

A constraint-free approach to optimal reinsurance

Hans U. Gerber^a, Elias S.W. Shiu^b, Hailiang Yang^c

^aDepartment of Statistics and Actuarial Science, The University of Hong Kong,
and Department of Actuarial Science, Faculty of Business and Economics,
University of Lausanne, CH-1015 Lausanne, Switzerland, e-mail: hgerber@unil.ch;

^bDepartment of Statistics and Actuarial Science, The University of Iowa,
Iowa City, Iowa 52242-1409, U.S.A., e-mail: elias-shiu@uiowa.edu;

^cDepartment of Statistics and Actuarial Science, The University of Hong Kong,
Pokfulam Road, Hong Kong, e-mail: hlyang@hku.hk

Abstract

Reinsurance is available for a reinsurance premium that is determined according to a convex premium principle H . The first insurer selects the reinsurance coverage that maximizes its expected utility. No conditions are imposed on the reinsurer's payment. The optimality condition involves the gradient of H . For several combinations of H and the first insurer's utility function, closed form formulas for the optimal reinsurance are given. If H is a zero utility principle (for example, an exponential principle or an expectile principle), it is shown, by means of Borch's Theorem, that the optimal reinsurer's payment is a function of the total claim amount and that this function satisfies the so-called 1-Lipschitz condition. Frequently, authors impose these two conclusions as hypotheses at the outset.

Keywords: Optimal reinsurance; expected utility; convex premium principle; Borch's theorem; Pareto-optimal risk exchange; constraint-free approach.

1 Introduction

Since the pioneering work by Borch (1960a, b, c, 1962) and Arrow (1963), there has been much research on optimal reinsurance. Almost all papers assume from the outset that there is an *indemnity function* (also called *ceded loss function* or *coverage function*). That is, there exists a function f such that if x is the realized loss, $f(x)$ is the indemnity paid by the reinsurer (many papers use the symbol $I(x)$ for this function). Some impose the restriction that $0 \leq f(x) \leq x$. Others assume that $f(0) = 0$ and

$$0 \leq f(x) - f(y) \leq x - y \quad \text{for } x \geq y \geq 0. \quad (1.1)$$

These constraints are imposed to avoid moral hazards. For example, if the second inequality sign in (1.1) were "greater than" for some x and y , $x > y \geq 0$, then the first

insurer would have an incentive to create an incremental loss, $x - y$, and, as a result, it could receive $f(x) - f(y)$.

Although these constraints are natural requirements, they can complicate the search for the optimal reinsurance. As one learns in calculus, unconstrained optimization is easier than constrained optimization. In this paper, we do not impose such assumptions *a priori*. A goal of this largely self-contained paper is to show, in a reader-friendly manner, that if the first insurer maximizes its expected utility and if the reinsurance premium is determined by a zero utility principle, the following conclusion is reached: There exists a function f such that if x is the realized loss, $f(x)$ is the optimal indemnity. Because the derivative of f is bounded between 0 and 1, constraint (1.1) is satisfied. See Section 5.

We should note that in certain papers, these assumptions are not made explicitly. See, for example, Moffet (1979) and Aase (2004b). They consider only reinsurance treaties that are equivalent to Pareto optimal risk exchanges (for which these assumptions are always satisfied). Also, under the assumption that the premium is determined by *Wang's premium principle*, Young (1999) has derived (1.1).

Buying reinsurance involves a compromise between safety and profitability. A rational reinsurance policy increases the safety of the first insurer at the expense of its profitability. There are two popular ways to reach a decision under these two conflicting objectives. One is to formulate a criterion for each objective, perhaps in the form of a score. Then, the first insurer would fix one score and maximize (or minimize) the other. The other is to measure the quality of reinsurance by means of expected utility, which combines safety and profitability in a single function. The first insurer would seek a reinsurance policy to maximize its expected utility. This is the approach considered in this paper.

In a classical paper, Arrow (1963) proved that stop loss is optimal under the criterion of maximizing the expected utility of the end-of-period wealth of the first insurer when the expected value principle is used to calculate the reinsurance premium. The ideas of Arrow (1963) have been extended in various directions. For example, Young (1999) has generalized Arrow's result to the situation where the premium is determined by Wang's premium principle. In each reinsurance contract, there are two parties, the first insurer and the reinsurer. As the two parties have conflicting interests, Borch (1969, p. 295) suggested that "the *optimal contract* must then appear as a reasonable compromise between these interests." Thus we incorporate the preferences of both the first insurer and the reinsurer in the optimization problem formulation. Earlier work in this direction includes Borch (1960c) and Gerber (1978); for recent studies, see Golubin (2008) and D'Ortona and Marcarelli (2017) and the references therein. For this class of problems, the Pareto-optimality setup seems natural. See also Asimit and Boonen (2018), Asimit et al. (2018), Cai et al. (2017), and Lo and Tang (2018). A brief review of Pareto-optimal risk exchanges can be found at the beginning of Section 5.

The literature on reinsurance in general, and optimal reinsurance in particular, has grown phenomenally in recent years. The latest book on reinsurance (Albrecher et al., 2017) has a list of 812 references. Although not every entry on the list is about reinsurance or optimal reinsurance, flipping through these 37 pages of titles is indeed a mind-expanding experience. The authors are to be commended for such an impressive book.

The heart of this paper is in Sections 4 and 5. Section 2 serves as a preparation in an attempt to make the paper largely self-contained. Premium principles are explained, in particular the expectile principle, which can be derived as a zero utility principle. It is noted that the underlying risk aversion function is a multiple of Dirac's delta function. Subsection 2.3 reviews an important tool for optimization, the "gradient" of a functional. Section 3 presents a result of independent interest: a new characterization of the expectile principle.

In Section 4, the objective is to maximize the expected utility of a first insurer, when the reinsurance premium is determined from a zero utility principle. The model can be considered as an extension of that in Arrow (1963), as it includes both insurer's utility function and the reinsurer's utility function. A general optimality criterion has been obtained in Deprez and Gerber (1985) under this setup. In four particular cases, closed-form expressions for the optimal reinsurer's payment and the reinsurer's premium are obtained.

Section 5 provides additional insight for an optimal reinsurance contract. Any reinsurance contract can be considered as a risk exchange between first insurer and reinsurer. An optimal reinsurance contract satisfies the optimality condition given in Deprez and Gerber (1985). But this is precisely the condition of *Borch's Theorem*. It follows that an optimal reinsurance contract is one particular Pareto-optimal risk exchange. Then we show that there exists a function f , with $0 \leq f(s) \leq s$ and $0 \leq f'(s) \leq 1$, such that the optimal reinsurer's payment is the image, under f , of the total claim amount. Thus, what many authors postulate as constraints are now obtained as a result of unconstrained optimization.

