A Novel LMI Condition for Stability of 2D Mixed Continuous-Discrete-Time Systems via Complex LFR and Lyapunov Functions

Graziano Chesi

Abstract

This paper addresses the problem of establishing stability of 2D mixed continuous-discrete-time systems. A novel linear matrix inequality (LMI) condition is proposed based on the introduction of a complex linear fractional representation (LFR) of the systems and on the use of complex Lyapunov functions depending rationally on a parameter. Promising results are obtained in terms of computational burden. Indeed, as shown by various examples with small and large dimensions, the computational burden of the proposed LMI condition may be rather smaller than that of other existing LMI conditions.

I. INTRODUCTION

It is well known that 2D systems play an important role in engineering. These systems are dynamical systems characterized by the presence of signals that evolve in two dimensions (e.g., space and time) through dynamics that mutually influence each other. These dynamics can be homogenous (i.e., both continuous-time or both discrete-time) or mixed (i.e., one continuous-time and the other discrete-time). See for instance [1], [11], [12], [15], [16], [18], [20], [24] and references therein. This paper considers the latter case, i.e., 2D mixed continuous-discrete-time systems.

2D mixed continuous-discrete-time systems have various applications, including irrigation channels [5], metal rolling processes [19], and vehicle platoons [14]. It is useful to mention that 2D mixed continuous-discrete-time systems are also known in the literature with other names, such as differential repetitive processes (where “differential” refers to the continuous dynamics, and “repetitive” refers to the discrete dynamics) and hybrid 2D systems (where “hybrid” is used to indicate the presence of inhomogeneous dynamics).

Department of Electrical and Electronic Engineering, The University of Hong Kong, Hong Kong. Email: chesi@eee.hku.hk

Like in any other class of dynamical systems, establishing stability represents a fundamental problem in 2D mixed continuous-discrete-time systems. This problem has received a number of contributions, based on techniques of various nature. One of the first contributions was proposed in [21] based on the eigenvalues of a matrix obtained through the Kronecker product. In order to provide conditions that could be possibly used in more general situations (e.g., presence of uncertainties, controller design, etc), lots of efforts have been spent by researchers for deriving conditions checkable with linear matrix inequalities (LMIs). These LMI conditions can be classified into two main groups. The first group exploits Lyapunov functions, in the time domain as done in the pioneering work [13], or in the frequency domain as done in [3], [4], [9], [10]. The second group does not use Lyapunov functions but instead eigenvalues combinations, see for instance [8], [9].

Unfortunately, the computational burden of the existing LMI conditions may be quite large. Indeed, this is the case of the LMI conditions that exploit Lyapunov functions, where the degree required by these functions for establishing stability or instability may be quite large. And this is the case of the LMI conditions that exploit eigenvalues combinations, where polynomials are constructed with Kronecker products or similar strategies, and the degree of these polynomials quickly grows with the dimensions of the systems.

This paper proposes a possible solution for this issue. Specifically, the paper addresses the problem of establishing stability of 2D mixed continuous-discrete-time systems. A novel LMI condition is proposed based on the introduction of a complex linear fractional representation (LFR) of the systems and on the use of complex Lyapunov functions depending rationally on a parameter. Promising results are obtained in terms of computational burden. Indeed, as shown by various examples with small and large dimensions, the computational burden of the proposed LMI condition may be rather smaller than that of other existing LMI conditions.

The paper is organized as follows. Section II introduces the notation and the problem formulation. Section III describes the proposed approach. Section IV presents the examples. Lastly, Section V reports the conclusions and future works.

