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On Gardner–Hartenstine's inequality

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Abstract

In the paper, we give a new generalization of Gardner–Hartenstine inequality and establish its integral form. As applications, we combine an important inequality and give some broader improvements.

MSC: 26D15; 52A40

Keywords: Hölder's inequality; Minkowski's inequality; Gardner–Hartenstine's inequality

1 Introduction

In [1], Gardner and Hartenstine established an interesting inequality. This inequality is crucial in their proof (as it was in [2]).

Theorem A *For $x_0, y_0 > 0$ and reals x_i, y_i, $i = 1, \ldots, n$, we have*

$$\frac{(\sum_{i=1}^n (x_i + y_i)^2)^{(n-1)/2}}{(x_0 + y_0)^{n-2}} \leq \frac{(\sum_{i=1}^n x_i^2)^{(n-1)/2}}{x_0^{n-2}} + \frac{(\sum_{i=1}^n y_i^2)^{(n-1)/2}}{y_0^{n-2}}, \tag{1.1}$$

with equality if and only if either $x_i = y_i = 0$ for $i = 1, 2, \ldots, n$ or $x_i = \alpha y_i$ for $i = 0, 1, \ldots, n$, and some $\alpha > 0$.

The first aim of this paper is to give a new generalization of the Gardner–Hartenstine inequality (1.1). Our result is given in the following theorem.

Theorem 1.1 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $x_{00}, y_{00} > 0$ and reals x_{ij}, y_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then*

$$\left(\frac{(\sum_{j=1}^m \sum_{i=1}^n (x_{ij} + y_{ij})^r)^{1/r}}{(x_{00} + y_{00})^{1/q}}\right)^p \leq \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{1/r}}{x_{00}^{1/q}}\right)^p + \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r)^{1/r}}{y_{00}^{1/q}}\right)^p, \tag{1.2}$$

with equality if and only if either $x_{ij} = y_{ij} = 0$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ or $x_{ij} = \alpha y_{ij}$ for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, and some $\alpha > 0$.

Remark 1.2 Let x_{ij} and y_{ij} change x_i and y_i, respectively, with appropriate transformation, and putting $m = 1$, $r = 2$, $p = n - 1$, and $q = (n-1)/(n-2)$ in (1.2), (1.2) becomes (1.1).

Another aim of this paper is to give an integral form of (1.2). Our result is given in the following theorem.

Theorem 1.3 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $u(x,y), v(x,y) > 0$ and $f(x,y)$, $g(x,y)$ are continuous functions on $[a,b] \times [c,d]$, then*

$$\begin{aligned}&\left(\frac{(\int_a^b \int_c^d (f(x,y)+g(x,y))^r\,dx\,dy)^{1/r}}{(u(x,y)+v(x,y))^{1/q}}\right)^p \\ &\quad\le \left(\frac{(\int_a^b \int_c^d f(x,y)^r\,dx\,dy)^{1/r}}{u(x,y)^{1/q}}\right)^p + \left(\frac{(\int_a^b \int_c^d g(x,y)^r\,dx\,dy)^{1/r}}{v(x,y)^{1/q}}\right)^p,\end{aligned} \tag{1.3}$$

with equality if and only if either $(\|f(x,y)\|_r^r, \|g(x,y)\|_r^r) = \alpha(\|u(x,y)\|_r^r, \|v(x,y)\|_r^r)$ for some $\alpha > 0$ or $\|f(x,y)\|_r^r = \|g(x,y)\|_r^r = 0$.

Let $f(x,y)$ and $g(x,y)$ change $f(x)$ and $g(x)$, respectively, with appropriate transformation, and putting $r = 2$, $p = n-1$ and $q = (n-1)/(n-2)$ in (1.3), (1.3) becomes the following result.

