a probability distribution, and, for every $i \in \{1, \dots, L\}$, $(\mathcal{N}_{1,i}, \dots, \mathcal{N}_{k,i}) \in \mathbf{Chan}(X_1, Y_1) \times \dots \times \mathbf{Chan}(X_k, Y_k)$ is a list of channels. Note that the convex combination $\sum_{i=1}^L p_i (\mathcal{N}_{1,i}, \dots, \mathcal{N}_{k,i})$ must satisfy a basic consistency requirement: if a channel in the list $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ is a convex combination of channels, say $\mathcal{N}_1 = \sum_i p_i \mathcal{N}_{1,i}$, then the list $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ should be equal to the corresponding convex combination $\sum_i p_i (\mathcal{N}_{1,i}, \mathcal{N}_2, \dots, \mathcal{N}_k)$. Requiring this consistency property to hold for every entry of the list implies that the convex combinations $\sum_{i=1}^L p_i (\mathcal{N}_{1,i}, \dots, \mathcal{N}_{k,i})$ can be represented as elements of the tensor product space $\mathbf{TP}(X_1, Y_1) \otimes \mathbf{TP}(X_2, Y_2) \otimes \dots \otimes \mathbf{TP}(X_k, Y_k)$, which consists of all linear combinations of the form

$$\mathcal{M} = \sum_{i=1}^L c_i \mathcal{N}_{1,i} \otimes \mathcal{N}_{2,i} \otimes \dots \otimes \mathcal{N}_{k,i}, \quad (\text{B2})$$

where $(c_i)_{i=1}^L$ are real coefficients, and each $\mathcal{N}_{j,k}$ is a trace-preserving map in $\mathbf{TP}(X_j, Y_j)$.

In summary, the unplaced channels can be regarded as elements of the tensor product space $\mathbf{TP}(X_1, Y_1) \otimes \mathbf{TP}(X_2, Y_2) \otimes \dots \otimes \mathbf{TP}(X_k, Y_k)$. Precisely, the list $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ can be regarded as the product channel $\mathcal{N}_1 \otimes \dots \otimes \mathcal{N}_k$. In this paper, we use the list notation $(\mathcal{N}_1, \dots, \mathcal{N}_k)$ and the tensor product notation $\mathcal{N}_1 \otimes \dots \otimes \mathcal{N}_k$, interchangeably, depending on which representation is more convenient.

As can be seen from equation (B2), the tensor product space $\mathbf{TP}(X_1, Y_1) \otimes \mathbf{TP}(X_2, Y_2) \otimes \dots \otimes \mathbf{TP}(X_k, Y_k)$ also contains channels that are not of the product form. The set of all such channels is in one-to-one correspondence with the set of k -partite no-signalling channels, i.e. channels $\mathcal{N} \in \mathbf{Chan}(X_1 \otimes \dots \otimes X_k, Y_1 \otimes \dots \otimes Y_k)$ with the additional property that the reduced state of any subset of the outputs depends only on the reduced state of the corresponding subset of inputs [23].

In the following, we use the notation $\mathbf{NSChan}[(X_1, Y_1), \dots, (X_k, Y_k)]$ to denote the set of all quantum channels in $\mathbf{TP}(X_1, Y_1) \otimes \mathbf{TP}(X_2, Y_2) \otimes \dots \otimes \mathbf{TP}(X_k, Y_k)$, possibly including channels of the non-product form. When the inputs and outputs are arbitrary, we will use the notation \mathbf{NSChan} .

B.2. Admissible supermaps

In order to specify the free operations in a resource theory, one has to first specify the broader set of operations from which the free operations are chosen. For a resource theory of communication, the operations are (1) supermaps from unplaced channels to unplaced channels, (2) supermaps from unplaced channels to placed channels, and (3) supermaps from placed channels to placed channels. This specification will be provided in the following.

(3) *Supermaps from placed channels to placed channels.* An admissible supermap transforming channels in $\mathbf{Chan}(A, B)$ into channels in $\mathbf{Chan}(A', B')$ is a linear transformation \mathcal{S} from the set $\mathbf{Map}(A, B)$ to the set $\mathbf{Map}(A', B')$, where A, B, A', B' are placed systems [38, 39].

Admissibility is the requirement that \mathcal{S} should transform unplaced channels into channels, even when acting locally on parts of larger devices. Mathematically, the admissibility condition can be formulated by introducing an additional system describing the environment: a supermap \mathcal{S} is admissible if, for every environment system E , and for every channel $\mathcal{C} \in \mathbf{Chan}(A, B \otimes E)$, the map $(\mathcal{S} \otimes \mathcal{I}_E)(\mathcal{C})$ belongs to $\mathbf{Chan}(A', B' \otimes E)$.

The admissible supermaps from $\mathbf{Chan}(A, B)$ to $\mathbf{Chan}(A', B')$ have been characterised in reference [38], which showed that any supermap from placed channels to placed channels can be obtained by sandwiching the input channel between a pre-processing channel and a post-processing channel, as in equation (2).