II. PROBLEM FORMULATION

The notation adopted in the paper is as follows. The natural numbers set (including zero), the real numbers set and the complex numbers set are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$. The imaginary unit is denoted by j. The symbols 0 and I denote null matrices and identity matrices of size specified by the context. The notations $\mathrm{Re}(\cdot)$, $\mathrm{Im}(\cdot)$ and $|\cdot|$ denote the real part, imaginary part and magnitude. The Euclidean norm is denoted by $\|\cdot\|_2$. The determinant is denoted by $\det(\cdot)$. The notation $A \otimes B$ denotes the Kronecker product of A and B. The complex conjugate, the transpose and the complex conjugate transpose of A are denoted by $\bar{A}$, A^T and A^H. A matrix A is said 1) Hermitian if $A^H = A$, 2) Hurwitz if all its eigenvalues have negative real part, and 3) Schur if all its eigenvalues have magnitude less than one. The symbol $\star$ denotes a corresponding block in Hermitian matrices. The notation $A > 0$ (respectively, $A \geq 0$) denotes a Hermitian positive definite (respectively, semidefinite) matrix A. The abbreviation "s.t." denotes "subject to".

In this paper we consider the 2D mixed continuous-discrete-time system described by

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}x_c(t,k) & = & A_{cc}x_c(t,k) + A_{cd}x_d(t,k) \\ x_d(t,k+1) & = & A_{dc}x_c(t,k) + A_{dd}x_d(t,k) \end{array} \right. \tag{1}$$

where $t \in \mathbb{R}$ and $k \in \mathbb{R}$ are independent variables, $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous state and the discrete state, and $A_{cc} \in \mathbb{R}^{n_c \times n_c}$, $A_{cd} \in \mathbb{R}^{n_c \times n_d}$, $A_{dc} \in \mathbb{R}^{n_d \times n_c}$ and $A_{dd} \in \mathbb{R}^{n_d \times n_d}$ are given matrices. Hereafter we introduce the definition of stability commonly considered in the literature for this system, see [2], [13], [17], [20], [22] for more information and for other types of stability.

Definition 1: The system (1) is said *stable* if, for all $(s,z) \in \mathbb{C} \times \mathbb{C} : \ \mathrm{Re(s)} \geq 0, \ |\mathrm{z}| \geq 1$, one has

$$\left\{ \begin{array}{rcl} 0 & \neq & \det(A_{cc} - sI_{n_c}) \\ 0 & \neq & \det(A_{dd} - zI_{n_d}) \\ 0 & \neq & \det \begin{pmatrix} A_{cc} - sI_{n_c} & A_{cd} \\ A_{dc} & A_{dd} - zI_{n_d} \end{pmatrix} \end{array} \right. . \tag{2}$$

□

The problem addressed in this paper is as follows.

Problem 1: Establish whether the system (1) is stable according to Definition 1. □

III. Proposed Approach

Let us start by recalling an important result in the literature, which provides an equivalent reformulation of the property of stability introduced in Definition 1 in terms of stability of a constant real matrix and of a parameter-dependent complex matrix.

Theorem 1 (see [20] and references therein): The system (1) is stable if and only if the following sub-conditions hold:

1) A_{cc} is Hurwitz;
2) $F(j\omega)$ is Schur for all $\omega \in \mathbb{R}$, where

$$F(s) = A_{dc}(sI - A_{cc})^{-1}A_{cd} + A_{dd}. \tag{3}$$

□

In order to present the proposed approach, let us introduce the following classes of matrix functions.

Definition 2: A matrix function $M : \mathbb{R} \to \mathbb{C}^{r\times r}$ is said to be *Hermitian* if

$$M(\omega) = M(\omega)^H \quad \forall\omega \in \mathbb{R}. \tag{4}$$

□

Definition 3: A matrix function $M : \mathbb{R} \to \mathbb{C}^{r_1\times r_2}$ is said to be *even* if

$$M(\omega) = \overline{M(-\omega)}. \tag{5}$$

□

Definition 4: A matrix polynomial $M : \mathbb{R} \to \mathbb{C}^{r\times r}$ is said to be a *sum of squares of matrix polynomials (SOS)* if there exist matrix polynomials $M_i : \mathbb{R} \to \mathbb{C}^{r\times r}$, $i = 1, \ldots, k$, such that

$$M(\omega) = \sum_{i=1}^{k} M_i(\omega)^H M_i(\omega). \tag{6}$$

□

For a matrix $V \in \mathbb{C}^{n_d \times n_d}$, with V Hermitian, let us define

$$G(V) = \begin{pmatrix} 0 & 0 \\ \star & V \end{pmatrix} - D' V D \tag{7}$$

where

$$D = \begin{pmatrix} A_{dc} & A_{dd} \end{pmatrix}. \tag{8}$$

Also, let us define and

$$H(\omega) = E(j\omega)^H E(j\omega) \tag{9}$$

where

$$E(s) = \begin{pmatrix} sI - A_{cc} & -A_{cd} \end{pmatrix}. \tag{10}$$

The approach proposed in this paper is as follows.