2 Preliminaries

2.1 Premium principles

Premium principles were originally introduced by Bühlmann (1970) under the term *principles of premium calculation*. A premium principle is a functional H that assigns a premium P to any risk X (a random variable with a known distribution), $P = H(X)$. The following three examples are popular.

1. *Variance principle*:

$$P = E[X] + \alpha \text{Var}(X), \quad (\alpha > 0).$$

The loading $P - E[X]$ is proportional to the variance of the risk.

2. *Standard deviation principle*:

$$P = E[X] + \beta \sqrt{\text{Var}(X)}, \quad (\beta > 0).$$

The loading is proportional to the standard deviation.

3. *Principles of zero utility*: Let $u(x)$ be a *utility function*, that is, a function that is strictly increasing and concave. Then P is determined as the solution of the equation

$$\mathbb{E}[u(P - X)] = u(0). \quad (2.1)$$

This is the condition that the expected utility with the new contract is the same as $u(0)$, the utility without the new contract. A prominent special case is where

$$u(x) = \frac{1}{a}(1 - e^{-ax}), \quad (a > 0). \quad (2.2)$$

Then (2.1) can be solved explicitly and we find that

$$P = \frac{1}{a} \ln \mathbb{E}[e^{aX}]. \quad (2.3)$$

This is known as the *exponential principle*. Another prominent special case will be discussed in the next section.

The *risk aversion function* associated to a utility function $u(x)$ is defined as

$$r(x) = -\frac{d \ln u'(x)}{dx}. \quad (2.4)$$

We note that the risk aversion function of (2.2) is $r(x) = a$, constant.

Remark 2.1: Families of premium principles and some of their properties are presented in Goovaerts et al. (1984), Young (2004) and Goovaerts et al. (2010). For the convex premium principles, see Deprez and Gerber (1985), and for the role of convexity in the context of mathematical finance, see Föllmer and Schied (2002, 2016).

2.2 The expectile principle

Let

$$u(x) = \begin{cases} (1 + \theta)x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases} \quad (2.5)$$

This is a “refracted” linear function. The parameter θ is positive. Condition (2.1) can now be written as

$$\mathbb{E}[(P - X)_+] = (1 + \theta)\mathbb{E}[(X - P)_+], \quad (2.6)$$

which has an appealing interpretation: The expected gain of the contract should be a multiple $(1 + \theta)$ of the expected loss of the contract. Condition (2.6) can be rewritten as

$$P = \mathbb{E}[X] + \theta\mathbb{E}[(X - P)_+]. \quad (2.7)$$

Thus the loading is proportional to the expected loss and the parameter θ has the role of a *loading factor*. We might call this principle the *expected loss principle*. However, we

prefer the name expectile principle for reasons that are explained in Remark 2.2 below. Furthermore, condition (2.6) can also be rewritten as

$$P = E[X] + \frac{\theta}{1 + \theta} E[(P - X)_+]. \quad (2.8)$$

In this sense, the principle could also be called the *expected gain principle*. We note that $u(x)$ in (2.5) is also $x - \theta x_-$ and $(1 + \theta)x - \theta x_+$. Upon substitution in (2.1), (2.7) and (2.8) are obtained.

What can be said about the risk aversion function $r(x)$? For $b < c$ and any utility function $u(x)$, we have

$$\int_b^c r(x) dx = -\ln u'(c) + \ln u'(b). \quad (2.9)$$

Let $u(x)$ as in (2.5) and suppose that $b < 0 < c$. Then

$$\int_b^c r(x) dx = \ln(1 + \theta). \quad (2.10)$$

This shows that

$$r(x) = \ln(1 + \theta)\delta(x), \quad (2.11)$$

where $\delta(x)$ is the *Dirac delta function*.

Remark 2.2: Let $0 < \alpha < 1/2$. The α -percentile is the number z that minimizes

$$\alpha E[(z - X)_+] + (1 - \alpha) E[(X - z)_+].$$

Similarly, the α -expectile is the number z that minimizes

$$\alpha E[(z - X)_+^2] + (1 - \alpha) E[(X - z)_+^2].$$

If we set the derivative equal to 0 we obtain (2.6) with $P = z$ and $1 + \theta = \frac{1 - \alpha}{\alpha}$. An early reference for this asymmetric least square value is Newey and Powell (1987).

Remark 2.3: Formula (2.7) reminds us of the *Dutch premium principle*, where however the loading is proportional to $E[(X - \alpha E[X])_+]$ for some $\alpha \geq 1$. See Young (2004). For $\alpha = 1$, the Dutch principle has been generalized by Fischer (2003). See McNeil et al. (2015, page 77). Similarly, the expectile principle can be generalized such that (2.7) is replaced by

$$P = E[X] + \theta_1 E[(X - P)_+]^{1/p}, \quad (2.12)$$

or (2.8) by

$$P = E[X] + \theta_2 E[(P - X)_+]^{1/p}. \quad (2.13)$$

Note that (2.12) and (2.13) are not equivalent unless $p = 1$.

Remark 2.4: For a general utility function $u(x)$, condition (2.1) can be written in the spirit of (2.6). Assume $u(0) = 0$. We set

$$w(x) = \frac{u(x)}{x}. \quad (2.14)$$

Then (2.1) can be written as

$$\mathbb{E}[(P - X)w(P - X)] = 0. \quad (2.15)$$

Note that $w(x)$ is the slope of the chord connecting the origin with the point $(x, u(x))$. Hence $w(x)$ is a positive and decreasing function. It can be interpreted as a *weight function*.

Remark 2.5: One might be tempted to start with any positive and decreasing function $w(x)$ and define P from (2.15). But, unless $w(x)$ is derived from a utility function as in (2.14), the resulting premium principle might have some undesirable properties. For example, if $w(x) = e^{-ax}$, $a > 0$, we obtain

$$P = \frac{\mathbb{E}[Xe^{aX}]}{\mathbb{E}[e^{aX}]}, \quad (2.16)$$

the *Esscher premium* with parameter a , which has been criticized for some of its properties. See Gerber (1981).