For channels placed in a sequence, the admissible supermaps transform quantum k -combs in $\mathbf{Comb}[(A_1, B_1), \dots, (A_k, B_k)]$ into quantum l -combs in $\mathbf{Comb}[(A'_1, B'_1), \dots, (A'_l, B'_l)]$. They are defined as linear transformations \mathcal{S} from the set $\mathbf{Map}(A_1 \otimes \dots \otimes A_k, B_1 \otimes \dots \otimes B_k)$ to the set $\mathbf{Map}(A'_1 \otimes \dots \otimes A'_l, B'_1 \otimes \dots \otimes B'_l)$ satisfying the property that, for every environment system E , and for every k -comb $\mathcal{C} \in \mathbf{Comb}[(A_1, B_1), \dots, (A_k, B_k \otimes E)]$, the map $(\mathcal{S} \otimes \mathcal{I}_E)(\mathcal{C})$ belongs to $\mathbf{Comb}[(A'_1, B'_1), \dots, (A'_l, B'_l \otimes E)]$. The admissible supermaps with $l = 1$ have been characterised in reference [39], and correspond to $(k + 1)$ -combs. The general form of such supermaps is shown in equation (7) in the special case of $k = 2$.

(2) *Supermaps from unplaced channels to placed channels.* These supermaps represent the possible placements of quantum channels. An admissible supermap transforming k unplaced channels into a single placed channel is a linear supermap \mathcal{S} from the set $\mathbf{Map}(X_1 \otimes \dots \otimes X_k, Y_1 \otimes \dots \otimes Y_k)$ to the set $\mathbf{Map}(A, B)$ where A and B are placed systems, and $X_1, \dots, X_k, Y_1, \dots, Y_k$ are unplaced systems. An admissible supermap \mathcal{S} should satisfy the condition that, for every environment system E , and for every channel

$\mathcal{C} \in \text{Chan}(X_1 \otimes \cdots \otimes X_k, Y_1 \otimes \cdots \otimes Y_k \otimes E)$ satisfying the condition $\text{Tr}_E \mathcal{C} \in \text{NSChan}[(X_1, Y_1), \dots, (X_k, Y_k)]$, the map $(\mathcal{S} \otimes \mathcal{I}_E)(\mathcal{C})$ belongs to $\text{Chan}(A, B \otimes E)$.

Examples of admissible supermaps are the basic placement supermap of equation (1), the parallel placement supermap of equation (4), and the sequential placement supermaps of equations (5) and (9). Note that, in fact, the parallel and sequential placements are just the products of many basic placements, and that the parallel or sequential nature of a given placement just depends on the causal structure of the spacetime points in which the inputs and outputs of the channels are placed.

Other examples of admissible placement supermaps are the quantum SWITCH placement (iii) of reference [23], and the superposition placement (iii*) of reference [19].

(1) *Supermaps from unplaced channels to unplaced channels.* An admissible supermap from unplaced channels in $\text{NSChan}[(X_1, Y_1), \dots, (X_k, Y_k)]$ to unplaced channels in $\text{NSChan}[(X'_1, Y'_1), \dots, (X'_l, Y'_l)]$ is a linear map \mathcal{S} from the set $\text{Map}(X_1 \otimes \cdots \otimes X_k, Y_1 \otimes \cdots \otimes Y_k)$ to the set $\text{Map}(X'_1 \otimes \cdots \otimes X'_l, Y'_1 \otimes \cdots \otimes Y'_l)$, where $(X_i)_{i=1}^k, (Y_i)_{i=1}^k, (X'_j)_{j=1}^l, (Y'_j)_{j=1}^l$ are unplaced systems. Admissibility is the condition that, for every environment system E , and for every channel $\mathcal{C} \in \text{Chan}(X_1 \otimes \cdots \otimes X_k, Y_1 \otimes \cdots \otimes Y_k \otimes E)$ satisfying the condition $\text{Tr}_E \mathcal{C} \in \text{NSChan}[(X_1, Y_1), \dots, (X_k, Y_k)]$, the map $\mathcal{C}' := (\mathcal{S} \otimes \mathcal{I}_E)(\mathcal{C})$ satisfies the condition $\text{Tr}_E \mathcal{C}' \in \text{NSChan}[(X'_1, Y'_1), \dots, (X'_l, Y'_l)]$.

An example of a supermap from unplaced channels to unplaced channels is the *discarding supermap*

$$\mathcal{S}_{\text{discard}}^m(\mathcal{N}_1, \dots, \mathcal{N}_k) := (\mathcal{N}_1, \dots, \mathcal{N}_{m-1}, \mathcal{N}_{m+1}, \dots, \mathcal{N}_k), \quad (\text{B3})$$

which discards the m th channel from a list of k channels. Intuitively, the discarding supermap should always be included in the set of free supermaps, as the communication provider can always decide to discard a communication device in the construction of a communication network. We say that the set of free supermaps from unplaced channels to unplaced channels is *trivial* if it consists only of discarding supermaps and identity supermaps.

