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# INDEX OPTIONS AND VOLATILITY DERIVATIVES IN A GAUSSIAN RANDOM FIELD RISK-NEUTRAL DENSITY MODEL

XIXUAN HAN
*Department of Statistics And Actuarial Science,*
*The University of Hong Kong, Hong Kong*
*xhan@hku.hk*

BOYU WEI
*Department of Statistics And Actuarial Science,*
*The University of Hong Kong, Hong Kong*
*u3001203@hku.hk*

HAILIANG YANG
*Department of Statistics And Actuarial Science,*
*The University of Hong Kong, Hong Kong*
*hlyang@hku.hk*

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We propose a risk-neutral forward density model using Gaussian random fields to capture different aspects of market information from European options and volatility derivatives of a market index. The well-structured model is built in the framework of the Heath- Jarrow-Morton philosophy and the Musiela parametrization with a user-friendly arbitrage-free condition. It reduces to the popular geometric Brownian motion model for the spot price of the market index and can be intuitively visualized to have a better view of the market trend. In addition, we develop theorems to show how the model drives local volatility and variance swap rates. Hence, volatility futures and options can be priced taking the forward density implied by European options as the initialization input. The model can be accordingly calibrated to the market prices of these volatility derivatives. An efficient algorithm is developed for both simulating and pricing, and a simulation study is conducted using market data.

*Keywords*: risk-neutral forward density; Heath-Jarrow-Morton (HJM) framework; Gaussian random field; market index; European options; volatility futures; volatility options.

## 1. Introduction

In an arbitrage-free market, actively traded standard European options provide interesting and valuable information about the trend and dispersion of the underlying asset in the future. The information reflects the expectations that market participants hold concerning the underlying asset as that of the market in general. And it may dramatically change as new releases come and unseen events happen over time.

One fairly common way to summarize the information is to construct option-implied risk-neutral forward densities. In the literature, they were also termed risk-neutral densities (RNDs), e.g. in Liu et al. (2007), Brunner & Hafner (2003), Bondarenko (2003) and Jackwerth (1999), state price densities (SPDs), e.g. in Yuan (2009), Zhang et al. (2009), Yatchew & Härdle (2006) and Aït-Sahalia & Lo (1998), or conditional densities in Filipović et al. (2012). At a certain time, the risk-neutral forward densities, from options of different strikes and maturities, exhibit the well-structured likelihood of where the underlying price may reach at a range of future times, visually forming a two-dimensional surface of strikes and future times as shown in Figure 1. In this paper, we focus on modelling of the surface in order to produce its time evolutions over time, which is particularly useful for joint calibration and pricing.

The financial community has had an extensive discussion on aspects of risk-neutral forward densities during the last three decades, including empirical properties, estimation, modelling, applications and so forth. Particularly the usefulness of risk-neutral forward densities have been well studied and verified. See, e.g., Härdle et al. (2015), Birru & Figlewski (2012), Kim & Kim (2003), Nikkinen (2003) and Dennis & Mayhew (2002) for related reference. Thus risk-neutral forward densities have been widely used in finance. Applications include testing market rationality (see Bondarenko 1997), measuring market participants' risk preference (see Aıt & Lo 2000), assessing market attitude towards coming events (see Söderlin 2000), estimating parameters of underlying stochastic models (see Bates 1996), guiding monetary policies (see Jondeau & Rockinger 2000), etc. Besides, the most important application is pricing complex exotic derivatives in order to be consistent with exchanged-traded options. Most of the research efforts in the field of risk-neutral forward densities are contributed to estimation techniques. We go through the major representatives to give a quick review. The first attempts of using European option prices to recover risk-neutral forward densities were shown in Ross (1976), Breeden & Litzenberger (1978) and Banz & Miller (1978), though pricing theory of contingent claims is the core part of them. Risk-neutral forward densities are demonstrated to have a proportional relationship with the second derivatives of option prices with respect to strikes. However, simply applying the finite difference method usually generates numerically unstable and inadequate results and as a result, the revealed relationship calls for more robust and powerful estimation methodologies. There are hundreds of research papers and a number of surveys emerged in this specific direction. Most of them could be classified into two categories according to a classic standard, i.e. parametric methods and non-parametric methods. To develop parametric methods, some interesting ideas come forward such as arbitrage-free interpolation of option prices, see Orosi (2015), cubic spline interpolation of implied volatilities, see Malz (2014), maximum entropy principle, see Rompolis (2010), wavelet method, see Haven et al. (2009), generalized gamma distribution based method, see Fabozzi & Albota (2009), C-type Gram-Charlier series expansion, see Rompolis & Tzavalis (2008) and cubic spline interpolation of option prices, see Monteiro et al. (2008).

Some classic techniques include spline and hyper-geometric functions based method, see Bu & Hadri (2007), generalized Student t-distribution, see Lim et al. (2005), GARCH-based estimation, see Fornari & Mele (2001), mixture density networks, see Schittenkopf & Dorffner (2001), flexible NLS pricing, see Rosenberg (1998) and stochastic model based method, see Heston (1993). Non-parametric methods also attract researchers' attention due to their model-free characteristics. For example, Yuan (2009) chooses the best pricing function from an admissible set whose risk-neutral forward densities were non-parametric mixtures of log-normals according to the least squares criterion, in order to fit risk-neutral forward densities. Yatchew & Härdle (2006) develops a nonparametric estimator of option pricing models via the least squares procedure to estimate risk-neutral forward densities. In Aït-Sahalia & Lo (1998), nonparametric kernel regression was employed to fit historical option prices and yielded estimations of risk-neutral forward densities. All of those estimation techniques boil down to pursue estimations that are non-negative, integrate to one and are guaranteed to be martingales. We call those the three essential requirements for estimation.

To our knowledge, little attention has been paid to modelling, though there have been a plenty of research achievements in estimation methods. Besides the three essential requirements for estimation, we further need to ensure that risk-neutral forward density dynamics converge to certain Dirac delta functions in some sense as time to expiration (maturity) shortens. Together we call them the four essential requirements for modelling. Even if challenging, a good risk-neutral forward density model is appealing in three perspectives. First, by estimating from historic data, we obtain a one-time risk-neutral forward density for a range of strikes and a range of future times. With this static surface, we are only able to price European style exotic derivatives and cannot cope with early exercises of derivatives. To be able to handle complicated contingent claims, dynamic information of risk-neutral forward densities is necessary. Second, a static risk-neutral forward density surface does not allow us to price volatility derivatives. Only with time evolutions of the surface, we are able to measure and assess the underlying's volatility in the future. Last but not least, a dynamic risk-neutral forward density model enables us to jointly consider index option prices and volatility option prices, and therefore accurately estimate prices of illiquid over-the-counter (OTC) derivative contracts. Since index options and volatility options have been actively traded in large volumes in exchanges, a model that is jointly calibrated to them may generate more adequate prices of OTC derivatives. This is because this model contains more information than models that are only calibrated to index options. To sum up, modelling risk-neutral forward densities is an important way to jointly model index options and volatility options. In Filipović et al. (2012), the authors propose a master equation to generally describe risk-neutral forward density dynamics and solve it using some various designs of volatility structure, yielding positive time evolutions that are martingales and integrate to unity. The idea is a partial success in that the model only depicts the risk-neutral forward densities at a specific time in the future. In other words, the

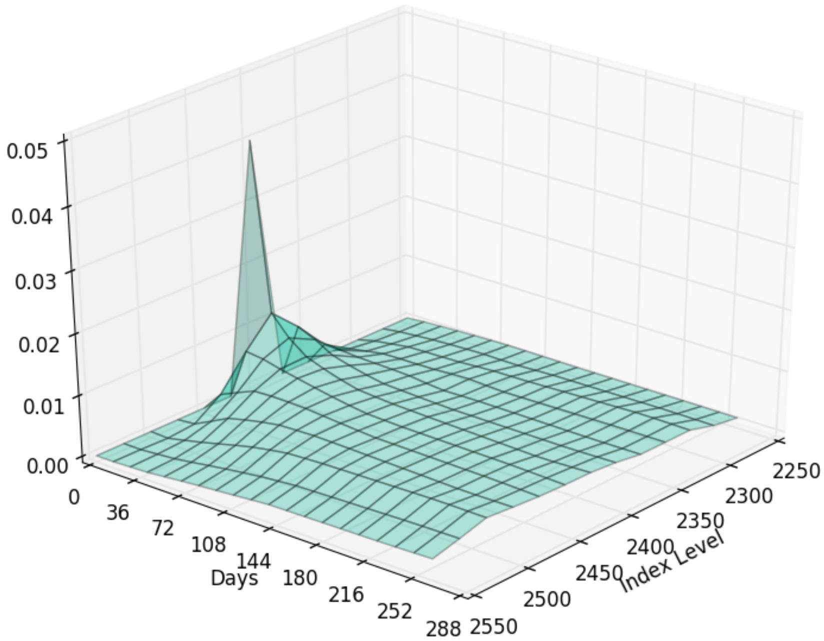


Fig. 1: Risk-neutral forward density surface implied by the SPX call options on 4 May 2017. The surface is calculated by firstly linearly interpolating option prices and then the finite difference method. The density will be used as the initial surface of our model in the simulation study in Section 6.

model could only price a part of exotic derivatives embedded with early exercise rights and could not price volatility derivatives (thus could not be calibrated to volatility derivatives). Furthermore, thir model is not clearly demonstrated to converge to a Dirac delta function in a certain sense and thus does not necessarily satisfy the last one of the four essential requirements for modelling.

To fill this gap, we propose a new model driven by Gaussian random fields, designed under the Heath-Jarrow-Morton (HJM) framework and constructed using the Musiela parametrization, in order to dynamically describe the time evolution of the whole risk-neutral forward density surface at a range of strikes and future times obtained from a cross-section of European options. Our model clearly satisfies the four essential requirements for modelling and most importantly, it enables us to price exotic derivatives with early exercise clauses and volatility derivatives. Our model also has a close relationship with the classic geometric Brownian motion models, which further shows its validity and rationality.

The HJM framework has received a large amount of attention in the field of interest rate curve modelling and has started to show advantages in modelling other financial instruments. The framework was firstly developed in the celebrated work of Heath et al. (1992). After that, it has been applied to evaluate stock options, e.g. see Schweizer & Wissel (2008) and Cont et al. (2002), variance swaps, e.g. see Buehler (2006), and credit derivatives, e.g. see Sidenius et al. (2008). A comprehensive survey of the applications of HJM framework is given in Carmona (2007). Furthermore, the framework is combined with Gaussian random fields, e.g. see Collin & Goldstein (2003), Goldstein (2000) and Pang (1998). Goldstein (2000) and Kennedy (1997) have extensive discussions on Gaussian random fields and their correlation structures in financial modelling.

The HJM framework and Gaussian random fields are the key foundations of our risk-neutral forward density model. The former enables us to construct models of what is unknown given what is known in the market information and the latter allows us to construct model surfaces rather than processes. Our model takes full advantage of the risk-neutral forward density from options, e.g. SPX options listed in Chicago Board Options Exchange, by treating it as the initial surface, which is an important condition that guarantees the convergence to Dirac delta functions as time to expiration goes to zero. To better incorporate the convergence, we adopt the Musiela parametrization when setting up our model in that the Musiela parametrization effectively put together different parts in the model and greatly simplifies the theoretical proofs. To ensure the model admits no arbitrage, we transform our model back to the normal parametrization and derive a complicated integral equation as the arbitrage-free condition. Then we identify a set of solutions to the integral equation. With one of the solutions, the dynamics of our model is a martingale and possesses a concise and compact representation. Based on this result, we link our model with the local volatility of the underlying asset, thus obtain the local volatility's dynamics. Finally, the local volatility's dynamics leads us to the dynamics of the instantaneous variance swap rates and the forward variance swap rates. Either of them may help us price volatility derivatives, e.g. VIX options traded in Chicago Board Options Exchange, via Monte Carlo simulations. The whole pricing procedure of VIX options encounters two computationally expensive steps. i.e. solving the martingale condition and calculating the increments in the dynamics of variance swap rates. To boost the simulation speed, we adopt graphic processing unit (GPU) acceleration, the most popular technology for parallel computing, and achieve acceptable running speed using two graphic cards in the simulation. After calibration, our model fits the market price data very well.

Besides the four essential requirements for modelling. Our model also enjoys the following advantages. First of all, the model takes risk-free rates and dividend rates as input and implies a time evolution for the spot price of the underlying asset. The time evolution has the drift of the difference between risk-free rates and dividend rates, which is consistent with existing models heavily used in the market. This property may increase the chance that our model fits market prices accurately after being calibrated. In addition, the model simultaneously describes the dynamics of the whole risk-neutral forward density and thus makes it possible for parameters to have term structures, i.e. to be time-varying in a deterministic manner. With this feature, it is feasible to calibrate our model to VIX options of different maturities simultaneously, and therefore it is able to capture a large amount of market information.

The paper is organized in the following way. In Section 2, we formally and mathematically give the definition of valid risk-neutral forward density models. The conditions in the definition can be viewed as the minimal requirements for a stochastic model to depict the dynamics of risk-neutral forward densities. After the definition, we introduce notations and build our model in the first half of Section 3. With the model, some assumptions are given to it in order to satisfy the definition of valid models in Section 2 as proved in the lemmas in this section. Focusing on volatility derivatives in the paper, we then connect risk-neutral forward densities to local volatility in Section 4, and present the dynamics of local volatility under our model. In Section 5, the dynamics of risk-neutral forward densities and local volatility are combined to obtain the dynamics of variance swap rates. To facilitate programming, we summarize the entire model by Algorithm 1 in Section 6 and give a convenient matrix representation to the increments of forward variance swap rates. The last part of this section shows a simulation study conducted on the market data on 4 May 2017. Finally Section 7 concludes the paper.

## 2. Valid Forward Density Models

In the beginning, we make a fundamental assumption about the market and the risk factors driving it. We consider the modeling in the probability space $(\Omega, \mathscr{F}, \mathbb{Q})$. In the following, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ represent the whole collection of positive real numbers and the whole collection of non-negative real numbers.