2.3 Directional derivatives of a premium principle

The “gradient” tells us how the premium reacts to small variations of the risk. It is a useful tool to analyze certain optimization problems. Let $H(X)$ be a premium principle. Let $H'(X)$ denote its gradient, if it exists. This random variable has the property that

$$\frac{d}{dt}H(X + tV) |_{t=0} = \mathbb{E}[H'(X)V]. \quad (2.17)$$

Let us revisit the three examples of Section 2. For the variance principle we have

$$H(X + tV) = \mathbb{E}[X + tV] + \alpha \text{Var}[X + tV]. \quad (2.18)$$

Because

$$\text{Var}[X + tV] = \text{Var}[X] + 2t\text{Cov}(X, V) + t^2\text{Var}[V] \quad (2.19)$$

and

$$\text{Cov}(X, V) = \mathbb{E}[(X - \mathbb{E}[X])V], \quad (2.20)$$

we find that

$$\frac{d}{dt}H(X + tV) |_{t=0} = \mathbb{E}[V] + 2\alpha\mathbb{E}[(X - \mathbb{E}[X])V]. \quad (2.21)$$

This shows that

$$H'(X) = 1 + 2\alpha(X - \mathbb{E}[X]). \quad (2.22)$$

Similarly, the gradient of the standard deviation principle turns out to be

$$H'(X) = 1 + \beta \frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}[X]}}. \quad (2.23)$$

For a principle of zero utility, we find that

$$H'(X) = \frac{u'(P - X)}{\mathbb{E}[u'(P - X)]}. \quad (2.24)$$

Thus the gradient of the exponential principle is

$$H'(X) = \frac{e^{aX}}{\mathbb{E}[e^{aX}]}, \quad (2.25)$$

and the gradient of the expectile principle assumes only two values,

$$H'(X) = \begin{cases} (1 + \theta)/\mathbb{E}[u'(P - X)] & \text{if } P < X, \\ 1/\mathbb{E}[u'(P - X)] & \text{if } P > X, \end{cases} \quad (2.26)$$

with a discontinuity at $P = X$.

For further discussion, see Promislow and Young (2005), especially Sections 4 and 5. The approach in their paper is mathematically rigorous and comprehensive.

3 A characterization of the expectile principle

We recall four properties that a premium principle might have.

- (i) translation invariance: $H(X + c) = H(X) + c$,
- (ii) monotonicity: $X \leq Y$ implies $H(X) \leq H(Y)$,
- (iii) positive homogeneity: $H(aX) = aH(X)$ if $a > 0$,
- (iv) subadditivity: $H(X + Y) \leq H(X) + H(Y)$.

If a principle satisfies all four properties, it is called *coherent* (Artzner et al., 1999).

Theorem: For each zero utility principle, the following three statements are equivalent.

- (a) It is an expectile principle.
- (b) It is coherent.
- (c) It is positively homogeneous.

Proof: (b) \Rightarrow (c) is obvious.

To show (a) \Rightarrow (b), we note that properties (i) and (ii) are satisfied by any zero utility principle, and that property (iii) is obvious from (2.6). To show property (iv), consider the functional

$$\phi(X, x) = E[X] + \theta E[(X - x)_+] - x, \quad (3.1)$$

with X being a risk, x a number, and $\theta > 0$. From the inequality $(a_1)_+ + (a_2)_+ \geq (a_1 + a_2)_+$, it follows that for any pair of risks X_1 and X_2 and any pair of numbers x_1 and x_2 ,

$$\phi(X_1, x_1) + \phi(X_2, x_2) \geq \phi(X_1 + X_2, x_1 + x_2). \quad (3.2)$$

Let H be the expectile principle with parameter θ . Then

$$\phi(X, H(X)) = 0 \quad (3.3)$$

for every X . Hence,

$$\begin{aligned} \phi(X_1 + X_2, H(X_1 + X_2)) &= \phi(X_1, H(X_1)) + \phi(X_2, H(X_2)) \\ &\geq \phi(X_1 + X_2, H(X_1) + H(X_2)) \end{aligned} \quad (3.4)$$

by (3.2). Because $\phi(X, x)$ is a decreasing function of x , we have property (iv),

$$H(X_1 + X_2) \leq H(X_1) + H(X_2).$$

To show that (c) \Rightarrow (a), we assume that $u(x)$ is a utility function such that the corresponding zero utility principle is positively homogeneous. Without loss of generality we assume $u(0) = 0$. The concavity of $u(x)$ is the condition that

$$u(x_1) - 2u(\bar{x}) + u(x_2) \leq 0 \quad (3.5)$$

for all x_1, x_2 , where $\bar{x} = (x_1 + x_2)/2$. Now consider a Bernoulli risk X with

$$\Pr(X = 1) = p, \quad \Pr(X = 0) = q, \quad (3.6)$$

($p + q = 1$). Then $P = H(X)$ is the solution of

$$pu(P - 1) + qu(P) = 0. \quad (3.7)$$

Note that $0 < P < 1$. From the positive homogeneity property it follows that

$$pu(aP - a) + qu(aP) = 0 \quad (3.8)$$

for all $a > 0$. We use this for $a = a_1$, $a = a_2$, and $a = \bar{a} = (a_1 + a_2)/2$ to see that

$$\begin{aligned} p\{u(a_1P - a_1) - 2u(\bar{a}P - \bar{a}) + u(a_2P - a_2)\} \\ + q\{u(a_1P) - 2u(\bar{a}P) + u(a_2P)\} = 0. \end{aligned} \quad (3.9)$$

Because of (3.5), both expressions within the braces must vanish. This shows that $u(x)$ is linear in x for $x < 0$ and $x > 0$. From $u(0) = 0$, the monotonicity and concavity of $u(x)$ it follows that

$$u(x) = \begin{cases} c_1x & \text{if } x < 0, \\ c_2x & \text{if } x \geq 0, \end{cases} \quad (3.10)$$

$$(3.11)$$

with $0 < c_2 \leq c_1$. This leads to the expectile principle with $\theta = (c_1 - c_2)/c_2$.

Remark 3.1: For another derivation of (a) \Rightarrow (b), see Proposition 8.25 on page 292 of McNeil et al. (2015).

Remark 3.2: Cheung et al. (2015a) and Goovaerts et al. (1984, page 135) characterize zero utility principles by the positive homogeneity property for the case where the utility function is not assumed to be concave.

Remark 3.3: One might argue that the expectile principle is a member of the family of disappointment aversion premium principles that were proposed by Cheung et al. (2015a)

4 Optimal purchase of reinsurance

We now turn to reinsurance. We shall not use the symbol X to denote risk. We consider a one-period model. The first insurer has to pay the *total claim amount* S (a positive random variable with a known distribution) at the end of the period. Of course it has received premiums for this obligation. However, the premiums will not play an explicit role in the following analysis, and we shall not introduce a symbol for them.