Corollary 1.4 *If $u(x), v(x) > 0$ and $f(x), g(x)$ are continuous functions on $[a,b]$, then*

$$\frac{(\int_a^b (f(x)+g(x))^2\,dx)^{(n-1)/2}}{(u(x)+v(x))^{n-2}} \le \frac{(\int_a^b f(x)^2\,dx)^{(n-1)/2}}{u(x)^{n-2}} + \frac{(\int_a^b g(x)^2\,dx)^{(n-1)/2}}{v(x)^{n-2}}, \tag{1.4}$$

with equality if and only if either $\|f(x)\|_r^r = \|g(x)\|_r^r = 0$ or $(\|f(x)\|_r^r, \|g(x)\|_r^r) = \alpha(\|u(x)\|_r^r, \|v(x)\|_r^r)$ for some $\alpha > 0$.

This is just an integral form of (1.1) established by Gardner and Hartenstine [1].

As applications, we combine another important inequality and give some broader improvements. Our results are given in the following theorems.

Theorem 1.5 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $x_{00}, y_{00}, a_{00}, b_{00} > 0$ and reals x_{ij}, y_{ij}, a_{ij}, b_{ij}, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, then*

$$\begin{aligned}&\frac{(\sum_{j=1}^m \sum_{i=1}^n [(x_{ij}+y_{ij})^r + (a_{ij}+b_{ij})^r])^p}{[(x_{00}+y_{00})^r + (a_{00}+b_{00})^r]^{p/q}} \\ &\quad\le \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right)^r \\ &\qquad + \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^m \sum_{i=1}^n b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right)^r\end{aligned} \tag{1.5}$$

with equality if and only if either $x_{ij} = y_{ij} = 0$ and $a_{ij} = b_{ij} = 0$ for $i = 1,\ldots,n$ and $j = 1,\ldots,m$ or $x_{ij} = \alpha y_{ij}$ and $a_{ij} = \beta b_{ij}$ for $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,m$ and some $\alpha, \beta > 0$, and

$$\begin{aligned}&\left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right) : \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^m \sum_{i=1}^n b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right) \\ &\quad= (x_{00}+y_{00}) : (a_{00}+b_{00}).\end{aligned}$$

Theorem 1.6 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $u(x,y), v(x,y), u'(x,y), v'(x,y) > 0$ and $f(x,y)$, $g(x,y), f'(x,y), g'(x,y)$ are continuous functions on $[a,b] \times [c,d]$, then*

$$\begin{aligned} &\frac{(\int_a^b \int_c^d [(f(x,y) + g(x,y))^r + (f'(x,y) + g'(x,y))^r]\,dx\,dy)^p}{[(u(x,y) + v(x,y))^r + (u'(x,y) + v'(x,y))^r]^{p/q}} \\ &\quad \le \left(\frac{(\int_a^b \int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}} \right)^r \\ &\qquad + \left(\frac{(\int_a^b \int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g'(x,y)^r\,dx\,dy)^{p/r}}{v'(x,y)^{p/q}} \right)^r \end{aligned} \tag{1.6}$$

with equality if and only if either $f(x,y) = g(x,y) = 0$ and $f'(x,y) = g'(x,y) = 0$ or $(f(x,y), g(x,y)) = \alpha(u(x,y), v(x,y))$ and $(f'(x,y), g'(x,y)) = \beta(u'(x,y), v'(x,y))$ and some $\alpha, \beta > 0$, and

$$\begin{aligned} &\left(\frac{(\int_a^b \int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}} \right) \\ &\quad : \left(\frac{(\int_a^b \int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g'(x,y)^r)^{p/r}}{v'(x,y)^{p/q}} \right) \\ &= \bigl(u(x,y) + v(x,y)\bigr) : \bigl(u'(x,y) + v'(x,y)\bigr). \end{aligned}$$

2 Generalizations

Our main results are given in the following theorems.

Theorem 2.1 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $x_{00}, y_{00} > 0$ and reals x_{ij}, y_{ij}, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, then*

$$\left(\frac{(\sum_{j=1}^m \sum_{i=1}^n (x_{ij} + y_{ij})^r)^{1/r}}{(x_{00} + y_{00})^{1/q}} \right)^p \le \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{1/r}}{x_{00}^{1/q}} \right)^p + \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r)^{1/r}}{y_{00}^{1/q}} \right)^p, \tag{2.1}$$

with equality if and only if either $x_{ij} = y_{ij} = 0$ for $i = 1,\ldots,n$, $j = 1,\ldots,m$ or $x_{ij} = \alpha y_{ij}$ for $i = 0,1,\ldots,n$, $j = 0,1,\ldots,m$, and some $\alpha > 0$.