**Assumption 2.1.** *There are an infinite number of risk factors in the market and can be classified into two groups. One group consists of correlated factors following a centered Gaussian random field* $\{W(t,K) : (t,K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{\geq 0}\}$*, in which* $K \in \mathbb{R}_{\geq 0}$ *represents the market index level, and the other group consists of a single factor independent of the previous group, following a Brownian motion* $\{B(t) : t \in \mathbb{R}_{\geq 0}\}$*.*

Note that the independence of the groups is made to simplify the analysis thereafter. Based on the risk factors, we define their generated information flow, i.e. the filtration $\{\mathscr{F}_t : t \in \mathbb{R}_{\geq 0}\}$, as

$$\mathscr{F}_t \triangleq \sigma\Big(\Big(\bigcup_{s\leq t}\bigcup_{k\in\mathbb{R}_{\geq 0}} \sigma(W(s,K))\Big) \bigcup \Big(\bigcup_{s\leq t} \sigma(B(s))\Big)\Big), \tag{2.1}$$

where $\sigma(\cdot)$ represents the generated sigma algebra of a random variable. Under the risk-neutral probability measure $\mathbb{Q}$, the risk-neutral forward density of an underlying asset, $S(t)$ for $t \in \mathbb{R}_{\geq 0}$, expressed in units of a base numeraire is defined as

$$p(t,\tau,K) \triangleq \mathbb{E}^{\mathbb{Q}}\left[\delta\left(S(t+\tau) - K\right) \mid \mathscr{F}_t\right] \tag{2.2}$$

for $t \in \mathbb{R}_{\geq 0}$ representing time, where $\tau \in \mathbb{R}_{\geq 0}$ and $K \in \mathbb{R}_{\geq 0}$ are differences in time, i.e. time lengths to expiration (maturity), and stock price levels. Let $\{r(t) : t \in \mathbb{R}_{\geq 0}\}$ denote a nonrandom risk-free interest rate process, $D(t,\tau) = \exp(-\int_t^{t+\tau} r(s)ds)$ the associated discounting factor, $C(t,\tau,K)$ the price of a European call option with maturity $t+\tau$ and strike $K$ and $P(t,\tau,K)$ the price of a European put option. Thus we have

$$C(t,\tau,K) = D(t,\tau) \int_{\mathbb{R}_{\geq 0}} (x-K)^+ p(t,\tau,x)dx \tag{2.3}$$

and

$$P(t,\tau,K) = D(t,\tau) \int_{\mathbb{R}_{\geq 0}} (K-x)^+ p(t,\tau,x)dx, \tag{2.4}$$

where $\tau \in \mathbb{R}_{\geq 0}$.

A key motivation behind such a risk-neutral forward density model is the joint modeling of index options and index volatility options, e.g. SPX options and VIX options. The risk-neutral forward density obtained from index options is fed into the model as the initial condition $p(0,\tau,K)$ for $\tau \in \mathbb{R}_{\geq 0}$ and $K \in \mathbb{R}_{\geq 0}$. And the volatility of the forward density dynamics can be calibrated using volatility options because it must be reflected in the volatility of the index volatility.

To have a valid and arbitrage-free market model, we give the following definition. It summarizes the four essential requirements that feasible models should incorporate to describe the time evolution of risk-neutral forward densities.

**Definition 2.1.** *A random surface* $\{p(t,\tau,K) \geq 0 : (t,\tau,K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}\}$ *must satisfy the following conditions to be a valid risk-neutral forward density model:*

*(1)* $\int_{\mathbb{R}_{\geq 0}} p(t,\tau,K)dK = 1$ *for all* $(t,\tau) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0}$*;*
*(2) the density process reduces to a Dirac delta function at the current underlying price, i.e.*

$$\lim_{\tau \to 0} p(t,\tau,K) = \delta(K - S(t)) \tag{2.5}$$

*for* $(t,K) \in \mathbb{R}^2_{\geq 0}$ *in a certain mode of convergence, where* $S(t)$ *represents the spot price of the underlying asset; and*
*(3) the stochastic process* $\{p(t,T-t,K) : 0 \leq t < T\}$*, according to Equation* (2.2)*, itself is a martingale under* $\mathbb{Q}$ *for all* $T \in \mathbb{R}_{>0}$ *and* $K \in \mathbb{R}_{\geq 0}$*.*

Note that the definition only contains the minimal requirements. In face, the real risk-neutral forward densities may be skewed to reflect skewed implied volatility. And there could be a significant difference between the characteristics of the risk-neutral forward densities of stock indices and the ones of individual stocks. It is not possible to put together all these details in a single model, so we focus on the most important four requirements to construct the modelling.

## 3. Random Field Forward Density Model

### 3.1. *Model Setting-up*

In this section, we construct a model to describe risk-neutral forward densities on $\mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$. Using the Musiela parametrization, we set up the model as

$$q(t, \tau, K) = q(0, \tau, Kk(t, \tau))f(t, \tau, K), \tag{3.1}$$

where $t \in \mathbb{R}_{\geq 0}$ for all $(\tau, K) \in \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$. The infinite-dimensional stochastic process $\{k(t, \tau) : (t, \tau) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0}\}$ that drives the surface in the direction of $K$ is defined as

$$dk(t, \tau) = (\sigma_k^2(t, \tau) - \mu_k(t, \tau))k(t, \tau)dt - \sigma_k(t, \tau)k(t, \tau)dB(t) \tag{3.2}$$

with the initial condition $k(0, \tau) = 1$ for $\tau \in \mathbb{R}_{>0}$, and the two-dimensional random field $\{f(t, \tau, K) : (t, \tau, K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}\}$ that causes shocks to the magnitude of densities follows

$$df(t, \tau, K) = \mu_f(t, \tau, K)f(t, \tau, K)dt + \sigma_f(t, \tau, K)f(t, \tau, K)dW(t, K) \tag{3.3}$$

with the initial surface $f(0, \tau, K) = 1$ for $(\tau, K) \in \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$. The initial surface $q(0, \tau, K)$, where $(\tau, K) \in \mathbb{R}^2_{\geq 0}$, represents the initial risk-neutral forward density at time 0 and has two roles. The first one is to enable the calibration to the index option market as mentioned previously. And the second one is to serve as a media to pass the shocks generated by $k(t, \tau)$ in the direction of $K$ to the whole density surface. In the above setting, $\mu_k(t, \tau) > 0$, $\sigma_k(t, \tau) > 0$, $\mu_f(t, \tau, K) > 0$ and $\sigma_f(t, \tau, K) > 0$ are infinite-dimensional $\mathscr{F}_t$-adapted processes indexed by $\tau$ and $K$, where $(t, \tau, K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$. Among them, $\sigma_k(t, \tau)$ and $\sigma_f(t, \tau, K)$ are deterministic and serve as parameters of our model. The correlation structure of $W(t, K)$ is assumed to be

$$dW(t, K_1)dW(t, K_2) = c_W(t, K_1, K_2)dt = c_W(t, K_2, K_1)dt \tag{3.4}$$

for $(K_1, K_2) \in \mathbb{R}^2_{\geq 0}$. Particularly, $c_W(t, K_1, K_2)$ is a continuous function satisfying the condition

$$c_W(t, K, K) = 1 \tag{3.5}$$

for $K \in \mathbb{R}_{\geq 0}$. Until now, $q(t, \tau, K)$ is not guaranteed to meet the condition

$$\int_{\mathbb{R}_{\geq 0}} q(t, \tau, K)dK = 1 \tag{3.6}$$

in Definition 2.1. So we apply the methodology in Cheung & Wei (2016), and further introduce a normalizing process

$$a(t, \tau) = \int_{\mathbb{R}_{\geq 0}} q(t, \tau, K)dK \tag{3.7}$$

and accordingly define a normalized process

$$p(t, \tau, K) = \frac{q(t, \tau, K)}{a(t, \tau)}. \tag{3.8}$$

The normalized process clearly satisfies the integration condition. Based on it, we add constraints to the model so that it is a valid risk-neutral forward density model according to Definition 2.1. To simplify the analysis and the presentation thereafter, defining

$$D(t,\tau,K,L) \triangleq \Big(\sigma_f^2(t,\tau,K) - 2c_W(t,K,L)\sigma_f(t,\tau,K)\sigma_f(t,\tau,L) + \sigma_f^2(t,\tau,L)\Big)^{\frac{1}{2}}, \tag{3.9}$$

we introduce another random field $\{Y(t,\tau,K,L) : t \in \mathbb{R}_{\geq 0}\}$ as

$$Y(t,\tau,K,L) \triangleq \begin{cases} \dfrac{\sigma_f(t,\tau,K)W(t,K) - \sigma_f(t,\tau,L)W(t,L)}{D(t,\tau,K,L)} & K \neq L \\ 0 & K = L \end{cases} \tag{3.10}$$

for $(\tau,K,L) \in \mathbb{R}^3_{\geq 0}$. When $K \neq L$ and $M \neq J$, the correlation structure is given by

$$\begin{aligned} dY(t,\tau,K,L)dY(t,\tau,M,J) = & \frac{\sigma_f(t,\tau,K)\sigma_f(t,\tau,M)c_W(t,K,M)}{D(t,\tau,K,L)D(t,\tau,M,J)}dt \\ & - \frac{\sigma_f(t,\tau,K)\sigma_f(t,\tau,J)c_W(t,K,J)}{D(t,\tau,K,L)D(t,\tau,M,J)}dt \\ & - \frac{\sigma_f(t,\tau,L)\sigma_f(t,\tau,M)c_W(t,L,M)}{D(t,\tau,K,L)D(t,\tau,M,J)}dt \\ & + \frac{\sigma_f(t,\tau,L)\sigma_f(t,\tau,J)c_W(t,L,J)}{D(t,\tau,K,L)E(t,\tau,M,J)}dt \\ \triangleq & \; c_Y(t,\tau,K,L,M,J)dt \end{aligned} \tag{3.11}$$

for $(K,L,M,J) \in \mathbb{R}^4_{\geq 0}$, otherwise the correlation is 0. Also note that $Y(t,\tau,K,L) = -Y(t,\tau,L,K)$.

### 3.2. *Model Assumptions*

We start with a discussion on the initial risk-neutral forward density $q(0,\tau,K)$ for $(\tau,K) \in \mathbb{R}^2_{\geq 0}$. Note that $q(0,\tau,K)$ has a definition at $\tau = 0$. Knowing either all call or all put options traded on all maturities and strikes plus the current price of the underlying, the initial density can be completely determined. However, in practice, only a few number of options are traded at the same time and we need to apply a method shown in Section 1 to obtain $q(0,\tau,K)$. What we have for sure is that at $\tau = 0$

$$q(0,0,K) = \delta(K - S(0)), \tag{3.12}$$

where $S(0)$ represents the underlying price at time 0. Therefore we make the following assumption of the initial density surface.

**Assumption 3.1.** *The initial surface $q(0,\tau,K)$ for $(\tau,K) \in \mathbb{R}^2_{\geq 0}$ is assumed*

*(1) to be continuous with respect to $\tau$ on $\mathbb{R}_{\geq 0}$ for all $K \in \mathbb{R}_{\geq 0}$ and*

*(2) to be thrice continuously differentiable with respect to* $\tau$ *and* $K$ *on* $\mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$.

The first condition of Assumption 3.1 and Equation (3.12) guarantee

$$\lim_{\tau \to 0} q(0, \tau, K) = \delta(K - S(0)) \tag{3.13}$$

for $K \in \mathbb{R}_{\geq 0}$, which is essential in developing Lemma 3.1. The model has four processes $\mu_k(t,\tau)$, $\sigma_k(t,\tau)$, $\mu_f(t,\tau,K)$ and $\sigma_f(t,\tau,K)$ involved. Since the derivation of dynamics in the rest of the paper requires Itó Lemma and the Leibniz rule of differentiation, it is also necessary to add assumptions to them so that Itó Lemma and Leibniz rule can be applied. In addition, we are modelling using the Musiela parametrization and thus hope that there are not any movements of the density surface in the vertical direction when $\tau = 0$, i.e. $f(t,\tau,K)$ treated as a stochastic process of $t$ slows down its volatility as $\tau$ decreases and stops moving when $\tau = 0$. To sum up, we make the following assumptions of $\mu_k(t,\tau)$, $\sigma_k(t,\tau)$, $\mu_f(t,\tau,K)$ and $\sigma_f(t,\tau,K)$.

**Assumption 3.2.** *The drift* $\{\mu_k(t,\tau) > 0 : (t,\tau) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0}\}$ *and volatility* $\{\sigma_k(t,\tau) > 0 : (t,\tau) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0}\}$ *of* $k(t,\tau)$ *are assumed*

*(1) to be thrice continuously differentiable with respect to* $\tau$ *on* $\mathbb{R}_{>0}$ *for all* $t \in \mathbb{R}_{\geq 0}$*;*
*(2) to satisfy the existence of* $\lim_{\tau \to 0} \mu_k(t,\tau) \triangleq \mu_k(t,0)$ *and* $\lim_{\tau \to 0} \sigma_k(t,\tau) \triangleq \sigma_k(t,0)$ *for all* $t \in \mathbb{R}_{\geq 0}$ *and*
*(3) to be Borel-measurable and continuous as functions of* $t$ *on* $\mathbb{R}_{\geq 0}$ *for all* $\tau \in \mathbb{R}_{>0}$.

*The drift* $\{\mu_f(t,\tau,K) > 0 : (t,\tau,K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}\}$ *and volatility* $\{\sigma_f(t,\tau,K) > 0 : (t,\tau,K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}\}$ *of* $f(t,\tau,K)$ *are assumed*

*(1) to be thrice continuously differentiable with respect to* $\tau$ *on* $\mathbb{R}_{>0}$ *for all* $(t,K) \in \mathbb{R}^2_{\geq 0}$*;*
*(2) to satisfy the existence of* $\lim_{\tau \to 0} \mu_f(t,\tau,K) \triangleq \mu_k(t,0,K)$ *for all* $(t,K) \in \mathbb{R}^2_{\geq 0}$ *and*

$$\lim_{\tau \to 0} \sigma_f(t,\tau,K) = 0 \tag{3.14}$$

*for all* $(t,K) \in \mathbb{R}^2_{\geq 0}$ *and*
*(3) to be Borel-measurable and continuous as a function of* $t$ *on* $\mathbb{R}_{\geq 0}$ *for all* $(\tau,K) \in \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$.

