An algebraic approach to the Siegel-Weil average for binary quadratic forms

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Quadratic forms

- Quad. form Q: $(a_{ij} \in \mathbb{Z})$

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- Genus gen(Q): $\mathcal{Q} \in$ gen(Q) iff $Q \sim_{\mathbb{Z}_p} \mathcal{Q}$ for all p (finite and infinite).

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- Example: $Q(x,y,z,w) = x^2+y^2+z^2+w^2$; Θ_Q in one-dim. space $\implies$ Lagrange's 4 □ Thm.

Siegel–Weil formula

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$$\Theta_{\mathrm{gen}(Q)}(z) := \frac{1}{\sum_{\mathcal{Q}\in\mathrm{gen}(Q)/\sim_{\mathbb{Z}}} \omega_{\mathcal{Q}}^{-1}} \sum_{\mathcal{Q}\in\mathrm{gen}(Q)/\sim_{\mathbb{Z}}} \frac{\Theta_{\mathcal{Q}}(z)}{\omega_{\mathcal{Q}}}.$$

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Question: Which one?

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- Holomorphicity.
- Bounded towards cusps.

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$$f\Big|_{k,\nu}\gamma(\tau) := \nu(\gamma)^{-1}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right).$$

Modifications

- Modular form f weight $k \in \frac{1}{2}\mathbb{Z}$ (half-integral weight):
- Modularity $f\Big|_{k,\nu}\gamma = f$ for $\gamma \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.
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Examples

- Eisenstein series ($\Gamma_\infty := \{\pm T^n : n \in \mathbb{Z}\},\ T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$):

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Holomorphic because 1 is holomorphic (and good convergence).

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- $n = 3$: Duke–Schulze-Pillot: m suff. large rep. by spinor genus containing Q $\implies$ m repr. by Q.

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$$\sum_{Q\in\mathcal{Q}_D/\sim_{\mathbb{Z}}} r_Q(n)=\omega_D\sum_{t|n}\left(\frac{D}{t}\right).$$
- For $n=0$, yields Dirichlet class number formula ($D<0$ fund.):
$$h(D)=\#\mathcal{Q}_D=\frac{\omega_D\sqrt{|D|}}{2\pi}L(\chi_D,1).$$

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- Group $= H$, princ. genus $= H^2$.

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Characters and Eisenstein series

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- Can compute

$$\chi(Q) = \chi_d(n) = \left(\frac{d}{n}\right),$$

where n prim. repres. by Q.

Characters and Eisenstein series (cont.)

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- Can plug in $E_{\chi_d,\chi_{d'}}$ (explicit).

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- Claim that
$$b(n) = \left(1 + \left(\frac{\Delta}{p}\right)\right) r_1(n).$$

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- For $p = \mathfrak{p}^2$ ramified:

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- Can consider similar questions for these.

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- Shimura:

$$\frac{1}{\sum_{M+\mu \in \mathrm{gen}(L+\nu)/\sim_{\mathbb{Z}}} \omega_{M+\mu}^{-1}} \sum_{M+\mu \in \mathrm{gen}(L+\nu)/\sim_{\mathbb{Z}}} \frac{r_{M+\mu}(n)}{\omega_{M+\mu}^{-1}}$$

is coeff. Eis. series./prod. local densities.

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- “Natural” F for Monster group, Mathieu groups, etc.?

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- Note that modularity not used (although Hecke operators natural).