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ORGANIC COMBINATIONS OF QUERMASSINTEGRALS AND DUAL QUERMASSINTEGRALS

CHANG-JIAN ZHAO AND WING-SUM CHEUNG

ABSTRACT. We establish Minkowski and Brunn-Minkowski inequalities for differences of quermassintegrals and dual quermassintegrals of convex and star bodies, respectively.

1. Introduction. The well-known classical Brunn-Minkowski inequality can be stated as follows:

If K and L are convex bodies in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1.1}$$

with equality if and only if K and L are homothetic.

The Brunn-Minkowski inequality has in recent decades dramatically extended its influence in many areas of mathematics. Various applications have surfaced, for example, to probability and multivariate statistics, shape of crystals, geometric tomography, elliptic partial differential equations, and combinatorics [**1, 5, 8, 10, 18**]. Several remarkable analogs have been established in other areas, such as potential theory and algebraic geometry [**3, 4, 6, 7, 12, 14, 15, 17, 19, 21**]. Reverse forms of the inequality are important in the local theory of Banach spaces [**18**]. An elegant survey on this inequality is provided by Gardner [**11**]. A general version of the Brunn-Minkowski's inequality holds: If K and L are convex bodies in $\mathbb{R}^n$ and $0 \leq i \leq n-1$, then

$$W_i(K+L)^{\frac{1}{n-1}} \geq W_i(K)^{\frac{1}{n-1}} + W_i(L)^{\frac{1}{n-1}}, \tag{1.2}$$

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with equality if and only if K and L are homothetic [**11**].

Leng [**13**] defined the quermassintegral difference function in 2004 by

$$Dw_i(K,D) = W_i(K) - W_i(D),$$

where K and D are convex bodies, $D \subset K$, and $0 \leq i \leq n-1$.

Inequality (1. 2)was extended to the quermassintegral difference of convex bodies as follows:

Theorem A. [**13**] *If K, L, and D are convex bodies in $\mathbb{R}^n$, $D \subset K$, and D' is a homothetic copy of D, then*

$$Dw_i(K+L, D+D')^{\frac{1}{n-1}} > Dw_i(K,D)^{\frac{1}{n-1}} + Dw_i(L,D')^{\frac{1}{n-1}}, \tag{1.3}$$

with equality for $0 \leq i < n-1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Lv [**24**] defined the dual quermassintegral difference function in 2010 by

where K and D are star bodies and $D \subset K$. Dual Brunn-Minkowski-type inequalities for the dual quermassintegral difference were also established in [**24**].

Motivated by the work of Leng and Lv, we give the following difference definition for the quermassintegral and dual quermassintegral:

Definition. Let K be a convex body and D be a star body in $\mathbb{R}^n$, with $D \subseteq K$. The *mixed quermassintegral difference function* of K and D is defined for $0 \leq i \leq n-1$ by

(1.4)

If $D = K$, then

$$Dw_i^*(K,D) = Dw_i^*(K,K) \geq 0,$$

since $W_i(K) \geq \tilde{W}_i(K)$ with equality if and only if K is a ball [**16**]. In view of $D \subset K$, we have $W_i(K) \geq \tilde{W}_i(K) \geq \tilde{W}_i(D)$. This shows the quantity $Dw_i^*(K,D)$ is negative. Taking $i = 0$ in (1.4) gives

$Dv^*(K,D) = V(K) - V(D)$, which is called mixed volume difference function of a convex body K and a star body D.

The first aim of this paper is to establish the Brunn-Minkowski inequality for the quermassintegral and the dual quermassintegral difference of convex and star bodies.

Theorem 1.1. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subset K$, $D' \subset L$ and $0 \leq i < n-1$, then*

$$(1.5) \quad (W_i(K+L) - \tilde{W}_i(D \tilde{+} D'))^{\frac{1}{n-1}} \geq (W(K) - \tilde{W}(D))^{\frac{1}{n-1}} + (W(L) - \tilde{W}(D'))^{\frac{1}{n-1}},$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(K), \tilde{W}_i(D)) = \mu(W_i(L), \tilde{W}_i(D'))$, where μ is a constant.

