AN ALGEBRAIC AND ANALYTIC APPROACH TO SPINOR EXCEPTIONAL BEHAVIOR IN TRANSLATED LATTICES

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ABSTRACT. In this announcement we discuss the representation problem for translations of positive-definite lattices via a discussion of representation by inhomogeneous quadratic polynomials. In particular, we give a survey of the extent to which algebraic and analytic methods are useful in determining how the behavior of the spinor genus contributes to failure of the local-global principle.

For a polynomial, f, with rational coefficients we say that f represents an integer, n, if $f(\vec{x}) = n$ has an integral solution. From the negative answer to Hilbert's 10th problem, we know that in general, there is no finite algorithm to determine the set of integers represented by f. However, in certain cases a solution can be determined. In this note, we will consider this so-called representation problem for inhomogeneous quadratic polynomials, that is, polynomials of the form

$$f(\vec{x}) = q(\vec{x}) + \ell(\vec{x}) + c,$$

where q is quadratic, ℓ is linear, and c is constant. Since it will not change the arithmetic of the problem, there is no harm in letting $c = 0$. The goal behind much of the work on this problem has been to find an integral analogue of Hasse's famous local-global principle.

We begin with a brief survey of results in the case when the linear part of f is identically 0. That is, $f(\vec{x})$ is just the homogenous quadratic polynomial $q(\vec{x})$. Then we let L be the quadratic lattice with associated quadratic map q, underlying rational quadratic space V and rank k. When L is indefinite, the representation problem is well understood; for details on this case, we direct the reader to a survey by Hsia [12]. Therefore, for the remainder of the note, we will assume that every quadratic map, q, is positive-definite. It is clear that n is represented locally by L at every prime p (including the infinite prime) when n is represented by L. Stated in the language of the arithmetic theory of quadratic forms, if n is represented by the global lattice L, then n is represented by $\mathrm{gen}(L)$, where $\mathrm{gen}(L)$ is the set of all lattices on V which are locally isometric to L at every prime p. However the converse of this statement does not necessarily hold. In the case $k \geq 4$, Tartakowsky [24] proved an asymptotic local-global principle (with some added primitivity conditions when $k = 4$). Later, in [14], Jöchner and Kitaoka show that for lattices with $k \geq 5$, there is an asymptotic local-global principle for representations which approximate a given (finite) set of local representations. For lattices of rank 4, a similar result is proved in [13] by Hsia and Jöchner, but in this case there are some additional primitivity conditions imposed on the local representations.

In the ternary case, representation by the genus of L is not sufficient to guarantee a global representation, not even if the representations are primitive. The adelic spin group of V acts on $\mathrm{gen}(L)$, and under this action, the genus of L is decomposed into finitely many spinor genera. The spinor genus containing L, denoted $\mathrm{spn}(L)$, plays a vital role in determining local representation behavior for ternary quadratic lattices. It it well known that an integer which is primitively

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represented by the genus of L will be represented primitively by either every spinor genus in the genus of L, or precisely half of the spinor genera. Integers satisfying the latter condition are called primitive spinor exceptions, which can be effectively determined by the work of Earnest and Hsia [7], Earnest, Hsia and Hung [8], and Kneser [16]. A theorem of Duke and Schulze-Pillot [5] says that there exists a constant C^*, depending only on L, such that an integer a will be represented by L if a is primitively represented by $\mathrm{spn}(L)$ and $a > C^*$. Unfortunately, the constant C^* is ineffective (it relies on Siegel's ineffective bound for the class numbers of imaginary quadratic fields [21]), so this does not lead to a full determination of the integers represented in the ternary case. However, it does lead to a determination of whether the local-global principle fails infinitely often [19], and hence leads to a measure of the extent of the failure of a local-global principle. For a comprehensive survey of the results mentioned above, the reader is directed to [20].

Armed with these results, we return to the question at hand, namely the question of representation of integers by inhomogeneous quadratic polynomials. In the positive-definite case, representation by an inhomogeneous quadratic polynomial is the same as representation of another integer (transformed due to completing a square) by a translated lattice, or lattice coset. Consequently, classical results for representation by lattices can be helpful in determining representation by inhomogeneous quadratic polynomials, and similarly, representations by translated lattices. When f is inhomogeneous with 4 or more variables then a solution to the representation problem follows immediately from the local-global representation with approximation from [13] and [14]. This is shown explicitly by Chan and Oh in [3]. Therefore only the ternary case remains of interest. For certain ternary inhomogeneous polynomials, specifically those that arise as sums of squares and triangular numbers, partial solutions are given in [1], [2], and [15]. In particular, these papers determine when such a polynomial represents all but finitely many positive integers; we call such a polynomial almost universal. Similar results in [9] and [10] determine when a polynomial is almost universal given that it satisfies a set of mild arithmetic conditions.