Theorem 2: Assume without loss of generality that A_{cc} is Hurwitz and A_{dd} is Schur. The system (1) is stable if there exist $\beta, \gamma \in \mathbb{R}$ and a matrix polynomial $V : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ Hermitian, even, and of degree $2h$, $h \in \mathbb{N}$, such that

$$\begin{cases} V(\omega) - \gamma(1+\omega^2)^h I \text{ is SOS} \\ W(\omega) - \gamma(1+\omega^2)^{d_1} I \text{ is SOS} \\ \gamma > 0 \end{cases} \tag{11}$$

where

$$W(\omega) = (1+\omega^2)^{d_2} G(V(\omega)) + \beta(1+\omega^2)^{d_3} H(\omega) \tag{12}$$

and

$$\begin{cases} d_1 = \max\{1, h\} \\ d_2 = \max\{0, 1-h\} \\ d_3 = \max\{0, h-1\}. \end{cases} \tag{13}$$

Proof. Let us suppose that the condition (11) holds for some $\beta, \gamma \in \mathbb{R}$ and for a matrix polynomial $V : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ Hermitian, even, and of degree $2h$, $h \in \mathbb{N}$. This implies that $V(\omega) - \gamma(1+\omega^2)^h I$

and $W(\omega) - \gamma(1+\omega^2)^{d_1} I$ are positive semidefinite for all $\omega \in \mathbb{R}$. For $\tilde{u}(k), \tilde{y}(k) \in \mathbb{C}^{n_c}$ and $\tilde{x}(k) \in \mathbb{C}^{n_d}$ let us define the auxiliary system

$$\left\{ \begin{array}{rcl} \tilde{x}(k+1) & = & A_{dd}\tilde{x}(k) + A_{dc}\tilde{u}(k) \\ j\omega\tilde{y}(k) & = & A_{cd}\tilde{x}(k) + A_{cc}\tilde{u}(k) \\ \tilde{u}(k) & = & \tilde{y}(k). \end{array} \right. \tag{14}$$

Since A_{cc} is Hurwitz, it follows that $F(j\omega)$ in (3) does exist for all $\omega \in \mathbb{R}$, and

$$\tilde{x}(k+1) = F(j\omega)\tilde{x}(k). \tag{15}$$

Let us define the candidate parameter-dependent quadratic Lyapunov function

$$\tilde{v}(\tilde{x}(k), \omega) = \tilde{x}(k)^H \tilde{V}(\omega) \tilde{x}(k) \tag{16}$$

where

$$\tilde{V}(\omega) = \frac{V(\omega)}{(1+\omega^2)^h}. \tag{17}$$

Since $V(\omega) - \gamma(1+\omega^2)^h I \geq 0$ and $\gamma > 0$, it follows that

$$\tilde{v}(\tilde{x}(k), \omega) > 0 \quad \forall \tilde{x}(k) \neq 0 \ \forall \omega \in \mathbb{R}. \tag{18}$$

Let us define

$$\tilde{z}(k) = \begin{pmatrix} \tilde{u}(k) \\ \tilde{x}(k) \end{pmatrix}. \tag{19}$$