The first insurer can buy a payment R (a random variable) from a reinsurer. The reinsurer's payment is made at the end of the period. Typically, it is a function of S ; however, we do not make this assumption *a priori*. For any R , the reinsurance premium P is determined according to a (re)insurance premium principle H , $P = H(R)$, that is known to the first insurer. For choosing R , the first insurer uses a utility function $u(x)$. Thus the problem is

$$\max_R \mathbb{E}[u(-S - H(R) + R)]. \quad (4.1)$$

We assume that H has a gradient and is translation invariant. Thus the quantity of interest is really $R - H(R)$, and a budget constraint on the reinsurance premium would not make sense in this context. It is natural to require that $R = 0$ if $S = 0$.

Let R^* be a solution of (4.1). Theorem 9 in Deprez and Gerber (1985) provides the following optimality criterion in terms of the gradient of H :

$$H'(R^*) = \frac{u'(-S - H(R^*) + R^*)}{\mathbb{E}[u'(-S - H(R^*) + R^*)]}. \quad (4.2)$$

We shall use the symbol P^* for $H(R^*)$. For a generalization of (4.2), see Corollary 2.4 in Kiesel and Rüschenendorf (2013).

Remark 4.1: In calculus, the unconstrained extrema of a differentiable function of several real variables can be found by the first-order condition: Equate the gradient of the function with the zero vector, and solve. By an analogous procedure, problem (4.1) can be treated. The functional to be maximized is

$$U(R) = \mathbb{E}[u(-S - H(R) + R)]. \quad (4.3)$$

To determine its gradient $U'(R)$, we note that, by (2.17),

$$\frac{d}{dt}U(R + tV)|_{t=0} = E[u'(-S - H(R) + R)(-E[H'(R)V] + V)]. \quad (4.4)$$

(For a rigorous derivation, see the paragraph around (4.4) in Promislow and Young (2005).) This shows that

$$U'(R) = -E[u'(-S - H(R) + R)]H'(R) + u'(-S - H(R) + R). \quad (4.5)$$

Setting $U'(R)$ equal to zero, we indeed obtain (4.2).

Remark 4.2: Note that the reinsurance premium $P = H(R)$ depends only on the distribution of R . This is different, if R is bought in the market, not necessarily from a particular reinsurer. See Section 10 in Gerber and Pafumi (1998). There, the assumption is that

$$P = H(R) = E[\Psi R] = E[R] + \text{Cov}(R, \Psi), \quad (4.6)$$

where the *price density* Ψ is a positive random variable with $E[\Psi] = 1$. Note that the optimality condition (141) in Gerber and Pafumi (1998) is similar to (4.2) because $H'(R) = \Psi$ in (4.6).

For the remainder of this section we consider four special cases. In each, we find a closed-form expression for R^* , the optimal reinsurer's payment. The graphs of R^* , as a function of S , are depicted in Figure 1.

For *Case 1*, we assume that $u(x)$ is the exponential utility function with parameter $a > 0$. Then the optimality condition reduces to

$$H'(R^*) = \frac{e^{a(S-R^*)}}{E[e^{a(S-R^*)}]} \quad (4.7)$$

We assume that the reinsurance premium is calculated according to the exponential principle with parameter $b > 0$,

$$P = \frac{1}{b} \ln E[e^{bR}]. \quad (4.8)$$

Then the optimality condition is

$$\frac{e^{bR^*}}{E[e^{bR^*}]} = \frac{e^{a(S-R^*)}}{E[e^{a(S-R^*)}]}, \quad (4.9)$$

showing that $R^* = \frac{a}{a+b}S + c$, where the constant c is the value of R^* when $S = 0$. Thus $c = 0$. Then

$$R^* = \frac{a}{a+b}S, \quad (4.10)$$

$$P^* = \frac{1}{b} \ln E[e^{\frac{ab}{a+b}S}]. \quad (4.11)$$

Hence, the optimal reinsurer's payment is a fixed fraction of S .

For *Case 2*, we continue to assume that the first insurer's utility function is exponential with parameter a , but now assume that the reinsurer uses the zero utility principle with

$$u_{re}(x) = \begin{cases} \frac{1}{b_1}(1 - e^{-b_1x}) & \text{if } x < 0, \\ \frac{1}{b_2}(1 - e^{-b_2x}) & \text{if } x > 0. \end{cases} \quad (4.12)$$

Here the risk aversion function is the constant b_1 if $x < 0$ and b_2 if $x > 0$. Hence, we assume that $0 < b_2 < b_1$. The optimality condition tells us that $u'_{re}(P^* - R^*)$ has to be proportional to $u'(-S - P^* + R^*)$ where $u'(x) = e^{-ax}$. From this and the requirement that $R^* = 0$ if $S = 0$, we find that

$$R^* = \begin{cases} \frac{a}{a+b_2}S & \text{if } S < \frac{a+b_2}{a}P^*, \\ \frac{a}{a+b_1}S + \frac{b_1-b_2}{a+b_1}P^* & \text{if } S > \frac{a+b_2}{a}P^*. \end{cases} \quad (4.13)$$

Finally P^* is determined as the solution of $E[u_{re}(P^* - R^*)] = 0$. Two limits are of interest, $b_1 = b_2$, which is Case 1, and $b_2 = 0$, where $u_{re}(x) = x$ if $x > 0$.

It is instructive to write (4.13) in the form

$$R^* - P^* = \frac{a}{a+b_1}(S - \frac{a+b_2}{a}P^*)_+ - \frac{a}{a+b_2}(\frac{a+b_2}{a}P^* - S)_+. \quad (4.14)$$

This has the following interpretation: The optimal reinsurance contract provides a payment that is the fraction $\frac{a}{a+b_1}$ of that of a stop-loss contract with deductible $\frac{a+b_2}{a}P^*$, for the stochastic "premium" $(P^* - \frac{a}{a+b_2}S)_+$, which is known only at the end of the period.

For *Case 3*, we assume that $u''(x) < 0$ so that $u'(x)$ is strictly decreasing. We assume that the reinsurance premium is determined according to the expectile principle with loading factor $\theta > 0$. Because of (2.26), the random variable on the RHS of (4.2) has one constant value if $P^* - R^* > 0$ and another constant value if $P^* - R^* < 0$. From this and the requirement that $R^* = 0$ if $S = 0$, we find that

$$\begin{aligned} S - R^* &= 0 & \text{if } P^* - R^* > 0, \\ S - R^* &= c & \text{if } P^* - R^* < 0. \end{aligned} \quad (4.15)$$

Here c is obtained from the condition that

$$\frac{u'(-c)}{u'(0)} = 1 + \theta. \quad (4.16)$$

From (4.15) it follows that

$$R^* = \begin{cases} S & \text{if } S < P^*, \\ P^* & \text{if } P^* < S < P^* + c, \\ S - c & \text{if } S > P^* + c. \end{cases} \quad (4.17)$$

Finally, the value of the junction P^* is determined from the condition that

$$(1 + \theta)E[(S - P^* - c)_+] = E[(P^* - S)_+]. \quad (4.18)$$

Let us write (4.17) in the form

$$R^* - P^* = (S - P^* - c)_+ - (P^* - S)_+. \quad (4.19)$$

In this sense, the optimal reinsurance contract provides the same payment as a stop-loss contract with deductible $P^* + c$. However, the “premium” $(P^* - S)_+$ is stochastic as in (4.14).