Proof From Minkowski's and Hölder's inequalities, we obtain

$$\begin{aligned} \left(\sum_{j=1}^m \sum_{i=1}^n (x_{ij} + y_{ij})^r \right)^{1/r} &\le \left(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r \right)^{1/r} + \left(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r \right)^{1/r} \\ &= \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{1/r}}{x_{00}^{1/q}} \right) x_{00}^{1/q} + \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r)^{1/r}}{y_{00}^{1/q}} \right) y_{00}^{1/q} \\ &\le \left\{ \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{1/r}}{x_{0}^{1/q}} \right)^p + \left(\frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ij}^r)^{1/r}}{y_{00}^{1/q}} \right)^p \right\}^{1/p} \\ &\quad \times \bigl((x_{00}^{1/q})^q + (y_{00}^{1/q})^q \bigr)^{1/q} \\ &= \left\{ \frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^m \sum_{i=1}^n y_{ijj}^r)^{p/r}}{y_{00}^{p/q}} \right\}^{1/p} (x_{00} + y_{00})^{1/q}. \end{aligned}$$

Rearranging, (2.1) follows.

The following is a discussion of the conditions for this equal sign to hold. Suppose that equality holds in (2.1). Then equality holds in Minkowski's inequality, which implies that $x_{ij} = \alpha y_{ij}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ and some $\alpha \geq 0$. Equality also holds in Hölder's inequality, implying that there are constants β and γ with $\beta^2 + \gamma^2 > 0$ such that

$$\beta\left(\frac{(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r)^{1/r}}{x_{00}^{1/q}}\right)^p = \gamma\left(x_{00}^{1/q}\right)^q,$$

or equivalently

$$\beta\left(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r\right)^{p/r} = \gamma x_{00}^p,$$

and the same equation with y_{ij} instead of x_{ij}, $i = 0, 1, \dots, n$; $j = 0, 1, \dots, m$. Therefore

$$\begin{aligned}\gamma x_{00}^p &= \beta\left(\sum_{j=1}^m \sum_{i=1}^n x_{ij}^r\right)^{p/r} \\ &= \beta\left(\sum_{j=1}^m \sum_{i=1}^n (\alpha y_{ij})^r\right)^{p/r} \\ &= \gamma(\alpha y_{00})^p.\end{aligned}$$

Obviously, if $\gamma = 0$, then $x_{ij} = y_{ij} = 0$ for $i = 1, \dots, n$; $j = 1, \dots, m$. If $\gamma \neq 0$, then $\alpha > 0$ and $x_{ij} = \alpha y_{ij}$ for $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, m$.

This proof is complete. □

Let x_{ij} become x_i with appropriate transformation, and $m = 1$, (2.2) reduces to the following result.

Corollary 2.2 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $x_0, y_0 > 0$ and reals x_i, y_i, $i = 1, 2, \dots, n$, then*

$$\left(\frac{(\sum_{i=1}^n (x_i + y_i)^r)^{1/r}}{(x_0 + y_0)^{1/q}}\right)^p \leq \left(\frac{(\sum_{i=1}^n x_i^r)^{1/r}}{x_0^{1/q}}\right)^p + \left(\frac{(\sum_{i=1}^n y_i^r)^{1/r}}{y_0^{1/q}}\right)^p,$$

with equality if and only if either $x_i = y_i = 0$ for $i = 1, \dots, n$ or $x_i = \alpha y_i$ for $i = 0, 1, \dots, n$, for some $\alpha > 0$.

