ROBUST OPTIMAL EXCESS-OF-LOSS REINSURANCE AND INVESTMENT STRATEGY FOR AN INSURER IN A MODEL WITH JUMPS

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Abstract. This paper considers a robust optimal excess-of-loss reinsurance-investment problem in a model with jumps for an ambiguity-averse insurer (AAI), who worries about ambiguity and aims to develop a robust optimal reinsurance-investment strategy. The AAI's surplus process is assumed to follow a diffusion model, which is an approximation of the classical risk model. The AAI is allowed to purchase excess-of-loss reinsurance and invest her surplus in a risk-free asset and a risky asset whose price is described by a jump-diffusion model. Under the criterion for maximizing the expected exponential utility of terminal wealth, optimal strategy and optimal value function are derived by applying the stochastic dynamic programming approach. Our model and results extend some of the existing results in the literature, and the economic implications of our findings are illustrated. Numerical examples show that considering ambiguity and reinsurance brings utility enhancements.

Keywords: Robust optimal control, Excess-of-loss reinsurance and investment, Jump-diffusion model, Utility maximization, Ambiguity-Averse Insurer.

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1. Introduction

Reinsurance is an effective risk-spreading approach, while investment is an increasingly important way of using insurers' surplus, and both are popular in the insurance industry. In recent years, the problem of optimal reinsurance-investment has attracted significant attention in the literature. A number of scholars have considered the problem of maximizing the expected utility of terminal wealth. For example, Yang & Zhang (2005) studies the optimal investment problem for an insurer in a jump-diffusion risk model. Lin & Li (2011) discusses the optimal reinsurance-investment problem in a jump-diffusion insurance risk model where the dynamics of the risky asset are governed by a constant elasticity of variance (CEV) model. Liang & Yuen (2016) derives the optimal reinsurance strategy for maximizing the expected exponential utility of terminal wealth in a risk model with dependent risks. Likewise, other optimization objectives are considered in the literature. We refer readers to Schmidli (2002) and Jang & Larsen (2015) for the criterion of ruin probability minimization, and Pressacco et al. (2011) and Bi et al. (2013) for the mean-variance criterion.

Although the problem of optimal reinsurance-investment has been widely investigated by many scholars, two aspects merit further exploration. The majority of the above-mentioned literature ignores ambiguity. However, it is a notorious fact that the return of risky assets is difficult to estimate with precision. Thus, some scholars have advocated and investigated the effect of ambiguity on portfolio selection, noting that in many cases, the parametric models used in theory, such as the Black-Scholes model, contain significant uncertainties in parameter estimates. Take drift parameter as an example. As the expected return of a risky asset is not known in a priori with any adequate precision, the investor must typically account for a significant level of error in drift parameter estimates. Compared with making ad-hoc decisions about how much error are contained in the estimates for the parameters of risky assets, investors may consider alternative models that are close to the estimated model. This method has been well accepted in quantitative finance to deal with portfolio selection and asset pricing in case of ambiguity or misspecification. For instance, Anderson et al. (1999) introduces ambiguity-aversion into the Lucas model and formulates alternative models. Uppal & Wang (2003) extends the model in Anderson et al. (1999) and develops a framework that allows investors to consider the level of ambiguity. Anderson et al. (2003) studies the continuous-time asset pricing model in which the investor takes the model misspecification into account. Maenhout (2004) optimizes an inter-temporal consumption problem with ambiguity, and derives the closed-form expressions of the optimal strategies under “homothetic robustness”. Maenhout (2006) obtains the optimal portfolio choice to maximize the expected power utility of the terminal wealth under ambiguity and stochastic premium. Flor & Larsen (2014) considers an investor who is ambiguous about the interest rate and stock returns models. For an insurer who manages her risk by purchasing reinsurance and investing her surplus in a financial market, ambiguity situation is identical to that of the above-mentioned investors. Moreover, the accurate estimation of an insurer’s surplus process can also be called into question. An ambiguity-averse insurer (AAI) would hope for a systematic and quantitative way to take ambiguity into account. For example, Korn et al. (2012) investigates the optimal reinsurance problem and the optimal reinsurance-investment problem with ambiguity by using the stochastic differential game approach. Yi et al. (2013) studies the problem of robust optimal reinsurance-investment under the Heston model for an AAI. Yi et al. (2015) obtains the robust optimal reinsurance-investment strategy under the benchmark and mean-variance criteria. Pun & Wong (2015) considers the problem of robust optimal reinsurance-investment with multi-scale stochastic volatility using a general concave utility function.

Although research on the robust optimal investment problem has been rapidly increasing in recent years, very few of these contributions deals with the problem in relation to ambiguity with jumps, which has a significant effect on the optimal strategy. Branger & Larsen (2013) analyzes the optimal portfolio selection problem for an ambiguity-averse investor who invests in a risky asset following a jump-diffusion process using the criterion of maximizing the expected power utility of the terminal wealth. Aït-Sahalia & Matthys (2014) considers the optimal consumption-portfolio selection problem in the presence of ambiguity where the price of the risky asset satisfies a Lévy process. Both Branger & Larsen (2013) and Aït-Sahalia & Matthys (2014) point out that the risks related to the uncertainty of the drift and the probability of jumps are fundamentally different in the portfolio selection problem, such that ignoring ambiguity with respect to (w.r.t.) the jump risk may result in large losses in the financial market. For robust optimal reinsurance-investment problem with jumps, Zeng et al. (2016) and Zheng et al. (2016) study the optimal proportional reinsurance-investment problem with ambiguity under criteria of mean-variance and expected utility maximization, respectively.

Unlike Yi et al. (2013), Yi et al. (2014), Pun & Wong (2015), Zeng et al. (2016) and Zheng et al. (2016), we are interested in the excess-of-loss reinsurance, which is preferred than proportional reinsurance in most situations (see Asmussen et al., 2000). Recently, more and more scholars focus on the optimal excess-of-loss reinsurance-investment problems, such as Gu et al. (2012), Zhao et al. (2013), and so on. To the best of our knowledge, this paper is a prior research on the robust optimal excess-of-loss reinsurance-investment problem with jumps for an AAI. In our model, the insurer's surplus process is assumed to be a Brownian motion with drift that can be considered as an approximation of the classical insurance risk models, and the insurer is allowed to purchase excess-of-loss reinsurance and invest her surplus in a risk-free asset and a risky asset whose price process is described by a jump-diffusion model. Given that the market (true model) may deviate from the estimated model (reference model) in reality, we incorporate ambiguity into our study, and assume that the insurer is ambiguity-averse to diffusion and jump risks. Following Maenhout (2004, 2006), the ambiguity level is chosen as inversely proportional to the optimal value function. Moreover, depending on the available information, the AAI may exhibit different levels of ambiguity to diffusion and jump risks. The infrequent nature of jumps in the price process for the risky asset makes it hard to estimate the intensity of jump risk, which indicates that the AAI is more ambiguity averse to the jump risk than to the diffusion risk, making it seem natural to have different levels of ambiguity aversion to diffusion and jump risks. Based on the above setup, we formulate a robust optimization problem with alternative models, and derive the explicit expressions of the robust optimal excess-of-loss reinsurance-investment strategy to maximize the expected exponential utility of terminal wealth. Some special cases of our model and results are provided, and the economic implications of our findings and utility enhancements from considering ambiguity and reinsurance are analyzed using numerical examples. The main contributions of this paper are as follows: (i) Ambiguity with jumps is introduced into the optimal excess-of-loss reinsurance-investment framework; (ii) utility enhancements from considering ambiguity and reinsurance are presented, which reveals that ambiguity and reinsurance should not be ignored; and (iii) some special cases of our model, such as the cases of investment-only, ambiguity-neutral insurer (ANI) and no jump, are provided, which demonstrates that our model are more general and can reduce to many special cases considered in the literature.

The remainder of this paper is organized as follows. Section 2 describes the formulation of the model. Section 3 derives the explicit expressions of the robust optimal reinsurance-investment strategy and the corresponding optimal value function. Section 4 provides some special cases of our model. Section 5 presents some numerical examples and sensitivity analysis of utility enhancements to illustrate our results. Section 6 concludes the paper.

2. General formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, where $T > 0$ is a finite constant representing the investment time horizon, $\mathcal{F}_t$ stands for the information available until time t, and $\mathbb{P}$ is a reference probability.

Suppose that an insurer's surplus process follows a diffusion model. To understand better that the diffusion model can be considered as an approximation of the classical insurance risk model, we start with the classical Cramér-Lundberg (C-L) model. In the C-L model, without reinsurance and investment, the surplus process of an insurer is described by

$$R(t) = x_0 + pt - \sum_{i=1}^{N_1(t)} Z_i,$$

where $x_0 \geqslant 0$ is the initial surplus; p is the premium rate; $\sum_{i=1}^{N_1(t)} Z_i$ is a compound Poisson process, representing the cumulative claims up to time t; $\{N_1(t)\}_{t\in[0,T]}$ is a homogeneous Poisson process with intensity $\lambda_1 > 0$; and the claim sizes Z_1, Z_2,... are assumed to be independent and identically distributed (i.i.d.) positive random variables with finite first moment $\mathrm{E}[Z_i] = \mu_Z$ and second moment $\mathrm{E}[Z_i^2] = \sigma_Z^2$. Z_1, Z_2,... are further assumed to be independent of $N_1(t)$ with common distribution $F(z)$. Denote by $D = \sup\{z : F(z) \leqslant 1\} < +\infty$, then $F(0) = 0, 0 < F(z) < 1$ for $0 < z < D$ and $F(z) = 1$ for $z \geqslant D$. Suppose that the premium rate p is calculated according to the expected value principle, i.e., $p = (1 + \eta)\lambda_1\mu_Z$, where $\eta > 0$ is the safety loading of the insurer.

To disperse the underlying insurance business risk, the insurer is allowed to purchase excess-of-loss reinsurance. Let a be a (fixed) excess-of-loss retention level and $Z_i^{(a)} = \min\{Z_i, a\}$ denote the part of the claims held by the insurer. Then, the surplus process of the insurer becomes

$$\bar{R}^{(a)}(t) = x_0 + p^{(a)}t - \sum_{i=1}^{N_1(t)} Z_i^{(a)},$$

where the premium rate

$$p^{(a)} = (1 + \eta)\lambda_1\mu_Z - (1 + \theta)\lambda_1(\mu_Z - \mathrm{E}[Z_i^{(a)}]) = (\eta - \theta)\lambda_1\mu_Z + \lambda_1(1 + \theta)\mathrm{E}[Z_i^{(a)}],$$

in which θ denotes the safety loading of the reinsurer. Suppose that $\theta > \eta$, which implies that the reinsurance is not cheap. According to Grandell (1991), the surplus process $\bar{R}^{(a)}(t)$ can be approximated by the following diffusion model

$$
\begin{aligned}
\mathrm{d}\bar{R}^{(a)}(t) &= (p^{(a)} - \lambda_1 \mathrm{E}[Z_i^{(a)}])\mathrm{d}t + \sqrt{\lambda_1 \mathrm{E}[(Z_i^{(a)})^2]}\mathrm{d}B_1(t) \\
&= \lambda_1[\theta\bar{\mu}(a) + (\eta - \theta)\mu_Z]\mathrm{d}t + \sqrt{\lambda_1}\bar{\sigma}(a)\mathrm{d}B_1(t),
\end{aligned}
$$

where $\{B_1(t)\}_{t\in[0,T]}$ is a standard Brownian motion and

$$
\bar{\mu}(a) = \mathrm{E}[Z_i^{(a)}] = \int_0^a (1 - F(s))\mathrm{d}s = \int_0^a \bar{F}(s)\mathrm{d}s,
$$

$$
(\bar{\sigma}(a))^2 = \mathrm{E}[(Z_i^{(a)})^2] = \int_0^a 2s(1 - F(s))\mathrm{d}s = \int_0^a 2s\bar{F}(s)\mathrm{d}s.
$$

Moreover, we assume that the insurer is allowed to invest in a financial market comprising a risk-free asset and a risky asset. The price process of the risk-free asset is described by

$$
\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \tag{2.1}
$$

where $r_0 > 0$ represents the risk-free interest rate. The price process of the risky asset follows a jump-diffusion process

$$
\mathrm{d}S(t) = S(t-)\left[\mu\mathrm{d}t + \sigma\mathrm{d}B_2(t) + \mathrm{d}\sum_{i=1}^{N_2(t)} Y_i\right], \tag{2.2}
$$

where μ and σ are constant; $\{B_2(t)\}_{t\in[0,T]}$ is a standard Brownian motion; $\{N_2(t)\}_{t\in[0,T]}$, representing the number of the risky asset price's jumps that occur during time interval $[0, t]$, is a homogeneous Poisson process with intensity λ_2; Y_i is the ith jump amplitude of the risky asset price; and Y_i, $i = 1, 2, ...$ are i.i.d. random variables with distribution function $G(y)$, finite first moment $\mathrm{E}[Y_i] = \mu_Y$ and second moment $\mathrm{E}[Y_i^2] = \sigma_Y^2$. Similar to Yi et al. (2013) and Pun & Wong (2015), we assume that $\{B_1(t)\}_{t\in[0,T]}$, $\{B_2(t)\}_{t\in[0,T]}$ and $\sum_{i=1}^{N_2(t)} Y_i$ are independent, and that $\mathrm{P}\{Y_i \geqslant -1 \text{ for all } i \geqslant 1\} = 1$ to ensure that the risky asset price remains positive. Generally, the expect return of the risky asset is larger than the risk-free interest rate, so we assume that $\mu + \lambda_2\mu_Y > r_0$. In Eq. (2.2), the diffusion term captures normal market movements, and the jumps describe sudden and unusually disastrous events. Next, we use a Poisson random measure $N(\cdot, \cdot)$ on $\Omega \times [0, T] \times [-1, \infty)$ to denote the compound Poisson process $\sum_{i=1}^{N_2(t)} Y_i$ as

$$
\sum_{i=1}^{N_2(t)} Y_i = \int_0^t \int_{-1}^{\infty} yN(\mathrm{d}s, \mathrm{d}y), \ \forall t \in [0, T].
$$