Remark. If D and D' are single points, (1.5) reduces to the classical Brunn-Minkowski inequality (1.2).

For two convex bodies K and L, an important inequality on the mixed volume is the well-known Minkowski inequality

(1.6)

with equality if and only if K and L are homothetic.

Inequality (1.6) was extended to volume differences in 2004:

Theorem B. [13] *Suppose that K and D are compact domains and L is a convex body. If $D \subset K$, $D' \subset L$, and D' is a homothetic copy of D, then*

$$(1.7) \quad (V_1(K,L) - V_1(D,D'))^n \geq (V(K) - V(D))^{n-1}(V(L) - V(D')),$$

with equality if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

The second aim of this paper is to establish the Minkowski inequality for the quermassintegral and the dual quermassintegral difference of convex and star bodies.

Theorem 1.2. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subseteq K, D' \subseteq L$, and $0 \leq i < n-1$, then*

$$(1.8) \quad (W_i(K,L) - \tilde{W}_i(D,D'))^{n-i} \geq (W_i(K) - \tilde{W}_i(D))^{n-i-1}(W_i(L) - \tilde{W}_i(D')),$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(K), \tilde{W}_i(D)) = \mu(W_i(L), \tilde{W}_i(D'))$, where μ is a constant.

Remark. If D and D' are single points, (1.8) reduces to the classical Minkowski inequality

$$(1.9) \qquad W_i(K,L)^{n-i} \geq W_i(K)^{n-i-1}W_i(L),$$

with equality if and only if K and L are homothetic. Taking $i = 0$ in (1.9) yields (1.6).

Please refer to the next section for the above interrelated notations, definitions and background.

2. Definitions and preliminaries. The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$, $n > 2$. Let $\mathcal{K}^n$ denote the set of convex bodies (that is compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors and the letter B for the unit ball centered at the origin. The boundary of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane through the origin that is orthogonal to u. For any $K \in \mathcal{K}^n$, we will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of the convex body K. The support function $h(K,\cdot)$ of $K \in \mathcal{K}^n$ is defined on $\mathbb{R}^n$ by $h(K,\cdot) = \max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$, namely, for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$ by $\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. The set of star bodies in $\mathbb{R}^n$ is denoted as $\mathcal{S}^n$.

2.1. Dual quermassintegrals. We introduce a vector addition on $\mathbb{R}^n$, which we call radial addition, as follows. If $x_1, \dots, x_r \in \mathbb{R}^n$, then $x_1 \tilde{+} \dots \tilde{+} x_r$ is defined to be the usual vector sum of $x_1, \dots, x_r$, provided that $x_1, \dots, x_r$ all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise.

If $K_1, \dots, K_r \in \mathcal{S}^n$ and $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, then *the radial Minkowski linear combination*, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$. It has the important property that for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$,

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot). \tag{2.1}$$

For $K_1, \dots, K_r \in \mathcal{S}^n$ and $\lambda_1, \dots, \lambda_r > 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i, defined by

$$V(\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1, \dots, i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.2}$$

where the sum is taken over all n-tuples $(i_1, \dots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (2.2) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1, \dots, i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \dots, K_{i_n}$. It is written as $\tilde{V}(K_{i_1}, \dots, K_{i_n})$ and is called the dual mixed volume of $K_{i_1}, \dots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volumes is written as $\tilde{V}_i(K, L)$. The dual mixed volume

$$\tilde{V}(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B, \dots, B}_{i}, L)$$

is written as $\tilde{W}_i(K, L)$, and $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$ and is called the dual quermassintegral.