Assumption 3.1 and Assumption 3.2 plus Equation (3.12) guarantee that the model reduces to a Dirac delta function as $\tau$ goes to 0 for all $(t,K) \in \mathbb{R}^2_{\geq 0}$, which is described in Lemma 3.1 below. And, as a special case, the result particularly indicates that

$$\lim_{\tau \to 0} \Big( \lim_{t \to 0} p(t,\tau,K) \Big) = \delta(K - S(0)), \tag{3.15}$$

which is consistent with Equation (3.12). Also we use $k(t,0)$ to denote $\exp\Big( \int_0^t \big( -\mu_k(s,0) + \frac{1}{2}\sigma_k^2(s,0) \big) ds - \int_0^t \sigma_k(s,0) dB(s) \Big)$ in Lemma 3.1.

**Lemma 3.1.** *When Assumption 3.1 and Assumption 3.2 holds and we assume*

$$a(t,0) \triangleq \lim_{\tau\to 0} a(t,\tau) = 1 \tag{3.16}$$

*almost surely, it follows that*

*(1)* $f(t,\tau,K)$ *converges to* $\exp(\int_0^t \mu_f(s,0,K)ds)$ *in probability under* $\mathbb{Q}$ *as* $\tau \to 0$ *for all* $(t,K) \in \mathbb{R}^2_{\geq 0}$*;*
*(2)* $k(t,\tau)$ *converges to* $k(t,0)$ *in probability under* $\mathbb{Q}$ *as* $\tau \to 0$ *for all* $t \in \mathbb{R}_{\geq 0}$*;*
*(3)* $q(0,\tau,\frac{S(0)}{k(t,0)}k(t,\tau))$ *converges to infinity as* $\tau \to 0$ *in probability under* $\mathbb{Q}$ *for all* $t \in \mathbb{R}_{\geq 0}$*; and*
*(4) we have*

$$\lim_{\tau\to 0} q(t,\tau,K) = \delta(K - \frac{S(0)}{k(t,0)}) \tag{3.17}$$

*for all* $(t,K) \in \mathbb{R}^2_{\geq 0}$ *in probability under* $\mathbb{Q}$.

**Proof.**

(1) When $t = 0$, it is trivial to see the result. Then we consider the case where $t > 0$. From Equation (3.3), it simply shows that

$$\begin{aligned} f(t,\tau,K) = \exp\Big(&\int_0^t \mu_f(s,\tau,K)ds - \frac{1}{2}\int_0^t \sigma_f^2(s,\tau,K)ds \\ &+ \int_0^t \sigma_f(s,\tau,K)dW(s,K)\Big). \end{aligned} \tag{3.18}$$

According to Assumption 3.2, $\mu_f(s,\tau,K)$ and $\sigma_f(s,\tau,K)$ are bounded as $\tau$ varies on $(0,\tau_0]$ for any $(s,K) \in [0,t] \otimes \mathbb{R}_{\geq 0}$, where $\tau_0 > 0$ is an arbitrary quantity. Thus there exists a finite function of $K$, denoted $M(K)$, such that

$$\begin{aligned} &\max_{(s,\tau)\in[0,t]\otimes(0,\tau_0]} \mu_f(s,\tau,K) \vee \sigma_f(s,\tau,K) \\ &= \max_{s\in[0,t]} \Big(\max_{\tau\in(0,\tau_0]} \mu_f(s,\tau,K) \vee \sigma_f(s,\tau,K)\Big) \triangleq M(K), \end{aligned} \tag{3.19}$$

where $K \in \mathbb{R}_{\geq 0}$. Thus by dominated convergence theorem, the first two deterministic terms in the exponent of Equation (3.18) follow

$$\lim_{\tau\to 0} \int_0^t \mu_f(s,\tau,K)ds = \int_0^t \mu_f(s,0,K)ds \tag{3.20}$$

and

$$\lim_{\tau\to 0} \int_0^t \sigma_f^2(s,\tau,K)ds = 0 \tag{3.21}$$

for all $K \in \mathbb{R}_{\geq 0}$, which also holds in the sense of $L^2(\mathbb{Q})$. And according to Corollary 3.1.8 of Øksendal (2003), the second random term in the exponent of

Equation (3.18) satisfies

$$\lim_{\tau\to 0}\int_0^t \sigma_f(s,\tau,K)dW(s,K)=0 \tag{3.22}$$

in $L^2(\mathbb{Q})$ for all $K\in\mathbb{R}_{\geq 0}$. Since

$$\begin{aligned}&\Big(-\frac{1}{2}\int_0^t \sigma_f^2(s,\tau,K)ds+\int_0^t \sigma_f(s,\tau,K)dW(s,K)\Big)^2\\ &\leq \frac{1}{2}\big(\int_0^t \sigma_f^2(s,\tau,K)ds\big)^2+2\big(\int_0^t \sigma_f(s,\tau,K)dW(s,K)\big)^2,\end{aligned} \tag{3.23}$$

we have that $-\frac{1}{2}\int_0^t \sigma_f^2(s,\tau,K)ds+\int_0^t \sigma_f(s,\tau,K)dW(s,K)$ converges to 0 in $L^2(\mathbb{Q})$. Hence the convergence holds in probability under $\mathbb{Q}$, and leads to the result that

$$\lim_{\tau\to 0} f(t,\tau,K)=\exp(\int_0^t \mu_f(s,0,K)ds) \tag{3.24}$$

for all $(t,K)\in\mathbb{R}_{>0}\otimes\mathbb{R}_{\geq 0}$ in probability under $\mathbb{Q}$ according to continuous mapping theorem. Therefore the conclusion holds for $(t,K)\in\mathbb{R}^2_{\geq 0}$.

(2) Similar to $f(t,\tau,K)$, we also have

$$\lim_{\tau\to 0} k(t,\tau)=k(t,0) \tag{3.25}$$

for all $t\in\mathbb{R}_{\geq 0}$ in probability under $\mathbb{Q}$ according to the existence of limits in Assumption 3.2.

(3) We seek to find a $\varepsilon_\tau>0$ such that

$$\mathbb{P}\left[q(0,\tau,\frac{S(0)}{k(t,0)}k(t,\tau))>\zeta\right]>1-\epsilon, \tag{3.26}$$

when $0<\tau<\varepsilon_\tau$, where $\zeta>0$ is arbitrarily large and $\epsilon>0$ is arbitrarily small. For an arbitrarily small $\delta>0$, according to Equation (3.12) and Assumption 3.1, there exists an $\varepsilon'_\tau>0$ such that

$$q(0,\tau)>\zeta+\delta \tag{3.27}$$

if $0<\tau<\varepsilon'_\tau$. Fixing a $\tau\in(0,\varepsilon'_\tau)$, according to Assumption 3.1, there exists an $\varepsilon_k>0$ such that

$$\mid q(0,\tau)-q(0,\tau,\frac{S(0)}{k(t,0)}x)\mid<\delta \tag{3.28}$$

if $\mid x-k(t,0)\mid<\varepsilon_k$. Thus, given $\tau\in(0,\varepsilon'_\tau)$ and $x\in(k(t,0)-\varepsilon_k,k(t,0)+\varepsilon_k)$, we have

$$q(0,\tau,\frac{S(0)}{k(t,0)}x)>\zeta. \tag{3.29}$$

Furthermore we find out the following probability:

$$\mathbb{P}\left[q(0,\tau,\frac{S(0)}{k(t,0)}k(t,\tau))>\zeta\middle|\mid k(t,\tau)-k(t,0)\mid<\varepsilon_k\right]=1 \tag{3.30}$$

when $\tau \in (0, \varepsilon'_\tau)$. In addition, the probability of our interest has the relationship

$$\begin{aligned} &\mathbb{P}\left[q(0,\tau,\frac{S(0)}{k(t,0)}k(t,\tau)) > \zeta\right] \\ \geq\; &\mathbb{P}\left[q(0,\tau,\frac{S(0)}{k(t,0)}k(t,\tau)) > \zeta \middle| \mid k(t,\tau)-k(t,0)\mid < \varepsilon_k\right] \\ &\mathbb{P}\left[\mid k(t,\tau)-k(t,0)\mid < \varepsilon_k\right] \\ =\; &\mathbb{P}\left[\mid k(t,\tau)-k(t,0)\mid < \varepsilon_k\right]. \end{aligned} \tag{3.31}$$

Knowing Equation (3.25), there exists an $\varepsilon''_\tau > 0$ such that

$$\mathbb{P}\left[\mid k(t,\tau)-k(t,0)\mid < \varepsilon_k\right] > 1-\epsilon \tag{3.32}$$

when $0 < \tau < \varepsilon''_\tau$. We take $\varepsilon_\tau = \min(\varepsilon'_\tau, \varepsilon''_\tau)$ and finally reach Equation (3.26).

(4) We note that $q(0,\tau,K)$ is continuous on $\mathbb{R}^2_{\geq 0}\backslash\{(0,S(0))\}$ from Assumption 3.1. Again by continuous mapping theorem, it follows that

$$\lim_{\tau\to 0} q(0,\tau,Kk(t,\tau)) = 0 \tag{3.33}$$

for all $K \in \mathbb{R}_{\geq 0}\backslash\{S(0))/k(t,0)\}$ in probability under $\mathbb{Q}$ for $t \in \mathbb{R}_{\geq 0}$. Considering the first three conclusions and

$$q(t,\tau,K) = q(0,\tau,Kk(t,\tau))f(t,\tau,K), \tag{3.34}$$

we therefore have

$$\lim_{\tau\to 0} q(t,\tau,K) = \delta(K - \frac{S(0)}{k(t,0)}) \tag{3.35}$$

for all $(t,K) \in \mathbb{R}^2_{\geq 0}$ in probability under $\mathbb{Q}$. With the assumption that $\lim_{\tau\to 0} a(t,\tau) = 1$, the conclusion is proved.

The Dirac delta function obtained in Lemma 3.1 therefore determines the spot process of the underlying asset $S(t)$ for $t \in \mathbb{R}_{\geq 0}$ in our model. In other words, the model reduces to

$$S(t) = \frac{S(0)}{k(t,0)} \tag{3.36}$$

for $t \in \mathbb{R}_{\geq 0}$ in probability under $\mathbb{Q}$ as $\tau$ goes to 0. The related process $k(t,\tau)$ has the form

$$k(t,\tau) = \exp\Big(\int_0^t \big(-\mu_k(s,\tau) + \frac{1}{2}\sigma_k^2(s,\tau)\big)ds - \int_0^t \sigma_k(s,\tau)dB(s)\Big) \tag{3.37}$$

for $(t,\tau) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0}$, which converges to

$$k(t,0) = \exp\Big(\int_0^t \big(-\mu_k(s,0) + \frac{1}{2}\sigma_k^2(s,0)\big)ds - \int_0^t \sigma_k(s,0)dB(s)\Big) \tag{3.38}$$

in probability under $\mathbb{Q}$ as $\tau \to 0$ according to Lemma 3.1. Since the dynamics of the spot process is completely driven by $k(t, 0)$. It is easy to see that our model reduces to the Geometric Brownian Motion model

$$\frac{dS(t)}{S(t)} = \mu_k(t, 0)dt + \sigma_k(t, 0)dB(t), \tag{3.39}$$

where $t \in \mathbb{R}_{\geq 0}$ and the initial condition is given by $S(0)$, in probability under $\mathbb{Q}$. Since we are modelling risk-neutral forward densities under the risk-neutral probability measure $\mathbb{Q}$, the spot process determined by our model should have the drift of the risk-free rate $r(t)$ for $t \in \mathbb{R}_{\geq 0}$. This tell us that

$$\lim_{\tau \to 0} \mu_k(t, \tau) = r(t) \tag{3.40}$$

for $t \in \mathbb{R}_{\geq 0}$. Note that, in practice, $r(t)$ may be replaced by the difference between the risk-free rate and the dividend rate of the underlying asset. Hence we not only need the existence of the limit of $\mu_k(t, \tau)$ when $\tau \to 0$ but also give the following assumption to it.

**Assumption 3.3.** *The drift $\mu_k(t, \tau)$ of $k(t, \tau)$ is assumed to satisfy*

$$\lim_{\tau \to 0} \mu_k(t, \tau) = r(t). \tag{3.41}$$

Furthermore, reducing to the GBM as $\tau$ goes to 0 indicates another useful connection, which is

$$\sigma_k(t, 0) = \sqrt{v(t)} = \sigma(t, 0, S(t)), \tag{3.42}$$

where $\{v(t) : t \in \mathbb{R}_{\geq 0}\}$ is the instantaneous variance swap rate and $\{\sigma^2(t, \tau, K) : (t, \tau, K) \in \mathbb{R}^3_{\geq 0}\}$ is the local volatility of the underlying asset. We will cover this part in details in Section 4.

It is not enough to have a constraint on the limit of $\mu_k(t, \tau)$ as $\tau$ goes to 0, we further need an assumption of $\mu_f(t, \tau, K)$ such that $\{p(t, T - t, K) : 0 \leq t < T\}$ is a martingale for each $T \in \mathbb{R}_{>0}$ and $K \in \mathbb{R}_{\geq 0}$.