Denote by $\nu(\mathrm{d}t, \mathrm{d}y) = \lambda_2\mathrm{d}t\mathrm{d}G(y)$, then

$$
\mathrm{E}\left[\sum_{i=1}^{N_2(t)} Y_i\right] = \int_0^t \int_{-1}^{\infty} y\nu(\mathrm{d}s, \mathrm{d}y), \ \forall t \in [0, T],
$$

where $\nu(\cdot,\cdot)$ is the compensator of the random measure $N(\cdot,\cdot)$. Thus, the compensated measure $\tilde{N}(\cdot,\cdot) = N(\cdot,\cdot) - \nu(\cdot,\cdot)$ is related to the compound Poisson process as follows

$$\int_0^t \int_{-1}^{\infty} y\tilde{N}(\mathrm{d}s, \mathrm{d}y) = \sum_{i=1}^{N_2(t)} Y_i - \mathrm{E}\left[\sum_{i=1}^{N_2(t)} Y_i\right], \ \forall t \in [0, T].$$

Let $\mathbf{u} := \{u(t) := (a(t), \pi(t))\}_{t\in[0,T]}$ be the reinsurance and investment strategy, $\pi(t)$ is the amount of money invested in the risky asset at time t, the remainder $X^{\mathbf{u}}(t) - \pi(t)$ is invested in the risk-free asset, $X^{\mathbf{u}}(t)$ is the wealth at time t associated with strategy $\mathbf{u}$. Then, the wealth process $\{X^{\mathbf{u}}(t)\}_{t\in[0,T]}$ can be described by

$$\begin{aligned} \mathrm{d}X^{\mathbf{u}}(t) &= \mathrm{d}\bar{R}^{(a)}(t) + (X^{\mathbf{u}}(t) - \pi(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi(t)\frac{\mathrm{d}S(t)}{S(t-)} \\ &= \{r_0 X^{\mathbf{u}}(t) + \lambda_1[\theta\bar{\mu}(a) + (\eta - \theta)\mu_Z] + \pi(t)(\mu - r_0)\}\,\mathrm{d}t \\ &\quad + \sqrt{\lambda_1}\bar{\sigma}(a)\mathrm{d}B_1(t) + \pi(t)\sigma\mathrm{d}B_2(t) + \int_{-1}^{\infty} \pi(t) y N(\mathrm{d}t, \mathrm{d}y). \end{aligned} \tag{2.3}$$

In the traditional reinsurance-investment model, the insurer is assumed to be ambiguity-neutral with objective function, as follows

$$\sup_{\mathbf{u}\in\tilde{\Pi}} \mathrm{E}_{t,x}[U(X^{\mathbf{u}}(T))] = \sup_{\mathbf{u}\in\tilde{\Pi}} \mathrm{E}[U(X^{\mathbf{u}}(T))|X^{\mathbf{u}}(t) = x], \tag{2.4}$$

where $\tilde{\Pi}$ is the set of all admissible strategies $\mathbf{u}$ in a given market. The utility function $U(x)$ is typically increasing and concave ($U''(x) < 0$). However, it is reasonable to assume that the insurer is ambiguity-averse and thus wants to guard herself against worst-case scenarios. We assume that the knowledge about ambiguity for the AAI is described by probability $\mathbb{P}$, namely, the reference probability (or model). However, she is sceptical about this reference model, and hopes to consider alternative models. Following Anderson et al. (1999), the AAI recognizes that the model under probability $\mathbb{P}$ is an approximation of the true model, thus she notes alternative models, broadly defined here as a class of probabilities that are equivalent to $\mathbb{P}$ as follows:

$$\mathcal{Q} := \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}.$$

Definition 2.1. *(Admissible strategy) A strategy $\boldsymbol{u} = \{u(t) := (a(t), \pi(t))\}_{t\in[0,T]}$ is said to be admissible if*
(i) $\forall t \in [0, T]$, $a(t) \in [0, D]$;
(ii) $\boldsymbol{u}$ is predictable w.r.t. $\{\mathcal{F}_t\}_{t\in[0,T]}$, and $\mathrm{E}^{\mathbb{Q}^*}\left[\int_0^T (a(t))^2 + (\pi(t))^2\mathrm{d}t\right] < \infty$;
(iii) $\forall (t, x) \in [0, T]\times\mathbb{R}$, Eq. (2.3) has a pathwise unique solution $\{X^{\boldsymbol{u}}(t)\}_{t\in[0,T]}$ with $\mathrm{E}_{t,x}^{\mathbb{Q}^*}[U(X^{\boldsymbol{u}}(T))] < +\infty$, where $\mathbb{Q}^*$ is the chosen model to describe the worst case and will be shown later.

Let Π be the set of all admissible strategies.

It is obvious that Π is not empty, since at least it contains deterministic controls. For such a class of controls, existence and path uniqueness of the solution to Eq. (2.3) is proved in Øksendal & Sulem (2007).

By Girsanov's Theorem, $\forall \mathbb{Q} \in \mathcal{Q}$, there exists $\Phi := \{\phi(t) := (\phi_1(t), \phi_2(t), \phi_3(t))\}_{t\in[0,T]}$[1] such that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \Lambda^{\Phi}(T),$$

where

$$\begin{aligned}\Lambda^{\Phi}(t) &= \exp\left\{\int_0^t \phi_1(s)\mathrm{d}B_1(s) - \frac{1}{2}\int_0^t (\phi_1(s))^2\mathrm{d}s + \int_0^t \phi_2(s)\mathrm{d}B_2(s) - \frac{1}{2}\int_0^t (\phi_2(s))^2\mathrm{d}s\right.\\ &\left.+\int_0^t\int_{-1}^{\infty} \ln\phi_3(s)N(\mathrm{d}s,\mathrm{d}y) + \int_0^t\int_{-1}^{\infty}(1-\phi_3(s))\nu(\mathrm{d}s,\mathrm{d}y)\right\}\end{aligned} \tag{2.5}$$

is a $\mathbb{P}$-martingale. Karatzas & Shreve (1988) can be consulted for this theorem. By Girsanov's theorem, under probability $\mathbb{Q}$,

$$\mathrm{d}B_1^{\mathbb{Q}}(t) = \mathrm{d}B_1(t) - \phi_1(t)\mathrm{d}t,$$

and

$$\mathrm{d}B_2^{\mathbb{Q}}(t) = \mathrm{d}B_2(t) - \phi_2(t)\mathrm{d}t$$

are Brownian motions. Following Branger & Larsen (2013), for tractability and ease of interpretation, the distribution of the claim Y_i is assumed to be known and is restricted to be identical under probabilities $\mathbb{P}$ and $\mathbb{Q}$, i.e., $\mathrm{E}^{\mathbb{Q}}[h(y)]=\mathrm{E}^{\mathbb{P}}[h(y)]$, where $h(\cdot)$ is a function of y. Under $\mathbb{Q}$, the random measure $N^{\mathbb{Q}}(\mathrm{d}t,\mathrm{d}y)$ has compensator measure given by $\lambda_2\phi_3(t)G(\mathrm{d}y)\mathrm{d}t$. Thus, the dynamic of the wealth process under probability $\mathbb{Q}$ is

$$\begin{aligned}\mathrm{d}X^{\mathbf{u}}(t) &= \left\{r_0X^{\mathbf{u}}(t) + \lambda_1[\theta\bar{\mu}(a) + (\eta-\theta)\mu_Z] + \pi(t)(\mu - r_0) + \sqrt{\lambda_1}\bar{\sigma}(a)\phi_1(t) + \pi(t)\sigma\phi_2(t)\right\}\mathrm{d}t\\ &+ \sqrt{\lambda_1}\bar{\sigma}(a)\mathrm{d}B_1^{\mathbb{Q}}(t) + \pi(t)\sigma\mathrm{d}B_2^{\mathbb{Q}}(t) + \int_{-1}^{\infty}\pi(t)yN^{\mathbb{Q}}(\mathrm{d}t,\mathrm{d}y).\end{aligned} \tag{2.6}$$

[1] $\Phi := \{\phi(t) := (\phi_1(t), \phi_2(t), \phi_3(t))\}_{t\in[0,T]}$ satisfies three conditions: (i) ϕ_1 and ϕ_2 are $\mathcal{F}_t$-adapted, and ϕ_3 is $\mathcal{F}_t$-predictable; (ii) $\phi_3(t) > 0$, for a.a. $(t,\omega) \in [0,T]\times\Omega$; and (iii) $\mathrm{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T(\phi_1^2(t)+\phi_2^2(t))\mathrm{d}t + \lambda_2\int_0^T(\phi_3(t)\ln\phi_3(t) - \phi_3(t)+1)\mathrm{d}t\Big\}\Big] < \infty$. We denote Θ for the space of all such processes Φ. Condition (iii) can be referred to Branger & Larsen (2013) and Zheng et al. (2016)

Suppose that the insurer tries to seek a robust optimal control which is the best choice in some worst-case models. Inspired by Maenhout (2004) and Branger & Larsen (2013), we formulate a robust control problem to modify problem (2.4) as follows

$$\sup_{\mathbf{u}\in\Pi}\inf_{\Phi\in\Theta} \mathrm{E}^{\mathbb{Q}}_{t,x}\left\{\int_0^T \Psi(s, X^{\mathbf{u}}(s), \phi(s))\mathrm{d}s + U(X^{\mathbf{u}}(T))\right\}, \tag{2.7}$$

where

$$\Psi(t, X^{\mathbf{u}}(t), \phi(t)) = \frac{(\phi_1(t))^2}{2\varphi^{B_1}(t)} + \frac{(\phi_2(t))^2}{2\varphi^{B_2}(t)} + \frac{\lambda_2(\phi_3(t)\ln\phi_3(t) - \phi_3(t) + 1)}{\varphi^J(t)},$$

and the expectation is calculated under the alternative probability $\mathbb{Q}$; $\varphi^{B_1}(t)$, $\varphi^{B_2}(t)$ and $\varphi^J(t)$ are strictly positive deterministic functions and represent the preference parameters for ambiguity-aversion, which measure the degree of confidence to the reference probability $\mathbb{P}$ at time t; and deviations from the reference model are penalized by the first three terms in the expectation, which depends on the relative entropy arising from the diffusion and jump risks. In Appendix A, we show that the increase in relative entropy from t to $t + \mathrm{d}t$ equals

$$\left[\frac{1}{2}(\phi_1(t))^2 + \frac{1}{2}(\phi_2(t))^2 + \lambda_2(\phi_3(t)\ln\phi_3(t) - \phi_3(t) + 1)\right]\mathrm{d}t. \tag{2.8}$$

According to Maenhout (2004), we know that the larger $\varphi^{B_1}(t)$, $\varphi^{B_2}(t)$ and $\varphi^J(t)$ are, the less the deviations from the reference model are penalized. Furthermore, the AAI has less faith in the reference model, and she is more likely to consider alternative models. Hence, the AAI's ambiguity aversion is increasing w.r.t. $\varphi^{B_1}(t)$, $\varphi^{B_2}(t)$ and $\varphi^J(t)$.

To solve problem (2.7), we define the optimal value function $V(t, x)$ as

$$V(t, x) = \sup_{\mathbf{u}\in\Pi}\inf_{\Phi\in\Theta} \mathrm{E}^{\mathbb{Q}}\left\{\int_t^T \Psi(s, X^{\mathbf{u}}(s), \phi(s))\mathrm{d}s + U(X^{\mathbf{u}}(T))\middle| X^{\mathbf{u}}_t = x\right\}. \tag{2.9}$$

Let $C^{1,2}([0, T] \times \mathbb{R})$ denote a class of functions that are continuously differentiable w.r.t. t on $[0, T]$, and twice continuously differentiable w.r.t. x on $\mathbb{R}$. Similar to Maenhout (2006) and Branger & Larsen (2013), for any $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$, according to the principle of dynamic programming, we can derive the HJB equation for problem (2.9):

$$\sup_{u\in[0,D]\times\mathbb{R}}\ \inf_{\phi\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+} \{\mathcal{A}^{\phi,u}V(t, x) + \Psi(t, x, \phi)\} = 0, \tag{2.10}$$

with the boundary condition $V(T, x) = U(x)$, where $u = (a, \pi)$, $\phi = (\phi_1, \phi_2, \phi_3)$ denote the values that $\mathbf{u}$ and Φ take, and

$$\begin{aligned} \mathcal{A}^{\phi,u}V(t, x) \quad = \quad & V_t + V_x\left[r_0 x + \lambda_1(\theta\bar{\mu}(a) + (\eta - \theta)\mu_Z) + \pi(\mu - r_0) + \sqrt{\lambda_1}\bar{\sigma}(a)\phi_1 + \pi\sigma\phi_2\right] \\ & + \frac{1}{2}V_{xx}[\lambda_1(\bar{\sigma}(a))^2 + \pi^2\sigma^2] + \lambda_2\phi_3\mathrm{E}^{\mathbb{Q}}[V(t, x + \pi y) - V(t, x)], \end{aligned}$$

here, V_t, V_x and V_{xx} represent the partial derivatives of $V(t, x)$ w.r.t. the corresponding variables.