Let us pre- and post-multiply $W(\omega) - \gamma(1+\omega^2)^{d_1} I$ times $\tilde{z}(k)^H$ and $\tilde{z}(k)$, respectively. Since $W(\omega) - \gamma(1+\omega^2)^{d_1} I \geq 0$, it follows that

$$\begin{array}{rcl} 0 & \leq & \tilde{z}(k)^H \left(W(\omega) - \gamma(1+\omega^2)^{d_1} I \right) \tilde{z}(k) \\ & = & (1+\omega^2)^{d_2-h} \left(\tilde{v}(\tilde{x}(k), \omega) - \tilde{v}(\tilde{x}(k+1), \omega) \right) \\ & & +\beta(1+\omega^2)^{d_3} \left\| E(j\omega)\tilde{z}(k) \right\|_2^2 - \gamma(1+\omega^2)^{d_1} \|\tilde{z}(k)\|_2^2. \end{array} \tag{20}$$

Let us observe that, along the trajectories of (14), one has

$$E(j\omega)\tilde{z}(k) = 0. \tag{21}$$

This implies

$$\begin{array}{rcl} 0 & \leq & (1+\omega^2)^{d_2-h} \left(\tilde{v}(\tilde{x}(k), \omega) - \tilde{v}(\tilde{x}(k+1), \omega) \right) \\ & & -\gamma(1+\omega^2)^{d_1} \|\tilde{z}(k)\|_2^2. \end{array} \tag{22}$$

Since $\gamma > 0$ and since $(1+\omega^2)^{d_2-h}$ and $(1+\omega^2)^{d_1}$ are positive for all $\omega \in \mathbb{R}$, it follows that

$$\tilde{v}(\tilde{x}(k+1),\omega) < \tilde{v}(\tilde{x}(k),\omega) \quad \forall \tilde{z}(k) \neq 0 \ \forall \omega \in \mathbb{R}. \tag{23}$$

Therefore, $\tilde{v}(\tilde{x}(k),\omega)$ is a Lyapunov function for (14) for all $\omega \in \mathbb{R}$, which implies from (15) that $F(j\omega)$ is Schur for all $\omega \in \mathbb{R}$. From Theorem 1 it follows that the system (1) is stable. □

Theorem 2 provides a novel condition for establishing whether the system (1) is stable. This condition is based on the introduction of a complex LFR of the system (1), obtained through the auxiliary system (14), and on the use of a complex Lyapunov function candidate depending rationally on a parameter, obtained through the matrix polynomial $V(\omega)$ and given by (17). Let us observe that A_{cc} and A_{dd} can be assumed to be Hurwitz and Schur without loss of generality due to Definition 1.

The condition provided by Theorem 2 requires to establish the existence of the scalars β, γ and of the complex matrix polynomial $V(\omega)$ (Hermitian, even, and of degree $2h$) such that the condition (11) holds. Let us observe that these decision variables are defined up to a positive scale factor. Also, let us observe that the condition (11) requires to establish whether two complex matrix polynomials depending affine linearly on the decision variables are SOS. This means that the condition (11) is equivalent to a system of LMIs.

Indeed, as explained in [9], these LMIs can be built as follows. Let $P : \mathbb{R} \to \mathbb{C}^{r\times r}$ be a complex matrix polynomial satisfying $P(\omega) = P(\omega)^H$, $\deg(P(\omega)) \leq 2q$, $q \in \mathbb{N}$. Let us define

$$\Phi(P(\omega)) = \begin{pmatrix} \mathrm{Re(P(\omega))} & \mathrm{Im(P(\omega))} \\ \star & \mathrm{Re(P(\omega))} \end{pmatrix}. \tag{24}$$

Let $b(\omega)$ be a vector whose entries are all the monomials in x of degree not greater than h, and let $S = S^T$ be a matrix that satisfies

$$\Phi(P(\omega)) = (b(\omega) \otimes I)^T S \, (b(\omega) \otimes I). \tag{25}$$

Let $L(\alpha)$ be a linear parametrization of the linear set

$$\mathcal{L} = \{\tilde{L} = \tilde{L}^T : \ (b(\omega) \otimes I)^T \tilde{L} \, (b(\omega) \otimes I) = 0\} \tag{26}$$

where α is a free vector with length equal to the dimension of $\mathcal{L}$. Then, $P(\omega)$ is SOS if and only if there exists α that satisfies the LMI

$$S + L(\alpha) \geq 0. \tag{27}$$

Also, whenever $P(\omega)$ is even, one may reduce the number of LMI scalar variables (i.e., the length of the vector α in this case) as explained in [10]. The reader is also referred to [6], [7] and references therein for more information about SOS matrix polynomials.