Appendix C. Placement of channels in an arbitrary (definite) causal structure

Communication through a network of $k \geq 2$ devices, connected via $r \leq k - 1$ intermediate parties, is described by specifying the causal structure of the communicating parties, and by considering supermaps that are compatible with that causal structure. In the case of a sender \mathbf{A} , receiver \mathbf{B} and a single repeater \mathbf{R} (where a boldface letter \mathbf{R} is identified with the list of input/output systems $(R_1, \dots, R_k, R'_1, \dots, R'_l)$ accessible to a given communicating party), the causal structure is implicitly given by the totally ordered set $\{\mathbf{A} \preceq \mathbf{R}, \mathbf{R} \preceq \mathbf{B}\}$, where $\mathbf{A} \preceq \mathbf{B}$ denotes that \mathbf{B} is in the future light cone of \mathbf{A} . In this case, it is clear that only placements between \mathbf{A} and \mathbf{R} , \mathbf{A} and \mathbf{B} , or \mathbf{R} and \mathbf{B} are allowed.

In the case of $r \geq 2$ repeaters, a general causal structure is described by a partially ordered set (poset), with a choice of possible relations between the intermediate parties $\{\mathbf{R}, \mathbf{S}, \dots, \mathbf{T}\}$. Formally, a poset is a set endowed with a binary relation, which is reflexive, antisymmetric and transitive. The latter two properties ensure that loops in the causal structure are not allowed, i.e. if \mathbf{A} precedes \mathbf{R} and \mathbf{R} precedes \mathbf{S} , then \mathbf{A} precedes \mathbf{S} , and therefore \mathbf{S} cannot also precede \mathbf{A} (unless $\mathbf{A} = \mathbf{R} = \mathbf{S}$). Physically, the description of causal structure as a poset is motivated by the structure of spacetime as described by special relativity [66].

Overall, the placement of communication devices between the communicating parties is described by a tensor product of basic placement supermaps, with the constraint that a placement from $\text{Chan}(X_i, Y_i)$ to $\text{Chan}(S'_i, T_i)$ is only possible if $\mathbf{S} \preceq \mathbf{T}$ in the causal structure.

We illustrate the scheme for a network of multiple repeaters with an example. Consider the communication scenario with a sender, a receiver, and $r = 3$ intermediate parties $\{\mathbf{R}, \mathbf{S}, \mathbf{T}\}$, arranged in a causal structure described by the poset $\{\mathbf{A} \preceq \mathbf{R}, \mathbf{R} \preceq \mathbf{B}, \mathbf{A} \preceq \mathbf{S}, \mathbf{S} \preceq \mathbf{T}, \mathbf{T} \preceq \mathbf{B}\}$. Suppose that the communicating parties have access to $k = 5$ devices, described by the list of unplaced channels $(\mathcal{N}_1, \dots, \mathcal{N}_5) \in \text{Chan}(X_1, Y_1) \times \cdots \times \text{Chan}(X_5, Y_5)$. The use of the devices is specified by placing them in a particular configuration between the sender, receiver, and repeaters. One possible placement is given by

$$\begin{aligned} \mathcal{S}_{\text{network}}^{A_1 R'_1 R'_2 B_2 A_3 S'_3 S'_4 T_4 T'_5 B_5}(\mathcal{N}_1, \dots, \mathcal{N}_5) &= \mathcal{S}_{\text{place}}^{A_1 R'_1}(\mathcal{N}_1) \otimes \mathcal{S}_{\text{place}}^{R'_2 B_2}(\mathcal{N}_2) \otimes \mathcal{S}_{\text{place}}^{A_3 S'_3}(\mathcal{N}_3) \otimes \mathcal{S}_{\text{place}}^{S'_4 T_4}(\mathcal{N}_4) \otimes \mathcal{S}_{\text{place}}^{T'_5 B_5}(\mathcal{N}_5) \\ &= \mathcal{W}^{R_1} \circ \mathcal{N}_1 \circ \mathcal{V}^{A_1} \otimes \mathcal{W}^{B_2} \circ \mathcal{N}_2 \circ \mathcal{V}^{R'_2} \otimes \mathcal{W}^{S_3} \circ \mathcal{N}_3 \circ \mathcal{V}^{A_3} \\ &\quad \otimes \mathcal{W}^{T_4} \circ \mathcal{N}_4 \circ \mathcal{V}^{S'_4} \otimes \mathcal{W}^{B_5} \circ \mathcal{N}_5 \circ \mathcal{V}^{T'_5}, \end{aligned} \quad (\text{C1})$$

essentially consisting of two sequences through repeaters, \mathbf{R} and (\mathbf{S}, \mathbf{T}) , placed in parallel between the sender and receiver, as illustrated in figure C.1(a).