Proposition 2.2. *If there exist a function $W(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$ and a Markovian control $(\Phi^*, \boldsymbol{u}^*) \in \Theta \times \Pi$, $\Phi^*(t) = \Phi^*(t, X^{\boldsymbol{u}^*}(t))$, $\boldsymbol{u}^*(t) = \boldsymbol{u}^*(t, X^{\boldsymbol{u}^*}(t))$ such that*

(i) for any $\phi \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$, $\mathcal{A}^{\phi,\boldsymbol{u}^*(t,x)}W(t,x) + \Psi(t,x,\phi) \geqslant 0$;

(ii) for any $u \in [0,D]\times\mathbb{R}$, $\mathcal{A}^{\Phi^*(t,x),u}W(t,x) + \Psi(t,x,\phi^*) \leqslant 0$;

(iii) $\mathcal{A}^{\Phi^*(t,x),\boldsymbol{u}^*(t,x)}W(t,x) + \Psi(t,x,\phi^*) = 0$;

(iv) for all $(\Phi,\boldsymbol{u}) \in \Theta\times\Pi$, $\lim_{t\to T-} W(t, X^{\boldsymbol{u}}(t)) = U(X^{\boldsymbol{u}}(T))$;

(v) $\{W(\tau, X^{\boldsymbol{u}}(\tau))\}_{\tau\in\mathfrak{T}}$ and $\{\Psi(\tau, X^{\boldsymbol{u}}(\tau), \phi(\tau))\}_{\tau\in\mathfrak{T}}$ are uniformly integrable, where $\mathfrak{T}$ denotes the set of stopping times $\tau \leqslant T$.

Then $W(t,x) = V(t,x)$ and $(\Phi^*, \boldsymbol{u}^*)$ is an optimal control.

Proof. The proof is similar to the proof of Theorem 3.2 in Mataramvura & Øksendal (2008). □

3. Explicit robust control: exponential utility

To derive explicit results, we need to make some assumptions regarding the AAI's utility. Suppose that the AAI has an exponential utility, i.e.,

$$U(x) = -\frac{1}{m}\mathrm{e}^{-mx}, \tag{3.1}$$

where $m > 0$ is a constant representing the absolute risk aversion coefficient. As we know, the exponential utility function plays an important role in insurance mathematics and actuarial practice. It is the only utility function under the principle of 'zero utility' giving a fair premium that is independent of the level of insurers' wealth (see Gerber, 1979).

Following Maenhout (2004), we assume that $\varphi^{B_1}(t)$, $\varphi^{B_2}(t)$ and $\varphi^J(t)$ are state-dependent as

$$\varphi^{B_1}(t) = -\frac{\gamma^{B_1}}{mV},\ \varphi^{B_2}(t) = -\frac{\gamma^{B_2}}{mV},\ \varphi^J(t) = -\frac{\gamma^J}{mV}, \tag{3.2}$$

where the ambiguity aversion coefficients γ^{B_1}, γ^{B_2} and γ^J are nonnegative and describe the insurer's attitudes to the model uncertainty. From Eq. (3.2), we find that $\varphi^{B_1}(t)$, $\varphi^{B_2}(t)$ and $\varphi^J(t)$ are increasing w.r.t. parameters γ^{B_1}, γ^{B_2} and γ^J, respectively.

Then, the robust optimal reinsurance-investment strategy and the corresponding optimal value function can be derived and summarized in the following theorem.

Theorem 3.1. *For problem (2.7) with exponential utility function (3.1) and assumption (3.2), the robust optimal reinsurance-investment strategy* $\boldsymbol{u}^* = \{(a^*(t), \pi^*(t))\}_{t\in[0,T]}$ *is given by*

$$a^*(t) = \begin{cases} \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1} + m} \cdot 1_{\left\{l\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right) \geqslant \max\{l(D),0\}\right\}} + D \cdot 1_{\left\{l(D) \geqslant \max\left\{l\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right),0\right\}\right\}}, & \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1} + m} \in [0, D], \\ D \cdot 1_{\{l(D)\geqslant 0\}}, & \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1} + m} \in [D, +\infty), \end{cases} \tag{3.3}$$

$$\pi^*(t) = \frac{e^{-r_0(T-t)}}{\gamma^{B_2} + m} \left\{ \frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2 E^{\mathbb{Q}}[y e^{-m\pi^*(t) y e^{r_0(T-t)}}]}{\sigma^2} e^{\frac{\gamma^J}{m} E^{\mathbb{Q}}[e^{-m\pi^*(t) y e^{r_0(T-t)}} - 1]} \right\}, \tag{3.4}$$

and the optimal value function is given by

$$V(t,x) = -\frac{1}{m} e^{-m[e^{r_0(T-t)}x - \bar{f}(t)]}, \tag{3.5}$$

where

$$l(a) = -\lambda_1 \theta \bar{\mu}(a) V m e^{r_0(T-t)} + \frac{1}{2}\lambda_1 m(\gamma^{B_1} + m)(\bar{\sigma}(a))^2 V e^{2r_0(T-t)}, \tag{3.6}$$

$$\bar{f}(t) = \frac{\lambda_1(\eta - \theta)\mu_Z}{r_0}(1 - e^{r_0(T-t)}) + \int_t^T \bar{l}_1(\omega) d\omega - \int_t^T \bar{l}_2(\omega) d\omega, \tag{3.7}$$

$$\bar{l}_1(\omega) = e^{r_0(T-\omega)} \int_0^{a^*(\omega)} [\lambda_1(\gamma^{B_1} + m) s e^{r_0(T-\omega)} - \lambda_1 \theta] \bar{F}(s) ds, \tag{3.8}$$

$$\bar{l}_2(\omega) = \pi^*(\omega)(\mu - r_0) e^{r_0(T-\omega)} - \frac{1}{2}(\gamma^{B_2} + m)\sigma^2(\pi^*(\omega))^2 e^{2r_0(T-\omega)}$$
$$+ \frac{\lambda_2}{\gamma^J}\left(1 - e^{\frac{\gamma^J}{m} E^{\mathbb{Q}}[e^{-m\pi^*(\omega) y e^{r_0(T-\omega)}} - 1]}\right). \tag{3.9}$$

Proof. See Appendix B. □

Proposition 3.2. *Eq. (3.4) has a unique positive root, i.e., there exists a unique* $\pi^*(t) \in [0, +\infty)$ *that satisfies Eq. (3.4).*

Proof. To proof the existence-uniqueness of $\pi^*(t)$, Eq. (3.4) can be transformed into

$$(\gamma^{B_2} + m)\pi^* \sigma^2 e^{r_0(T-t)} = \mu - r_0 + \lambda_2 E^{\mathbb{Q}}[y e^{-m\pi^* y e^{r_0(T-t)}}] e^{\frac{\gamma^J}{m} E^{\mathbb{Q}}[e^{-m\pi^* y e^{r_0(T-t)}} - 1]}.$$

Suppose that

$$h(\pi) = \mu - r_0 + \lambda_2 E^{\mathbb{Q}}[y e^{-m\pi y e^{r_0(T-t)}}] e^{\frac{\gamma^J}{m} E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}} - 1]} - (\gamma^{B_2} + m)\pi \sigma^2 e^{r_0(T-t)}.$$

We obtain

$$
\begin{aligned}
h'(\pi) &= -\lambda_2 m e^{r_0(T-t)}\mathrm{E}^{\mathbb{Q}}[y^2 e^{-m\pi y e^{r_0(T-t)}}] e^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]} \\
&\quad -\lambda_2\gamma^J e^{r_0(T-t)}\left(\mathrm{E}^{\mathbb{Q}}[y e^{-m\pi y e^{r_0(T-t)}}]\right)^2 e^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]} - (\gamma_{B_2}+m)\sigma^2 e^{r_0(T-t)} < 0,
\end{aligned}
$$

which implies that $h(\pi)$ is a decreasing function w.r.t. π. Furthermore, we have $h(0) = \mu - r_0 + \lambda_2\mathrm{E}^{\mathbb{Q}}[y] > 0$. Also, we can find that if $\pi > \frac{\mu - r_0 + \lambda_2\mathrm{E}^{\mathbb{Q}}[y]}{(\gamma^{B_2}+m)\sigma^2 e^{r_0(T-t)}} > 0$, we have $h(\pi) < 0$. Therefore, Eq. (3.4) has a unique positive root. $\square$

Remark 3.3. *If the distribution of claim size satisfies*

$$
\begin{cases}
\theta\bar{\mu}(D) - \dfrac{(\gamma^{B_1}+m)e^{r_0 T}(\bar{\sigma}(D))^2}{2} > 0, \\[2ex]
\min\{\psi(T), \psi(0)\} > 0,
\end{cases}
\tag{3.10}
$$

where $\psi(t) = \theta\bar{\mu}\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right) - \frac{(\gamma^{B_1}+m)e^{r_0(T-t)}}{2}\left(\bar{\sigma}\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right)\right)^2 - \theta\bar{\mu}(D) + \frac{(\gamma^{B_1}+m)e^{r_0(T-t)}(\bar{\sigma}(D))^2}{2}$, *for example, exponential distribution and uniform distribution with certain parameters, etc., the robust optimal reinsurance-investment strategy becomes*

(i) if $D > \frac{\theta}{\gamma^{B_1}+m}$, *the robust optimal reinsurance-investment strategy is given by*

$$
a^*(t) = \frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m},\ 0 \leqslant t \leqslant T,
\tag{3.11}
$$

$$
\pi^*(t) = \frac{e^{-r_0(T-t)}}{\gamma^{B_2}+m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2\mathrm{E}^{\mathbb{Q}}[y e^{-m\pi^*(t) y e^{r_0(T-t)}}]}{\sigma^2} e^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[e^{-m\pi^*(t) y e^{r_0(T-t)}}-1]}\right\},\ 0 \leqslant t \leqslant T,
\tag{3.12}
$$

and the optimal value function is given by

$$
V(t,x) = -\frac{1}{m}e^{-m[e^{r_0(T-t)}x - f_1(t)]},\ 0 \leqslant t \leqslant T;
\tag{3.13}
$$

(ii) if $D \leqslant \frac{\theta}{\gamma^{B_1}+m}$, *the robust optimal reinsurance-investment strategy is given by*

$$
a^*(t) = \begin{cases}
\dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}, & 0 \leqslant t \leqslant T + \dfrac{1}{r_0}\ln\dfrac{D(\gamma^{B_1}+m)}{\theta}, \\[2ex]
D, & T + \dfrac{1}{r_0}\ln\dfrac{D(\gamma^{B_1}+m)}{\theta} < t \leqslant T,
\end{cases}
\tag{3.14}
$$

$$
\pi^*(t) = \frac{e^{-r_0(T-t)}}{\gamma^{B_2}+m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2\mathrm{E}^{\mathbb{Q}}[y e^{-m\pi^*(t) y e^{r_0(T-t)}}]}{\sigma^2} e^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[e^{-m\pi^*(t) y e^{r_0(T-t)}}-1]}\right\},\ 0 \leqslant t \leqslant T,
\tag{3.15}
$$

and the optimal value function is given by

$$
V(t,x) = \begin{cases}
-\dfrac{1}{m}e^{-m[e^{r_0(T-t)}x - f_2(t)]}, & 0 \leqslant t \leqslant T + \dfrac{1}{r_0}\ln\dfrac{D(\gamma^{B_1}+m)}{\theta}, \\[2ex]
-\dfrac{1}{m}e^{-m[e^{r_0(T-t)}x - f_3(t)]}, & T + \dfrac{1}{r_0}\ln\dfrac{D(\gamma^{B_1}+m)}{\theta} < t \leqslant T,
\end{cases}
\tag{3.16}
$$

where

$$f_1(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-t)}) + \int_t^T l_1(\omega)\mathrm{d}\omega - \int_t^T l_2(\omega)\mathrm{d}\omega, \tag{3.17}$$

$$f_2(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(\mathrm{e}^{r_0(T-k)} - \mathrm{e}^{r_0(T-t)}) + \int_t^k l_1(\omega)\mathrm{d}\omega - \int_t^T l_2(\omega)\mathrm{d}\omega$$

$$+\frac{\lambda_1\eta\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-k)}) - \frac{\lambda_1\sigma_Z^2(\gamma^{B_1}+m)}{4r_0}(1-\mathrm{e}^{2r_0(T-k)}), \tag{3.18}$$

$$f_3(t) = \frac{\lambda_1\eta\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-t)}) - \frac{\lambda_1\sigma_Z^2(\gamma^{B_1}+m)}{4r_0}(1-\mathrm{e}^{2r_0(T-t)}) - \int_t^T l_2(\omega)\mathrm{d}\omega, \tag{3.19}$$

$$l_1(\omega) = \mathrm{e}^{r_0(T-\omega)}\int_0^{\frac{\theta\mathrm{e}^{-r_0(T-\omega)}}{\gamma^{B_1}+m}} [\lambda_1(\gamma^{B_1}+m)s\mathrm{e}^{r_0(T-\omega)} - \lambda_1\theta]\bar{F}(s)\mathrm{d}s, \tag{3.20}$$

$$l_2(\omega) = \pi^*(\omega)(\mu-r_0)\mathrm{e}^{r_0(T-\omega)} - \frac{1}{2}(\gamma^{B_2}+m)\sigma^2(\pi^*(\omega))^2\mathrm{e}^{2\mathrm{r}_0(\mathrm{T}-\omega)}$$

$$+\frac{\lambda_2}{\gamma^J}\left(1-\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^*(\omega)y\mathrm{e}^{r_0(T-\omega)}}-1]}\right). \tag{3.21}$$

Proof. See Appendix B. □

Remark 3.4. *From Eq. (3.11), we find that the robust optimal reinsurance strategy decreases w.r.t. the ambiguity aversion coefficient γ^{B_1}, which provides the same insight as the intuition that an AAI with a higher ambiguity aversion level is prone to purchasing more reinsurance. This property is also shown in Yi et al. (2013). Analogously, we can derive the effects of ambiguity aversion coefficients on the robust optimal investment strategy, and we further analyze the effects using numerical examples. Eq. (3.12) illustrates that the first term of the robust optimal investment strategy is the speculative demand, which depends on the expected excess return. The second term arises from the jump risk in the risky asset price process, and it is not present when the risky asset's price process has a continuous sample paths. In addition, we find that the robust optimal reinsurance strategy is a function of the current time t, and is independent of the parameters of the risky asset, while the robust optimal investment strategy is independent of parameters of the insurance business.*

Special cases of our results in Theorem 3.1 can be found in the literature. If $\gamma^{B_1} = \gamma^{B_2} = \gamma^J = 0$, problem (2.7) reduces to problem (2.4), and the optimal excess-of-loss reinsurance strategy reduces to that in Bai & Guo (2010). If $\lambda_2 = 0$, there is no jump in the price process of the risky asset and the optimal investment strategy is similar to that in Maenhout (2004), which considers the robust portfolio selection maximizing a power utility.