IV. EXAMPLES

In this section we present some numerical examples. The condition proposed in Theorem 2 is compared with two existing LMI conditions based on the use of Lyapunov functions:

- the first existing LMI condition is our previous one proposed in [10], which improves the one in [9];
- the second existing LMI condition is the one proposed in [3], which improves the one in [4]. It should be noticed that this condition allows one to address more general problems than Problem 1, indeed, it can be used for 2D systems with mixed dynamics as well as for 2D systems with non-mixed dynamics (i.e., continuous-continuous or discrete-discrete). The results reported for this condition in this paper are based on our implementation of the condition.

The toolbox SeDuMi [23] for Matlab is adopted to test all the mentioned LMI conditions on a standard computer with Windows 10, Intel Core i7, 3.4 GHz, 8 GB RAM.

In these examples, we report the number of independent LMI scalar variables. For the condition proposed in Theorem 2, this number is 2 (for β and γ), plus the number of independent scalar coefficients of $V(\omega)$, plus the number of the entries in the vectors α_{even} needed to convert the SOS conditions into LMI conditions as done in (27), minus 1 (since all these variables are defined up to a positive scale factor).

Before proceeding, it is useful to remark that there exist also LMI conditions not based on the use of Lyapunov functions, such as [8], [9] which exploit eigenvalues combinations. These LMI conditions present a larger computational burden (measured in terms of number of LMI scalar variables) in the examples considered in this paper where the problem is to establish stability. However, these LMI conditions not based on the use of Lyapunov functions may be useful for establishing instability.

A. Example 1

The first example is borrowed from [9], [10]. Let us consider the system (1) with the matrices

$$\begin{cases} A_{cc} & = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{pmatrix} \\ A_{cd} & = \begin{pmatrix} 0 & 0.5 & 0 \\ -0.5 & 1.5 & 1 \\ -0.5 & 0 & \delta \end{pmatrix} \\ A_{dc} & = \begin{pmatrix} 0.2 & 0 & 0.4 \\ 0 & -0.3 & 0 \\ 0 & 0 & 0.3 \end{pmatrix} \\ A_{dd} & = \begin{pmatrix} -0.4 & 0 & 0 \\ 0.2 & 0 & 0.3 \\ 0 & -0.4 & -0.2 \end{pmatrix} \end{cases}$$

where $\delta \in \mathbb{R}$ is a parameter. It turns out that, for $\delta = 1$, the system is stable. This can be verified using the condition proposed in Theorem 2. Indeed, this condition holds with $h = 1$, and the number of independent LMI scalar variables is 43. The found $V(\omega)$ is

$$V(\omega) = \begin{pmatrix} 0.817\omega^2 + 0.507 & -0.111\omega^2 - 0.224 - j0.033\omega & -0.57\omega^2 - 0.235 - j0.261\omega \\ \star & 0.266\omega^2 + 0.494 & 0.001\omega^2 + 0.139 - j0.196\omega \\ \star & \star & 0.607\omega^2 + 0.242 \end{pmatrix}.$$

For comparison, let us consider our previous method in [10]. For establishing stability in this case, the number of independent LMI scalar variables is 99 (which corresponds to the choice $d = 2$ in [10], where d denotes the degree of the complex matrix Lyapunov function used).

Also, we consider the method in [3]. For establishing stability in this case, the number of independent LMI scalar variables is 69 (which corresponds to the choice $\alpha = 1$ in [3], where α denotes the degree of the complex matrix Lyapunov function used).