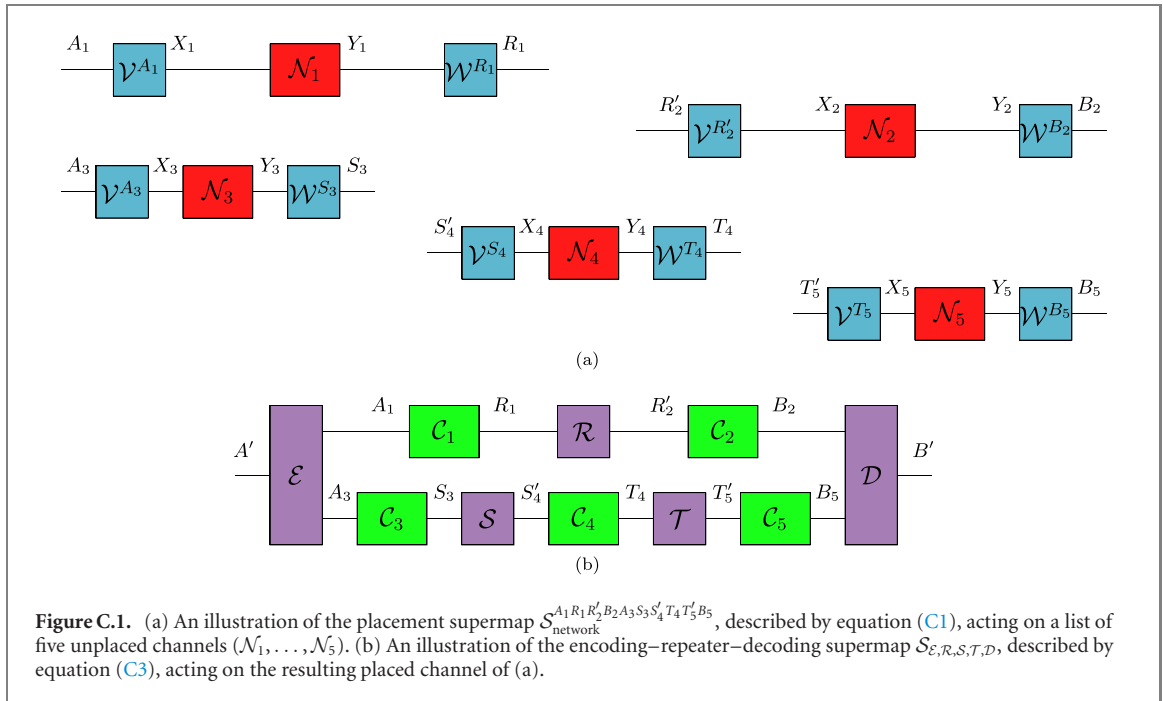


Figure C.1. (a) An illustration of the placement supermap $\mathcal{S}_{\text{network}}^{A_1 R_1 R'_2 B_2 A_3 S_3 S'_4 T_4 T'_5 B_5}$, described by equation (C1), acting on a list of five unplaced channels $(\mathcal{N}_1, \dots, \mathcal{N}_5)$. (b) An illustration of the encoding–repeater–decoding supermap $\mathcal{S}_{\mathcal{E},\mathcal{R},\mathcal{S},\mathcal{T},\mathcal{D}}$, described by equation (C3), acting on the resulting placed channel of (a).

Note, that here the different intermediate parties are labelled $\mathbf{R}, \mathbf{S}, \dots, \mathbf{T}$. The subscript i (j) of the placed system R_i (R'_j) at the communicating party \mathbf{R} labels which input system X_i (output system Y_j) it corresponds to. In contrast, in the main text where each party only had access to a single system, R_i (R_i) denoted the single input (output) system of the i th repeater party.

With the devices placed within the network of communicating parties, we once again consider the free operations on the placed channels. Consider a subset of $l \leq k$ placed channels $\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_l \in \text{Chan}(\cdot, R_1) \times \dots \times \text{Chan}(\cdot, R_m) \times \text{Chan}(R'_{m+1}, \cdot) \times \dots \times \text{Chan}(R'_l, \cdot)$, where the first $m \leq l$ channels have output systems at \mathbf{R} (and any arbitrary placed input systems), and the remaining $l - m$ channels have input systems at \mathbf{R} (and any arbitrary placed output systems). The final $k - l$ channels have neither input nor output systems at \mathbf{R} . The free operations that can be performed at \mathbf{R} are taken to be those of the form

$$\mathcal{S}_{\mathcal{R}}(\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_k) := [(\mathcal{C}_{m+1} \otimes \dots \otimes \mathcal{C}_l) \circ \mathcal{R} \circ (\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_m)] \otimes (\mathcal{C}_{l+1} \otimes \dots \otimes \mathcal{C}_k), \quad (\text{C2})$$

where $\mathcal{R} \in \text{Chan}(R_1 \otimes \dots \otimes R_m, R'_{m+1} \otimes \dots \otimes R'_l)$. This includes as a special case the free operations $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{S}_{\mathcal{D}}$ that can be performed by the sender and receiver, respectively, in which case $m = 0$ or $m = l$. Overall, the choice of free operations on placed channels is taken to be any sequential or parallel composition of (local) party supermaps of the form of equation (C2). When two supermaps $\mathcal{S}_{\mathcal{R}}$ and $\mathcal{S}_{\mathcal{T}}$ commute, we use the shorthand $\mathcal{S}_{\mathcal{R},\mathcal{T}} := \mathcal{S}_{\mathcal{T}} \circ \mathcal{S}_{\mathcal{R}} = \mathcal{S}_{\mathcal{R}} \circ \mathcal{S}_{\mathcal{T}}$.