Remark 3.5. *(Proportional reinsurance case). If the AAI can purchase proportional reinsurance or acquire new business (by acting as a reinsurer for other insurers, for example) to manage her insurance business risk, the reinsurance level at any time t is associated with the value $1 - p(t)$, where $p(t) \in [0, +\infty)$ can be regarded as the value of the risk exposure. Then, $\boldsymbol{u}_p := \{(p(t), \pi(t))\}_{t\in[0,T]}$ is the reinsurance-investment strategy, and the wealth process of the AAI under probability $\mathbb{Q}$ is*

$$\begin{aligned} \mathrm{d}X^{\boldsymbol{u}_p}(t) &= \Big\{r_0 X^{\boldsymbol{u}_p}(t) + \lambda_1(\eta - \theta + \theta p(t))\mu_Z + \pi(t)(\mu - r_0) + \sqrt{\lambda_1}p(t)\sigma_Z\phi_1(t) + \pi(t)\sigma\phi_2(t)\Big\}\mathrm{d}t \\ &+ \sqrt{\lambda_1}p(t)\sigma_Z \mathrm{d}B_1^{\mathbb{Q}}(t) + \pi(t)\sigma \mathrm{d}B_2^{\mathbb{Q}}(t) + \int_{-1}^{\infty}\pi(t)yN^{\mathbb{Q}}(\mathrm{d}t, \mathrm{d}y). \end{aligned} \tag{3.22}$$

Similarly, we can obtain the robust optimal proportional reinsurance-investment strategy $\boldsymbol{u}_p^* := \{(p^*(t), \pi^*(t))\}_{t\in[0,T]}$ as

$$p^*(t) = \frac{\theta\mu_Z \mathrm{e}^{-r_0(T-t)}}{(\gamma^{B_1} + m)\sigma_Z^2},\ 0 \leqslant t \leqslant T, \tag{3.23}$$

$$\pi^*(t) = \frac{\mathrm{e}^{-r_0(T-t)}}{\gamma^{B_2} + m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2 \mathrm{E}^{\mathbb{Q}}[y\mathrm{e}^{-m\pi^*(t)y\mathrm{e}^{r_0(T-t)}}]}{\sigma^2}\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^*(t)y\mathrm{e}^{r_0(T-t)}}-1]}\right\},\ 0 \leqslant t \leqslant T, \tag{3.24}$$

and the optimal value function $V^p(t, x)$ as

$$V^p(t, x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_4(t)]},\ 0 \leqslant t \leqslant T, \tag{3.25}$$

where

$$f_4(t) = \frac{\lambda_1(\eta - \theta)\mu_Z}{r_0}(1 - \mathrm{e}^{r_0(T-t)}) - \frac{\lambda_1\theta^2\mu_Z^2}{2(\gamma^{B_1} + m)\sigma_Z^2}(T - t) - \int_t^T l_2(\omega)\mathrm{d}\omega \tag{3.26}$$

with $l_2(\omega)$ given by Eq. (3.21).

In addition, from Eqs. (3.23) and (3.24), we find that the effects of the model parameters on the robust optimal proportional reinsurance-investment strategy are similar to those in an excess-of-loss reinsurance case. However, according to the results of the optimal value functions with excess-of-loss and proportional reinsurance given in Table 1 with the values of the parameters given in Table 2, we find that $V(t, x) \geqslant V^p(t, x)$, which implies that excess-of-loss reinsurance is preferred than proportional reinsurance in our model.

Table 1. Optimal value functions with excess-of-loss and proportional reinsurance.

t	0	1	2	3	4	5	6	7
$V(t,x)$	-0.1488	-0.1754	-0.2066	-0.2431	-0.2857	-0.3353	-0.3932	-0.4606
$V^p(t,x)$	-0.1673	-0.1946	-0.2262	-0.2627	-0.3049	-0.3537	-0.4099	-0.4749

4. Special cases

This section provides some special cases of our model: investment-only, ambiguity-neutral insurer (ANI) and no jump. We can derive the corresponding robust optimal strategies and optimal value functions for these cases similar to the expression of Theorem 3.1, however, to make the robust optimal strategies simple and intuitive, we consider these special cases under condition (3.10).

4.1. **Investment-only case.**

If there is no reinsurance in our model, i.e., $a(t) \equiv D$, the wealth process of an AAI under the probability $\mathbb{Q}$ reduces to

$$\begin{aligned} \mathrm{d}X^{\mathbf{u}}(t) &= \left\{r_0 X^{\mathbf{u}}(t) + \lambda_1\eta\mu_Z + \pi(t)(\mu - r_0) + \sqrt{\lambda_1}\sigma_Z\phi_1(t) + \pi(t)\sigma\phi_2(t)\right\}\mathrm{d}t \\ &\quad + \sqrt{\lambda_1}\sigma_Z \mathrm{d}B_1^{\mathbb{Q}}(t) + \pi(t)\sigma\mathrm{d}B_2^{\mathbb{Q}}(t) + \int_{-1}^{\infty}\pi(t)yN^{\mathbb{Q}}(\mathrm{d}t,\mathrm{d}y), \end{aligned} \tag{4.1}$$

and the corresponding HJB equation becomes

$$\begin{aligned} &\sup_{\pi\in\mathbb{R}}\inf_{(\phi_1,\phi_2,\phi_3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+}\Big\{\tilde{V}_t + \tilde{V}_x\left[r_0x + \lambda_1\eta\mu_Z + \pi(\mu - r_0) + \sqrt{\lambda_1}\sigma_Z\phi_1 + \pi\sigma\phi_2\right] \\ &+\frac{1}{2}\tilde{V}_{xx}[\lambda_1\sigma_Z^2 + \pi^2\sigma^2] + \lambda_2\phi_3\mathrm{E}^{\mathbb{Q}}[\tilde{V}(t, x+\pi y) - \tilde{V}(t,x)] \\ &+\frac{\phi_1^2}{2\varphi^{B_1}(t)} + \frac{\phi_2^2}{2\varphi^{B_2}(t)} + \frac{\lambda_2(\phi_3\ln\phi_3 - \phi_3 + 1)}{\varphi^J(t)}\Big\} = 0, \end{aligned} \tag{4.2}$$

where $\tilde{V}$ is a short notation for $\tilde{V}(t,x)$ representing the optimal value function of the investment-only problem with the boundary condition $\tilde{V}(T,x) = U(x)$. Similar to the derivations of the reinsurance-investment case, we have the robust optimal investment strategy and the corresponding optimal value function as follows.

Proposition 4.1. *For the investment-only problem, i.e., $a(t) \equiv D$, under assumptions (3.1) and (3.2), the robust optimal investment strategy is given by*

$$\pi^*(t) = \frac{\mathrm{e}^{-r_0(T-t)}}{\gamma^{B_2} + m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2\mathrm{E}^{\mathrm{Q}}[y\mathrm{e}^{-m\pi^*(t)y\mathrm{e}^{r_0(T-t)}}]}{\sigma^2}\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathrm{Q}}[\mathrm{e}^{-m\pi^*(t)y\mathrm{e}^{r_0(T-t)}} - 1]}\right\},\ 0 \leqslant t \leqslant T, \tag{4.3}$$

and the optimal value function is given by

$$\tilde{V}(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_3(t)]},\ 0 \leqslant t \leqslant T, \tag{4.4}$$

where $f_3(t)$ is given by Eq. (3.19).

In addition, if we do not consider the insurance business, i.e., $\lambda_1 = 0$, then the robust optimal investment strategy is given by Eq. (4.3), and the optimal value function is

$$\check{V}(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_5(t)]},\ 0 \leqslant t \leqslant T, \tag{4.5}$$

with

$$f_5(t) = -\int_t^T l_2(\omega)\mathrm{d}\omega, \tag{4.6}$$

where $l_2(\omega)$ is given by Eq. (3.21).

4.2. **Ambiguity-neutral insurer (ANI) case.** If all of the ambiguity aversion coefficients equal 0, i.e, $\gamma^{B_1} = \gamma^{B_2} = \gamma^J = 0$, our model reduces to an optimization problem for an ANI. Then, the ANI's wealth process under probability $\mathbb{P}$ is described by Eq. (2.3) and the objective function is given by Eq. (2.4). Denote the optimal value function by

$$\hat{V}(t,x) = \sup_{\hat{\mathbf{u}}\in\tilde{\Pi}} \mathrm{E}[U(X^{\hat{u}}(T))], \tag{4.7}$$

where $\hat{\mathbf{u}} = \{(\hat{a}(t), \hat{\pi}(t))\}_{t\in[0,T]}$. The corresponding HJB equation is

$$\begin{aligned}&\sup_{(\hat{a},\hat{\pi})\in[0,D]\times\mathbb{R}} \Big\{\hat{V}_t + \hat{V}_x\left[r_0 x + \lambda_1(\theta\bar{\mu}(\hat{a}) + (\eta-\theta)\mu_Z) + \hat{\pi}(\mu - r_0)\right] \\ &+\frac{1}{2}\hat{V}_{xx}[\lambda_1(\bar{\sigma}(\hat{a}))^2 + \hat{\pi}^2\sigma^2] + \lambda_2\mathrm{E}[\hat{V}(t,x+\hat{\pi}y) - \hat{V}(t,x)]\Big\} = 0,\end{aligned} \tag{4.8}$$

where $\hat{V}$ is a short notation for $\hat{V}(t,x)$ with $\hat{V}(T,x) = U(x)$.

Proposition 4.2. *For problem (4.7) of an ANI who ignores ambiguity with utility (3.1), (i) if $D > \frac{\theta}{m}$, the optimal reinsurance-investment strategy is given by*

$$\hat{a}^*(t) = \frac{\theta\mathrm{e}^{-r_0(T-t)}}{m},\ 0 \leqslant t \leqslant T, \tag{4.9}$$

$$\hat{\pi}^*(t) = \frac{\mathrm{e}^{-r_0(T-t)}}{m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2\mathrm{E}[y\mathrm{e}^{-m\hat{\pi}^*(t)y\mathrm{e}^{r_0(T-t)}}]}{\sigma^2}\right\},\ 0 \leqslant t \leqslant T, \tag{4.10}$$

and the optimal value function $\hat{V}(t,x)$ is given by

$$\hat{V}(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_6(t)]},\ 0 \leqslant t \leqslant T; \tag{4.11}$$

(ii) if $D \leqslant \frac{\theta}{m}$, the optimal reinsurance-investment strategy is given by

$$\hat{a}^*(t) = \begin{cases} \dfrac{\theta e^{-r_0(T-t)}}{m}, & 0 \leqslant t \leqslant T + \dfrac{1}{r_0}\ln\dfrac{Dm}{\theta}, \\ D, & T + \dfrac{1}{r_0}\ln\dfrac{Dm}{\theta} < t \leqslant T, \end{cases} \tag{4.12}$$

$$\hat{\pi}^*(t) = \frac{e^{-r_0(T-t)}}{m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2 \mathrm{E}[y e^{-m\hat{\pi}^*(t) y e^{r_0(T-t)}}]}{\sigma^2}\right\},\ 0 \leqslant t \leqslant T, \tag{4.13}$$

and the optimal value function $\hat{V}(t,x)$ is given by

$$\hat{V}(t,x) = \begin{cases} -\dfrac{1}{m} e^{-m[e^{r_0(T-t)}x - f_7(t)]}, & 0 \leqslant t \leqslant T + \dfrac{1}{r_0}\ln\dfrac{Dm}{\theta}, \\ -\dfrac{1}{m} e^{-m[e^{r_0(T-t)}x - f_8(t)]}, & T + \dfrac{1}{r_0}\ln\dfrac{Dm}{\theta} < t \leqslant T, \end{cases} \tag{4.14}$$

where

$$f_6(t) = \frac{\lambda_1(\eta - \theta)\mu_Z}{r_0}(1 - e^{r_0(T-t)}) + \int_t^T l_3(\omega)\mathrm{d}\omega - \int_t^T l_4(\omega)\mathrm{d}\omega, \tag{4.15}$$

$$f_7(t) = \frac{\lambda_1(\eta - \theta)\mu_Z}{r_0}(e^{r_0(T-k)} - e^{r_0(T-t)}) + \int_t^k l_3(\omega)\mathrm{d}\omega - \int_t^T l_4(\omega)\mathrm{d}\omega$$

$$+\frac{\lambda_1 \eta \mu_Z}{r_0}(1 - e^{r_0(T-k)}) - \frac{\lambda_1 \sigma_Z^2 m}{4r_0}(1 - e^{2r_0(T-k)}), \tag{4.16}$$

$$f_8(t) = \frac{\lambda_1 \eta \mu_Z}{r_0}(1 - e^{r_0(T-t)}) - \frac{\lambda_1 \sigma_Z^2 m}{4r_0}(1 - e^{2r_0(T-t)}) - \int_t^T l_4(\omega)\mathrm{d}\omega, \tag{4.17}$$

$$l_3(\omega) = e^{r_0(T-\omega)} \int_0^{\frac{\theta e^{-r_0(T-\omega)}}{m}} [\lambda_1 m s e^{r_0(T-\omega)} - \lambda_1 \theta]\bar{F}(s)\mathrm{d}s, \tag{4.18}$$

$$l_4(\omega) = \hat{\pi}^*(\omega)(\mu - r_0)e^{r_0(T-\omega)} - \frac{1}{2}m\sigma^2(\hat{\pi}^*(\omega))^2 e^{2r_0(T-\omega)}$$

$$-\frac{\lambda_2}{m}\mathrm{E}[e^{-m\hat{\pi}^*(\omega) y e^{r_0(T-\omega)}} - 1]. \tag{4.19}$$

Proposition 4.2 shows that the optimal reinsurance strategy in Eq. (4.9) becomes the result in Bai & Guo (2010), i.e., our model extends the optimal excess-of-loss reinsurance strategy in Bai & Guo (2010) to the case of robust optimal formulation.