This merits a numerical example. We assume an exponential utility function with parameter $a > 0$. Then (4.16) yields

$$c = \frac{\ln(1 + \theta)}{a}. \quad (4.20)$$

Furthermore, we assume that S is exponentially distributed with mean one. Then (4.18) is the condition that

$$(1 + \theta)e^{-(P^*+c)} = P^* - 1 + e^{-P^*}. \quad (4.21)$$

Because of (4.20), the LHS is

$$(1 + \theta)^{1+a}e^{-P^*}. \quad (4.22)$$

The resulting equation is solved for P^* . Table 1 shows the values of the junctions P^* and P^*+c for various values of θ and a . We note that when the loading factor θ increases, the reinsurance premium P^* also increases; this is intuitively obvious. When θ is small, an increase of a will lead to an increase of P^* . However, when θ is large, this is not always true. This can be explained as follows: when θ is small (the reinsurance is relatively inexpensive), a risk-averse first insurer is willing to have a reinsurance contract which covers more loss, even if the premium will be larger. When θ is large enough, a risk-averse first insurer may not be always willing to buy a reinsurance contract covering more loss due to the cost of the reinsurance premium.

Table 1: Values of the junctions: P^* and $P^* + c$

	$a = 0.1$	$a = 0.2$	$a = 0.3$	$a = 0.5$	$a = 0.7$	$a = 0.9$
$\theta = 0.01$	1.0040 1.1035	1.0044 1.0542	1.0048 1.0380	1.0055 1.0254	1.0062 1.0204	1.0070 1.0181
$\theta = 0.05$	1.0199 1.5078	1.0217 1.2657	1.0235 1.1861	1.0272 1.1248	1.0309 1.1006	1.0345 1.0887
$\theta = 0.1$	1.0391 1.9921	1.0427 1.5193	1.0463 1.3640	1.0536 1.2442	1.0609 1.1971	1.0682 1.1741
$\theta = 0.2$	1.0757 2.8989	1.0828 1.9944	1.0899 1.6976	1.1043 1.4689	1.1187 1.3792	1.1333 1.3359
$\theta = 0.3$	1.1102 3.7338	1.1207 2.4325	1.1312 2.0057	1.1523 1.6770	1.1738 1.5486	1.1955 1.4870
$\theta = 0.4$	1.1428 4.5102	1.1565 2.8389	1.1703 2.2919	1.1981 1.8710	1.2264 1.7071	1.2552 1.6291
$\theta = 0.5$	1.1738 5.2285	1.1906 3.2179	1.2075 2.5591	1.2418 2.0527	1.2768 1.8550	1.3124 1.7629
$\theta = 0.6$	1.2033 5.9033	1.2230 3.5730	1.2430 2.8097	1.2836 2.2236	1.3251 1.9965	1.3675 1.8897
$\theta = 0.8$	1.2583 7.1362	1.2838 4.2227	1.3096 3.2689	1.3623 2.5379	1.4164 2.2561	1.4717 2.1248
$\theta = 1.0$	1.3089 8.2404	1.3398 4.8055	1.3712 3.6817	1.4353 2.8216	1.5012 2.4914	1.5690 2.3392
$\theta = 2.0$	1.5158 12.5019	1.5697 7.0628	1.6247 5.2867	1.7389 3.9352	1.8567 3.4262	1.9776 3.1983
$\theta = 10$	2.3006 26.2796	2.4488 14.4383	2.6012 10.5942	2.9179 7.7137	3.2489 6.6745	3.5927 6.2570

For *Case 4*, we make the opposite assumptions in some sense. We assume that the reinsurer determines its premium using the zero utility principle according to a utility function $u_{re}(x)$ such that $u'_{re}(x)$ is strictly decreasing. The first insurer's utility function is the refracted linear function

$$u(x) = \begin{cases} (1 + \theta)(x + \kappa) & \text{if } x < -\kappa, \\ x + \kappa & \text{if } x > -\kappa. \end{cases} \quad (4.23)$$

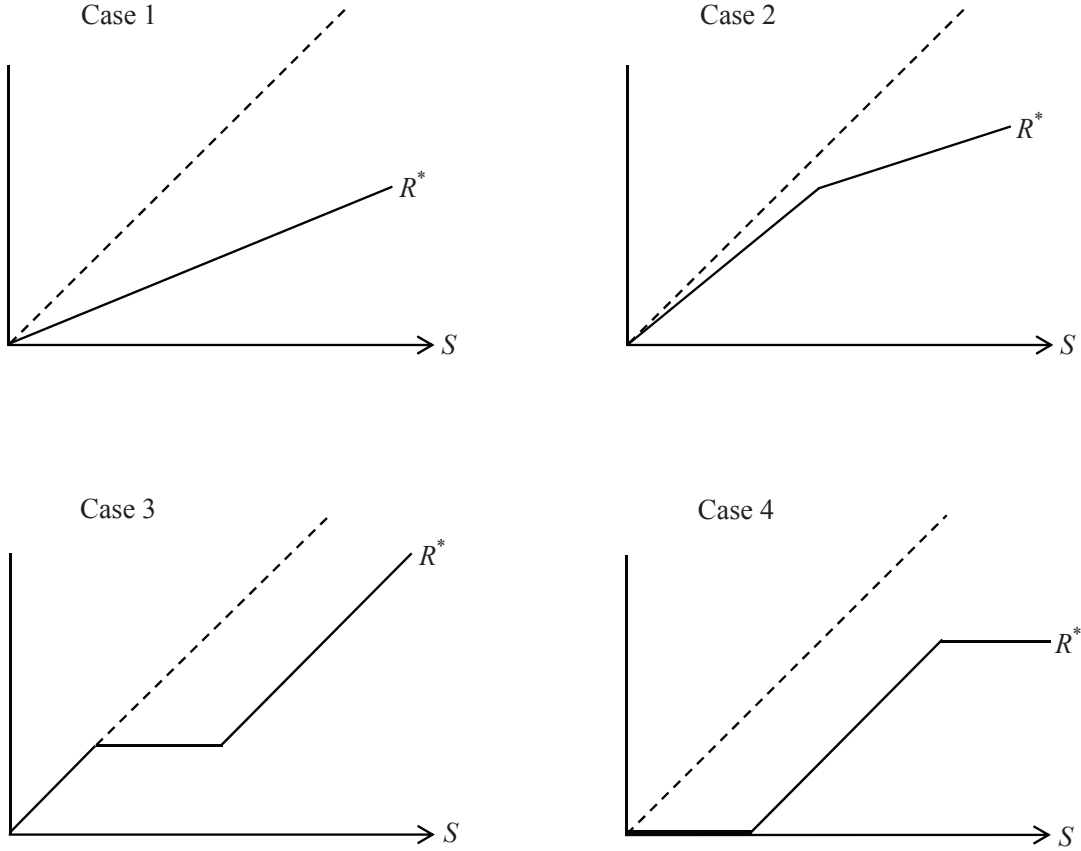
Here $\theta > 0$; κ is the amount that has been set aside to meet the obligation. From the optimality condition we see that $u'_{re}(P^* - R^*)$ must be a multiple of $u'(-S - P^* + R^*)$. Thus $P^* - R^*$ has one constant value if $-S - P^* + R^* > -\kappa$, and another constant value if $-S - P^* + R^* < -\kappa$. From this and the requirement that $R^* = 0$ when $S = 0$, we conclude that