Theorem 2.3 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $u(x, y), v(x, y) > 0$ and $f(x, y)$, $g(x, y)$ are continuous functions on $[a, b] \times [c, d]$, then*

$$\begin{aligned}&\left(\frac{(\int_a^b \int_c^d (f(x,y) + g(x,y))^r \, dx \, dy)^{1/r}}{(u(x,y) + v(x,y))^{1/q}}\right)^p \\ &\quad\leq \left(\frac{(\int_a^b \int_c^d f(x,y)^r \, dx \, dy)^{1/r}}{u(x,y)^{1/q}}\right)^p + \left(\frac{(\int_a^b \int_c^d g(x,y)^r \, dx \, dy)^{1/r}}{v(x,y)^{1/q}}\right)^p, \qquad (2.2)\end{aligned}$$

with equality if and only if either $(\|f(x,y)\|_r^r, \|g(x,y)\|_r^r) = \alpha(\|u(x,y)\|_r^r, \|v(x,y)\|_r^r)$ for some $\alpha > 0$ or $\|f(x,y)\|_r^r = \|g(x,y)\|_r^r = 0$.

Proof From Minkowski's and Hölder's integral inequalities, we obtain

$$
\begin{aligned}
&\left(\int_a^b \int_c^d \bigl(f(x,y)+g(x,y)\bigr)^r \,dx\,dy\right)^{1/r} \\
&\quad\le \left(\int_a^b \int_c^d f(x,y)^r \,dx\,dy\right)^{1/r} + \left(\int_a^b \int_c^d g(x,y)^r \,dx\,dy\right)^{1/r} \\
&\quad= \left(\frac{(\int_a^b \int_c^d f(x,y)^r \,dx\,dy)^{1/r}}{u(x,y)^{1/q}}\right) u(x,y)^{1/q} \\
&\qquad+ \left(\frac{(\int_a^b \int_c^d f(x,y)^r \,dx\,dy)^{1/r}}{v(x,y)^{1/q}}\right) v(x,y)^{1/q} \\
&\quad\le \left\{\left(\frac{(\int_a^b \int_c^d f(x,y)^r \,dx\,dy)^{1/r}}{u(x,y)^{1/q}}\right)^p \right.\\
&\qquad\left.+ \left(\frac{(\int_a^b \int_c^d g(x,y)^r \,dx\,dy)^{1/r}}{v(x,y)^{1/q}}\right)^p\right\}^{1/p} \\
&\qquad\times \Bigl(\bigl(u(x,y)^{1/q}\bigr)^q + \bigl(v(x,y)^{1/q}\bigr)^q\Bigr)^{1/q} \\
&\quad= \left\{\frac{(\int_a^b \int_c^d f(x,y)^r \,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g(x,y)^r \,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right\}^{1/p} \\
&\qquad\times \bigl(u(x,y)+v(x,y)\bigr)^{1/q}.
\end{aligned}
$$

Rearranging, (2.2) follows.

The following is a discussion of the conditions for this equal sign to hold. Suppose that equality holds in (2.2). Then equality holds in Minkowski's inequality, which implies that $f(x,y) = \alpha g(x,y)$ and some $\alpha \ge 0$. Equality also holds in Hölder's inequality, implying that there are constants β and γ with $\beta^2 + \gamma^2 > 0$ such that

$$
\beta\left(\frac{(\int_a^b \int_c^d f(x,y)^r \,dx)^{1/r}}{u(x,y)^{1/q}}\right)^p = \gamma\bigl(u(x,y)^{1/q}\bigr)^q,
$$

or equivalently

$$
\beta\left(\int_a^b \int_c^d f(x,y)^r \,dx\,dy\right)^{p/r} = \gamma\, u(x,y)^p,
$$

and the same equation with $g(x,y)$ instead of $f(x,y)$. Therefore

$$
\begin{aligned}
\gamma\, u(x,y)^p &= \beta\left(\int_a^b \int_c^d f(x,y)^r \,dx\,dy\right)^{p/r} \\
&= \beta\left(\int_a^b \int_c^d \bigl(\alpha g(x,y)\bigr)^r \,dx\,dy\right)^{p/r} \\
&= \gamma\bigl(\alpha v(x,y)\bigr)^p.
\end{aligned}
$$

Obviously, if $\gamma = 0$, then $\|f(x,y)\|_r^r = \|g(x,y)\|_r^r = 0$. If $\gamma \ne 0$, then $\alpha > 0$ and $(\|f(x,y)\|_r^r, \|g(x,y)\|_r^r) = \alpha(\|u(x,y)\|_r^r, \|v(x,y)\|_r^r)$.