Our present work endeavors to solve the question of almost universality for ternary inhomogeneous f whose corresponding quadratic part has class number one. As a test case for this, we consider representations by $f = P_m$ where

$$P_m(x,y,z) := \frac{(m-2)x^2-(m-4)x}{2} + \frac{(m-2)y^2-(m-4)y}{2} + \frac{(m-2)z^2-(m-4)z}{2},$$

the sum of three generalized (i.e., $x, y, z \in \mathbb{Z}$) m-gonal numbers where m is even. An easy calculation shows that an integer n is represented by P_m if and only if

$$t_n := 3\left(\frac{(m-4)}{2}\right)^2 + 2(m-2)n$$

is represented by the translated lattice $L+\nu$, where $L \cong \langle (m-2)^2, (m-2)^2, (m-2)^2\rangle$ in a basis $\{e_1, e_2, e_3\}$ and

$$\nu := \frac{(m-4)(e_1+e_2+e_3)}{2(m-2)}.$$

Here the underlying lattice L always has class number one, meaning that the genus, spinor genus, and class of L correspond, so any failure of local global principle by $L+\nu$ can't simply be reduced to a failure at the lattice level. As in the case with lattices, it is reasonable to expect that any infinite family of t_n for which the local-global principle fails results as a consequence of exceptional integers which are not represented by the spinor genus of $L+\nu$. However, the spinor theory in this case does not yet appear to be developed enough to conclude this claim. Motivated by the assumption that the local-global principle is governed by the spinor genus of $L+\nu$, we investigate the local-global principle via the analytic theory of modular forms. From an analytic perspective, we can consider representations of t_n by $L+\nu$ by considering the theta function $\Theta_{L+\nu}$ which is a weight $\frac{3}{2}$ modular form; the function $\Theta_{L+\nu}$ is the generating function for elements of $L+\nu$ of a given norm (introduced in [22] and investigated further in [6]). We can decompose the theta function as

$$\Theta_{L+\nu} = \mathcal{E}_{L+\nu} + \mathcal{U}_{L+\nu} + f_{L+\nu} \tag{1}$$

where $\mathcal{E}_{L+\nu}$ is in the space spanned by Eisenstein series, $\mathcal{U}_{L+\nu}$ is in the space spanned by unary theta functions, and $f_{L+\nu}$ is a cusp form orthogonal to unary theta functions. If our motivation holds true, these components of the theta function can be interpreted as contributions to the representation by the genus, spinor genus and class of the translated lattice $L+\nu$.

When $m \equiv 0 \mod 4$, it is always possible to find an entire congruence class of integers which fail to be represented locally by P_m and consequently almost universality is out of the questions. The case when $m \not\equiv 0 \mod 4$ then requires a careful melding of algebraic and analytic techniques; we are able to determine when local representations correspond to representations by the spinor genus and ultimately representations by the global lattice. To explain why a combination of these methods is necessary, we briefly contrast the strengths and weaknesses of the algebraic and analytic techniques. A primary contribution from the analytic side comes from the fact that Duke [4] has shown that the coefficients of $f_{L+\nu}$ grow slowly; this was a necessary ingredient needed by Duke and Schulze-Pillot [5] to overcome deficiencies on the algebraic side in the case of lattices. Using purely analytic techniques, one may be able to show that $\mathcal{U}_{L+\nu}$ vanishes identically for certain L and ν, in which case the slow-growing coefficients of $f_{L+\nu}$ imply that the local-global principle holds for t_n sufficiently large (with some primitivity condition). On the other hand, when $\mathcal{U}_{L+\nu}$ does not vanish identically, the analytic techniques fall short of providing an answer; namely, although the coefficients of $f_{L+\nu}$ are small, it is a historically difficult question to determine when these coefficients vanish. In cases where $\mathcal{U}_{L+\nu}$ does not vanish, we fall back on the motivation from spinor genera and combine with algebraic techniques to determine that certain coefficients of $f_{L+\nu}$ vanish.