As an example, consider the placed channels given in equation (C1) and let $\mathcal{C}_i = \mathcal{W} \circ \mathcal{N}_i \circ \mathcal{V}$. Then the action of the most general free supermap on the placed channels $\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_5$ is given by

$$\mathcal{S}_{\mathcal{E},\mathcal{R},\mathcal{S},\mathcal{T},\mathcal{D}}(\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_5) = \mathcal{D} \circ [(\mathcal{C}_2 \circ \mathcal{R} \circ \mathcal{C}_1) \otimes (\mathcal{C}_5 \circ \mathcal{T} \circ \mathcal{C}_4 \circ \mathcal{S} \circ \mathcal{C}_3)] \circ \mathcal{E}, \quad (\text{C3})$$

and is illustrated in figure C.1(b).

Appendix D. Comparison with other frameworks

Our framework is based on the approach of Coecke, Fritz, and Spekkens [29], where the set of free operations is taken as the starting point from which the notion of resource is defined. An alternative approach is to start from a set of ‘zero resources’ and to define the free operations as those that preserve this set. For resource theories of quantum channels, this approach was adopted in references [32–34], where free channels were specified first, and free operations were defined as those supermaps that transform free channels into free channels.

In standard quantum Shannon theory, a natural choice for the set of free channels is the set of constant channels: no communication protocol in standard quantum Shannon theory can achieve communication using only constant channels. The set of supermaps that transform constant channels into constant

channels was characterised in reference [34], where the authors showed that a supermap preserves the set of constant channels if and only if it is of the form $\mathcal{S}(\mathcal{N}) = \sum_i c_i \mathcal{D}_i \circ \mathcal{N} \circ \mathcal{E}_i$, where \mathcal{E}_i and \mathcal{D}_i are suitable channels, and (c_i) are real (possibly negative) coefficients, such that the map $\sum_i c_i \mathcal{E}_i \otimes \mathcal{D}_i$ is a quantum channel. Physically, these supermaps correspond to the transformations that can be achieved with the assistance of free no-signalling channels between the sender's and receiver's locations.

Going from standard quantum Shannon theory to its extensions, it is not clear whether constant channels should still be regarded as free. Clearly, a placed constant channel is useless for communication, because it does not transfer any information from the sender's laboratory to the receiver's laboratory. Hence, placed constant channels should still be considered as free. On the other hand, an *unplaced* constant channel may still be useful, depending on how it interacts with the placements allowed by the theory. This is indeed what happens when the allowed placements include the quantum SWITCH [9].

One might insist that operations that transform constant channels (placed or unplaced) into non-constant channels should not be allowed in a resource theory of communication. This requirement would amount to the following:

Condition 1' (No activation of constant channels). In a resource theory of communication, no free operation $\mathcal{S} \in \mathbf{M}_{\text{free}}$ should be able to transform a constant channel into a non-constant channel.

Note that condition 1' (no activation of constant channels) is stronger than condition 1 (no side-channel generation). If a supermap violated condition 1, by allowing the sender and receiver to communicate independently of the input channels, then in particular it would allow the sender and receiver to communicate with constant channels, thus violating condition 1'. In fact, condition 1' is *strictly stronger* than condition 1. The quantum SWITCH placement transforms two completely depolarising channels into a non-constant channel [9], thereby violating condition 1'. On the other hand, the quantum SWITCH placement does not permit the sender and receiver to communicate independently of the input channels: for example, if the input channels are the completely depolarising channel and the identity, the quantum SWITCH placement outputs the channel

$$\mathcal{S}_{\text{SWITCH}}^{A,B,\omega}(\mathcal{N}_{\text{dep}}, \mathcal{I}) = \mathcal{N}_{\text{dep}} \otimes \omega, \quad (\text{D1})$$

which is constant and does not permit any communication. Hence, the quantum SWITCH placement satisfies condition 1, while it violates condition 1'.

One motivation for assuming condition 1' would be the idea that the communication provider could 'break' some of the available devices, by turning them into constant channels, *before placing them between the sender and receiver*. This pre-placement operation would be described by a *constant supermap*, of the form

$$\mathcal{S}^{\mathcal{N}_0} : \mathcal{N} \mapsto \mathcal{N}_0 \quad \forall \mathcal{N} \in \text{Chan}(X, Y), \quad (\text{D2})$$

where \mathcal{N}_0 is a constant channel. If such constant supermaps were allowed, then conditions 1 and 1' would become equivalent: a placement supermap that transforms some constant channel into a non-constant channel could be preceded by a constant supermap, thus enabling communication independently of the input channels. However, it is not obvious why constant supermaps should be regarded as free. Ultimately, assuming constant supermaps to be free is equivalent to assuming *by fiat* that constant channels are zero-resource channels, and therefore can be generated for free.

In summary, it is important to distinguish between two requirements: (a) constant channels should not be transformed into non-constant channels, and (b) it should not be possible to communicate independently of the input devices. While requirement (b) may still be too weak to guarantee that a resource theory of communication is interesting, it appears that there are interesting resource theories of communication that violate the requirement (a) and still lead to non-trivial Shannon-theoretic structures.