$$R^* = \begin{cases} 0 & \text{if } S < \kappa - P^*, \\ S - (\kappa - P^*) & \text{if } \kappa - P^* < S < \kappa + c, \\ c + P^* & \text{if } S > \kappa + c. \end{cases} \quad (4.24)$$

Here c and P^* must satisfy the two conditions

$$\frac{u'_{re}(-c)}{u'_{re}(P^*)} = 1 + \theta \quad (4.25)$$

Figure 1. R^* in the four special cases



and

$$\mathbb{E}[u_{re}(P^* - R^*)] = 0. \quad (4.26)$$

Thus we find that it is now optimal to obtain full coverage for the *layer* between $\kappa - P^*$ and $\kappa + c$. Here a stop-loss payment with deductible $\kappa - P^*$ is capped at the level $c + P^*$. Therefore, such a contract might be called a *trimmed stop-loss* contract, or, in the language of Kaluszka and Okolewski (2008), a *limited stop-loss* contract.

We illustrate this with a numerical example. We assume that $u_{re}(x)$ is the exponential utility function with parameter $b > 0$. Then (4.25) is the condition that

$$P^* + c = \frac{\ln(1 + \theta)}{b}. \quad (4.27)$$

Again, we assume that S is exponentially distributed with mean one. Then (4.26) is the condition that

$$e^{bP^*} = 1 - e^{-A} + \frac{e^{-A}}{1 - b} [1 - e^{-(1-b)(B-A)}] + e^{-B+b(B-A)}, \quad (4.28)$$

where $A = \kappa - P^*$, $B = \kappa + c$. By (4.27), this condition becomes

$$e^{bP^*} = 1 + \frac{b}{1-b} [1 - (1+\theta)^{(b-1)/b}] e^{-\kappa+P^*}. \quad (4.29)$$

Table 2 (for $\kappa = 1.1$) and Table 3 (for $\kappa = 1.5$) show the values of P^* and of the junctions, $A = \kappa - P^*$ and $B = \kappa + c$, for various values of b and θ . Note that, similar to that in Table 1, when θ increases, P^* increases. In this example, the reinsurer has a constant risk aversion function that takes the value b . For small θ , when b increases, a risk aversion reinsurer would like to have a contract which covers less loss, so the premium is smaller. For large θ this may not be the case. In the large θ case, the first insurer is more risk-averse, so he may be willing to pay higher premium. The reinsurer may be willing to accept a contract which covers more loss.

Table 2: Values of P^* , $\kappa - P^*$ and $\kappa + c$ when $\kappa = 1.1$

	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 1$
$b = 0.05$	0.4581	0.6231	0.6664	0.6765	0.6789	0.6798
	0.6419	0.4769	0.4336	0.4235	0.4211	0.4202
	2.5481	4.1233	5.6809	7.1529	8.5304	14.2831
$b = 0.1$	0.2772	0.4628	0.5825	0.6566	0.7010	0.7589
	0.8228	0.6372	0.5175	0.4434	0.3990	0.3411
	1.7759	2.4604	3.1411	3.8981	4.4537	7.2726
$b = 0.2$	0.1511	0.2762	0.3809	0.4694	0.5451	0.8019
	0.9489	0.8238	0.7191	0.6306	0.5549	0.2981
	1.4255	1.7354	2.0309	2.3130	2.5822	3.7638
$b = 0.5$	0.0635	0.1215	0.1751	0.2251	0.2719	0.4738
	1.0365	0.9785	0.9249	0.8749	0.8281	0.6262
	1.2271	1.3431	1.4496	1.5478	1.6390	2.0125

Table 3: Values of P^* , $\kappa - P^*$ and $\kappa + c$ when $\kappa = 1.5$

	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.5$	$\theta = 1$
$b = 0.05$	0.2509	0.3069	0.3181	0.3205	0.3211	0.3213
	1.2491	1.1931	1.1819	1.1795	1.1789	1.1787
	3.1553	4.8395	6.4292	7.9089	9.2882	15.0416
$b = 0.1$	0.1674	0.2545	0.2980	0.3196	0.3305	0.3426
	1.3326	1.2455	1.2020	1.1804	1.1695	1.1574
	2.2857	3.0687	3.8259	4.5451	5.2242	8.0889
$b = 0.2$	0.0964	0.1680	0.2212	0.2609	0.2907	0.3622
	1.4036	1.3320	1.2788	1.2391	1.2093	1.1378
	1.8802	2.2436	2.5906	2.9215	3.2366	4.6036
$b = 0.5$	0.0419	0.0789	0.1120	0.1418	0.1688	0.2738
	1.4581	1.4211	1.3880	1.3582	1.3312	1.2262
	1.6487	1.7857	1.9127	2.0311	2.1421	2.6125

Remark 4.3: Case 4 might be compared with the development in Cai and Weng (2016). There, the first insurer's goal is to minimize the expected loss used as a risk measure.

This is done under a set of constraints and the mathematics is substantially more complicated. Yet, in some cases it is also found that the optimal reinsurer's payment, R^* , is of a single layer type, just as in (4.24).

Remark 4.4: Case 4 should also be compared with Cheung et al. (2015b). In this paper, the premium is a convex combination of the $E[R]$ and the supremum of R . Using optimality criteria that are motivated by disappointment theory, and additional assumptions, it is shown that single layer reinsurance payments are optimal.