**Assumption 3.4.** *The drift* $\{\mu_f(t,\tau,K) : t \in \mathbb{R}_{\geq 0}\}$ *is assumed to satisfy*

$$\begin{aligned}
&\int_{\mathbb{R}_{\geq 0}} p(t,\tau,L)\mu_f(t,\tau,L)dL - \mu_f(t,\tau,K) \\
=& -(\ln q)'_2(0,\tau,Kk(t,\tau)) \\
&+ K(\ln q)'_3(0,\tau,Kk(t,\tau))\sigma_k^2(t,\tau)k(t,\tau) \\
&- K(\ln q)'_3(0,\tau,Kk(t,\tau))\mu_k(t,\tau)k(t,\tau) \\
&- K(\ln q)'_3(0,\tau,Kk(t,\tau))k'_2(t,\tau) \\
&+ \frac{1}{2}K^2\big((\ln q)''_{33} + ((\ln q)'_3)^2\big)(0,\tau,Kk(t,\tau))\sigma_k^2(t,\tau)k^2(t,\tau) \\
&- (\ln f)'_2(t,\tau,K) \\
&+ \int_{\mathbb{R}_{\geq 0}} (\ln q)'_2(0,\tau,Lk(t,\tau))p(t,\tau,L)dL \\
&- \sigma_k^2(s,\tau)k(t,\tau)\int_{\mathbb{R}_{\geq 0}} L(\ln q)'_3(0,\tau,Lk(t,\tau))p(t,\tau,L)dL \\
&+ \mu_k(t,\tau)k(t,\tau)\int_{\mathbb{R}_{\geq 0}} L(\ln q)'_3(0,\tau,Lk(t,\tau))p(t,\tau,L)dL \\
&+ k'_2(t,\tau)\int_{\mathbb{R}_{\geq 0}} L(\ln q)'_3(0,\tau,Lk(t,\tau))p(t,\tau,L)dL \\
&- \frac{1}{2}\sigma_k^2(t,\tau)k^2(t,\tau)\int_{\mathbb{R}_{\geq 0}} L^2\big((\ln q)''_{33} + ((\ln q)'_3)^2\big)(0,\tau,Lk(t,\tau))p(t,\tau,L)dL \\
&+ \int_{\mathbb{R}_{\geq 0}} p(t,\tau,L)(\ln f)'_2(t,\tau,L)dL \\
&+ \sigma_k^2(t,\tau)k^2(t,\tau)\Big(\int_{\mathbb{R}_{\geq 0}} L(\ln q)'_3(0,\tau,Lk(t,\tau))p(t,\tau,L)dL\Big)^2 \\
&+ \int_{\mathbb{R}^2_{\geq 0}} p(t,\tau,L)p(t,\tau,J)\sigma_f(t,\tau,L)\sigma_f(t,\tau,J)c_W(t,L,J)dLdJ \\
&- K(\ln q)'_3(0,\tau,Kk(t,\tau))\sigma_k^2(t,\tau)k^2(t,\tau)\int_{\mathbb{R}_{\geq 0}} L(\ln q)'_3(0,\tau,Lk(t,\tau))p(t,\tau,L)dL \\
&- \sigma_f(t,\tau,K)\int_{\mathbb{R}_{\geq 0}} p(t,\tau,L)\sigma_f(t,\tau,L)c_W(t,L,K)dL
\end{aligned} \tag{3.43}$$

*for each* $(\tau,K) \in \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$.

Though Assumption 3.4 looks complicated, it has an infinite number of solutions and we indeed find some. How to solve it will be discussed in Section 6.1. Now, with this assumption, we proceed to prove that $\{p(t,T-t,K) : t \in [0,T)\}$ is a martingale when $T \in \mathbb{R}_{>0}$ and $K \in \mathbb{R}_{\geq 0}$ are fixed. To find out the dynamics of $p(t,T-t,K)$, we need to know the dynamics of $k(t,T-t)$ and $f(t,T-t,K)$. Note that the dynamics of $k(t,\tau)$ and $f(t,\tau,K)$ with $\tau$ fixed are different from the dynamics of $k(t,T-t)$

and $f(t, T-t, K)$ with $T$ fixed. Indeed, Equation (3.37) indicates that

$$k(t,T-t) = \exp\Big(\int_0^t \big(-\mu_k(s,T-t) + \frac{1}{2}\sigma_k^2(s,T-t)\big)ds - \int_0^t \sigma_k(s,T-t)dB(s)\Big), \tag{3.44}$$

where $t \in [0,T)$ and $T \in \mathbb{R}_{>0}$ is fixed. Thus, with Assumption 3.2, we have

$$\begin{aligned}
&dk(t,T-t)\\
=\ & k(t,T-t)\Big(\sigma_k^2(s,T-t) - \mu_k(t,T-t)\Big)dt\\
&+ k(t,T-t)\int_0^t \mu'_{k2}(s,T-t) - \sigma_k(s,T-t)\sigma'_{k2}(s,T-t)ds\ dt\\
&+ k(t,T-t)\int_0^t \sigma'_{k2}(s,T-t)dB(s)\ dt\\
&- k(t,T-t)\sigma_k(t,T-t)dB(t)\\
=\ & k(t,T-t)\Big(\sigma_k^2(s,T-t) - \mu_k(t,T-t)\Big)dt\\
&- k'_2(t,T-t)dt - k(t,T-t)\sigma_k(t,T-t)dB(t).
\end{aligned} \tag{3.45}$$

Similarly the dynamics of $f(t,T-t,K)$ is

$$\begin{aligned}
&df(t,T-t,K)\\
=\ & f(t,T-t,K)\mu_f(t,T-t,K)dt\\
&+ f(t,T-t,K)\int_0^t \sigma_f(s,T-t,K)\sigma'_{f2}(s,T-t,K)\\
&\quad - \mu'_{f2}(t,T-t,K)ds\ dt\\
&- f(t,T-t,K)\int_0^t \sigma'_{f2}(s,T-t,K)dW(s,K)\ dt\\
&+ f(t,T-t,K)\sigma_f(t,T-t,K)dW(t,K)\\
=\ & \mu_f(t,T-t,K)f(t,T-t,K)dt - f'_2(t,T-t,K)dt\\
&+ \sigma_f(t,T-t,K)f(t,T-t,K)dW(t,K).
\end{aligned} \tag{3.46}$$

With the help of the above dynamics, we show that our model with Assumption 3.2 and Assumption 3.4 satisfies the martingale condition in Definition 2.1. To further simplify the notations, we define

$$\begin{aligned}
&E(t,T-t,K,L)\\
\triangleq\ & \Big(K(\ln q)'_3(0,T-t,Kk(t,T-t)) - L(\ln q)'_3(0,T-t,Lk(t,T-t))\Big)\\
&\sigma_k(t,T-t)k(t,T-t).
\end{aligned} \tag{3.47}$$

Note that $E(t,T-t,K,L) = -E(t,T-t,L,K)$.

**Lemma 3.2.** *With Assumptions 3.2 and 3.4, the stochastic process* $\{p(t, T-t, K) : t \in [0, T)\}$ *is a martingale for all* $T \in \mathbb{R}_{>0}$ *and* $K \in \mathbb{R}_{\geq 0}$ *with the dynamics*

$$\begin{aligned} &\frac{dp(t, T-t, K)}{p(t, T-t, K)} \\ =& -\int_{\mathbb{R}_{\geq 0}} p(t, T-t, L) E(t, T-t, K, L) dL \; dB(t) \\ &+ \int_{\mathbb{R}_{\geq 0}} p(t, T-t, L) D(t, T-t, K, L) dY(t, T-t, K, L) \; dL. \end{aligned} \tag{3.48}$$

**Proof.** Clearly, we know $\tau = T - t$ for each fixed $T \in \mathbb{R}_{>0}$ and $t \in [0, T)$, and insert it into $q(t, \tau, K)$. This gives us

$$q(t, T-t, K) = q(0, T-t, Kk(t, T-t)) f(t, T-t, K). \tag{3.49}$$

Then, considering the independence between $W(t, K)$ and $B(t)$, we find out the dynamics of $q(t, T-t, K)$ as

$$\begin{aligned} & dq(t, T-t, K) \\ =& - q_2'(0, T-t, Kk(t, T-t)) f(t, T-t, K) dt \\ &+ K q_3'(0, T-t, Kk(t, T-t)) dk(t, T-t) f(t, T-t, K) \\ &+ \frac{1}{2} K^2 q_{33}''(0, T-t, Kk(t, T-t)) \\ &\quad dk(t, T-t) dk(t, T-t) f(t, T-t, K) \\ &+ q(0, T-t, Kk(t, T-t)) df(t, T-t, K) \\ =& - q_2'(0, T-t, Kk(t, T-t)) f(t, T-t, K) dt \\ &+ K q_3'(0, T-t, Kk(t, T-t)) \\ &\quad f(t, T-t, K) \sigma_k^2(t, T-t) k(t, T-t) dt \\ &- K q_3'(0, T-t, Kk(t, T-t)) \\ &\quad f(t, T-t, K) \mu_k(t, T-t) k(t, T-t) dt \\ &- K q_3'(0, T-t, Kk(t, T-t)) f(t, T-t, K) k_2'(t, T-t) dt \\ &+ \frac{1}{2} K^2 q_{33}''(0, T-t, Kk(t, T-t)) f(t, T-t, K) \\ &\quad \sigma_k^2(t, T-t) k^2(t, T-t) dt \\ &+ q(t, T-t, K) \mu_f(t, T-t, K) dt \\ &- q(0, T-t, Kk(t, T-t)) f_2'(t, T-t, K) dt \\ &- K q_3'(0, T-t, Kk(t, T-t)) f(t, T-t, K) \\ &\quad \sigma_k(t, T-t) k(t, T-t) dB(t) \\ &+ q(t, T-t, K) \sigma_f(t, T-t, K) dW(t, K). \end{aligned} \tag{3.50}$$

Similarly, the normalizing process has the dynamics

$$
\begin{aligned}
& da(t,T-t) \\
= & -\int_{\mathbb{R}_{\geq 0}} q_2'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dt \\
& + \sigma_k^2(s,T-t)k(t,T-t)\int_{\mathbb{R}_{\geq 0}} Lq_3'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dt \\
& - \mu_k(t,T-t)k(t,T-t)\int_{\mathbb{R}_{\geq 0}} Lq_3'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dt \\
& - k_2'(t,T-t)\int_{\mathbb{R}_{\geq 0}} Lq_3'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dt \\
& + \frac{1}{2}\sigma_k^2(t,T-t)k^2(t,T-t)\int_{\mathbb{R}_{\geq 0}} L^2 q_{33}''(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dt \\
& + \int_{\mathbb{R}_{\geq 0}} q(t,T-t,L)\mu_f(t,T-t,L)dL\ dt \\
& - \int_{\mathbb{R}_{\geq 0}} q(0,T-t,Lk(t,T-t))f_2'(t,T-t,L)dL\ dt \\
& - \sigma_k(t,T-t)k(t,T-t)\int_{\mathbb{R}_{\geq 0}} Lq_3'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dB(t) \\
& + \int_{\mathbb{R}_{\geq 0}} q(t,T-t,L)\sigma_f(t,T-t,L)dW(t,L)\ dL.
\end{aligned}
\tag{3.51}
$$

To facilitate the development of the dynamics of $dp(t,T-t,K)$, we note that

$$
\begin{aligned}
& da(t,T-t)da(t,T-t) \\
= & \sigma_k^2(t,T-t)k^2(t,T-t) \\
& \quad \Big(\int_{\mathbb{R}_{\geq 0}} Lq_3'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\Big)^2\ dt \\
& + \int_{\mathbb{R}^2_{\geq 0}} q(t,T-t,L)q(t,T-t,J) \\
& \quad \sigma_f(t,T-t,L)\sigma_f(t,T-t,J)c_W(t,L,J)dLdJ\ dt
\end{aligned}
\tag{3.52}
$$

and

$$
\begin{aligned}
& dq(t,T-t,K)da(t,T-t) \\
= & Kq_3'(0,T-t,Kk(t,T-t))f(t,T-t,K)\sigma_k^2(t,T-t)k^2(t,T-t) \\
& \quad \int_{\mathbb{R}_{\geq 0}} Lq_3'(0,T-t,Lk(t,T-t))f(t,T-t,L)dL\ dt \\
& + q(t,T-t,K)\sigma_f(t,T-t,K) \\
& \quad \int_{\mathbb{R}^2_{\geq 0}} q(t,T-t,L)\sigma_f(t,T-t,L)c_W(t,L,K)dL\ dt.
\end{aligned}
\tag{3.53}
$$

Finally, the dynamics of $p(t,T-t,K)$ follows

$$\begin{aligned}
&dp(t,T-t,K)\\
=&\frac{dq(t,T-t,K)}{a(t,T-t)}+q(t,T-t,K)\Big(-\frac{da(t,T-t)}{a^2(t,T-t)}+\frac{da(t,T-t)da(t,T-t)}{a^3(t,T-t)}\Big)\\
&+dq(t,T-t,K)\Big(-\frac{da(t,T-t)}{a^2(t,T-t)}+\frac{da(t,T-t)da(t,T-t)}{a^3(t,T-t)}\Big)\\
=&\frac{dq(t,T-t,K)}{a(t,T-t)}+q(t,T-t,K)\Big(-\frac{da(t,T-t)}{a^2(t,T-t)}+\frac{da(t,T-t)da(t,T-t)}{a^3(t,T-t)}\Big)\\
&-\frac{dq(t,T-t,K)da(t,T-t)}{a^2(t,T-t)}.
\end{aligned} \tag{3.54}$$

In details, its dynamics is given by

$$dp(t,T-t,K)=\mathrm{Drift}_p(t,T-t,K)+\mathrm{Diffusion}_p(t,T-t,K), \tag{3.55}$$

where the diffusion terms are given by

$$\begin{aligned}
&\mathrm{Diffusion}_p(t,T-t,K)\\
=&-K(\ln q)'_3(0,T-t,Kk(t,T-t))\\
&\quad p(t,T-t,K)\sigma_k(t,T-t)k(t,T-t)dB(t)\\
&+p(t,T-t,K)\sigma_f(t,T-t,K)dW(t,K)\\
&+p(t,T-t,K)\sigma_k(t,T-t)k(t,T-t)\\
&\quad\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dB(t)\\
&-p(t,T-t,K)\int_{\mathbb{R}_{\geq 0}}p(t,T-t,L)\sigma_f(t,T-t,L)dW(t,L)\ dL
\end{aligned} \tag{3.56}$$

and the drift terms are given by

$$
\begin{aligned}
&\mathrm{Drift}_p(t,T-t,K)\\
=&-(\ln q)'_2(0,T-t,Kk(t,T-t))p(t,T-t,K)dt\\
&+K(\ln q)'_3(0,T-t,Kk(t,T-t))p(t,T-t,K)\sigma_k^2(t,T-t)k(t,T-t)dt\\
&-K(\ln q)'_3(0,T-t,Kk(t,T-t))p(t,T-t,K)\mu_k(t,T-t)k(t,T-t)dt\\
&-K(\ln q)'_3(0,T-t,Kk(t,T-t))p(t,T-t,K)k'_2(t,T-t)dt\\
&+\frac{1}{2}K^2\big((\ln q)''_{33}+((\ln q)'_3)^2\big)(0,T-t,Kk(t,T-t))\\
&\quad p(t,T-t,K)\sigma_k^2(t,T-t)k^2(t,T-t)dt\\
&+p(t,T-t,K)\mu_f(t,T-t,K)dt\\
&-p(t,T-t,K)(\ln f)'_2(t,T-t,K)dt\\
&+p(t,T-t,K)\int_{\mathbb{R}_{\geq 0}}(\ln q)'_2(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dt\\
&-p(t,T-t,K)\sigma_k^2(s,T-t)k(t,T-t)\\
&\quad\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dt\\
&+p(t,T-t,K)\mu_k(t,T-t)k(t,T-t)\\
&\quad\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dt\\
&+p(t,T-t,K)k'_2(t,T-t)\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dt\\
&-\frac{1}{2}p(t,T-t,K)\sigma_k^2(t,T-t)k^2(t,T-t)\\
&\quad\int_{\mathbb{R}_{\geq 0}}L^2\big((\ln q)''_{33}+((\ln q)'_3)^2\big)(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dt\\
&-p(t,T-t,K)\int_{\mathbb{R}_{\geq 0}}p(t,T-t,L)\mu_f(t,T-t,L)dL\ dt\\
&+p(t,T-t,K)\int_{\mathbb{R}_{\geq 0}}p(t,T-t,L)(\ln f)'_2(t,T-t,L)dL\ dt\\
&+p(t,T-t,K)\sigma_k^2(t,T-t)k^2(t,T-t)\\
&\quad\Big(\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\Big)^2\ dt\\
&+p(t,T-t,K)\int_{\mathbb{R}^2_{\geq 0}}p(t,T-t,L)p(t,T-t,J)\\
&\quad\sigma_f(t,T-t,L)\sigma_f(t,T-t,J)c_W(t,L,J)dLdJ\ dt\\
&-K(\ln q)'_3(0,T-t,Kk(t,T-t))p(t,T-t,K)\sigma_k^2(t,T-t)k^2(t,T-t)\\
&\quad\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dt\\
&-p(t,T-t,K)\sigma_f(t,T-t,K)\\
&\quad\int_{\mathbb{R}_{\geq 0}}p(t,T-t,L)\sigma_f(t,T-t,L)c_W(t,L,K)dL\ dt.
\end{aligned}
\tag{3.57}
$$