4.3. **No jump case.** If the intensity of the jump in the price process of the risky asset equals 0, i.e., $\lambda_2 = 0$, our model reduces to a case without jumps. Then, the wealth process of the AAI under probability $\mathbb{Q}$ with the strategy $\bar{\mathbf{u}} = \{(\bar{a}(t), \bar{\pi}(t))\}_{t\in[0,T]}$ is

$$\begin{aligned} \mathrm{d}X^{\bar{\mathbf{u}}}(t) &= \Big\{ r_0 X^{\bar{\mathbf{u}}}(t) + \lambda_1[\theta\bar{\mu}(\bar{a}) + (\eta-\theta)\mu_Z] + \bar{\pi}(t)(\mu - r_0) + \sqrt{\lambda_1}\bar{\sigma}(\bar{a})\phi_1(t) + \bar{\pi}(t)\sigma\phi_2(t)\Big\}\mathrm{d}t \\ &+ \sqrt{\lambda_1}\bar{\sigma}(\bar{a})\mathrm{d}B_1^{\mathbb{Q}}(t) + \bar{\pi}(t)\sigma \mathrm{d}B_2^{\mathbb{Q}}(t), \end{aligned} \tag{4.20}$$

and the corresponding HJB equation is

$$\begin{aligned} \sup_{(\bar{a},\bar{\pi})\in[0,D]\times\mathbb{R}} \inf_{(\phi_1,\phi_2)\in\mathbb{R}\times\mathbb{R}} &\Big\{ \bar{V}_t + \bar{V}_x \Big[r_0 x + \lambda_1(\theta\bar{\mu}(\bar{a}) + (\eta-\theta)\mu_Z) + \bar{\pi}(\mu - r_0) + \sqrt{\lambda_1}\bar{\sigma}(\bar{a})\phi_1 + \bar{\pi}\sigma\phi_2 \Big] \\ &+ \frac{1}{2}\bar{V}_{xx}[\lambda_1(\bar{\sigma}(\bar{a}))^2 + \bar{\pi}^2\sigma^2] + \frac{\phi_1^2}{2\varphi^{B_1}(t)} + \frac{\phi_2^2}{2\varphi^{B_2}(t)} \Big\} = 0, \end{aligned} \tag{4.21}$$

where $\bar{V}$ is a short notation for $\bar{V}(t,x)$ representing the optimal value function of the no jump case with the boundary condition $\bar{V}(T,x) = U(x)$. Similarly, we can derive the robust optimal reinsurance-investment strategy and the corresponding optimal value function, explicitly.

Proposition 4.3. *If the risky asset price does not have jumps, under assumptions (3.1) and (3.2), the robust optimal reinsurance-investment strategy and the optimal value function reduce to the follows: (1) if $D > \frac{\theta}{\gamma^{B_1}+m}$, the robust optimal reinsurance-investment strategy is given by*

$$\bar{a}^*(t) = \frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1} + m}, \ 0 \leqslant t \leqslant T, \tag{4.22}$$

$$\bar{\pi}^*(t) = \frac{(\mu - r_0)e^{-r_0(T-t)}}{\sigma^2(\gamma^{B_2} + m)}, \ 0 \leqslant t \leqslant T, \tag{4.23}$$

and the optimal value function is given by

$$\bar{V}(t,x) = -\frac{1}{m} e^{-m[e^{r_0(T-t)}x - f_9(t)]}, \ 0 \leqslant t \leqslant T; \tag{4.24}$$

(2) if $D \leqslant \frac{\theta}{\gamma^{B_1}+m}$, the robust optimal reinsurance-investment strategy is given by

$$\bar{a}^*(t) = \begin{cases} \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1} + m}, & 0 \leqslant t \leqslant k, \\ D, & k < t \leqslant T, \end{cases} \tag{4.25}$$

$$\bar{\pi}^*(t) = \frac{(\mu - r_0)e^{-r_0(T-t)}}{\sigma^2(\gamma^{B_2} + m)}, \ 0 \leqslant t \leqslant T, \tag{4.26}$$

and the optimal value function is given by

$$\bar{V}(t,x) = \begin{cases} -\dfrac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_{10}(t)]}, & 0 \leqslant t \leqslant k, \\ -\dfrac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_{11}(t)]}, & k < t \leqslant T, \end{cases} \tag{4.27}$$

with

$$f_9(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-t)}) + \int_t^T l_1(\omega)\mathrm{d}\omega - \int_t^T l_5(\omega)\mathrm{d}\omega, \tag{4.28}$$

$$f_{10}(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(\mathrm{e}^{r_0(T-k)} - \mathrm{e}^{r_0(T-t)}) + \int_t^k l_1(\omega)\mathrm{d}\omega - \int_t^T l_5(\omega)\mathrm{d}\omega$$

$$+\frac{\lambda_1\eta\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-k)}) - \frac{\lambda_1\sigma_Z^2(\gamma^{B_1}+m)}{4r_0}(1-\mathrm{e}^{2r_0(T-k)}), \tag{4.29}$$

$$f_{11}(t) = \frac{\lambda_1\eta\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-t)}) - \frac{\lambda_1\sigma_Z^2(\gamma^{B_1}+m)}{4r_0}(1-\mathrm{e}^{2r_0(T-t)}) - \int_t^T l_5(\omega)\mathrm{d}\omega, \tag{4.30}$$

$$l_5(\omega) = \bar{\pi}^*(\omega)(\mu - r_0)\mathrm{e}^{r_0(T-\omega)} - \frac{1}{2}(\gamma^{B_2}+m)\sigma^2(\bar{\pi}^*(\omega))^2\mathrm{e}^{2\mathrm{r}_0(\mathrm{T}-\omega)}, \tag{4.31}$$

where $l_1(\omega)$ is given in Eq. (3.20).

Proposition 4.3 illustrates that if there is no jump in our model, the robust optimal investment strategy is similar to that in Maenhout (2004), which considers the robust portfolio selection maximizing a power utility. Furthermore, if we do not consider the insurance business and jumps, i.e., $\lambda_1 = \lambda_2 = 0$, then the robust optimal investment strategy is given by Eq. (4.23), and the optimal value function is given by

$$\acute{V}(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_{12}(t)]},\ 0 \leqslant t \leqslant T, \tag{4.32}$$

with

$$f_{12}(t) = -\int_t^T l_5(\omega)\mathrm{d}\omega \tag{4.33}$$

where $l_5(\omega)$ is given by Eq. (4.31).

5. Numerical examples and utility enhancements

This section provides some numerical examples to illustrate the effects of model parameters on the robust optimal reinsurance-investment strategy and utility enhancements under condition (3.10). We take the case $D > \frac{\theta}{\gamma^{B_1}+m}$ as an example, and the analysis for the case of $D \leqslant \frac{\theta}{\gamma^{B_1}+m}$ is similar. Moreover, suppose that both claim size Z_i and jump size Y_i follow exponential distributions with parameters λ_Z and λ_Y, respectively, i.e., the density functions of claim size Z_i and jump size Y_i are $f(z) = \lambda_Z e^{-\lambda_Z z}$, $z \geqslant 0$ and $g(y) = \lambda_Y e^{-\lambda_Y (y+1)}$, $y \geqslant -1$. Throughout numerical analysis, unless otherwise stated, the basic parameters are given in the following table.

TABLE 2. Parameters of model in numerical examples

r_0	μ	σ	η	θ	m	γ^{B_1}	γ^{B_2}	γ^J	λ_1	λ_2	λ_Z	λ_Y	T	t
0.03	0.08	0.25	0.10	0.30	0.2	0.2	0.5	0.7	1	2	2	2	10	0

5.1. **Effects of model parameters on the robust optimal reinsurance-investment strategy.** From Eq. (3.11), we can obtain partial derivatives of the robust optimal reinsurance strategy $a^*(t)$ w.r.t. different parameters. Table 3 shows that: (i) $a^*(t)$ increases as a function of t and the reinsurer's safety loading θ. As the remaining time decreases, the AAI becomes less risk averse and undertakes more risks by herself, whereas when θ increases, to decrease the expensive payment to reinsurance, the AAI prefers to take more insurance business and raise the retention level of reinsurance; (ii) $a^*(t)$ decreases w.r.t. the risk-free interest rate r_0, the insurer's risk aversion coefficient m and the ambiguity aversion coefficient γ^{B_1}. The main reason for this is that with the increase of r_0, the risk-free asset is more attractive, so the AAI is more likely to invest more wealth in the risk-free asset instead of purchasing more reinsurance. Note that the larger m is, the more risk averse the AAI is, so she purchases more reinsurance to spread risk. Moreover, an AAI with a higher ambiguity aversion level γ^{B_1} is prone to purchasing more reinsurance to disperse the underlying insurance business risks.

TABLE 3. Partial derivatives of $a^*(t)$ in the case of $D > \frac{\theta}{\gamma^{B_1}+m}$.

derivatives	$\frac{\partial a^*(t)}{\partial t}$	$\frac{\partial a^*(t)}{\partial r_0}$	$\frac{\partial a^*(t)}{\partial m}$	$\frac{\partial a^*(t)}{\partial \theta}$	$\frac{\partial a^*(t)}{\partial \gamma^{B_1}}$
value	> 0	< 0	< 0	> 0	< 0

Eq. (3.12) reveals that the robust optimal investment strategy $\pi^*(t)$ is implicit and depends on different parameters in a very complicated way. It is difficult to analyze the effects of different parameters on $\pi^*(t)$ analytically through the partial derivatives of $\pi^*(t)$. We perform the sensitivity analysis of $\pi^*(t)$ w.r.t. different parameters through numerical simulations.

FIGURE 1. Effects of parameters γ^{B_2}, γ^J, λ_2 and λ_Y on $\pi^*(t)$.

Figure 1 shows the effects of the ambiguity aversion coefficients γ^{B_2} and γ^J, the jump intensity of the risky asset λ_2 and the parameter λ_Y of distribution function $G(y)$ on the robust optimal investment strategy $\pi^*(t)$. We find that $\pi^*(t)$ is a decreasing function of γ^{B_2} and γ^J; that is, the more ambiguity aversion the AAI is, the less risky asset she purchases. In addition, the robust optimal investment strategy $\pi^*(t)$ is more sensitive to γ^{B_2} and γ^J when γ^{B_2} and γ^J are small than that when they are large, which means that the marginal effect of increasing the amount invested in the risky asset is declining. The robust optimal investment strategy $\pi^*(t)$ also decreases w.r.t. λ_2 and λ_Y, because that as the jump intensity λ_2 increases, the risky asset becomes more risky and less attractive. Moreover, a larger λ_Y implies that the expectation and variance of Y_i decrease, prompting the AAI to invest more wealth in the risky asset.

5.2. **Effects of model parameters on the utility enhancements.** This subsection discusses the AAI's utility enhancements by numerical illustration. Without loss of generality, we consider the case of $D > \frac{\theta}{m} > \frac{\theta}{\gamma^{B_1}+m}$, and the analyzes of other cases are similar.

First, we study the effect of ambiguity aversion on utility enhancement. We show that an insurer who suffers from ambiguity aversion follows a suboptimal strategy. Suppose that an AAI does not take the optimal strategy $\mathbf{u}^* = \{(a^*(t), \pi^*(t))\}_{t\in[0,T]}$ given in Theorem 3.1, but instead makes her decisions as if she were an ANI, i.e., the AAI follows the strategy $\hat{\mathbf{u}}^* = \{(\hat{a}^*(t), \hat{\pi}^*(t))\}_{t\in[0,T]}$ given in Proposition 4.2. The optimal value function for the AAI following strategy $\hat{\mathbf{u}}^*$ is defined by

$$\begin{aligned}
&\check{V}(t,x)\\
&= \inf_{\Phi\in\Theta} \mathrm{E}^{\mathbb{Q}} \left\{ \int_t^T \left[\frac{(\phi_1(s))^2}{2\varphi^{B_1}(s)} + \frac{(\phi_2(s))^2}{2\varphi^{B_2}(s)} + \frac{\lambda_2(\phi_3(s)\ln\phi_3(s) - \phi_3(s) + 1)}{\varphi^J(s)} \right] \mathrm{d}s + U(X^{\hat{u}^*}(T)) \right\}.
\end{aligned} \tag{5.1}$$

Note that $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$, which describe the alternative model, are determined endogenously and depend on the reinsurance-investment strategy. Unlike in the optimal case, the reinsurance-investment strategy is now pre-specified. The functions $\varphi^{B_1}(t)$, $\varphi^{B_2}(t)$ and $\varphi^{J}(t)$ are defined analogously to Eq. (3.2).