B. Example 2

In this second example let us consider the system (1) with the matrices

$$\left\{\begin{array}{rcl} A_{cc} & = & \begin{pmatrix} -2 & -1.3 & 1.7 \\ -0.9 & -2 & -0.8 \\ -0.3 & -0.3 & -3 \end{pmatrix} \\ A_{cd} & = & \begin{pmatrix} 1 & 0.6 & 0.7 \\ -0.2 & 0 & 0.3 \\ 0 & 1 & -1.2 \end{pmatrix} \\ A_{dc} & = & \begin{pmatrix} 0.3 & 0 & 0.5 \\ 0 & -0.2 & 0 \\ 0.8 & -1 & -0.7 \end{pmatrix} \\ A_{dd} & = & \begin{pmatrix} 0.5 & 0 & 0.4 \\ -0.4 & 0.9 & 0.3 \\ 0.7 & 0.3 & -0.4 \end{pmatrix} \end{array}\right. .$$

It turns out that the system is stable. For establishing this property, the number of independent LMI scalar variables is:

- 43 in the condition proposed in Theorem 2 (corresponding to the choice $h = 1$);
- 99 in our previous method in [10] (corresponding to the choice $d = 2$);
- 168 in the method in [3] (corresponding to the choice $\alpha = 2$).

The found $V(\omega)$ in the condition proposed in Theorem 2 is

$$V(\omega) = \begin{pmatrix} 0.137\omega^2 + 0.227 & -0.108\omega^2 - 0.339 - j0.202\omega & -0.004\omega^2 - 0.031 - j0.031\omega \\ \star & 0.193\omega^2 + 0.785 & 0.003\omega^2 + 0.075 + j0.056\omega \\ \star & \star & 0.022\omega^2 + 0.008 \end{pmatrix}.$$

C. Example 3

In this third example we consider two systems analogous to the system considered in Example 2 but having matrices with larger size. Specifically, let A_{cc}, A_{cd}, A_{dc} and A_{dd} be defined as in

Example 2, and let us redefine them as

$$\begin{cases} A_{cc} & \to & B \otimes A_{cc} \\ A_{cd} & \to & B \otimes A_{cd} \\ A_{dc} & \to & B \otimes A_{dc} \\ A_{dd} & \to & B \otimes A_{dd} \end{cases}$$

where B is an identity matrix. Let us consider the following two cases.

1) B is the identity matrix 2×2: in this case, the matrices A_{cc}, A_{cd}, A_{dc} and A_{dd} have size 6×6. It turns out that the system is stable. For establishing this property, the number of independent LMI scalar variables is:

- 157 in the condition proposed in Theorem 2 (corresponding to the choice $h = 1$);
- 981 in our previous method in [10] (corresponding to the choice $d = 2$);
- 651 in the method in [3] (corresponding to the choice $\alpha = 2$).

2) B is the identity matrix 3×3: in this case, the matrices A_{cc}, A_{cd}, A_{dc} and A_{dd} have size 9×9. It turns out that the system is stable. For establishing this property, the number of independent LMI scalar variables is:

- 343 in the condition proposed in Theorem 2 (corresponding to the choice $h = 1$);
- 4266 in our previous method in [10] (corresponding to the choice $d = 2$);
- 1449 in the method in [3] (corresponding to the choice $\alpha = 2$).

V. CONCLUSIONS

This paper has addressed the problem of establishing stability of 2D mixed continuous-discrete-time systems. A novel LMI condition has been proposed based on the introduction of a complex LFR of the systems and on the use of complex Lyapunov functions depending rationally on a parameter. Promising results have been obtained in terms of computational burden. Indeed, as shown by various examples with small and large dimensions, the computational burden of the proposed LMI condition may be rather smaller than that of other existing LMI conditions.

Several directions can be considered in future works. For instance, one direction could be the extension of the proposed condition to the case of 2D mixed continuous-discrete-time systems affected by uncertainties in order to establish robust stability.

ACKNOWLEDGEMENTS

The author would like to thank Rick Middleton for many instructive discussions about the problem addressed in this paper. Also, the author would like to thank the Associate Editor and the Reviewers for their useful comments.

REFERENCES

[1] B. D. O. Anderson, P. Agathoklis, E. I. Jury, and M. Mansour. Stability and the matrix Lyapunov equation for discrete 2D systems. *IEEE Transactions on Circuits and Systems*, 33(3):261–267, 1986.