Therefore, with Assumption 3.2, the above dynamics reduces to

$$\begin{aligned}
& dp(t,T-t,K) \\
= & -K(\ln q)_3'(0,T-t,Kk(t,T-t))p(t,T-t,K) \\
& \quad \sigma_k(t,T-t)k(t,T-t)dB(t) \\
& + p(t,T-t,K)\sigma_f(t,T-t,K)dW(t,K) \\
& + p(t,T-t,K)\sigma_k(t,T-t)k(t,T-t) \\
& \quad \int_{\mathbb{R}_{\geq 0}} L(\ln q)_3'(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\ dB(t) \\
& - p(t,T-t,K)\int_{\mathbb{R}_{\geq 0}} p(t,T-t,L)\sigma_f(t,T-t,L)dW(t,L)\ dL.
\end{aligned} \tag{3.58}$$

With this, the conclusion of the lemma is clear.

Another thing that is easy to verify is whether or not $p(t,T-t,K)$ always has the integration of 1. This should hold because we have normalized $q(t,T-t,K)$ to obtain $p(t,T-t,K)$. Since we have proved Lemma 3.2, we may check the proof by studying the dynamics of $\int_{\mathbb{R}_{\geq 0}} p(t,T-t,K)dK$. Considering Equation (3.58), it follows

$$d\int_{\mathbb{R}_{\geq 0}} p(t,T-t,K)dK = 0 \tag{3.59}$$

for $T \in \mathbb{R}_{>0}$ and $t \in [0,T)]$. It tells us that

$$\begin{aligned}
\int_{\mathbb{R}_{\geq 0}} p(t,T-t,K)dK &= \int_{\mathbb{R}_{\geq 0}} p(0,T,K)dK \\
&= \int_{\mathbb{R}_{\geq 0}} q(0,T,K)dK \\
&= 1,
\end{aligned} \tag{3.60}$$

which is what we expect.

## 4. From Risk-Neutral Forward Density to Local Volatility

According to Definition 2.2, the martingale property of the discounted price of the underlying $S(t)$ is natural in that we are modeling under $\mathbb{Q}$. By martingale representation theorem and the definition of $\{\mathscr{F}_t : t \in \mathbb{R}_{\geq 0}\}$ in Section 2, there exists at least one Brownian motion $\{Z(t) : t \in \mathbb{R}_{\geq 0}\}$ adapted to $\{\mathscr{F}_t\}$ such that

$$\frac{dS(t)}{S(t)} = r(t)dt + \sqrt{v(t)}dZ(t), \tag{4.1}$$

where $v(t)$ is a $\mathscr{F}_t$-adapted process and is usually termed as instantaneous variance swap rate. Indeed, such a Brownian motion does exist and is not unique. As discussed in Section 3.2, we may let $\{B(t) : t \in \mathbb{R}_{\geq 0}\}$ and $\{Z(t) : t \in \mathbb{R}_{\geq 0}\}$ be the same Brownian motion. To gain a closer look at $v(t)$, we first consider variance swap rates. The variance swap rate $V(t,T-t)$ observed at $t$ for the maturity $T$ is defined in Broadie & Jain (2008) as

$$V(t,T-t) \triangleq \frac{1}{T-t}\mathbb{E}^{\mathbb{Q}}\left[[\ln S](T) - [\ln S](t) \mid \mathscr{F}_t\right]. \tag{4.2}$$

Then the instantaneous variance rate $v(t)$ at $t$ is connected to the variance swap rate $V(t,T-t)$ at $t$ via

$$v(t) = \lim_{T\to t} V(t,T-t). \tag{4.3}$$

Following the classic work in Derman & Kani (1994), for each future time $T \geq t$ and each market level $K \in \mathbb{R}_{\geq 0}$, the local volatility $\sigma(t,T-t,K)$ at $t \in \mathbb{R}_{\geq 0}$ is defined as

$$\sigma^2(t,T-t,K) \triangleq \mathbb{E}^{\mathbb{Q}}\left[v(T) \mid \mathscr{F}_t, S(T) = K\right]. \tag{4.4}$$

Therefore it is trivial to see

$$v(t) = \sigma^2(t,0,S(t)) \tag{4.5}$$

and

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma(t,0,S(t))dZ(t). \tag{4.6}$$

As demonstrated in Dupire (1994) and Derman & Kani (1994), the relationship between the local volatility $\sigma(t,T-t,K)$ and call options (2.3) of different maturities and strikes is given by the foward partial differential equation

$$C_2'(t,T,K) + r(T)KC_3'(t,T,K) - \frac{1}{2}\sigma^2(t,T-t,K)K^2C_{33}''(t,T,K) = 0 \tag{4.7}$$

for $T \geq t \geq 0$ and $K \in \mathbb{R}_{\geq 0}$ with the initial condition $C(t,0,K) = (S(t)-K)^+$. We, however, are more interested in the Fokker-Planck equation indicated of Equation (4.6), which is

$$\begin{aligned}\frac{\partial}{\partial T}p(t,T-t,K) &+ r(T)\frac{\partial}{\partial K}\Big(Kp(t,T-t,K)\Big)\\ &-\frac{1}{2}\frac{\partial^2}{\partial K^2}\Big(\sigma^2(t,T-t,K)K^2p(t,T-t,K)\Big) = 0,\end{aligned} \tag{4.8}$$

and is used in the procedure of deriving Dupire's formula (Equation (4.7)), where $T \geq t \geq 0$ and $K \in \mathbb{R}_{\geq 0}$ and the initial condition follows $p(t,0,K) = \delta(K - S(t))$. Integrating on both sides of Equation (4.8) twice gives us

$$\sigma^2(t,\tau,K) = 2\frac{\int_K^\infty \int_y^\infty p_2'(t,\tau,x)dxdy}{K^2p(t,\tau,K)} + 2r(t+\tau)\frac{\int_K^\infty yp(t,\tau,y)dy}{K^2p(t,\tau,K)} \tag{4.9}$$

for $t \in \mathbb{R}_{\geq 0}$ and $\tau \in \mathbb{R}_{>0}$. To facilitate the presentation of the following theorem, we introduce the following notations

$$\begin{aligned} & H(t,\tau,K,M) \\ \triangleq\, & p(t,\tau,K)p(t,\tau,M)\int_{\mathbb{R}^2_{\geq 0}} p(t,\tau,L)p(t,\tau,J) \\ & \Big(E(t,\tau,K,L)E(t,\tau,M,J) \\ & + D(t,\tau,K,L)D(t,\tau,M,J)c_Y(t,\tau,K,L,M,J)\Big)dLdJ, \end{aligned} \tag{4.10}$$

where $(t,\tau,K,M) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{\geq 0}$. Given the definition of $H(t,\tau,K,M)$, we further define

$$\begin{aligned} I(t,\tau,K) \triangleq & -\frac{2}{K^2p^2(t,\tau,K)}\int_K^\infty \int_y^\infty \frac{p''_{22}(t,\tau,x)}{p'_2(t,\tau,x)}H(t,\tau,x,K)dxdy \\ & + \frac{\sigma^2(t,\tau,K)}{p^2(t,\tau,K)}H(t,\tau,K,K) \\ & - \frac{2r(T)}{K^2p^2(t,\tau,K)}\int_K^\infty yH(t,\tau,y,K)dy, \end{aligned} \tag{4.11}$$

$$\begin{aligned} & N(t,\tau,K) \\ \triangleq\, & \frac{1}{K^2p(t,\tau,K)}\int_K^\infty \int_y^\infty \frac{p'''_{222}(t,\tau,x)p'_2(t,\tau,x) - (p''_{22}(t,\tau,x))^2}{(p'_2(t,\tau,x))^3}H(t,\tau,x,x)dxdy \\ & + I(t,\tau,K), \end{aligned} \tag{4.12}$$

$$\begin{aligned} & O(t,\tau,K) \\ \triangleq\, & \sigma^2(t,\tau,K)\int_{\mathbb{R}_{\geq 0}} p(t,\tau,L)E(t,\tau,K,L)dL \\ & - \frac{2r(T)}{K^2p(t,\tau,K)}\int_K^\infty \int_{\mathbb{R}_{\geq 0}} yp(t,\tau,y)p(t,\tau,L)E(t,\tau,y,L)dLdy \\ & - \frac{2}{K^2p(t,\tau,K)}\int_K^\infty \int_y^\infty \int_{\mathbb{R}_{\geq 0}} \frac{p''_{22}(t,\tau,x)}{p'_2(t,\tau,x)}p(t,\tau,x)p(t,\tau,L)E(t,\tau,x,L)dLdxdy \end{aligned} \tag{4.13}$$

and

$$R(t,\tau,M,L) \triangleq p(t,\tau,M)p(t,\tau,L)D(t,\tau,M,L). \tag{4.14}$$

**Theorem 4.1.** *Assume Assumption 3.1 and Assumption 3.2, the dynamics of* $\sigma^2(t, T-t, K)$ *is given by*

$$\begin{aligned}
&d\sigma^2(t,T-t,K)\\
&= N(t,T-t,K)dt + O(t,T-t,K)dB(t)\\
&- \frac{\sigma^2(t,T-t,K)}{p(t,T-t,K)}\int_{\mathbb{R}_{\geq 0}} R(t,T-t,K,L)dY(t,T-t,K,L)\ dL\\
&+ \frac{2r(T)}{K^2p(t,T-t,K)}\int_K^\infty\int_{\mathbb{R}_{\geq 0}} yR(t,T-t,y,L)dY(t,T-t,y,L)\ dLdy\\
&+ \frac{2}{K^2p(t,T-t,K)}\int_K^\infty\int_y^\infty\int_{\mathbb{R}_{\geq 0}} \frac{p''_{22}(t,T-t,x)}{p'_2(t,T-t,x)}R(t,T-t,x,L)\\
&\quad dY(t,T-t,x,L)\ dLdxdy,
\end{aligned} \tag{4.15}$$

*where* $t \in [0,T]$*,* $(T,K) \in \mathbb{R}_{>0} \otimes \mathbb{R}_{\geq 0}$*. And the initial condition is* $\sigma^2(0,T,K)$*.*

**Proof.** Applying Itô Lemma to Equation (4.9), we have

$$\begin{aligned}
&d\sigma^2(t,T-t,K)\\
&= \frac{2}{K^2p(t,T-t,K)}\int_K^\infty\int_y^\infty dp'_2(t,T-t,x)dxdy\\
&\quad + \frac{2r(T)}{K^2p(t,T-t,K)}\int_K^\infty ydp(t,T-t,y)dy\\
&\quad - \frac{2}{K^2p^2(t,T-t,K)}\int_K^\infty\int_y^\infty dp'_2(t,T-t,x)dp(t,T-t,K)dxdy\\
&\quad - \frac{2r(T)}{K^2p^2(t,T-t,K)}\int_K^\infty ydp(t,T-t,y)dp(t,T-t,K)dy\\
&\quad - \frac{\sigma^2(t,T-t,K)}{p(t,T-t,K)}dp(t,T-t,K)\\
&\quad + \frac{\sigma^2(t,T-t,K)}{p^2(t,T-t,K)}dp(t,T-t,K)dp(t,T-t,K).
\end{aligned} \tag{4.16}$$

The only thing unknown to us in the above equation is the dynamics of $p'_2(t, T-t, K)$, which follows

$$\begin{aligned}
&dp'_2(t,T-t,K)\\
&= \frac{p''_{22}(t,T-t,K)}{p'_2(t,T-t,K)}dp(t,T-t,K)\\
&\quad + \frac{1}{2}\frac{p'''_{222}(t,T-t,K)p'_2(t,T-t,K) - (p''_{22}(t,T-t,K))^2}{(p'_2(t,T-t,K))^3}\\
&\quad dp(t,T-t,K)dp(t,T-t,K),
\end{aligned} \tag{4.17}$$

as

$$\frac{d}{dp}p'_2(t,T-t,K) = \frac{p''_{22}(t,T-t,K)}{p'_2(t,T-t,K)} \tag{4.18}$$

and

$$\frac{d^2}{dp^2}p_2'(t,T-t,K) = \frac{p_{222}'''(t,T-t,K)p_2'(t,T-t,K) - (p_{22}''(t,T-t,K))^2}{(p_2'(t,T-t,K))^3}. \quad (4.19)$$