It is obvious that value function $\check{V}(t,x)$ defined in Eq. (5.1) is smaller than $V(t,x)$ in Eq. (2.9). Bases on $\check{V}(t,x)$, we define the utility enhancement from considering ambiguity as follows

$$UE_1(t) := 1 - \frac{V(t,x)}{\check{V}(t,x)} = 1 - \mathrm{e}^{m(f_1(t)-f_0(t))}, \tag{5.2}$$

where $f_1(t)$ and $f_0(t)$ are given by Eq. (3.17) and

$$f_0(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-t)}) + \int_t^T l_6(\omega)\mathrm{d}\omega - \int_t^T l_7(\omega)\mathrm{d}\omega, \tag{5.3}$$

$$l_6(\omega) = \mathrm{e}^{r_0(T-\omega)} \int_0^{\frac{\theta \mathrm{e}^{-r_0(T-\omega)}}{m}} [\lambda_1(\gamma^{B_1}+m)s\mathrm{e}^{r_0(T-\omega)} - \lambda_1\theta]\bar{F}(s)\mathrm{d}s, \tag{5.4}$$

$$l_7(\omega) = \hat{\pi}^*(\omega)(\mu-r_0)\mathrm{e}^{r_0(T-\omega)} - \frac{1}{2}(\gamma^{B_2}+m)\sigma^2(\hat{\pi}^*(\omega))^2\mathrm{e}^{2\mathrm{r}_0(\mathrm{T}-\omega)}$$

$$+\frac{\lambda_2}{\gamma^J}\left(1-\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\hat{\pi}^*(\omega)y\mathrm{e}^{r_0(T-\omega)}}-1]}\right). \tag{5.5}$$

Derivation of suboptimal value function are given in Appendix C.

TABLE 4. Effects of γ^{B_2} and γ^J on the utility enhancement $UE_1(t)$.

$\gamma^{B_2} \diagdown \gamma^J$	0	1	2	3	4	5	6	7
0	0.0775	0.1153	0.1932	0.2719	0.3370	0.3876	0.4262	0.4558
1	0.3028	0.3912	0.5021	0.5786	0.6312	0.6678	0.6941	0.7134
2	0.5289	0.6387	0.7199	0.7704	0.8030	0.8250	0.8404	0.8515
3	0.6867	0.7981	0.8484	0.8781	0.8967	0.9091	0.9176	0.9237
4	0.7928	0.8907	0.9196	0.9362	0.9464	0.9531	0.9577	0.9610
5	0.8633	0.9420	0.9579	0.9669	0.9724	0.9760	0.9784	0.9801
6	0.9099	0.9696	0.9782	0.9830	0.9859	0.9877	0.9890	0.9899
7	0.9407	0.9842	0.9888	0.9913	0.9928	0.9938	0.9944	0.9949

Table 4 shows the effects of ambiguity aversion coefficients for diffusion and jump risks on the utility enhancement $UE_1(t)$. We find that $UE_1(t)$ increases w.r.t γ^{B_2} and γ^J, and the change trends are also clearly showed in Figure 2. Particularly, the first column ($\gamma^J = 0$) is the case of ambiguity without jumps. We find that the value of $UE_1(t)$ with ambiguity for jump-diffusion risks ($\gamma^J \neq 0$) is much larger than that without jumps. Thus, the utility enhancement from ambiguity for the jump risk cannot be ignored.

Figure 2 discloses the utility enhancement $UE_1(t)$ as an increasing function of γ^{B_1}, γ^{B_2}, γ^J and the remaining time $T-t$. $UE_1(t)$ is higher for the AAI with less information about the model $\mathbb{P}$ (higher γ^{B_1}, γ^{B_2} and γ^J) than that for the AAI with more information (smaller γ^{B_1}, γ^{B_2} and γ^J). Moreover, $UE_1(t)$ has a remarkable upward trend as $T-t$ increases. Figure 3 reveals that $UE_1(t)$ increases w.r.t. λ_2 and λ_Y. A larger λ_2 or λ_Y implies more uncertainties for the price of the risky asset, such that taking robust optimal strategy for an AAI brings larger utility enhancement.

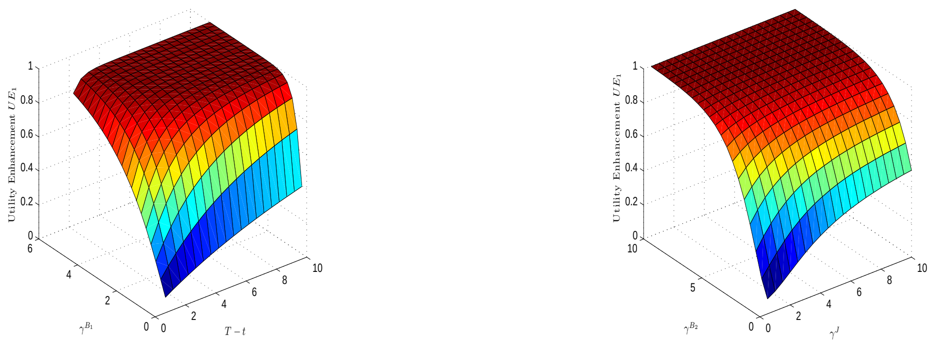

FIGURE 2. Effects of $T-t$, γ^{B_1}, γ^{B_2} and γ^J on the utility enhancement $UE_1(t)$.

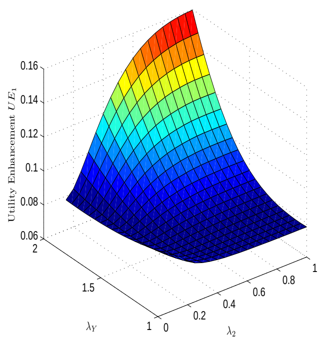

FIGURE 3. Effects of λ_2 and λ_Y on the utility enhancement $UE_1(t)$.

Next, we analyze the utility enhancement from considering reinsurance. As stated in some researches about optimal reinsurance-investment problem, reinsurance can optimize the utility. This is also true in the optimal reinsurance-investment problem with ambiguity. Bases on $\tilde{V}(t,x)$ (the optimal value function without reinsurance), we define the utility enhancement from considering reinsurance as follows

$$UE_2(t) := 1 - \frac{V(t,x)}{\tilde{V}(t,x)} = 1 - \mathrm{e}^{m(f_1(t)-f_3(t))}, \tag{5.6}$$

where $f_1(t)$ and $f_3(t)$ are given in Eqs. (3.17) and (3.19).

Figure 4 illustrates the effects of the remaining time $T-t$, the ambiguity aversion coefficient γ^{B_1}, the reinsurer's safety loading θ and the insurer's risk aversion coefficient m on the utility enhancement $UE_2(t)$. From Figure 4, we find that $UE_2(t)$ increases w.r.t. $T-t$. A reasonable explanation is that the AAI faces more uncertainty when $T-t$ is longer. In addition, $UE_2(t)$ is increasing w.r.t. γ^{B_1}, which means that the AAI with a higher ambiguity aversion level is prone to purchasing more reinsurance to disperse the insurance business risk. Then, the utility enhancement from considering reinsurance increases. As shown in Figure 4, $UE_2(t)$ decreases w.r.t. θ, which can be explained by the fact that a higher θ increases the cost of the AAI's purchase of reinsurance, so she cedes less risk to the reinsurer. Intuitively, an extremely high reinsurance premium provides a favored position for acquiring business, causing utility enhancement if new business acquisition is prohibited. Moreover, Figure 4 shows that $UE_2(t)$ increases w.r.t. m, which indicates that the AAI with a higher risk aversion coefficient is more likely to purchase more reinsurance for risk-spreading.

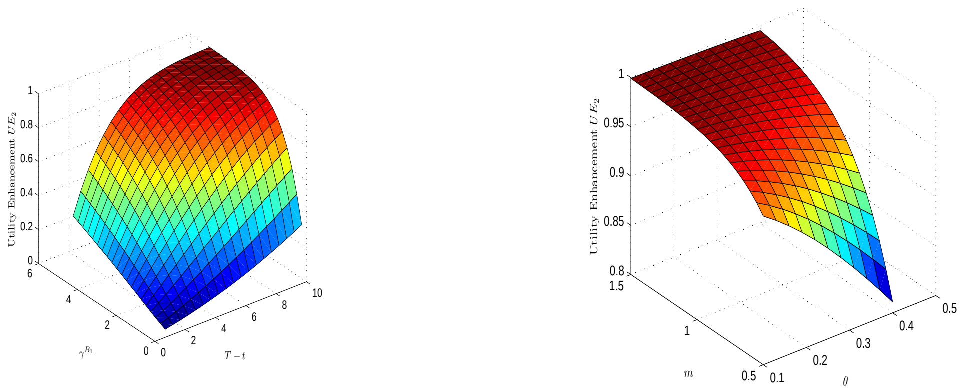

FIGURE 4. Effects of $T-t$, γ^{B_1}, θ and m on the utility enhancement $UE_2(t)$.

As shown in Figure 5, the utility enhancement $UE_2(t)$ increases w.r.t. the jump intensity of claim size λ_1. A possible reason for this is that a higher λ_1 results in a more severe fluctuation in the AAI's surplus process, making her more likely to cede more reinsurance to the reinsurer for dispersing risks. Consequently, the utility enhancement from considering reinsurance increases. In addition, from Figure 5, we find that the effect of λ_z on $UE_2(t)$ is small.

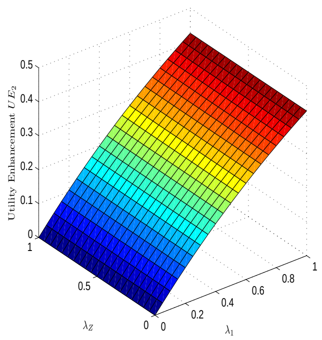

FIGURE 5. Effects of λ_1 and λ_Z on the utility enhancement $UE_2(t)$.

6. CONCLUSION

In this paper, we consider a robust optimal excess-of-loss reinsurance and investment problem with jumps for an AAI, who worries about ambiguity and aims to develop robust optimal strategies. The insurer's surplus is assumed to follow a Brownian motion with drift, and the insurer is allowed to purchase excess-of-loss reinsurance and invest her surplus in a risk-free asset and a risky asset whose price dynamics is described by a jump-diffusion model. Meanwhile, the insurer may lack full confidence on the model describing the economy, in which case we formulate a systematic analysis of the robust reinsurance-investment problem. By applying the stochastic dynamic programming approach, explicit expressions for the robust optimal reinsurance-investment strategy to maximize the expected exponential utility of terminal wealth and the corresponding optimal value function are obtained. Some special cases of our model and results are provided, and the economic implications of our findings and utility enhancements from considering ambiguity and reinsurance are analyzed using numerical examples. We find that (i) the effect of ambiguity by jump risk on the robust optimal investment strategy and the optimal value function is significant, and utility enhancement from considering ambiguity in the case of jump-diffusion risks is much larger than that without jumps; (ii) the robust optimal reinsurance-investment strategy for the AAI is affected by the attitude towards ambiguity, so the AAI facing model uncertainty has a smaller optimal strategy than an ANI; (iii) reinsurance brings large utility enhancement for the AAI, which implies that reinsurance is very important in risk management; and (iv) for the robust optimal reinsurance-investment problem, the optimal value function with excess-of-loss reinsurance is preferred than that with proportional reinsurance.

In future research, more complicated models, such as stochastic volatility with jumps, can be taken into account, although doing so may make it difficult to obtain the closed-form solution. Thus, other methods, such as asymptotic, or other practical methods may be introduced to deal with the robust optimal reinsurance-investment problem.

Appendix A.

Derivation of relative entropy.

The relative entropy is defined as the expectation under the alternative probability of the log Radon-Nikodym derivative defined in Eq. (2.5). Using Itô's formula, we have

$$\begin{aligned} \mathrm{d}\ln\Lambda^{\Phi} &= \phi_1(t)\mathrm{d}B_1(t) + \phi_2(t)\mathrm{d}B_2(t) + \lambda_2(1-\phi_3(t))\mathrm{d}t \\ &-\frac{1}{2}(\phi_1(t))^2\mathrm{d}t - \frac{1}{2}(\phi_2(t))^2\mathrm{d}t + \int_{-1}^{\infty}\ln\phi_3(t)N(\mathrm{d}t,\mathrm{d}y). \end{aligned} \tag{A.1}$$

The relative entropy over the interval from t to $t+\varepsilon$ is given by

$$\begin{aligned} &\mathrm{E}^{\mathbb{Q}}\left[\ln\frac{\Lambda^{\Phi}(t+\varepsilon)}{\Lambda^{\Phi}(t)}\right] = \mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\varepsilon}\phi_1(s)(\mathrm{d}B_1^{\mathbb{Q}}(s)+\phi_1(s)\mathrm{d}s) + \int_t^{t+\varepsilon}\phi_2(s)(\mathrm{d}B_2^{\mathbb{Q}}(s)+\phi_2(s)\mathrm{d}s)\right. \\ &\left.+\int_t^{t+\varepsilon}[\lambda_2(1-\phi_3(s)) - \frac{1}{2}(\phi_1(s))^2 - \frac{1}{2}(\phi_2(s))^2]\mathrm{d}s\right] \\ &+\int_t^{t+\varepsilon}\int_{-1}^{\infty}\ln\phi_3(s)\nu(\mathrm{d}s,\mathrm{d}y) + \int_t^{t+\varepsilon}\lambda_2\phi_3(s)\ln\phi_3(s)\mathrm{d}s \\ &= \mathrm{E}^{\mathbb{Q}}\left[\int_t^{t+\varepsilon}\left(\frac{1}{2}(\phi_1(s))^2 + \frac{1}{2}(\phi_2(s))^2 + \lambda_2(\phi_3(s)\ln\phi_3(s) - \phi_3(s) + 1)\right)\right]. \end{aligned} \tag{A.2}$$

Let $\varepsilon\to 0$ and we have the continuous-time limit of the relative entropy given by Eq. (2.8).

Appendix B.

Proof of Theorem 3.1 and Remark 3.3.