This proof is complete. □

Let $f(x,y)$ become $f(x)$ with appropriate transformation, (2.2) reduces to the following result.

Corollary 2.4 *For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $r>1$. If $u(x),v(x)>0$ and $f(x)$, $g(x)$ are continuous functions on $[a,b]$, then*

$$\left(\frac{(\int_a^b(f(x)+g(x))^r\,dx)^{1/r}}{(u(x)+v(x))^{1/q}}\right)^p \le \left(\frac{(\int_a^b f(x)^r\,dx)^{1/r}}{u(x)^{1/q}}\right)^p + \left(\frac{(\int_a^b g(x)^r\,dx)^{1/r}}{v(x)^{1/q}}\right)^p, \tag{2.3}$$

with equality if and only if either $\|f(x)\|_r^r = \|g(x)\|_r^r = 0$ or $(\|f(x)\|_r^r, \|g(x)\|_r^r) = \alpha(\|u(x)\|_r^r, \|v(x)\|_r^r)$ for some $\alpha>0$.

3 Improvements

We need the following lemma to prove our main results.

Lemma 3.1 ([3] p.39) *If $a_i\ge 0$, $b_i>0$, $i=1,\ldots,m$, and $\sum_{i=1}^m \alpha_i = 1$, then*

$$\left(\prod_{i=1}^m(a_i+b_i)\right)^{\alpha_i} \ge \left(\prod_{i=1}^m a_i\right)^{\alpha_i} + \left(\prod_{i=1}^m b_i\right)^{\alpha_i}, \tag{3.1}$$

with equality if and only if $a_1/b_1=\cdots=a_m/b_m$.

Theorem 3.2 *For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $r>1$. If $x_{00},y_{00},a_{00},b_{00}>0$ and reals x_{ij}, y_{ij}, a_{ij}, b_{ij}, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, then*

$$\begin{aligned}
&\frac{(\sum_{j=1}^m\sum_{i=1}^n[(x_{ij}+y_{ij})^r+(a_{ij}+b_{ij})^r])^p}{[(x_{00}+y_{00})^r+(a_{00}+b_{00})^r]^{p/q}} \\
&\quad\le \left(\frac{(\sum_{j=1}^m\sum_{i=1}^n x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^m\sum_{i=1}^n y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right)^r \\
&\qquad + \left(\frac{(\sum_{j=1}^m\sum_{i=1}^n a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^m\sum_{i=1}^n b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right)^r
\end{aligned} \tag{3.2}$$

with equality if and only if either $x_{ij}=y_{ij}=0$ and $a_{ij}=b_{ij}=0$ for $i=1,\ldots,n$ and $j=1,\ldots,m$ or $x_{ij}=\alpha y_{ij}$ and $a_{ij}=\beta b_{ij}$ for $i=0,1,\ldots,n$ and $j=0,1,\ldots,m$ and some $\alpha,\beta>0$, and

$$\begin{aligned}
&\left(\frac{(\sum_{j=1}^m\sum_{i=1}^n x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^m\sum_{i=1}^n y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right) : \left(\frac{(\sum_{j=1}^m\sum_{i=1}^n a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^m\sum_{i=1}^n b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right) \\
&\quad = (x_{00}+y_{00}) : (a_{00}+b_{00}).
\end{aligned}$$