5 Optimal reinsurance as a Pareto-optimal risk exchange

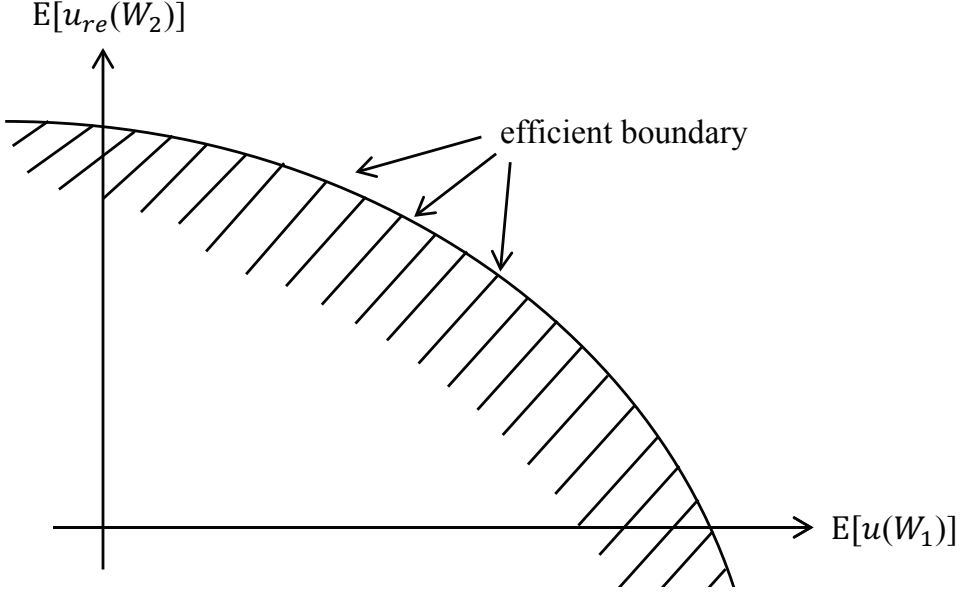
As preparation, we begin with a review of the theory of *risk exchanges* that was developed by Karl Borch. Consider n companies with a combined wealth W (a random variable) at the end of the period. As a result of a risk exchange, company i will have wealth W_i at the end of the period ($W_1 + \dots + W_n = W$). Company i uses a risk-averse utility function $u_i(x)$ and hence is interested in $E[u_i(W_i)]$. The set of all points ($E[u_1(W_1)], \dots, E[u_n(W_n)]$) is a convex set in the n -dimensional Euclidean space; for $n = 2$ see Figure 2. A risk exchange is *Pareto-optimal*, if the corresponding point is on the *efficient boundary*. Borch's Theorem states that a risk exchange is Pareto-optimal if and only if there are constants $k_i > 0$ such that $k_i u'_i(W_i)$ is the same random variable for all i . (Note that the vector (k_1, \dots, k_n) is orthogonal to the tangent plane of the efficient boundary). For a Pareto-optimal risk exchange, let dW_i denote the infinitesimal increment of company i 's wealth that is implied by an infinitesimal increment dW of combined wealth. It is known that dW_i is inversely proportional to the risk aversion function of company i ; see, for example, formula (101) in Gerber and Pafumi (1998).

Now suppose that the reinsurer uses a zero utility principle, say, according to a utility function $u_{re}(x)$. Then the optimality condition (4.2) becomes

$$\frac{u'_{re}(P^* - R^*)}{E[u'_{re}(P^* - R^*)]} = \frac{u'(-S - P^* + R^*)}{E[u'(-S - P^* + R^*)]}. \quad (5.1)$$

Thus the condition of Borch's Theorem is satisfied. Here, the combined "wealth" W is $-S$. We conclude that the optimal reinsurance contract must be the result of one particular Pareto-optimal risk exchange between reinsurer and first insurer. The geometric interpretation of these findings is as follows. See Figure 2. Consider a risk exchange W_1, W_2 ($W_1 + W_2 = -S$), where W_1 is the wealth of the first insurer and W_2 the wealth of the reinsurer after the exchange. Each risk exchange is represented by a point ($E[u(W_1)], E[u_{re}(W_2)]$). The Pareto-optimal risk exchanges correspond to points on the north-east boundary, the efficient boundary. Only risk exchanges with $E[u_{re}(W_2)] = E[u_{re}(H(R) - R)] = u_{re}(0)$ are permissible because the reinsurer uses a zero utility principle; these are represented by the horizontal line in the middle of Figure 3. Thus it is clear that first insurer will choose the risk exchange represented by the point furthest to the right on this horizontal line. That is, the optimal reinsurance is a Pareto-optimal risk exchange.

Figure 2. Pareto-optimal risk exchanges



Furthermore, dW_2 must be inversely proportional to the reinsurer's risk aversion function $r_{re}(x)$,

$$dW_2 = \frac{r(-S - P^* + R^*)}{r(-S - P^* + R^*) + r_{re}(P^* - R^*)} dW, \quad (5.2)$$

or

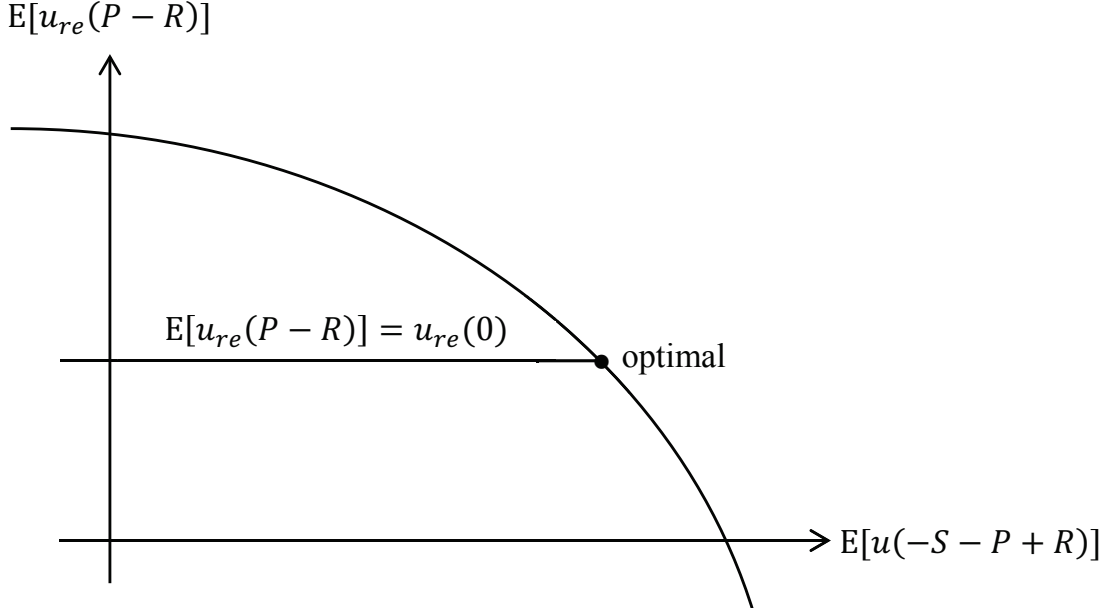
$$dR^* = \frac{r(-S - P^* + R^*)}{r(-S - P^* + R^*) + r_{re}(P^* - R^*)} dS. \quad (5.3)$$