For any $K \in \mathcal{S}^n$, an integral corresponding form of the dual quermassintegral is

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{2.3}$$

2.2. Quermassintegrals. For $K_1, \dots, K_r \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_r \geq 0$, the volume of the Minkowski linear combination $\lambda_1 K_1 + \dots + \lambda_r K_r$ is a homogeneous polynomial of degree n in the λ_i, given by

$$V(\lambda_1 K_1 + \ldots + \lambda_r K_r) = \sum V_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.4}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (2.4) to be symmetric in their arguments, then they are uniquely determined. The coefficient $V_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as $V(K_{i_1}, \ldots, K_{i_n})$ and is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the mixed volume is written as $V_i(K, L)$. The mixed volume

$$V(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i}, L)$$

is written as $V_i(K, L)$, and $V_i(K, B)$ is written as $W_i(K)$ and is called the quermassintegral.

3. Main results. In order to prove main results, we need some lemmas.

Lemma 3.1. [2] *Let*

$$\phi(x) = (x_1^p - x_2^p - \cdots - x_n^p)^{1/p}, \quad p > 1,$$

with x_i in the region $\mathbb{R}$ defined by

$$x_i \geq 0 \quad \text{and} \quad x_1 \geq (x_2^p + x_3^p + \cdots + x_n^p)^{1/p}. \tag{3.1}$$

For $x, y \in \mathbb{R}$ satisfying (3.1), *we have*

$$\phi(x+y) \geq \phi(x) + \phi(y), \tag{3.2}$$

with equality if and only if $x = \mu y$ where μ is a constant.

Lemma 3.2. [23] *Let $a, b, c, d > 0, 0 < \alpha < 1, 0 < \beta < 1$ and $\alpha + \beta = 1$. If $a > b$ and $c > d$, then*

$$a^\alpha c^\beta - b^\alpha d^\beta \geq (a-b)^\alpha (c-d)^\beta, \tag{3.3}$$

with equality if and only if $ad = bc$.

Theorem 3.3. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subset K$, $D' \subset L$ and $0 \leq i < n-1$, then*

$$(3.4)\quad Dw_i^*(K+L, D\tilde{+}D')^{\frac{1}{n-i}} > Dw_i^*(K,D)^{\frac{1}{n-i}} + Dw_i^*(L,D')^{\frac{1}{n-i}},$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(K), \tilde{W}_i(D)) = \mu(W_i(L), \tilde{W}_i(D'))$, where μ is a constant.

Proof. From the Brunn-Minkowski and dual Brunn-Minkowski inequalities [**14, 22**], we have for $K, L \in \mathcal{K}^n$,

$$W_i(K+L)^{\frac{1}{n-i}} > W_i(K)^{\frac{1}{n-i}} + W_i(L)^{\frac{1}{n-i}}$$

with equality if and only if K and L are homothetic. If $D, D' \in \mathcal{S}^n$, then

$$\tilde{W}_i(D\tilde{+}D')^{\frac{1}{n-i}} < \tilde{W}_i(D)^{\frac{1}{n-i}} + \tilde{W}_i(D')^{\frac{1}{n-i}},$$

with equality if and only if D and D' are dilates.

Hence, we obtain

$$(3.5)\quad Dw_i^*(K+L, D\tilde{+}D') > \left[W_i(K)^{\frac{1}{n-i}} + W_i(L)^{1/(i-1)}\right]^{n-i} - \left[\tilde{W}_i(D)^{\frac{1}{n-i}} + \tilde{W}_i(D')^{\frac{1}{n-i}}\right]^{n-i},$$

with equality if and only if K and L are homothetic, and D and D' are dilates.

By using Lemma 3.1, we have

$$\begin{aligned}
&(Dw_i^*(K+L, D\tilde{+}D'))^{\frac{1}{n-i}} \\
&\quad \geq \left\{\left[W_i(K)^{\frac{1}{n-i}} + W_i(L)^{\frac{1}{n-i}}\right]^{n-i} - \left[\tilde{W}_i(D)^{\frac{1}{n-i}} + \tilde{W}_i(D')^{\frac{1}{n-i}}\right]^{n-i}\right\}^{\frac{1}{n-i}} \\
&\quad \geq (W_i(K) - \tilde{W}_i(D))^{\frac{1}{n-i}} + (W_i(L) - \tilde{W}_i(D'))^{\frac{1}{n-i}} \\
&\quad = Dw_i^*(K,D)^{\frac{1}{n-i}} + Dw_i^*(L,D')^{\frac{1}{n-i}}.
\end{aligned}$$