Proof From (2.1), we have

$$\begin{aligned}
&\left(\sum_{j=1}^m\sum_{i=1}^n(x_{ij}+y_{ij})^r\right)^{1/r} \\
&\quad\le \left\{\frac{(\sum_{j=1}^m\sum_{i=1}^n x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^m\sum_{i=1}^n y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right\}^{1/p}(x_{00}+y_{00})^{1/q},
\end{aligned} \tag{3.3}$$

with equality if and only if either $x_{ij} = y_{ij} = 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ or $x_{ij} = \alpha y_{ij}$ for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$ and some $\alpha > 0$, and

$$\left(\sum_{j=1}^{m}\sum_{i=1}^{n}(a_{ij}+b_{ij})^r\right)^{1/r} \leq \left\{\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right\}^{1/p}(a_{00}+b_{00})^{1/q}, \tag{3.4}$$

with equality if and only if either $a_{ij} = b_{ij} = 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ or $a_{ij} = \alpha b_{ij}$ for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$ and some $\alpha > 0$.

From (3.1), (3.3), and (3.4), we obtain

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{i=1}^{n}\left[(x_{ij}+y_{ij})^r + (a_{ij}+b_{ij})^r\right] \\
&\quad\leq \left\{\left(\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right)^r\right\}^{1/p}\left((x_{00}+y_{00})^r\right)^{1/q} \\
&\qquad + \left\{\left(\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right)^r\right\}^{1/p}\left((a_{00}+b_{00})^r\right)^{1/q} \\
&\quad\leq \left\{\left(\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right)^r\right. \\
&\qquad \left. + \left(\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right)^r\right\}^{1/p}\left[(x_{00}+y_{00})^r + (a_{00}+b_{00})^r\right]^{1/q}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}[(x_{ij}+y_{ij})^r + (a_{ij}+b_{ij})^r])^p}{[(x_{00}+y_{00})^r + (a_{00}+b_{00})^r]^{p/q}} &\leq \left(\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}x_{ij}^r)^{p/r}}{x_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}y_{ij}^r)^{p/r}}{y_{00}^{p/q}}\right)^r \\
&\quad + \left(\frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}^r)^{p/r}}{a_{00}^{p/q}} + \frac{(\sum_{j=1}^{m}\sum_{i=1}^{n}b_{ij}^r)^{p/r}}{b_{00}^{p/q}}\right)^r.
\end{aligned}$$

From the equality conditions of (3.3), (3.4), and (3.1), we easily get the equality in (3.2). □

Remark 3.3 Let $a_{ij} = b_{ij} = 0$, (3.2) becomes a similar form of (2.1). Putting $x_{ij} = a_{ij}$, $y_{ij} = b_{ij}$ in (3.2), where $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$, (3.2) reduces to (2.1).

Theorem 3.4 *For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$. If $u(x,y), v(x,y), u'(x,y), v'(x,y) > 0$ and $f(x,y)$, $g(x,y), f'(x,y), g'(x,y)$ are continuous functions on $[a,b] \times [c,d]$, then*

$$\begin{aligned}
&\frac{(\int_a^b\int_c^d[(f(x,y)+g(x,y))^r + (f'(x,y)+g'(x,y))^r]\,dx\,dy)^p}{[(u(x,y)+v(x,y))^r + (u'(x,y)+v'(x,y))^r]^{p/q}} \\
&\quad\leq \left(\frac{(\int_a^b\int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right)^r \\
&\qquad + \left(\frac{(\int_a^b\int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g'(x,y)^r\,dx\,dy)^{p/r}}{v'(x,y)^{p/q}}\right)^r
\end{aligned} \tag{3.5}$$

with equality if and only if either $f(x,y) = g(x,y) = 0$ *and* $f'(x,y) = g'(x,y) = 0$ *or* $(f(x,y), g(x,y)) = \alpha(u(x,y), v(x,y))$ *and* $(f'(x,y), g'(x,y)) = \beta(u'(x,y), v'(x,y))$ *and some* $\alpha, \beta > 0$, *and*

$$\begin{aligned}
&\left(\frac{(\int_a^b\int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right) \\
&\quad : \left(\frac{(\int_a^b\int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g'(x,y)^r)^{p/r}}{v'(x,y)^{p/q}}\right) \\
&= \big(u(x,y) + v(x,y)\big) : \big(u'(x,y) + v'(x,y)\big).
\end{aligned}$$