Hence we may simplify Equation (4.16) to have

$$\begin{aligned}
&d\sigma^2(t,T-t,K)\\
&= \frac{2}{K^2p(t,T-t,K)}\int_K^\infty\int_y^\infty \frac{p_{22}''(t,T-t,x)}{p_2'(t,T-t,x)}dp(t,T-t,x)dxdy\\
&+ \frac{1}{K^2p(t,T-t,K)}\int_K^\infty\int_y^\infty \frac{p_{222}'''(t,T-t,x)p_2'(t,T-t,x) - (p_{22}''(t,T-t,x))^2}{(p_2'(t,T-t,x))^3}\\
&\quad dp(t,T-t,x)dp(t,T-t,x)dxdy\\
&+ \frac{2r(T)}{K^2p(t,T-t,K)}\int_K^\infty ydp(t,T-t,y)dy\\
&- \frac{2}{K^2p^2(t,T-t,K)}\int_K^\infty\int_y^\infty \frac{p_{22}''(t,T-t,x)}{p_2'(t,T-t,x)}dp(t,T-t,x)dp(t,T-t,K)dxdy\\
&- \frac{2r(T)}{K^2p^2(t,T-t,K)}\int_K^\infty ydp(t,T-t,y)dp(t,T-t,K)dy\\
&- \frac{\sigma^2(t,T-t,K)}{p(t,T-t,K)}dp(t,T-t,K)\\
&+ \frac{\sigma^2(t,T-t,K)}{p^2(t,T-t,K)}dp(t,T-t,K)dp(t,T-t,K).
\end{aligned} \quad (4.20)$$

From Lemma 3.2, we may obtain the interactions

$$\begin{aligned}
&dp(t,T-t,K)dp(t,T-t,K)\\
&= p^2(t,T-t,K)\Big(\int_{\mathbb{R}_{\geq 0}} p(t,T-t,L)E(t,T-t,K,L)dL\Big)^2\ dt\\
&+ p^2(t,T-t,K)\int_{\mathbb{R}^2_{\geq 0}} p(t,T-t,L)p(t,T-t,J)\\
&\quad D(t,T-t,K,L)D(t,T-t,K,J)c_Y(t,T-t,K,L,K,J)dLdJ\ dt.\\
&= H(t,T-t,K,K)\ dt
\end{aligned} \quad (4.21)$$

and

$$dp(t,T-t,K)dp(t,T-t,M) = H(t,T-t,K,M)\ dt, \quad (4.22)$$

where $(K,M) \in \mathbb{R}_{\geq 0}$. Thus we finally obtain the result shown in the theorem.

## 5. Pricing of Volatility Options

Variance swap rates are closely connected with European options. Following the approach in Carr & Madan (1998) and Demeterfi et al. (1999), we have

$$\begin{aligned} V(t,\tau) &= \frac{2}{\tau}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{dS(t)}{S(t)} - \ln(\frac{S(T)}{S(t)})\Big|\mathscr{F}_t\right] \\ &= \frac{2}{\tau}D^{-1}(t,\tau)\Big[\int_0^{F(t,\tau)}\frac{1}{K^2}P(t,\tau,K)dK + \int_{F(t,\tau)}^{+\infty}\frac{1}{K^2}C(t,\tau,K)dK\Big], \end{aligned} \tag{5.1}$$

where $F(t,\tau) \triangleq \mathbb{E}\left[S(t+\tau) \mid \mathscr{F}_t\right]$ represents the forward price of the underlying asset. Since $\{p(t,T-t,K) : t \in [0,T]\}$ is a martingale for any fixed $T \in \mathbb{R}_{>0}$ and $K \in \mathbb{R}_{\geq 0}$, the discounted process of $C(t,T-t,K)$ also trivially satisfies the martingale property because

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[D(0,t)C(t,T-t,K) \mid \mathscr{F}_s\right] \\ &= D(0,T)\int_{\mathbb{R}_{\geq 0}}(x-K)^+\mathbb{E}^{\mathbb{Q}}\left[p(t,T-t,x) \mid \mathscr{F}_s\right]dx \\ &= D(0,T)\int_{\mathbb{R}_{\geq 0}}(x-K)^+p(s,T-s,K)dx \\ &= D(0,s)C(s,T-s,K), \end{aligned} \tag{5.2}$$

where $0 < s \leq t$. Another way to verify this fact uses the classic result from Derman & Kani (1994), Dupire (1994) and Dupire (1997):

$$p(t,T-t,K) = D(t,T-t)^{-1}\frac{\partial^2}{\partial k^2}C(t,T-t,K). \tag{5.3}$$

However, to price volatility options using Equation (5.1), we need to find the dynamics of $F(t,\tau)$, $C(t,\tau,K)$ and $P(t,\tau,K)$. To do so, we have to derive their dynamics from the dynamics of $p(t,\tau,K)$. This procedure could be very complicated and we show the following way instead.

Similar to the definition of variance swap rate (4.2), the forward variance swap rate observed at $t$ for the period from $T_1$ to $T_2$ is defined as

$$\begin{aligned} V(t,T_1-t,T_2-t) &\triangleq \frac{1}{T_2-T_1}\mathbb{E}^{\mathbb{Q}}\left[[\ln S](T_2) - [\ln S](T_1) \mid \mathscr{F}_t\right] \\ &= \frac{(T_2-t)V(t,T_2-t)-(T_1-t)V(t,T_1-t)}{T_2-T_1}. \end{aligned} \tag{5.4}$$

There is a close relationship between the forward variance swap rate and the local volatility defined by Equation (4.4) according to Cheung et al. (2016), which is

$$V(t,T_1-t,T_2-t) = \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\int_0^{\infty}\sigma^2(t,s-t,K)p(t,s-t,K)dKds. \tag{5.5}$$

As a special case, it also indicates

$$V(t,\tau) = \frac{1}{\tau}\int_t^{t+\tau}\int_0^{\infty}\sigma^2(t,s-t,K)p(t,s-t,K)dKds. \tag{5.6}$$

To evaluate volatility options, we need to know the dynamics of $V(t, T-t, \tau+T-t)$ or $V(t,\tau)$ for $T \geq t \geq 0$ and $\tau > 0$. Both of them represent the volatility of the underlying asset from $T$ to $T+\tau$ when $t = T$. Then, with the dyamics developed in Lemma 3.2 and Theorem 4.1, we have the following.

**Theorem 5.1.** *Assume Assumptions 3.1 and 3.2 and the initial conditions* $V(0, T, T+\tau)$ *and* $V(0,\tau)$*,*

*(1) the dynamics of* $V(t, T-t, \tau+T-t)$ *follows*

$$\begin{aligned}
& dV(t, T-t, T+\tau-t) \\
&= \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} \Big(N(t, s-t, K)p(t, s-t, K) - I(t, s-t, K)\Big) dsdK\ dt \\
&+ \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} O(t, s-t, K)p(t, s-t, K) dsdK\ dB(t) \\
&- \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} \int_{\mathbb{R}_{\geq 0}} \sigma^2(t, s-t, K)p(t, s-t, K)p(t, s-t, L) \\
&\quad E(t, s-t, K, L) dLdsdK\ dB(t) \\
&+ \frac{2}{\tau}\int_0^\infty \int_T^{T+\tau} \int_K^\infty \int_{\mathbb{R}_{\geq 0}} \frac{r(s)y}{K^2} R(t, s-t, y, L) \\
&\quad dY(t, s-t, y, L)\ dLdydsdK \\
&+ \frac{2}{\tau}\int_0^\infty \int_T^{T+\tau} \int_K^\infty \int_y^\infty \int_{\mathbb{R}_{\geq 0}} \frac{p''_{22}(t, s-t, x)}{K^2 p'_2(t, s-t, x)} R(t, s-t, x, L) \\
&\quad dY(t, s-t, x, L)\ dLdxdydsdK,
\end{aligned} \tag{5.7}$$

*where* $T \geq t \geq 0$ *and* $\tau \in \mathbb{R}_{>0}$.

*(2) the dynamics of* $V(t,\tau)$ *follows*

$$\begin{aligned}
dV(t,\tau) = dV(t, 0, \tau) &+ \frac{1}{\tau}\int_0^\infty \sigma^2(t, \tau, K)p(t, \tau, K) dK\ dt \\
&- \frac{1}{\tau}\int_0^\infty \sigma^2(t, 0, K)p(t, 0, K) dK\ dt,
\end{aligned} \tag{5.8}$$

*where* $(t,\tau) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{>0}$.

**Proof.**

(1) Since the definition of forward variance swap rate gives us

$$V(t, T-t, T+\tau-t) = \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} \sigma^2(t, s-t, K)p(t, s-t, K) dsdK, \tag{5.9}$$

by Itô Lemma, we know

$$\begin{aligned} & dV(t,T-t,T+\tau-t) \\ &= \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} d\sigma^2(t,s-t,K)p(t,s-t,K)dsdK \\ &+ \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} \sigma^2(t,s-t,K)dp(t,s-t,K)dsdK \\ &+ \frac{1}{\tau}\int_0^\infty \int_T^{T+\tau} d\sigma^2(t,s-t,K)dp(t,s-t,K)dsdK. \end{aligned} \tag{5.10}$$

Thus the dynamics is obtained according to Theorem 4.1.

(2) Equation (5.6) tells us that

$$V(t,\tau) = \frac{1}{\tau}\int_0^\tau \int_0^\infty \sigma^2(t,s,K)p(t,s,K)dKds. \tag{5.11}$$

Using Itô Lemma, we have

$$\begin{aligned} dV(t,\tau) = dV(t,0,\tau) &+ \frac{1}{\tau}\int_0^\infty \sigma^2(t,\tau,K)p(t,\tau,K)dK\ dt \\ &- \frac{1}{\tau}\int_0^\infty \sigma^2(t,0,K)p(t,0,K)dK\ dt. \end{aligned} \tag{5.12}$$

The result in Theorem 5.1 can be verified by inserting Equation (4.20) into Equation (5.10) as well. By doing so, it gives us

$$\begin{aligned} & dV(t,T-t,T+\tau-t) \\ &= \frac{2}{\tau}\int_0^\infty \frac{1}{K^2}\int_T^{T+\tau}\int_K^\infty\int_y^\infty \frac{p''_{22}(t,s-t,x)}{p'_2(t,s-t,x)}dp(t,s-t,x)dxdydsdK \\ &+ \frac{1}{\tau}\int_0^\infty \frac{1}{K^2}\int_T^{T+\tau}\int_K^\infty\int_y^\infty \frac{p'''_{222}(t,s-t,x)p'_2(t,s-t,x)-(p''_{22}(t,s-t,x))^2}{(p'_2(t,s-t,x))^3} \\ & dp(t,s-t,x)dp(t,s-t,x)dxdydsdK \\ &+ \frac{2}{\tau}\int_0^\infty \frac{1}{K^2}\int_T^{T+\tau} r(s)\int_K^\infty ydp(t,s-t,y)dydsdK. \end{aligned} \tag{5.13}$$

The equation builds a connection between the dynamics of forward variance swap rates and the dynamics of the risk-neutral forward density, which simplifies calculation a lot and benefits the simulation of variance swap rates greatly. Furthermore, inserting the dynamics of $p(t,T-t,K)$ from Lemma 3.2 also leads to the result in

Theorem 5.1 using the relationship

$$\begin{aligned}
&O(t,s-t,K)p(t,s-t,K)-\sigma^2(t,s-t,K)p(t,s-t,K)\\
&\int_{\mathbb{R}_{\geq 0}} p(t,s-t,L)E(t,s-t,K,L)dL\\
=&-\frac{2r(s)}{K^2}\int_K^\infty\int_{\mathbb{R}_{\geq 0}} yp(t,s-t,y)p(t,s-t,L)E(t,s-t,y,L)dLdy\\
&-\frac{2}{K^2}\int_K^\infty\int_y^\infty\int_{\mathbb{R}_{\geq 0}}\frac{p''_{22}(t,s-t,x)}{p'_2(t,s-t,x)}\\
&p(t,s-t,x)p(t,s-t,L)E(t,s-t,x,L)dLdxdy.
\end{aligned} \tag{5.14}$$

Then we proceed to the valuation of volatility derivatives. The VIX call and put options can be priced by

$$\begin{aligned}
&VC(t,T-t,T+\tau-t,K_{VIX})\\
\triangleq\ &D(t,T-t)\mathbb{E}^{\mathbb{Q}}\left[\left(V(T,\tau)^{\frac{1}{2}}-K_{VIX}\right)^+|\mathscr{F}_t\right]
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
&VP(t,T-t,T+\tau-t,K_{VIX})\\
\triangleq\ &D(t,T-t)\mathbb{E}^{\mathbb{Q}}\left[\left(K_{VIX}-V(T,\tau)^{\frac{1}{2}}\right)^+|\mathscr{F}_t\right],
\end{aligned} \tag{5.16}$$

where $T\geq t\geq 0$, $\tau\in\mathbb{R}_{>0}$ and $K_{VIX}\in\mathbb{R}_{\geq 0}$. In this way of pricing VIX options, we basically consider the dynamics of the volatility from $t$ to $t+\tau$ as time $t$ evolves. In other words, the window, of which the volatility is considered, moves as time passes by and finally the left boundary of the window, $t$, approaches $T$. We should consider the underlying dynamics in this moving window. Another way to look at VIX options could be

$$\begin{aligned}
&VC(t,T-t,T+\tau-t,K_{VIX})\\
=\ &D(t,T-t)\mathbb{E}^{\mathbb{Q}}\left[\left(V(T,0,\tau)^{\frac{1}{2}}-K_{VIX}\right)^+|\mathscr{F}_t\right]
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
&VP(t,T-t,T+\tau-t,K_{VIX})\\
=\ &D(t,T-t)\mathbb{E}^{\mathbb{Q}}\left[\left(K_{VIX}-V(T,0,\tau)^{\frac{1}{2}}\right)^+|\mathscr{F}_t\right],
\end{aligned} \tag{5.18}$$

which is different from the first way in that the window $[T,T+\tau]$ does not move and we should consider the dynamics of the forward variance swap rate in a fixed window as time $t$ approaches $T$. Similar to the pricing of volatility options, we still have two ways to price volatility futures. The most often used pricing formula of VIX futures is

$$VF(t,T-t,T+\tau-t)\triangleq\mathbb{E}^{\mathbb{Q}}\left[V(T,\tau)^{\frac{1}{2}}\mid\mathscr{F}_t\right], \tag{5.19}$$

where $T \geq t$ and $\tau \in \mathbb{R}_{>0}$. However, since we have a simpler dynamics of the forward variance swap rate shown in Equation (5.13), we prefer to use

$$VF(t, T-t, T+\tau-t) = \mathbb{E}^{\mathbb{Q}}\left[V(T, 0, \tau)^{\frac{1}{2}} \mid \mathscr{F}_t\right]. \tag{5.20}$$

Thus, we are able to obtain $VC(t, T-t, T+\tau-t, K_{VIX})$, $VP(t, T-t, T+\tau-t, K_{VIX})$ and $VF(t, T-t, T+\tau-t)$ numerically by simulating either $V(t, T-t, T+\tau-t)$ or $V(t,\tau)$ and following the Monte Carlo philosophy.