From (5.3) it follows that $R^* = f(S)$ for some function $f(s)$ with $0 \leq f'(s) \leq 1$. From this and $f(0) = 0$, we see that $0 \leq f(s) \leq s$. The function f is usually called an *indemnity function*. It is also known as a *coverage function* (Raviv 1979) and a *ceded loss function* (Cai and Weng 2016; Lo 2017). Many papers only consider reinsurance payments R of this type; this can complicate the search for the optimum. In this paper we did not impose this restriction *a priori*.

Let us revisit the four special cases in Section 4. In Case 1, the risk aversion functions are constant, and

$$dR^* = \frac{a}{a+b} dS \quad (5.4)$$

Figure 3. Optimal reinsurance



by (5.3), which leads directly to (4.10). Similarly, in Case 2,

$$dR^* = \begin{cases} \frac{a}{a+b_2}dS & \text{if } P^* - R^* > 0, \\ \frac{a}{a+b_1}dS & \text{if } P^* - R^* < 0, \end{cases} \quad (5.5)$$

which explains (4.13).

In Cases 3 and 4, one of the risk aversion functions is a multiple of the Dirac delta function or its translation. As a consequence, the factor in front of dS in (5.3) is either 0 or 1. In Case 3, $r_{re}(x)$ is a multiple of the Dirac delta function. Hence,

$$dR^* = \begin{cases} 0 & \text{if } P^* - R^* = 0, \\ dS & \text{if } P^* - R^* \neq 0. \end{cases} \quad (5.6)$$

This explains (4.17). In Case 4, $r(x + \kappa)$ is a multiple of the Dirac delta function. Hence

$$dR^* = \begin{cases} dS & \text{if } -S - P^* + R^* + \kappa = 0, \\ 0 & \text{if } -S - P^* + R^* + \kappa \neq 0. \end{cases} \quad (5.7)$$

This explains (4.24)

Remark 5.1: Formula (5.3) resembles formula (10), with $c = 0$, in Raviv (1979). They are different for two reasons: The premium P in (10) is fixed, and (10) results from constrained optimization.

Remark 5.2: Formula (5.3) should also be compared with Theorem 3 in Aase (2004b), which is credited to Moffet (1979). Consider any Pareto-optimal risk exchange between reinsurer and first insurer, whereby the reinsurer receives p and pays $R = f(S)$. Then

$$f'(s) = \frac{r(-s - p + f(s))}{r(-s - p + f(s)) + r_{re}(p - f(s))}. \quad (5.8)$$

Note that the premium p has the role of a *side payment* and is not determined in this context. The efficient boundary in Figures 2 and 3 can be parameterized by p .

Remark 5.3: *A priori*, the concept of Pareto-optimality makes sense for risk exchanges between two (or more) cooperative insurers, but not so much for risk exchanges between a first insurer and a reinsurer, who have different roles.

6 Conclusion

We study an optimal reinsurance problem in which the first insurer maximizes its expected utility and the premium principle used by the reinsurer is known. We do not, *a priori*, assume that there is an indemnity function f . Natural constraints such as (1.1) are not imposed. A key tool is (4.2), an optimality condition from Deprez and Gerber (1985).

With the reinsurance premium being determined by a zero utility principle, we study four cases and obtain closed-form expressions for the optimal reinsurance payment R^* and the reinsurance premium P^* . Figure 1 exhibits R^* as a function of the total claim amount S . In *Case 1*, both the first insurer and the reinsurer use an exponential utility function. Then R^* is proportional to S . In *Case 2*, the reinsurer's utility function is generalized to (4.12); hence the reinsurer's risk aversion function is a constant for $x < 0$ and another constant for $x > 0$. This translates to a discontinuity of the slope of R^* . In *Case 3*, the reinsurance premium is determined according to the expectile principle. It is interesting to note that the resulting optimal reinsurance contract can be interpreted as a stop-loss contract, where however the "premium" is stochastic. In *Case 4*, the first insurer's utility function is given by (4.23). The resulting contract might be called a trimmed or limited stop-loss contract.

We further investigated the problem from the viewpoint of a risk exchange between first insurer and reinsurer. It follows from (4.2) and Borch's Theorem that an optimal reinsurance contract is one particular Pareto-optimal risk exchange. From this, we see that there exists a function $f(s)$ with $R^* = f(S)$, $0 \leq f(s) \leq s$ and $0 \leq f'(s) \leq 1$. Thus, what many authors postulate as constraints are now obtained as a result of unconstrained optimization.

A referee has noted out that Cases 3 and 4 have an optimal indemnity function f with $f'(s) = 1$ for some s and that some practitioners have criticized such indemnity functions because they may provoke moral hazard for the reinsurer. Perhaps a solution is to impose the constraint,

$$0 \leq f(s) \leq c < 1,$$

for some constant c . Alternatively, we assume that the reinsurer has a risk aversion function that is strictly positive everywhere. Then the quotient in (5.3) is less than one.

Aase (2004a) pointed out that Karl Borch's "pioneering work on Pareto-optimal risk exchanges in reinsurance opened a new area of actuarial science, which has been in continuous growth. This research field offers a deeper understanding of the preferences and behavior of the parties in an insurance markets." It seems fitting to end this paper with

Borch's (1969) words: "[T]here are two parties to a reinsurance contract, and that these parties have conflicting interests. The *optimal contract* must then appear as a reasonable compromise between these interests. To me the most promising line of research seems to be the study of contracts, which in different ways can be said to be optimal from the point of view of both parties."

Acknowledgments

The authors are grateful to Hansjoerg Albrecher, Vali Asimit, Ka Chun Cheung, Ambrose Lo, Phillip Yam and two anonymous referees for their valuable comments. We acknowledge with thanks the support from Principal Financial Group, Research Grants Council of the Hong Kong Special Administrative Region (project No. HKU 17330816), and Society of Actuaries' Centers of Actuarial Excellence Research Grants.

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