Proof From (2.1), we have

$$\begin{aligned}
&\left(\int_a^b \int_c^d \big(f(x,y) + g(x,y)\big)^r\,dx\,dy\right)^{1/r} \\
&\quad \le \left\{\frac{(\int_a^b\int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right\}^{1/p} \\
&\qquad \times \big(u(x,y) + v(x,y)\big)^{1/q},
\end{aligned} \tag{3.6}$$

with equality if and only if either $f(x,y) = g(x,y) = 0$ or $(f(x,y), g(x,y)) = \alpha(u(x,y), v(x,y))$ for some $\alpha > 0$. And

$$\begin{aligned}
&\left(\int_a^b \int_c^d \big(f'(x,y) + g'(x,y)\big)^r\,dx\,dy\right)^{1/r} \\
&\quad \le \left\{\frac{(\int_a^b\int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g'(x,y)^r\,dx\,dy)^{p/r}}{v'(x,y)^{p/q}}\right\}^{1/p} \\
&\qquad \times \big(u'(x,y) + v'(x,y)\big)^{1/q},
\end{aligned} \tag{3.7}$$

with equality if and only if either $f'(x,y) = g'(x,y) = 0$ or $(f'(x,y), g'(x,y)) = \beta(u'(x,y), v'(x,y))$ and for some $\beta > 0$,

From (3.1), (3.6), and (3.7), we obtain

$$\begin{aligned}
&\int_a^b \int_c^d \big[\big(f(x,y) + g(x,y)\big)^r + \big(f'(x,y) + g'(x,y)\big)^r\big]\,dx\,dy \\
&\quad \le \left\{\left(\frac{(\int_a^b\int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right)^r\right\}^{1/p} \big(\big(u(x,y) + v(x,y)\big)^r\big)^{1/q} \\
&\qquad + \left\{\left(\frac{(\int_a^b\int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g'(x,y)^r\,dx\,dy)^{p/r}}{v'(x,y)^{p/q}}\right)^r\right\}^{1/p} \\
&\qquad \times \big(\big(u'(x,y) + v'(x,y)\big)^r\big)^{1/q} \\
&\quad \le \left\{\left(\frac{(\int_a^b\int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right)^r\right. \\
&\qquad \left. + \left(\frac{(\int_a^b\int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b\int_c^d g'(x,y)^r\,dx\,dy)^{p/r}}{v'(x,y)^{p/q}}\right)^r\right\}^{1/p} \\
&\qquad \times \big[\big(u(x,y) + v(x,y)\big)^r + \big(u'(x,y) + v'(x,y)\big)^r\big]^{1/q}.
\end{aligned}$$

Hence

$$\frac{(\int_a^b \int_c^d [(f(x,y)+g(x,y))^r + (f'(x,y)+g'(x,y))^r]\,dx\,dy)^p}{[(u(x,y)+v(x,y))^r + (u'(x,y)+v'(x,y))^r]^{p/q}}$$
$$\le \left(\frac{(\int_a^b \int_c^d f(x,y)^r\,dx\,dy)^{p/r}}{u(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g(x,y)^r\,dx\,dy)^{p/r}}{v(x,y)^{p/q}}\right)^r$$
$$+ \left(\frac{(\int_a^b \int_c^d f'(x,y)^r\,dx\,dy)^{p/r}}{u'(x,y)^{p/q}} + \frac{(\int_a^b \int_c^d g'(x,y)^r\,dx\,dy)^{p/r}}{v'(x,y)^{p/q}}\right)^r.$$

From the equality conditions of (3.6), (3.7), and (3.1), we easily get the equality in (3.2). □

Remark 3.5 Let $f'(x,y) = g'(x,y) = 0$, (3.3) becomes a similar form of (2.2). Putting $f(x,y) = f'(x,y)$, $g(x,y) = g'(x,y)$, $u(x,y) = u'(x,y)$ and $v(x,y) = v'(x,y)$ in (3.5), (3.5) reduces to (2.2).

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