[2] O. Bachelier, T. Cluzeau, R. David, and N. Yeganefar. Structural stabilization of linear 2D discrete systems using equivalence transformations. *Multidimensional Systems and Signal Processing*, pages 1–24, 2016.

[3] O. Bachelier, W. Paszke, N. Yeganefar, and D. Mehdi. Comments on "on stabilization of 2D Roesser models". *IEEE Transactions on Automatic Control*, 63(8):2745–2749, 2018.

[4] O. Bachelier, W. Paszke, N. Yeganefar, D. Mehdi, and A. Cherifi. LMI necessary and sufficient stability conditions for 2D Roesser models. *IEEE Transactions on Automatic Control*, 61(3):766–770, 2016.

[5] M. Cantoni, E. Weyer, Y. Li, S.K. Ooi, I. Mareels, and M. Ryan. Control of large-scale irrigation networks. *Proceeding of IEEE*, 95:75–91, 2007.

[6] G. Chesi. LMI techniques for optimization over polynomials in control: a survey. *IEEE Transactions on Automatic Control*, 55(11):2500–2510, 2010.

[7] G. Chesi. On the complexity of SOS programming and applications in control systems. *Asian Journal of Control*, 20(5):2005–2013, 2018.

[8] G. Chesi. Stability test for complex matrices over the complex unit circumference via LMIs and applications in 2D systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 66(5):1960–1969, 2019.

[9] G. Chesi and R. H. Middleton. Necessary and sufficient LMI conditions for stability and performance analysis of 2D mixed continuous-discrete-time systems. *IEEE Transactions on Automatic Control*, 59(4):996–1007, 2014.

[10] G. Chesi and R. H. Middleton. H-infinity and H-two norms of 2D mixed continuous-discrete-time systems via rationally-dependent complex Lyapunov functions. *IEEE Transactions on Automatic Control*, 60(10):2614–2625, 2015.

[11] S. Foda and P. Agathoklis. Stability of differential multipass processes. *Electronics Letters*, 24(16):1016–1017, 1988.

[12] E. Fornasini and G. Marchesini. Doubly-indexed dynamical systems: state-space models and structural properties. *Mathematical Systems Theory*, 12:59–72, 1978.

[13] K. Galkowski, W. Paszke, E. Rogers, S. Xu, and J. Lam. Stability and control of differential linear repetitive processes using an LMI setting. *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, 50(9):662–666, 2003.

[14] S. Knorn and R. H. Middleton. Stability of two-dimensional linear systems with singularities on the stability boundary using LMIs. *IEEE Transactions on Automatic Control*, 58(10):2579–2590, 2013.

[15] W. S. Lu and E. B. Lee. Stability analysis of two-dimensional systems via a Lyapunov approach. *IEEE Transactions on Circuits and Systems*, 32:61–68, 1985.

[16] D. H. Owens. Asymptotic stability of differential multipass processes. *Electronics Letters*, 15(15):446–447, 1979.

[17] L. Pandolfi. Exponential stability of 2-D systems. *Systems and Control Letters*, 4(6):381–385, 1984.

[18] R. P. Roesser. A discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control*, 20(1):1–10, 1975.

[19] E. Rogers, K. Galkowski, and D. H. Owens. *Stability Analysis for Linear Repetitive Processes*, volume 349 of *Lecture Notes in Control And Information Sciences Series*. Springer, 2007.

[20] E. Rogers and D. H. Owens. *Stability Analysis for Linear Repetitive Processes*, volume 175 of *Lecture Notes in Control And Information Sciences Series*. Springer, 1992.

[21] E. Rogers and D. H. Owens. Kronecker product based stability tests and performance bounds for a class of 2D continuous-discrete linear systems. *Linear Algebra and its Applications*, 353(1):33–52, 2002.

[22] C. W. Scherer. Lossless $\mathcal{H}_\infty$-synthesis for 2D systems. *Systems and Control Letters*, 95:35–45, 2016.

[23] J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999.

[24] M. E. Valcher. Characteristic cones and stability properties of two-dimensional autonomous behaviors. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 47(3):290–302, 2000.