To solve Eq. (2.10), we conjecture that the optimal value function is

$$V(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x-\bar{f}(t)]} \tag{B.1}$$

with the boundary condition $f_1(T) = 0$. A direct calculation yields

$$
\begin{aligned}
&V_t = -Vm[-r_0 x e^{r_0(T-t)} - \bar{f}_t],\ V_x = -Vm e^{r_0(T-t)},\ V_{xx} = Vm^2 e^{2r_0(T-t)}, \\
&V(t, x+\pi y) - V(t,x) = V(e^{-m\pi y e^{r_0(T-t)}} - 1),
\end{aligned}
\tag{B.2}
$$

where $\bar{f} = \bar{f}(t)$ and $\bar{f}_t$ stands for the partial derivative of $\bar{f}(t)$ w.r.t. time t for short.

According to the first-order optimality conditions, the functions $\phi_1^*(t)$, $\phi_2^*(t)$ and $\phi_3^*(t)$, which realize the infimum part of Eq. (2.10) are given by

$$\phi_1^*(t) = -\gamma^{B_1}\sqrt{\lambda_1}\bar{\sigma}(a)e^{r_0(T-t)}, \tag{B.3}$$

$$\phi_2^*(t) = -\gamma^{B_2}\pi\sigma e^{r_0(T-t)}, \tag{B.4}$$

$$\phi_3^*(t) = e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]}. \tag{B.5}$$

Substituting Eqs. (B.2)-(B.5) into Eq. (2.10), we have

$$
\begin{aligned}
\sup_{(a,\pi)\in[0,D]\times\mathbb{R}} \Bigg\{ & V_t + r_0 x V_x + \lambda_1(\theta\bar{\mu}(a) + (\eta-\theta)\mu_Z)V_x + \pi(\mu - r_0)V_x + \frac{\lambda_1(\bar{\sigma}(a))^2\gamma^{B_1}V_x^2}{2mV} + \frac{\sigma^2\pi^2\gamma^{B_2}V_x^2}{2mV} \\
& + \frac{1}{2}\lambda_1(\bar{\sigma}(a))^2 V_{xx} + \frac{1}{2}\pi^2\sigma^2 V_{xx} - \frac{mV\lambda_2}{\gamma^J}\left(1 - e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]}\right)\Bigg\} = 0.
\end{aligned}
\tag{B.6}
$$

The first-order optimality condition gives the optimal reinsurance strategy

$$\lambda_1\theta V_x\frac{d\bar{\mu}(a)}{da} - \frac{\lambda_1\varphi^{B_1}(t)V_x^2}{2}\cdot\frac{d(\bar{\sigma}(a))^2}{da} + \frac{\lambda_1 V_{xx}}{2}\cdot\frac{d(\bar{\sigma}(a))^2}{da} = 0,$$

i.e.,

$$\bar{F}(a_0^*)(\theta V_x - \varphi^{B_1}(t)V_x^2 a_0^* + V_{xx}a_0^*) = 0.$$

Since $0 < \bar{F}(a_0^*) \leqslant 1$, we have

$$a_0^*(t) = \frac{\theta V_x}{\varphi^{B_1}(t)V_x^2 - V_{xx}} = \frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1} + m}. \tag{B.7}$$

Furthermore, differentiating Eq. (B.6) w.r.t. $\pi(t)$ implies that a nonlinear equation for the robust optimal investment strategy $\pi^*(t)$ is

$$\pi^*(t) = \frac{e^{-r_0(T-t)}}{\gamma^{B_2} + m}\left\{\frac{\mu - r_0}{\sigma^2} + \frac{\lambda_2 E^{\mathbb{Q}}[y e^{-m\pi^*(t) y e^{r_0(T-t)}}]}{\sigma^2} e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi^*(t) y e^{r_0(T-t)}}-1]}\right\}. \tag{B.8}$$

We first justify that π^* given in Eq. (B.8) derived by the first-order conditions is the optimal investment strategy. Let

$$g(\pi) = \pi(\mu - r_0)V_x + \frac{\sigma^2\pi^2\gamma^{B_2}V_x^2}{2mV} + \frac{1}{2}\pi^2\sigma^2 V_{xx} + \frac{mV\lambda_2}{\gamma^J}e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]},$$

which gather the term of π in the left side of Eq. (B.6). Furthermore,

$$g_\pi(\pi) = (\mu - r_0)V_x + \frac{\sigma^2\pi\gamma^{B_2}V_x^2}{mV} + \pi\sigma^2 V_{xx} - mV\lambda_2 e^{r_0(T-t)} e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]} E^{\mathbb{Q}}[y e^{-m\pi y e^{r_0(T-t)}}],$$

$$g_{\pi\pi}(\pi) = \frac{\sigma^2\gamma^{B_2}V_x^2}{mV} + \sigma^2 V_{xx} + m\gamma^J V\lambda_2 e^{2r_0(T-t)} e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]} \left(E^{\mathbb{Q}}[y e^{-m\pi y e^{r_0(T-t)}}]\right)^2$$

$$+m^2 V\lambda_2 e^{2r_0(T-t)} e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi y e^{r_0(T-t)}}-1]} E^{\mathbb{Q}}[y e^{-m\pi y e^{r_0(T-t)}}].$$

Since $V < 0$, $V_x > 0$ and $V_{xx} < 0$, it is obvious that $g_{\pi\pi}(\pi) < 0$ for any admissible π. Therefore, the first-order optimality condition gives the optimal investment strategy.

Next, similarly, let

$$l(a) = \lambda_1\theta\bar{\mu}(a)V_x + \frac{\lambda_1(\bar{\sigma}(a))^2\gamma^{B_1}V_x^2}{2mV} + \frac{1}{2}\lambda_1(\bar{\sigma}(a))^2 V_{xx},$$

which gathers the term of a in the left side of Eq. (B.6). Since $l(a)$ is a continuous function of a ($a \in [0, D]$), the optimal reinsurance strategy a^* will appears at $a_0^* = \frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}$ such that $l_a(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}) = 0$ or the two end points of the interval 0 and D. Therefore, the robust optimal reinsurance strategy is given by

$$a^*(t) = \begin{cases} \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m} \cdot 1_{\left\{l\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right) \geqslant \max\{l(D),0\}\right\}} + D \cdot 1_{\left\{l(D) \geqslant \max\left\{l\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right),0\right\}\right\}}, & \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m} \in [0, D], \\ D \cdot 1_{\{l(D) \geqslant 0\}}, & \dfrac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m} \in [D, +\infty), \end{cases}$$

and the optimal value function is given by

$$V(t, x) = -\frac{1}{m} e^{-m[e^{r_0(T-t)}x - \bar{f}(t)]};$$

where

$$\bar{f}(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(1 - e^{r_0(T-t)}) + \int_t^T \bar{l}_1(\omega)d\omega - \int_t^T \bar{l}_2(\omega)d\omega,$$

$$\bar{l}_1(\omega) = e^{r_0(T-\omega)}\int_0^{a^*(\omega)} [\lambda_1(\gamma^{B_1} + m)s e^{r_0(T-\omega)} - \lambda_1\theta]\bar{F}(s)ds,$$

$$\bar{l}_2(\omega) = \pi^*(\omega)(\mu - r_0)e^{r_0(T-\omega)} - \frac{1}{2}(\gamma^{B_2} + m)\sigma^2(\pi^*(\omega))^2 e^{2r_0(T-\omega)}$$

$$+\frac{\lambda_2}{\gamma^J}\left(1 - e^{\frac{\gamma^J}{m}E^{\mathbb{Q}}[e^{-m\pi^*(\omega)y e^{r_0(T-\omega)}}-1]}\right).$$

In addition, if the distribution of claim size satisfies condition (3.10), we can derive $l\left(\frac{\theta e^{-r_0(T-t)}}{\gamma^{B_1}+m}\right) > l(D) > l(0) = 0$, $\forall t \in [0, T]$. Eq. (B.7) shows that $a^*(t) \in [0, D]$ when $t \leqslant k := T + \frac{1}{r_0}\ln\frac{D(\gamma^{B_1}+m)}{\theta}$. We try to find the solution to Eq. (B.6) in the following cases.

(i) If $D > \frac{\theta}{\gamma^{B_1}+m}$, we have $k > T$. Then, $0 \leqslant t \leqslant T < k$, and the robust optimal reinsurance-investment strategy is shown in Eqs. (B.7) and (B.8). Similarly, we assume that the optimal value function is

$$V(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_1(t)]},\ 0 \leqslant t \leqslant T.$$

Plugging Eqs. (B.7) and (B.8) into Eq. (B.6), we have

$$\begin{aligned} &f_{1t} - \lambda_1(\eta-\theta)\mu_Z \mathrm{e}^{r_0(T-t)} + \mathrm{e}^{r_0(T-t)}\int_0^{\frac{\theta \mathrm{e}^{-r_0(T-t)}}{\gamma^{B_1}+m}} [\lambda_1(\gamma^{B_1}+m)s\mathrm{e}^{r_0(T-t)} - \lambda_1\theta]\bar{F}(s)\mathrm{d}s \\ &-\pi^*(\mu - r_0)\mathrm{e}^{r_0(T-t)} + \frac{1}{2}(\gamma^{B_2}+m)\sigma^2(\pi^*)^2\mathrm{e}^{2\mathrm{r}_0(\mathrm{T-t})} - \frac{\lambda_2}{\gamma^J}\left(1 - \mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^* y \mathrm{e}^{r_0(T-t)}}-1]}\right) = 0. \end{aligned} \tag{B.9}$$

Considering the boundary condition $f_1(T) = 0$, the solution to Eq. (B.9) is

$$f_1(t) = \frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(1 - \mathrm{e}^{r_0(T-t)}) + \int_t^T l_1(\omega)\mathrm{d}\omega - \int_t^T l_2(\omega)\mathrm{d}\omega, \tag{B.10}$$

where

$$l_1(\omega) = \mathrm{e}^{r_0(T-\omega)}\int_0^{\frac{\theta \mathrm{e}^{-r_0(T-\omega)}}{\gamma^{B_1}+m}} [\lambda_1(\gamma^{B_1}+m)s\mathrm{e}^{r_0(T-\omega)} - \lambda_1\theta]\bar{F}(s)\mathrm{d}s, \tag{B.11}$$

$$\begin{aligned} l_2(\omega) &= \pi^*(\omega)(\mu - r_0)\mathrm{e}^{r_0(T-\omega)} - \frac{1}{2}(\gamma^{B_2}+m)\sigma^2(\pi^*(\omega))^2\mathrm{e}^{2\mathrm{r}_0(\mathrm{T}-\omega)} \\ &+\frac{\lambda_2}{\gamma^J}\left(1 - \mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^*(\omega) y \mathrm{e}^{r_0(T-\omega)}}-1]}\right). \end{aligned} \tag{B.12}$$

(ii) If $D \leqslant \frac{\theta}{\gamma^{B_1}+m}$, we have $k \leqslant T$. Thus, $0 \leqslant t < k \leqslant T$. In the case of $0 \leqslant t \leqslant k$, the derivations of the solution to Eq. (B.6) are similar to those of case (i), and we assume that the optimal value function is

$$V(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_2(t)]},\ 0 \leqslant t \leqslant k,$$

where $f_2(t)$ needs to be determined.

In the case of $k < t \leqslant T$, choosing $a^*(t) = D$, and Eq. (B.6) becomes

$$\begin{aligned} &V_t + r_0xV_x + \lambda_1\eta\mu_Z V_x + \pi^*(\mu - r_0)V_x + \frac{\lambda_1\sigma_Z^2\gamma^{B_1}V_x^2}{2mV} + \frac{\sigma^2(\pi^*)^2\gamma^{B_2}V_x^2}{2mV} + \frac{1}{2}\lambda_1\sigma_Z^2V_{xx} \\ &+\frac{1}{2}(\pi^*)^2\sigma^2V_{xx} - \frac{mV\lambda_2}{\gamma^J}\left(1 - \mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^* y \mathrm{e}^{r_0(T-t)}}-1]}\right) = 0. \end{aligned} \tag{B.13}$$

Similarly, we have the solution to Eq. (B.13) as follows

$$V(t,x) = -\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x - f_3(t)]},\ k < t \leqslant T.$$

Taking into account the boundary condition $f_3(T) = 0$ and the continuity of $V(t, x)$ at time $t = k$, we have

$$\begin{aligned} f_2(t) = & \frac{\lambda_1(\eta - \theta)\mu_Z}{r_0}(\mathrm{e}^{r_0(T-k)} - \mathrm{e}^{r_0(T-t)}) + \int_t^k l_1(\omega)\mathrm{d}\omega - \int_t^T l_2(\omega)\mathrm{d}\omega \\ & + \frac{\lambda_1\eta\mu_Z}{r_0}(1 - \mathrm{e}^{r_0(T-k)}) - \frac{\lambda_1\sigma_Z^2(\gamma^{B_1} + m)}{4r_0}(1 - \mathrm{e}^{2r_0(T-k)}), \end{aligned} \tag{B.14}$$

$$f_3(t) = \frac{\lambda_1\eta\mu_Z}{r_0}(1 - \mathrm{e}^{r_0(T-t)}) - \frac{\lambda_1\sigma_Z^2(\gamma^{B_1} + m)}{4r_0}(1 - \mathrm{e}^{2r_0(T-t)}) - \int_t^T l_2(\omega)\mathrm{d}\omega, \tag{B.15}$$

where $l_1(\omega)$ and $l_2(\omega)$ are given in Eqs. (B.11) and (B.12).

Summarizing the above analysis, we can derive the robust optimal reinsurance-investment strategy and the optimal value function explicitly.

Next, conditions (i)-(v) in Proposition 2.2 will be checked. We first give two lemmas.