Algebraic methods suffice in determining local representations by $L+\nu$. This can be done by considering representation of t_n by the so-called enveloping lattice $M := L+\mathbb{Z}\nu$ and comparing the the spinor genus and genus of M to that of $L+\nu$. In the case when m is even and $m \not\equiv 0 \mod 12$ the algebraic approach is even sufficient to determine that every t_n is represented primitively by $\mathrm{spn}(L+\nu)$ but the lack of any lattice coset analogue of Duke and Schulze-Pillot's result [5, Corollary] prevents further progress in this direction. In essence, the algebraic methods fails to gain any handle on the behavior of the spinor genus and the presence of something analogous to spinor exceptions, although heuristic evidence suggests that the locally represented integers which fail to be globally represented almost always fall inside one of finitely many square classes.

To understand the contribution of the spinor genus to the representation problem we turn to analytic method, specifically we look at the Siegel-Weil mass formula. One formulation of this formula states that

$$\mathcal{E}_{\mathrm{gen}(L)} := \frac{1}{\sum_{M\in \mathrm{gen}(L)} \omega_M^{-1}} \sum_{M\in \mathrm{gen}(L)} \frac{\theta_M}{\omega_M} \tag{2}$$

is an Eisenstein series, where θ_L is the generating function for elements of L with a given norm, and ω_L is the number of automorphs of L. Due to work of Shimura [23], an analogue to equation (2) exists in the cases of translated lattices,

$$\mathcal{E}_{\mathrm{gen}(L+\nu)} := \frac{1}{\sum_{M+\nu'\in \mathrm{gen}(L+\nu')} \omega_{M+\nu'}^{-1}} \sum_{M+\nu'\in \mathrm{gen}(L+\nu)} \frac{\theta_{M+\nu'}}{\omega_{M+\nu'}}. \tag{3}$$

On the right-hand side of equation (3), the ${t_n}^{th}$ term in each summand counts (up to some scaling by the number of automorphs) the number of representations of n by a given lattice in the genus of $L+\nu$, and whether or not a coefficient of this average is positive can be completely determined using algebraic methods. So we already see a connection here between the local algebraic theory and its utility in building up the analytic theory. Summing instead over representations by the spinor genus, we obtain

$$\frac{1}{\sum_{M\in \mathrm{spn}(L)}\omega_M^{-1}}\sum_{M\in \mathrm{spn}(L)}\frac{\theta_M}{\omega_M}=\mathcal{E}_{\mathrm{gen}(L)}+\mathcal{U}_{\mathrm{spn}(L)} \tag{4}$$

where $\mathcal{U}_{\mathrm{spn}(L)}$ is a linear combination of unary theta functions, [17, 18]. The functions on the right-hand side are simply modular forms whose Fourier coefficients can be explicitly computed, but we point out that the role of $\mathcal{U}_{\mathrm{spn}(L)}$ is to count the excesses and deficiencies of representations by the spinor genus, as compared to representations by the weighted average of the genus. On the other hand, the left-hand side of the equation is a weighted average of sums of theta series, all of whose coefficients are positive. Therefore a zero ${t_n}^{th}$ coefficient in the left-hand side can only occur if the ${t_n}^{th}$ coefficient of each θ_M is zero. In this way, equation (4) illuminates the connection between the analytic theory and the algebraic theory of spinor exceptions and hints at how one would hope to make explicit the decomposition in (1).

This leads the authors to make the following conjecture regarding the representation of integers by the spinor genus of a lattice coset.

Conjecture 1. *We have*

$$\theta_{\mathrm{spn}(L+\nu)}:=\frac{1}{\sum_{M+\nu'\in \mathrm{spn}(L+\nu)}\omega_{M+\nu'}^{-1}}\sum_{M+\nu'\in \mathrm{spn}(L+\nu)}\frac{\theta_{M+\nu'}}{\omega_{M+\nu'}}=\mathcal{E}_{\mathrm{gen}(L+\nu)}+\mathcal{U}_{\mathrm{spn}(L+\nu)},$$

where $\mathcal{U}_{\mathrm{spn}(L+\nu)}$ *is a linear combination of unary theta functions.*

A proof of a special cases of this conjecture for an infinite family of lattice cosets appears in the authors' recent publication [11].

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