## 6. Simulation

### 6.1. *Algorithm*

To do the simulation for our model, the most important thing is to keep the martingale condition shown in Assumption 3.4. Since we could give further functional assumptions about the shape of $\mu_k(t,\tau,K)$ and obtain $\sigma_k(t,\tau)$ and $\sigma_f(t,\tau,K)$ by calibrating our model to the market prices of volatility options, we assume $\mu_k(t,\tau,K)$, $\sigma_k(t,\tau)$ and $\sigma_f(t,\tau,K)$ are known in this subsection. Therefore Assumption 3.4 becomes

$$\begin{aligned}&\int_{\mathbb{R}_{\geq 0}} p(t, T-t, L)\mu_f(t, T-t, L)dL - \mu_f(t, T-t, K)\\ &= F(t, T-t, K) + G(t, T-t),\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}&F(t, T-t, K)\\ \triangleq &-(\ln q)'_2(0, T-t, Kk(t, T-t))\\ &+K(\ln q)'_3(0, T-t, Kk(t, T-t))\sigma_k^2(t, T-t)k(t, T-t)\\ &-K(\ln q)'_3(0, T-t, Kk(t, T-t))\mu_k(t, T-t)k(t, T-t)\\ &-K(\ln q)'_3(0, T-t, Kk(t, T-t))k'_2(t, T-t)\\ &+\frac{1}{2}K^2\big((\ln q)''_{33} + ((\ln q)'_3)^2\big)(0, T-t, Kk(t, T-t))\\ &\quad\sigma_k^2(t, T-t)k^2(t, T-t)\\ &-(\ln f)'_2(t, T-t, K)\\ &-K(\ln q)'_3(0, T-t, Kk(t, T-t))\sigma_k^2(t, T-t)k^2(t, T-t)\\ &\quad\int_{\mathbb{R}_{\geq 0}} L(\ln q)'_3(0, T-t, Lk(t, T-t))p(t, T-t, L)dL\\ &-\sigma_f(t, T-t, K)\int_{\mathbb{R}_{\geq 0}} p(t, T-t, L)\sigma_f(t, T-t, L)c_W(t, L, K)dL\end{aligned} \tag{6.2}$$

and

$$
\begin{aligned}
&G(t,T-t)\\
\triangleq\ &\int_{\mathbb{R}_{\geq 0}}(\ln q)'_2(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\\
&-\sigma_k^2(s,T-t)k(t,T-t)\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\\
&+\mu_k(t,T-t)k(t,T-t)\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\\
&+k'_2(t,T-t)\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\\
&-\frac{1}{2}\sigma_k^2(t,T-t)k^2(t,T-t)\\
&\quad\int_{\mathbb{R}_{\geq 0}}L^2\big((\ln q)''_{33}+((\ln q)'_3)^2\big)(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\\
&+\int_{\mathbb{R}_{\geq 0}}p(t,T-t,L)(\ln f)'_2(t,T-t,L)dL\\
&+\sigma_k^2(t,T-t)k^2(t,T-t)\Big(\int_{\mathbb{R}_{\geq 0}}L(\ln q)'_3(0,T-t,Lk(t,T-t))p(t,T-t,L)dL\Big)^2\\
&+\int_{\mathbb{R}^2_{\geq 0}}p(t,T-t,L)p(t,T-t,J)\sigma_f(t,T-t,L)\sigma_f(t,T-t,J)c_W(t,L,J)dLdJ.
\end{aligned}
\tag{6.3}
$$

Noting $\int_{\geq 0}F(t,T-t,L)p(t,T-t,L)dL = -G(t,T-t)$, it can be found that $-F(t,T-t,K)$ simply is a solution to Equation (6.1). In addition, Equation (6.1) definitely have numerous solutions, for example $-F(t,T-t,K)+C$, where $C\in\mathbb{R}$. In general, $C$ may affect the overall drifting rate of the surface $f$. According to our experience in simulation, $C$ does not affect the result much if it is not really large. So we simply use $-F(t,T-t,K)$ in the simulation. Applying the solution discussed above, we derive the following simulation algorithm for the discretized version of our model. Fixing a $T\in\mathbb{R}_{>0}$, we assume there are $n+1$ points between 0 and $T$ denoted as

$$0=t_0<t_1<t_2<\cdots<t_{n-1}<t_n=T \tag{6.4}$$

with the distance between adjacent points being a fixed value $\Delta t\triangleq t_{i+1}-t_i$. And fixing a $K_{\max}\in\mathbb{R}_{>0}$, we assume there are $m+1$ points denoted as

$$0=K_0<K_1<K_2<\cdots<K_{m-1}<K_m=K_{\max}. \tag{6.5}$$

Therefore, at $t_0$, we can obtain $\mu_f(t_0,T-t_0,K_i)$ for $i=0,1,2,\ldots,m$ by $-F(t_0,T-t_0,K_i)+C$ because $k(t_0,T-t_0)$, $f(t_0,T-t_0,K_i)$ and $p(t_0,T-t_0,K_i)$ are given by initial conditions and $\mu_k(t_0,T-t_0)$, $\sigma_k(t_0,T-t_0)$ and $\sigma_f(t_0,T-t_0,K_i)$ are known as parameters. After $\mu_f(t_0,T-t_0,K_i)$ is obtained, the drift of the dynamics of

$p(t, T-t, K_i)$ at $t_0$ becomes 0. We do the sampling for $B(t_1)-B(t_0)$ and $W(t_1, K_i)-W(t_0, K_i)$ for $i = 0, 1, \dots, m$. Then, by applying Equation (3.58), we may find out $p(t_1, T-t_1, K_i) - p(t_0, T-t_0, K_i)$ for $i = 0, 1, 2, \dots, m$ and thus $p(t_1, T-t_1, K_i)$ using $k(t_0, T-t_0)$, $f(t_0, T-t_0, K_i)$, $\sigma_k(t_0, T-t_0)$ and $\sigma_f(t_0, T-t_0, K_i)$. Another way is to use Equations (3.1) and (3.8) to directly obtain $p(t_1, T-t_1, K_i)$, which uses $f(t_1, T-t_1, K_i)$ calculated by $\mu_f(t_0, T-t_0, K_i)$. We adopt the latter one in our algorithm. With the input parameters, it is straightforward to calculate $k(t_1, T-t_1)$. With $k(t_1, T-t_1)$, $f(t_1, T-t_1, K_i)$ and $p(t_1, T-t_1, K_i)$, we proceed to do the simulation at time $t_1$. For the rest of time points, repeating the above procedure gives us the wanted result.

With Algorithm 1, we can simulate the processes $\{p(t_i, T-t_i, K_j) : j = 0, 1, 2, \dots, m\}$ for each $i = 1, 2, 3, \dots, n$. To have a view of the entire surface and how it changes over time, we use the discretization of $[0, \Gamma]$:

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{l-1} < \tau_l = \Gamma \tag{6.6}$$

with the distance between adjacent points being $\Delta t$ defined above and we are supposed to construct a sequence of surfaces $\{\{p(t_i, \tau_k, K_j) : j = 0, 1, 2, \dots, m,\ k = 0, 1, 2, \dots, l\} : i = 0, 1, 2, \dots, n\}$. This can be done by using the initial surface and Algorithm 1 to simulate

$$\{p(t_i, t_h - t_i, K_j) : i = 0, 1, 2, \dots, h,\ j = 0, 1, 2, \dots, m\} \tag{6.7}$$

for $h = 0, 1, 2, \dots, n$, and by setting

$$p(t_i, \tau_k, K_j) = \begin{cases} p(t_i, t_{k+i} - t_i, K_j) & \text{if } k + i \le n \\ 0 & \text{otherwise} \end{cases} \tag{6.8}$$

for $i = 0, 1, 2, \dots, n$, $j = 0, 1, 2, \dots, m$ and $k = 0, 1, 2, \dots, l$. In the above equation, we set the density at the points of maturities beyond $T$ to 0.

With the densities at different time points simulated by Algorithm 1, the local volatility at those time points can be simply derived via Equation (4.9). Since it involves the first-order partial derivative $p_2'(t, T-t, K)$, we adopt the centered finite differences defined by

$$p_2'(t_i, T-t_i, K_j) \approx \frac{p(t_{i+1}, T-t_{i+1}, K_j) - p(t_{i-1}, T-t_{i-1}, K_j)}{2\Delta t} \tag{6.9}$$

for the points that are not on the boundaries and the finite differences defined by

$$p_2'(t_0, T-t_0, K_j) \approx \frac{p(t_1, T-t_1, K_j) - p(t_0, T-t_0, K_j)}{\Delta t} \tag{6.10}$$

and

$$p_2'(t_n, T-t_n, K_j) \approx \frac{p(t_n, T-t_n, K_j) - p(t_{n-1}, T-t_{n-1}, K_j)}{\Delta t} \tag{6.11}$$

for the points on the boundaries. Therefore we obtain a sequence of local volatility surfaces $\{\sigma^2(t_i, \tau_k, K_j) : j = 0, 1, 2, \dots, m,\ k = 0, 1, 2, \dots, l\}$ for $i = 0, 1, 2, \dots, n$.

**Algorithm 1:** Algorithm of Simulating the risk-neutral forward density model (3.8)

**input** : $C$, $p(t_0, t_i, K_j)$, $k(t_0, t_i) = 1$, $f(t_0, t_i, K_j) = 1$, $\mu_k(t_i, t_l)$, $\sigma_k(t_i, t_l)$ and $\sigma_f(t_i, t_l, K_j)$ for $i = 0, 1, 2, \dots, n$, $l = 0, 1, 2, \dots, n$ and $j = 0, 1, 2, \dots, m$

**output:** $p(t_i, t_h - t_i, K_j)$ and $\Delta p(t_i, t_h - t_i, K_j)$ for $i = 0, 1, 2, 3, \dots, h$, $h = 0, 1, 2, 3, \dots, n$ and $j = 0, 1, 2, \dots, m$

**for** $h \in \{0, 1, 2, \dots, n\}$ **do**
  **for** $j \in \{0, 1, 2, \dots, m\}$ **do**
    $\mu_f(t_0, t_h - t_0, K_j) \leftarrow -F(t_0, t_h - t_0, K_j) + C$
  **end**
  **for** $i \in \{0, 1, 2, \dots, h-1\}$ **do**
    Sample $\Delta B(t_i) \leftarrow B(t_{i+1}) - B(t_i)$
    $$k(t_{i+1}, t_h - t_{i+1}) \leftarrow k(t_0, t_h - t_{i+1}) \prod_{l=0}^{i} \exp\Big(\big(\frac{1}{2}\sigma_k^2(t_l, t_h - t_{i+1}) - \mu_k(t_l, t_h - t_{i+1})\big)\Delta t - \sigma_k(t_l, t_h - t_{i+1})\Delta B(t_l)\Big)$$
    **for** $j \in \{0, 1, 2, \dots, m\}$ **do**
      Sample $\Delta W(t_i, K_j) \leftarrow W(t_{i+1}, K_j) - W(t_i, K_j)$
      $$f(t_{i+1}, t_h - t_{i+1}, K_j) \leftarrow f(t_0, t_h - t_{i+1}, K_j) \prod_{l=0}^{i} \exp\Big(\big(\mu_f(t_l, t_h - t_{i+1}, K_j) - \frac{1}{2}\sigma_f^2(t_l, t_h - t_{i+1}, K_j)\big)\Delta t + \sigma_f(t_l, t_h - t_{i+1}, K_j)\Delta W(t_l, K_j)\Big)$$
      $p(t_{i+1}, t_h - t_{i+1}, K_j) \leftarrow q(0, t_h - t_{i+1}, K_j k(t_{i+1}, t_h - t_{i+1})) f(t_{i+1}, t_h - t_{i+1}, K_j)$
      $\mu_f(t_{i+1}, t_h - t_{i+1}, K_j) \leftarrow -F(t_{i+1}, t_h - t_{i+1}, K_j) + C$
    **end**
    $$p(t_{i+1}, t_h - t_{i+1}, K_j) \leftarrow \frac{p(t_{i+1}, t_h - t_{i+1}, K_j)}{\sum_{j=0}^{m} p(t_{i+1}, t_h - t_{i+1}, K_j)}$$
  **end**
**end**

Then we proceed to the simulation of forward variance swap rates defined by Equation (5.4) and instantaneous variance swap rates defined by Equation (4.2). As shown in Theorem 5.1, the dynamics of $V(t,\tau)$ can be obtained from the dynamics of $V(t,T-t,T+\tau-t)$ with $T=t$ and the dynamics of the local volatility $\sigma^2(t,0,K)$ and $\sigma^2(t,\tau,K)$. Thus we focus on the simulation of $\{V(t_i,T-t_i,T+\tau-t_i) : i=0,1,2,\ldots,n\}$, where $T$ is the maturity and $\tau$ is the time duration. To do so, we apply Equation (5.13) instead of the result in Theorem 5.1 because Equation (5.13) enables us to directly use the dynamics of the risk-neutral forward density surfaces. Equation (5.13) has second-order and third-order partial derivatives in it. We approximate second-order and third-order derivatives by running the the finite difference method of first-order derivatives for corresponding times.