Lemma B.**1**. The following expectation is finite

$$J(T) := \mathrm{E}\left[\exp\left\{\int_0^T \left(\frac{(\phi_1^*(t))^2}{2} + \frac{(\phi_2^*(t))^2}{2} + \lambda_2(\phi_3^*(t)\ln\phi_3(t) - \phi_3(t) + 1)\right)\mathrm{d}t\right\}\right]. \tag{B.16}$$

Proof. Substituting Eqs. (B.3)-(B.5) into Eq. (B.16), we have

$$\begin{aligned} & \mathrm{E}\left[\exp\left\{\int_0^T \left(\frac{(\phi_1^*(t))^2}{2} + \frac{(\phi_2^*(t))^2}{2} + \lambda_2(\phi_3^*(t)\ln\phi_3(t) - \phi_3(t) + 1)\right)\mathrm{d}t\right\}\right] \\ & = \mathrm{E}\left[\exp\left\{\int_0^T \left(\frac{1}{2}\gamma_{B_1}^2\lambda_1(\sigma(\bar{a}^*))^2\mathrm{e}^{2r_0(T-t)} + \frac{1}{2}\gamma_{B_2}^2(\pi^*)^2\sigma^2\mathrm{e}^{2r_0(T-t)}\right.\right.\right. \\ & \left.\left.\left.\lambda_2\left(\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^* y\mathrm{e}^{r_0(T-t)}}-1]}\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^* y\mathrm{e}^{r_0(T-t)}} - 1] - \mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\pi^* y\mathrm{e}^{r_0(T-t)}}-1]} + 1\right)\right)\mathrm{d}t\right\}\right]. \end{aligned} \tag{B.17}$$

Since a^* and π^* are finite, the right side of Eq. (B.16) is finite. □

Lemma B.**2**. The optimal strategy $\mathbf{u}^*$ and the corresponding function $W(t, X^{\mathbf{u}^*}(t))$ have the following properties:

(a) $\mathbf{u}^*$ is an admissible strategy;

(b) $\mathrm{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}|W(t, X^{\mathbf{u}^*}(t))|^4\right) < \infty$;

(c)$\mathrm{E}^{\mathbb{Q}^*}\left(\sup_{t\in[0,T]}\left|\frac{(\phi_1^*(t))^2}{2\varphi^{B_1}(t)} + \frac{(\phi_2^*(t))^2}{2\varphi^{B_2}(t)} + \frac{\lambda_2(\phi_3^*(t)\ln\phi_3^*(t) - \phi_3^*(t) + 1)}{\varphi^J(t)}\right|^2\right) < \infty$.

Proof. (a). From the process of solving HJB equation, we know condition (i) in Definition 2.1 holds, and the optimal strategy $\mathbf{u}^*$ is deterministic and state-independent, thus condition (ii) in Definition 2.1 is satisfied. Condition (iii) in Definition 2.1 can be obtained by property (b).

(b). Substituting Eqs. (B.3)-(B.5), (B.7)-(B.8) into Eq. (2.6), we have

$$\begin{aligned} X^{\mathbf{u}^*}(t) & = x_0\mathrm{e}^{r_0 t} + \int_0^t A\mathrm{d}s + \int_0^t \sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathbb{Q}^*}(s) + \int_0^t \pi^*(s)\sigma\mathrm{d}B_2^{\mathbb{Q}^*}(s) \\ & + \int_0^t\int_{-1}^{\infty} \pi^*(s)yN(\mathrm{d}s, \mathrm{d}y), \end{aligned} \tag{B.18}$$

where $A=\lambda_1(\theta\bar{\mu}(a^*)+(\eta-\theta)\mu_Z)+(\mu-r_0)\pi^*(s)-\gamma^{B_1}\lambda_1(\bar{\sigma}(a^*))^2\mathrm{e}^{r_0(T-t)}-\gamma^{B_2}\sigma^2(\pi^*(s))^2$. Given that $\mathbf{u}^*$ is deterministic, A is bounded. Inserting Eq. (B.18) into candidate value function (3.13), we obtain the following upper boundary with appropriate constants $K>0$,

$$
\begin{aligned}
&|W(t,X^{\mathbf{u}^*}(t))^4|=\left|\frac{1}{m^4}\mathrm{e}^{-4m(\mathrm{e}^{r_0(T-t)}X^{\mathbf{u}^*}(t)-f(t))}\right|=\left|\frac{1}{m^4}\mathrm{e}^{-4m\mathrm{e}^{r_0(T-t)}X^{\mathbf{u}^*}(t)+4mf(t)}\right|\\
&\leqslant K\mathrm{e}^{-4m\mathrm{e}^{r_0(T-t)}X^{\mathbf{u}^*}(t)}\\
&=K\mathrm{e}^{-4m\mathrm{e}^{r_0(T-t)}(x_0\mathrm{e}^{r_0t}+\int_0^t A\mathrm{d}s+\int_0^t\sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathrm{Q}^*}(s)+\int_0^t\pi^*(s)\sigma\mathrm{d}B_2^{\mathrm{Q}^*}(s)+\int_0^t\int_{-1}^{\infty}\pi^*(s)yN(\mathrm{d}s,\mathrm{d}y))}\\
&\leqslant\bar{K}\mathrm{e}^{-4m(\int_0^t\sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathrm{Q}^*}(s)+\int_0^t\pi^*(s)\sigma\mathrm{d}B_2^{\mathrm{Q}^*}(s))},
\end{aligned}
\tag{B.19}
$$

where $\bar{K}$ is a constant satisfying $\bar{K}>K\mathrm{e}^{-4m\mathrm{e}^{r_0(T-t)}(x_0\mathrm{e}^{r_0t}+\int_0^t A\mathrm{d}s+\int_0^t\int_{-1}^{\infty}\pi^*(s)yN(\mathrm{d}s,\mathrm{d}y))}$. The first inequality in Eq. (B.19) is valid, because $f(t)$ is deterministic and bounded, and the second inequality follows from the fact that $x_0\mathrm{e}^{r_0t}$, $\mathrm{e}^{r_0(T-t)}$, $\int_0^t A\mathrm{d}s$ and $\int_0^t\int_{-1}^{\infty}\pi^*(s)yN(\mathrm{d}s,\mathrm{d}y)$ are deterministic and bounded. Now, we consider the integral $\mathrm{e}^{\int_0^t-4m\sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathrm{Q}^*}(s)}$.

$$
\mathrm{e}^{\int_0^t-4m\sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathrm{Q}^*}(s)}=\underbrace{\mathrm{e}^{\int_0^t 8m^2\lambda_1(\bar{\sigma}(a^*))^2\mathrm{d}s}}_{const.}\cdot\underbrace{\mathrm{e}^{-\int_0^t 8m^2\lambda_1(\bar{\sigma}(a^*))^2\mathrm{d}s+\int_0^t-4m\sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathrm{Q}^*}(s)}}_{martingale}.
$$

Thus,

$$
\mathrm{E}^{\mathrm{Q}^*}\left[\mathrm{e}^{\int_0^t-4m\sqrt{\lambda_1}\bar{\sigma}(a^*)\mathrm{d}B_1^{\mathrm{Q}^*}(s)}\right]<\infty.
$$

Similarly, $\mathrm{E}^{\mathrm{Q}^*}\left[\mathrm{e}^{\int_0^t-4m\pi^*(s)\sigma\mathrm{d}B_2^{\mathrm{Q}^*}(s)}\right]<\infty$. Consequently,

$$
\mathrm{E}^{\mathrm{Q}^*}\left[\sup_{t\in[0,T]}|W(t,X^{\mathbf{u}^*}(t))|^4\right]<\infty.
$$

(c). Let $\Gamma(t)=\frac{(\phi_1^*(t))^2m}{2\gamma^{B_1}}+\frac{(\phi_2^*(t))^2m}{2\gamma^{B_2}}+\frac{\lambda_2(\phi_3^*(t)\ln\phi_3^*(t)-\phi_3^*(t)+1)m}{\gamma^J}$, which is obviously bounded, and according to Eq. (3.2) with W instead of V, we have

$$
\begin{aligned}
&\mathrm{E}^{Q^*}\left(\sup_{t\in[0,T]}\left|\frac{(\phi_1^*(t))^2}{2\varphi^{B_1}(t)}+\frac{(\phi_2^*(t))^2}{2\varphi^{B_2}(t)}+\frac{\lambda_2(\phi_3^*(t)\ln\phi_3^*(t)-\phi_3^*(t)+1)}{\varphi^J(t)}\right|^2\right)\\
&\leqslant\mathrm{E}^{Q^*}\left(\sup_{t\in[0,T]}|\Gamma(t)|^2|W(t,X^{\mathbf{u}^*}(t))|^2\right)\\
&\leqslant\left(\mathrm{E}^{Q^*}\sup_{t\in[0,T]}|\Gamma(t)|^4\right)^{\frac{1}{2}}\left(\mathrm{E}^{Q^*}\sup_{t\in[0,T]}|W(t,X^{\mathbf{u}^*}(t))|^4\right)^{\frac{1}{2}}<\infty.
\end{aligned}
$$

The first inequality follows from Cauchy-Schwarz inequality, and the second inequality follows from property (b). □

From the above derivations, it is easy to check that conditions (i)-(iv) in Proposition 2.2 hold for $W(t,x)$. By Lemma B.2, condition (v) in Proposition 2.2 also holds for $W(t,x)$. Then, $W(t,x)$ is the optimal value function of problem (2.9), i.e., $W(t,x)=V(t,x)$, and $\mathbf{u}^*=\{(a^*(t),\pi^*(t))\}_{t\in[0,T]}$ is the optimal strategy. The proof can also be referred to Corollary 1.2 in

Kraft (2004), Theorem 3.1 in Øksendal & Sulem (2007) and Theorem 8.1 in Fleming & Soner (2006). □

Appendix C.

Derivation of suboptimal value function.

The optimal value function $\check{V}(t,x)$ associated with $\hat{\mathbf{u}}^*$ solves the infimum problem

$$\inf_{(\phi_1,\phi_2,\phi_3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+}\Big\{\check{V}_t+\check{V}_x\Big[r_0x+\lambda_1(\theta\bar{\mu}(\hat{a}^*)+(\eta-\theta)\mu_Z)+\hat{\pi}^*(\mu-r_0)+\sqrt{\lambda_1}\bar{\sigma}(\hat{a}^*)\phi_1+\hat{\pi}^*\sigma\phi_2\Big]$$
$$+\frac{1}{2}\check{V}_{xx}[\lambda_1(\bar{\sigma}(\hat{a}^*))^2+(\hat{\pi}^*)^2\sigma^2]+\lambda_2\phi_3\mathrm{E}^{\mathbb{Q}}[\check{V}(t,x+\hat{\pi}^*y)-\check{V}(t,x)]$$
$$+\frac{\phi_1^2}{2\varphi^{B_1}(t)}+\frac{\phi_2^2}{2\varphi^{B_2}(t)}+\frac{\lambda_2(\phi_3\ln\phi_3-\phi_3+1)}{\varphi^J(t)}\Big\}=0, \tag{C.1}$$

where $\check{V}$ is a short notation for $\check{V}(t,x)$ with the boundary condition $\check{V}(T,x)=U(x)$. The first-order conditions w.r.t. ϕ_1, ϕ_2 and ϕ_3 show that the alternative model is given by ϕ_1^*, ϕ_2^* and ϕ_3^* in Eqs. (B.3)-(B.5) with $\hat{a}^*$ and $\hat{\pi}^*$ substituted for a and π, respectively. By inserting Eqs. (B.3)-(B.5) into Eq. (C.1), we have

$$\check{V}_t+r_0x\check{V}_x+\lambda_1(\theta\bar{\mu}(\hat{a}^*)+(\eta-\theta)\mu_Z)\check{V}_x+\hat{\pi}^*(\mu-r_0)\check{V}_x+\frac{\lambda_1(\bar{\sigma}(\hat{a}^*))^2\gamma^{B_1}\check{V}_x^2}{2m\check{V}}+\frac{\sigma^2(\hat{\pi}^*)^2\gamma^{B_2}\check{V}_x^2}{2m\check{V}}$$
$$+\frac{1}{2}\lambda_1(\bar{\sigma}(\hat{a}^*))^2\check{V}_{xx}+\frac{1}{2}(\hat{\pi}^*)^2\sigma^2\check{V}_{xx}-\frac{m\check{V}\lambda_2}{\gamma^J}\left(1-\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\hat{\pi}^*y\mathrm{e}^{r_0(T-t)}}-1]}\right)=0. \tag{C.2}$$

We try to find the solution to Eq. (C.2) in the following way

$$\check{V}(t,x)=-\frac{1}{m}\mathrm{e}^{-m[\mathrm{e}^{r_0(T-t)}x-f_0(t)]}. \tag{C.3}$$

Plugging the relevant derivatives into Eq. (C.2), we derive

$$f_0(t)=\frac{\lambda_1(\eta-\theta)\mu_Z}{r_0}(1-\mathrm{e}^{r_0(T-t)})+\int_t^T l_6(\omega)\mathrm{d}\omega-\int_t^T l_7(\omega)\mathrm{d}\omega, \tag{C.4}$$

where

$$l_6(\omega)=\mathrm{e}^{r_0(T-\omega)}\int_0^{\frac{\theta\mathrm{e}^{-r_0(T-\omega)}}{m}}[\lambda_1(\gamma^{B_1}+m)s\mathrm{e}^{r_0(T-\omega)}-\lambda_1\theta]\bar{F}(s)\mathrm{d}s, \tag{C.5}$$

$$l_7(\omega)=\hat{\pi}^*(\omega)(\mu-r_0)\mathrm{e}^{r_0(T-\omega)}-\frac{1}{2}(\gamma^{B_2}+m)\sigma^2(\hat{\pi}^*(\omega))^2\mathrm{e}^{2\mathrm{r}_0(\mathrm{T}-\omega)}$$
$$+\frac{\lambda_2}{\gamma^J}\left(1-\mathrm{e}^{\frac{\gamma^J}{m}\mathrm{E}^{\mathbb{Q}}[\mathrm{e}^{-m\hat{\pi}^*(\omega)y\mathrm{e}^{r_0(T-\omega)}}-1]}\right). \tag{C.6}$$

Then we have the expressions $\check{V}(t,x)$.

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