Appendix E. Discussion of the notion of side-channel proposed in reference [28]

The authors of reference [28] write 'The abstract way to know whether a process contains a "side-channel" is to look at the reduced process', which they define as the quantum channel obtained by inserting completely depolarising channels into the supermap under consideration. However, it is not clear why the reduced process should be defined in terms of completely depolarising channels, instead of arbitrary constant channels. The criterion for 'side-channels' proposed by the authors of reference [28] seems to be an incomplete version of condition 1' of the previous appendix: instead of having all constant channels, as in condition 1', they consider only completely depolarising channels.

After stating their criterion for side-channels, the authors of reference [28] continue by writing ‘We see that in the case of the quantum switch, the reduced process [...] is a quantum channel which allows for direct communication [...], no matter what noisy operations are being applied [...]’. While the first half of the sentence is correct, it is unclear how the second half should be interpreted: since the reduced channel was defined by applying completely depolarising channels, the fact that it allows for direct communication does not imply that the quantum SWITCH allows for communication ‘no matter what operations are being applied’. In fact, this statement is false: when a completely depolarising channel and the identity channel are applied, the reduced process of the quantum SWITCH is a constant channel, and does not allow for any communication, as shown by equation (D1) of the previous appendix.

Appendix F. Proof of proposition 3

Here we provide a proof of proposition 3. The proof follows from the following lemmas, proven at the end of this appendix.

Lemma 1. Let $\mathcal{N}_1 \in \text{Chan}(X)$ and $\mathcal{N}_2 \in \text{Chan}(X)$ be two quantum channels, and let $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$ be their vacuum extensions. Then, $\tilde{\mathcal{N}}_1 \circ \tilde{\mathcal{N}}_2$ is a vacuum extension of $\mathcal{N}_1 \circ \mathcal{N}_2$, and its vacuum interference operator is $F_1 F_2$, where F_1 (F_2) is the vacuum interference operator of \mathcal{N}_1 (\mathcal{N}_2).

Lemma 2. Let $F \in L(\mathcal{H}_X)$ be the vacuum interference operator associated to a generic vacuum-extended channel $\tilde{\mathcal{C}} \in \text{Chan}(\tilde{X})$. Then, one has $\|F\|_\infty \leq 1$.

Lemma 3. Let $\mathcal{C} \in \text{Chan}(X)$ be a quantum channel on a quantum system of dimension $d \geq 2$, and let $\tilde{\mathcal{C}} \in \text{Chan}(\tilde{X})$ be an arbitrary vacuum extension of \mathcal{C} . If the Choi operator

$$C := \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{C}(|i\rangle\langle j|), \quad (\text{F1})$$

has full rank, then the vacuum interference operator F associated to $\tilde{\mathcal{C}}$ satisfies the strict inequality $\|F\|_\infty < 1$.

Proof of proposition 3. Let $\tilde{\mathcal{N}}_{\text{dep}}$ be an arbitrary vacuum extension of the completely depolarising channel \mathcal{N}_{dep} . By lemma 1, the relation

$$\tilde{\mathcal{N}}_{\text{dep}} \circ \tilde{\mathcal{N}}_{\text{dep}} = \tilde{\mathcal{N}}_{\text{dep}} \quad (\text{F2})$$

implies the relation $F^2 = F$, where F is the vacuum interference operator of $\tilde{\mathcal{N}}_{\text{dep}}$. In turn, the relation $F^2 = F$ implies the relation $F = F^n$ for every integer $n \in \mathbb{N}$. In terms of the norm, this condition yields the bound

$$\begin{aligned} \|F\|_\infty &= \|F^n\|_\infty \\ &\leq \|F\|_\infty^n \quad \forall n \in \mathbb{N}. \end{aligned} \quad (\text{F3})$$

Now, the Choi operator of the completely depolarising channel \mathcal{N}_{dep} is $N = I \otimes I/d$ and has full rank. Hence, lemma 3 implies $\|F\|_\infty < 1$. Hence, equation (F3) implies $\|F\|_\infty = 0$, and therefore $F = 0$. In summary, the only vacuum extension satisfying the condition (F2) is the incoherent one. \square

Proof of lemma 1. Using equation (21) for the vacuum-extended channels $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$, we obtain the relation

$$(\tilde{\mathcal{N}}_1 \circ \tilde{\mathcal{N}}_2)(\rho) = (\mathcal{N}_1 \circ \mathcal{N}_2)(P_X \rho P_X) + \langle \text{vac} | \rho | \text{vac} \rangle |\text{vac}\rangle\langle \text{vac}| + F_1 F_2 \rho |\text{vac}\rangle\langle \text{vac}| + |\text{vac}\rangle\langle \text{vac}| \rho F_2^\dagger F_1^\dagger, \quad (\text{F4})$$

valid for every $\rho \in \text{St}(\tilde{X})$. From equation (F4) one can deduce that $\tilde{\mathcal{N}}_1 \circ \tilde{\mathcal{N}}_2$ is a vacuum extension of $\mathcal{N}_1 \circ \mathcal{N}_2$ (conditions (18) and (19) are satisfied). Moreover, comparison of equation (F4) with (21) shows that the vacuum interference operator of $\tilde{\mathcal{N}}_1 \circ \tilde{\mathcal{N}}_2$ is $F_1 F_2$. \square

Proof of lemma 2. By definition (22), the vacuum interference operator can be expressed as $F = \sum_i \bar{\gamma}_i C_i$, where $\{\tilde{C}_i := C_i \oplus \gamma_i |\text{vac}\rangle\langle \text{vac}|\}$ is an arbitrary Kraus representation of the vacuum-extended channel $\tilde{\mathcal{C}}$.