In simulation of variance swap rates, we follow the discretization schemes of $[0,T]$ in Equation (6.4) for $[0,K_{\max}]$, Equation (6.5) and $[0,\Gamma]$ shown in Equation (6.6), but suppose $\tau=\Gamma$ and $l<n$, which is not set in the simulation of $p(t,\tau,K)$. With $l<n$, we only need the discretized density values and their derivatives with maturities from $t_{n-l}$ to $t_n$ for simulating the dynamics of the forward variance swap rate with the maturity at $t_{n-l}$ and of the time duration $\Gamma$ for the time from $t_0$ to $t_{n-l-1}$. To facilitate the presentation and the simulation, we define the following functions

$$\alpha_i(x,y) \triangleq \frac{p''_{22}(t_i,x-t_i,y)}{p'_2(t_i,x-t_i,y)}\Delta p(t_i,x-t_i,y)), \tag{6.12}$$

$$\begin{aligned}\beta_i(x,y) \triangleq & \frac{p'''_{222}(t_i,x-t_i,y)p'_2(t_i,x-t_i,y)-(p''_{22}(t_i,x-t_i,y))^2}{(p'_2(t_i,x-t_i,y))^3} \\ & dp(t_i,x-t_i,y)dp(t_i,x-t_i,y)\end{aligned} \tag{6.13}$$

and

$$\gamma_i(x,y) \triangleq r(x)ydp(t_i,x-t_i,y) \tag{6.14}$$

for the integrands in Equation (5.13), where $i=0,1,2,\ldots,n-l-1$. Then we define a functional $\mathbf{\Theta}(\cdot) : \mathbb{R}^{\mathbb{R}^2_{\geq 0}} \to \mathbb{R}^{(l+1)\times m}$ by

$$\mathbf{\Theta}(f) = \begin{bmatrix} f(t_{n-l},K_0) & f(t_{n-l},K_1) & \cdots & f(t_{n-l},K_m) \\ f(t_{n-l+1},K_0) & f(t_{n-l+1},K_1) & \cdots & f(t_{n-l+1},K_m) \\ \vdots & \vdots & \ddots & \vdots \\ f(t_n,K_0) & f(t_n,K_1) & \cdots & f(t_n,K_m) \end{bmatrix}_{(l+1)\times m}, \tag{6.15}$$

where $f \in \mathbb{R}^{\mathbb{R}^2_{\geq 0}}$, a function $\mathbf{\Pi}(\cdot) : \mathbb{R}_{>0} \to \mathbb{R}^{m\times m}_{>0}$ by

$$\mathbf{\Pi}(x) = \begin{bmatrix} x & 0 & \cdots & 0 & 0 \\ x & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & \cdots & x & 0 \\ x & x & \cdots & x & x \end{bmatrix}_{m\times m} \tag{6.16}$$

and a function $\mu(\cdot) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}^{(l+1)\times 1}$ by

$$\mu(x) = \begin{bmatrix} x \ x \ \cdots \ x \end{bmatrix}_{1\times(l+1)}^T . \tag{6.17}$$

where $x \in \mathbb{R}_{>0}$. Note that the square of $\Pi(x)$ follows a useful and compact form

$$\mathbf{\Pi}^2(x) = \begin{bmatrix} x^2 & 0 & 0 & \cdots & 0 & 0 \\ 2x^2 & x^2 & 0 & \cdots & 0 & 0 \\ 3x^2 & 2x^2 & x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (m-1)x^2 & (m-2)x^2 & (m-3)x^2 & \cdots & x^2 & 0 \\ mx^2 & (m-1)x^2 & (m-2)x^2 & \cdots & 2x^2 & x^2 \end{bmatrix}_{m\times m} . \tag{6.18}$$

Last but not least, a $m \times 1$ vectors is introduced as

$$\lambda = \begin{bmatrix} \frac{1}{K_0^2} \ \frac{1}{K_1^2} \cdots \frac{1}{K_m^2} \end{bmatrix}_{1\times m}^T \tag{6.19}$$

With the above defined notations, we rewrite Equation (5.13) to a much more user-friendly form

$$\Delta V(t_i, t_{n-l} - t_i, t_n - t_i) \tag{6.20}$$

$$= \frac{1}{\Gamma}\mu(\Delta t)^T \Big(\Big(2\mathbf{\Theta}(\alpha_i) + \mathbf{\Theta}(\beta_i)\Big)\mathbf{\Pi}^2(1) + 2\mathbf{\Theta}(\gamma_i)\mathbf{\Pi}(1)\Big)\lambda \tag{6.21}$$

for $i = 0, 1, 2, \ldots, n-l-1$, with which we can simulate $V(t_{n-l}, 0, t_n - t_{n-l})$ as many times as we need.

### 6.2. *Calibration*

With the algorithm described in the previous section, we are able to simulate variance swap rates as many times as we want. With simulated variance swap rates and by Monte-Carlo method, the price of a volatility derivative can be determined, which is called the model price. By comparing the model price with the market price, we determine the parameters in pre-specified parameter structures of the drift and volatility terms in our model. To facilitate the simulation and simplify the model, we apply the following parameter structures:

$$\mu_k(t, \tau) = \psi_s t + r(\tau), \tag{6.22}$$

$$\sigma_k(t, \tau) = \pi_s \tau + \pi_i, \tag{6.23}$$

$$\sigma_f(t, \tau) = \omega_s \tau, \tag{6.24}$$

and

$$c_W(t, K_1, K_2) = \exp(\rho \mid K_1 - K_2 \mid). \tag{6.25}$$

In these parameter structures, $\sigma_k(t,\tau)$, $\sigma_f(t,\tau)$ and $c_W(t,K_1,K_2)$ are not dependent on $t$. In practice, $r(\tau)$ represents the difference between the risk-free rate and the dividend rate implied by index futures. We apply a Gaussian random field for $\{W(t,K)\}$, and thus the structure for $c_W(t,K_1,K_2)$ is from Kennedy (1997), which gives the necessary conditions of a random field to be Markovian, stationary and Gaussian. We summarize the parameters to be calibrated in Table 1.

| Parameter Notation | Parameter Meaning |
|---|---|
| $\psi_s$ | the slope of the drift term $\mu_k(t,\tau)$ in $k(t,\tau)$ |
| $\pi_s$ | the slope of the volatility term $\sigma_k(t,\tau)$ in $k(t,\tau)$ |
| $\pi_i$ | the intercept of the volatility term $\sigma_k(t,\tau)$ in $k(t,\tau)$ |
| $\omega_s$ | the slope of the volatility term $\sigma_f(t,\tau)$ in $f(t,\tau)$ |
| $\rho$ | the coefficient in the correlation structure $c_W(t,K_1,K_2)$ of the Gaussian random field $\{W(t,K)\}$ |

Table 1: Summary of the Parameters for Calibration

In the simulation and calibration, we apply the data extracted from a Bloomberg terminal on 4 May 2017. The data include prices of 13 S&P 500 index call options (Ticker: SPX) with strikes uniformly distributed from 2250 to 2550 and with maturities being 19 May 2017, 16 Jun 2017, 21 Jul 2017, 18 Aug 2017, 15 Sep 2017 and 15 Dec 2017. These option prices give us the initial risk-neutral forward density surface. In addition, the risk-free rates and futures-implied dividend rates are from 5 May 2017 to 15 Dec 2017. We mainly focus on the VIX call options (Ticker: VIX) that expire on 21 Jun 2017 and 19 Jul 2017. For each maturity, we extract the data for five strikes. We downloaded the data of forward variance swap rates with the duration of 30 days from 4 Jun 2017 to 4 Dec 2017. To facilitate the calibration, middle prices are derived from the ask and bid prices. The above data is discrete. Between data points, we employ linear interpolation.

In the calibration, we apply the philosophy of grid-search to select optimal parameters. Starting with initial values, the model prices of all combinations of parameter values are calculated and the best one is chosen using the middle prices of the ten VIX call options of two maturities according to the mean squared percentage error

$$\text{MSPE} \triangleq \frac{\sum_{i=1}^{n} 10\Big(\frac{\text{price}_i^{\text{model}} - \text{price}_i^{\text{middle}}}{\text{price}_i^{\text{middle}}}\Big)^2}{n}. \tag{6.26}$$

Then we set up a finer and smaller grid centered at this best combination of parameter values and proceed to find the best one in the new grid. After repeating the whole procedure for several times, we stop and achieve a fairly optimal set of parameter values. In calibration, the number of Monte Carlo evaluations used is set

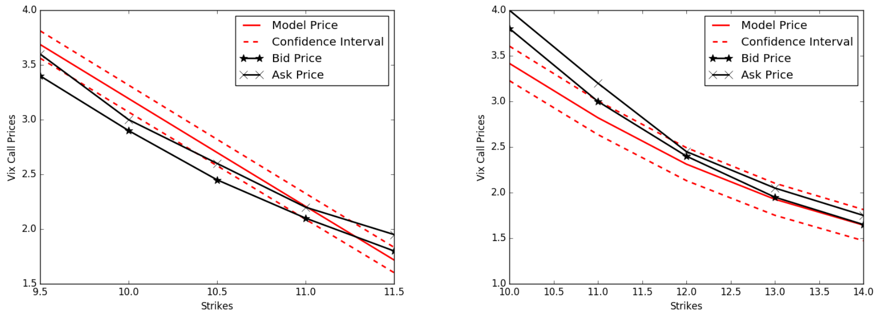


Fig. 2: Model prices versus market prices of VIX call options on 4 May 2017. In the left panel, we compare the model prices and the market prices that matures on 21 Jun 2017. In the right panel, the maturity is on 19 Jul 2017. The confidence interval has the radius of two standard deviations. The model prices and corresponding standard deviations are obtained from 6000 Monte Carlo evaluations.

to 3000.

| Parameter | $\psi_s$ | $\pi_s$ | $\pi_i$ | $\omega_s$ | $\rho$ | MSPE |
|---|---|---|---|---|---|---|
| Value | 0.009750 | 0.041 | 0.000833 | 0.102900 | -0.013407 | 0.003540 |

Table 2: Calibrated Parameter Values

Our model involves the complicated calculation steps shown in Algorithm 1. To accelerate the simulation, we take advantage of the GPU acceleration by CUDA C/C++ and apply cuBLAS to further boost the speed of matrix operations. In the simulation, we apply OpenMP to use multiple CPU threads to operate two graphic cards. In Figure 2, a comparison between the model prices calculated by the calibrated model and the market prices is shown. To gain a clearer view on how the density surface evolves, we show four snapshots in Figure 3. The whole video can be found at https://youtu.be/NtGM_hMiNKs.

## 7. Conclusions

In this paper, we developed a new model to dynamically describe the time evolution of risk-neutral forward densities. Since it took risk-neutral forward densities derived from European options as inputs, our model was naturally calibrated to European options. Also, the model was able to price many various kinds of financial derivatives and we took VIX options as examples. Therefore, we illustrated how to price VIX options by our model and jointly calibrated it to SPX options and VIX options.

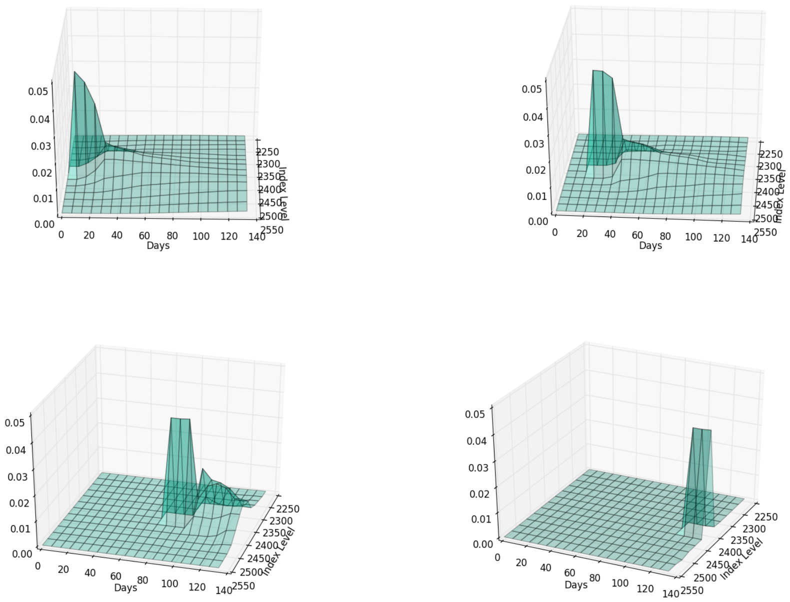


Fig. 3: Time evolution of the risk-neutral forward density using our model. The evolution is simulated using the parameter values shown in Table 2. We show four snapshots of a single simulation above and a video of fifteen times of simulation can be found at https://youtu.be/NtGM_hMiNKs.

After calibration, the model fitted the market prices of VIX options very well, demonstrating its capability to capture market information.

The model has many advantages as shown in Section 1. On the other hand, it has some disadvantages as well. The model is fairly complicated in terms of its structure and therefore leads to complex formulas for simulation and calibration. To mitigate this negative effect, we adopt several assumptions, e.g. the independence assumption between $\{W(t, K) : (t, K) \in \mathbb{R}_{\geq 0} \otimes \mathbb{R}_{\geq 0}\}$ and $\{B(t) : t \in \mathbb{R}_{\geq 0}\}$, to simplify the model. However, it still has a complex martingale condition, though the condition luckily admits friendly and explicit solutions. The complex structure of our model also complicates the model assumptions. Besides the assumptions aimed at simplifying the model, we also make many other assumptions in Section 3.2 to guarantee and satisfy, for example, the existence of partial derivatives, the convergence to Dirac delta functions and conditions of referred theorems. In addition to complicated theoretical analysis, the complicated model structure slows simulations. Though this can be solved by the fast development of computer technologies, it still calls for more efforts to pursue a better formulation of risk-neutral forward density models.

Another challenge is the numerical instability of finite difference methods when approximating partial derivatives. Our model involves partial derivatives in many places. In simulations, finite difference methods significantly lower the stability of our model and have to be taken care of very carefully. In Figure 3 and the video we show at the end of Section 6.2, the surfaces converge to Dirac delta functions too fast and violates empirical observations more or less. This is because the partial derivatives obtained by finite different methods are very large in scales when time to expiration is close to zero.

To sum up, our model makes a good start for modelling the surfaces of risk-neutral forward densities. It puts together some elements, including the HJM framework, Gaussian random fields, and the Musiela parametrization, that are highly useful for developing better models of risk-neutral forward densities. The model achieves satisfactory performance in terms of theoretical properties and capabilities of fitting market prices and has the potential of being improved with better formulations in the future.

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