In view of the equality conditions of inequalities (3.2) and (3.5), it follows that (3.4) holds if and only if K and L are homothetic, D and D' are dilates, and $(W_i(K), \tilde{W}_i(D)) = \mu(W_i(L), \tilde{W}_i(D'))$, where μ is a constant. $\square$

Taking $i = 0$ in inequality (3.4), we have:

Corollary. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subset K$ and $D' \subset L$, then*

$$Dv^*(K+L, D\tilde{+}D')^{1/n} \geq Dv^*(K,D)^{1/n} + Dv^*(L,D')^{1/n}, \tag{3.6}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

If D and D' are single points in (3.6), then (3.6) reduces to the classical Brunn-Minkowski inequality (1.1).

Theorem 3.4. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D \subseteq K$, $D' \subseteq L$ and $0 \leq i < n-1$, then*

$$(W_i(K,L) - \tilde{W}_i(D,D'))^{n-i} \geq Dw_i^*(K,D)^{n-i-1} Dw_i^*(L,D'), \tag{3.7}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(W_i(K), \tilde{W}_i(D)) = \mu(W_i(L), \tilde{W}_i(D'))$, where μ is a constant.

Proof. From the Minkowski and dual Minkowski inequalities, we have

with equality if and only if K and L are homothetic. If $D, D' \in \mathcal{S}^n$, then

with equality if and only if D and D' are dilates.

Hence

$$(W_i(K,L) - \tilde{W}_i(D,D')) \geq W_i(K)^{\frac{n-i-1}{n-1}} W_i(L)^{\frac{1}{n-1}} \tag{3.8}$$

with equality if and only if K and L are homothetic and D and D' are dilates.

By using Lemma 3.2, we have

$$\begin{aligned}
&(W_i(K,L)-\tilde{W}_i(D,D'))^{n-i}\\
&\qquad \geq \left[W_i(K)^{\frac{n-i-1}{n-1}}W_i(L)^{\frac{1}{n-1}}-\tilde{W}_i(D)^{\frac{n-i-1}{n-1}}\tilde{W}_i(D')^{\frac{1}{n-1}}\right]^{n-i}\\
&\qquad \geq (W_i(K)-\tilde{W}_i(D))^{n-i-1}(W_i(L)-\tilde{W}_i(D'))\\
&\qquad = Dw_i^*(K,D)^{n-i-1}Dw_i^*(L,D').
\end{aligned}$$

In view of the equality conditions of inequality (3.3) and inequality (3.8), it follows that (3.7) holds if and only if K and L are homothetic, D and D' are dilates, and $(W_i(K),\tilde{W}_i(D))=\mu(W_i(L),\tilde{W}_i(D'))$, where μ is a constant. □

Taking $i=0$ in Theorem 3.4, we get the following result.

Corollary. *Let K and L be convex bodies, and D and D' be star bodies in $\mathbb{R}^n$. If $D\subseteq K$ and $D'\subseteq L$, then*

$$(V_1(K,L)-\tilde{V}_1(D,D'))^n \geq Dv^*(K,D)^{n-1}Dv^*(L,D'), \tag{3.9}$$

with equality if and only if K and L are homothetic, D and D' are dilates, and $(V(K),V(D))=\mu(V(L),V(D'))$, where μ is a constant.

If D and D' are single points, (3.9) reduces to the classical Minkowski inequality (1.6).

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DEPARTMENT OF MATHEMATICS, CHINA JILIANG UNIVERSITY, HANGZHOU 310018, P. R. CHINA
Email address: **chjzhao@163.com, chjzhao@aliyun.com**

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF HONG KONG, POKFULAM ROAD, HONG KONG
Email address: **wscheung@hku.hk**