By definition, one has

$$\|F\|_\infty := \max_{\{|\phi\rangle \in \mathcal{H}_X, \|\phi\|=1\}} \|F|\phi\rangle\|. \quad (\text{F5})$$

Let $|\phi\rangle \in \mathcal{H}_X$ be a unit vector such that $\|F\|_\infty = \|F|\phi\rangle\|$. Then, one has the following series of (in)equalities:

$$\begin{aligned}
\|F\|_\infty^2 &= \|F|\phi\rangle\|^2 \\
&= \langle\phi|F^\dagger F|\phi\rangle \\
&= \sum_{ij} \gamma_i \bar{\gamma}_j \langle\phi|C_i^\dagger C_j|\phi\rangle \\
&\leq \sum_{ij} |\gamma_i| |\gamma_j| \left| \langle\phi|C_i^\dagger C_j|\phi\rangle \right| \\
&\leq \sum_{ij} |\gamma_i| |\gamma_j| \sqrt{\langle\phi|C_i^\dagger C_i|\phi\rangle \langle\phi|C_j^\dagger C_j|\phi\rangle} \\
&= \left(\sum_i |\gamma_i| \sqrt{\langle\phi|C_i^\dagger C_i|\phi\rangle} \right)^2 \\
&\leq \left(\sum_i \langle\phi|C_i^\dagger C_i|\phi\rangle \right)^2 \\
&= 1,
\end{aligned} \tag{F6}$$

where the first inequality is the triangle inequality for the modulus, and the second and third inequalities are Cauchy–Schwarz inequalities. \square

Proof of lemma 3. Lemma 2 shows that the norm of F is smaller than or equal to 1. The equality $\|F\|_\infty = 1$ holds if and only if all the inequalities in equation (F6) hold with the equality sign. In the following we will show that saturating the second inequality is impossible when C has full rank.

The second inequality in equation (F6) is saturated if and only if

$$C_i|\phi\rangle \propto C_j|\phi\rangle \quad \forall i, j, \tag{F7}$$

that is, if and only if

$$C_i|\phi\rangle = \lambda_i|\phi_0\rangle \quad \forall i, \tag{F8}$$

where $|\phi_0\rangle \in \mathcal{H}_X$ is a fixed unit vector, and $\{\lambda_i\}$ are complex numbers.

Let $A = \sum_i \alpha_i C_i$ be an arbitrary linear combination of the operators $\{C_i\}$, with complex coefficients $\{\alpha_i\}$. Then, one has

$$\begin{aligned}
A|\phi\rangle &= \sum_i \alpha_i C_i|\phi\rangle \\
&= \left(\sum_i \alpha_i \lambda_i \right) |\phi_0\rangle.
\end{aligned} \tag{F9}$$

In other words, every linear combination of the operators $\{C_i\}$ must map $|\phi\rangle$ into a vector proportional to $|\phi_0\rangle$.

Now, the Choi operator C has full rank if and only if the operators C_i are a spanning set for the vector space $L(\mathcal{H}_X)$. This means, in particular, that there exist coefficients $\{\alpha_i\}$ such that $A = \sum_i \alpha_i C_i = |\phi_0^\perp\rangle\langle\phi|$, where $|\phi_0^\perp\rangle$ is a unit vector orthogonal to $|\phi_0\rangle$ (such a vector exists because the Hilbert space \mathcal{H}_X is at least two-dimensional). In this case, one has $A|\phi\rangle = |\phi_0^\perp\rangle$, meaning that equation (F9) cannot be satisfied. This implies that equation (F7) cannot be satisfied, and that the bound $\|F\|_\infty \leq 1$ cannot hold with the equality sign. \square

Appendix G. Reply to the interferometric arguments of Guérin *et al*

Reference [28] provides an interferometric implementation of the SDPP supermap \mathcal{F} defined by

$$\mathcal{F}(\mathcal{N}_1, \mathcal{N}_2)(\rho) = [(\mathcal{N}_2 \circ \mathcal{N}_1) \otimes \mathcal{I}_C] \circ \mathcal{U}^{\text{CNOT}}(\rho \otimes \omega),$$

arguing that in this implementation it cannot be said that the map \mathcal{F} transfers information from the message to the control before the noisy channels are applied.

The proposal of reference [28] is shown on the right-hand side of figure G.1. It consists of an interferometer with two spatial models 0 and 1. On mode 0, the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$ is applied right away, while in mode 1 it is applied after a NOT gate.

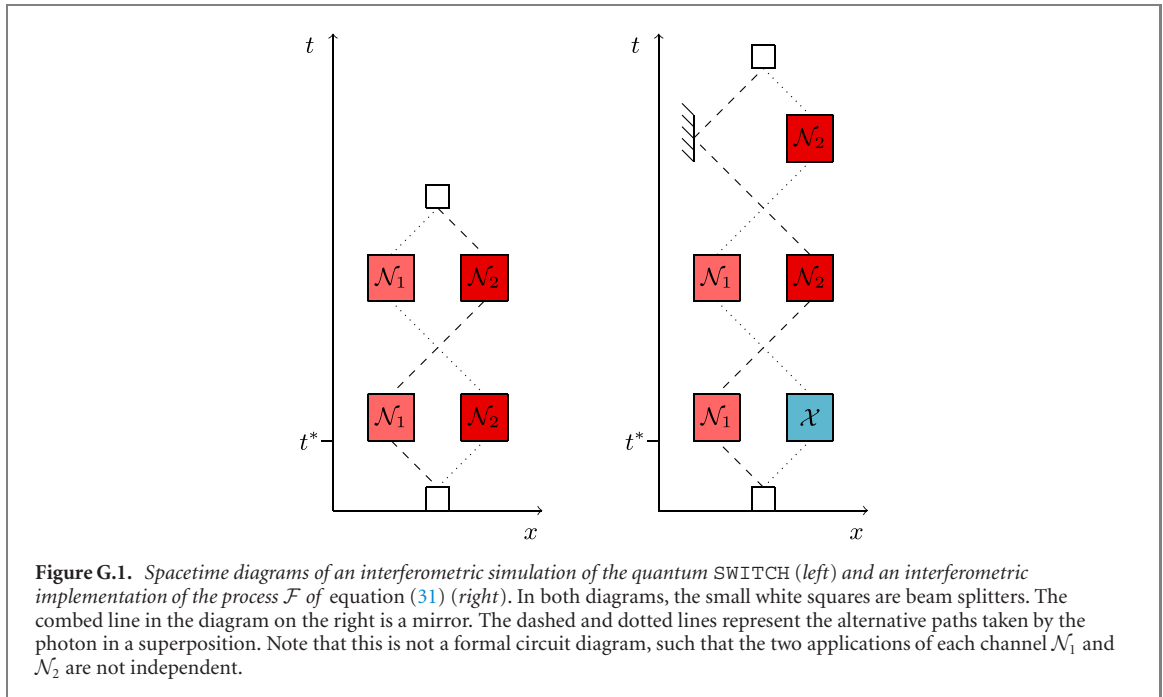


Figure G.1. Spacetime diagrams of an interferometric simulation of the quantum SWITCH (left) and an interferometric implementation of the process \mathcal{F} of equation (31) (right). In both diagrams, the small white squares are beam splitters. The combed line in the diagram on the right is a mirror. The dashed and dotted lines represent the alternative paths taken by the photon in a superposition. Note that this is not a formal circuit diagram, such that the two applications of each channel \mathcal{N}_1 and \mathcal{N}_2 are not independent.

Note that the time of application of the channel $\mathcal{N}_2 \circ \mathcal{N}_1$ depends on the mode: the channel is applied at an earlier time on mode 0, and at a later time on mode 1. Since the control is in a superposition state, the message is sent through both modes in a coherent superposition, and therefore the time of application of the channel $\mathcal{N}_2 \circ \mathcal{N}_1$ ends up in a coherent superposition.

In this particular implementation, the NOT gate on mode 1 takes place at the same time as the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$ on mode 0. One could also arrange the setup in such a way that the NOT gate on mode 1 takes place before or after the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$ on mode 0. The authors of reference [28] argue that ‘this already shows an ambiguity regarding whether [the NOT gate] should be considered as part of the “encoding” or not’.

We point out, however, that it is misleading to compare the time when the NOT gate takes place *on mode* 1 with the time when the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$ takes place *on mode* 0. Instead, one should compare the times on the same mode: in all possible implementations, the NOT gate on mode 1 takes place before the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$.

The authors of reference [28] appear to have missed the fact that a quantum particle can be sent through a noisy channel at a superposition of different times, and therefore, the encoding operations performed before the transmission can also take place at a superposition of different times. The times of application of the encoding operations and of the noisy channel can be different in different branches of the superposition, but the fact that the encoding causally precedes the noisy channel is true in all branches, and is independent of the specific implementation of the SDPP supermap \mathcal{F} .

The authors of reference [28] insist that the ‘encoding’ should be defined as the set of operations that take place before a given time, *in all branches of the superposition* (specifically, they choose the time denoted as t^* in figure G.1). Starting from this premise, they compare the interferometric implementation of the SDPP supermap \mathcal{F} with an interferometric simulation of the quantum SWITCH, shown on the left-hand side of figure G.1, and claim that ‘the encoding [for the supermap \mathcal{F}] is the same as for the switch’. Instead, if one takes into account that the transmission through the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$ happens at a superposition of two different times, depending on the modes, then the encoding operations are completely different:

- For the quantum SWITCH, one has the encoding

$$\mathcal{E}(\rho) = \rho \otimes \omega, \quad (\text{G1})$$

which does not transfer any information from the message to the control,

- For the supermap \mathcal{F} , one has the encoding

$$\mathcal{E}(\rho) = U^{\text{CNOT}}(\rho \otimes \omega)U^{\text{CNOT}}, \quad (\text{G2})$$

which transfers information from the message to the control, thereby exploiting the control as a side-channel that completely bypasses the noisy channel $\mathcal{N}_2 \circ \mathcal{N}_1$.

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