Optimal proportional reinsurance to minimize the probability of drawdown under thinning-dependence structure

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Abstract

In this paper, we consider the optimal proportional reinsurance problem in a risk model with the thinning-dependence structure, and the criterion is to minimize the probability that the value of the surplus process drops below some fixed proportion of its maximum value to date which is known as the probability of drawdown. The thinning dependence assumes that stochastic sources related to claim occurrence are classified into different groups, and that each group may cause a claim in each insurance class with a certain probability. By the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, the optimal reinsurance strategy and the corresponding minimum probability of drawdown are derived not only for the expected value principle but also for the variance premium principle. Finally, some numerical examples are presented to show the impact of model parameters on the optimal results.

Keywords: Proportional reinsurance; Stochastic optimal control; Probability of drawdown; Thinning-dependence structure

1. Introduction

In recent years, research on insurance risk processes with correlated classes of business has attracted a great deal of attention in the actuarial literature. To depict such a dependence structure among several classes of insurance business, the so-called common-shock risk model are often used. The problem of optimal reinsurance with common shock dependence has been studied in the past few years. Bai et al. (2013) sought the optimal excess of loss reinsurance to minimize the ruin probability for the diffusion approximation risk model. Liang & Yuen (2016) considered the objective of maximizing the expected exponential utility with variance premium principle, and derived the optimal reinsurance strategy not only for the diffusion approximation risk model but also for the compound Poisson risk model. Yuen et al. (2015) extended their work to the case with more than two correlated classes and premiums determined using the expected value principle. Liang et al. (2016, 2017) investigated the optimal reinsurance-investment problems in a financial market with jump-diffusion risky asset, where the jumps in both the risky asset and insurance risk process are correlated through a common shock.

In addition to the common-shock dependence, there exists other risk models with dependence among claim-number processes in the literature. Yuen & Wang (2002) proposed a continuous-time risk model with thinning dependence, in which claims in one class may induce in other classes with certain probabilities. A typical example is that a severe car accident may cause not only the loss of the damaged car but also the medical expenses of injured driver and passengers. Inspired by the work of Yuen & Wang (2002), Wu & Yuen (2003) studied the thinning relation in discrete-time case. Wang & Yuen (2005) extended the thinning-dependence structure into a more general framework, and derived some basic properties of the risk process as well as investigated the impact of thinning dependence on ruin probability. It is worth noting that the common-shock risk model is a special case of the thinning risk model of Wang & Yuen (2005).

The problem of controlling risk exposures to reach a certain goal is another important research topic, and has been studied extensively in the past few decades. For example, see Pestien & Sudderth (1985), Browne (1997), Young (2004), Moore et al. (2006), Wang & Young (2012), Yener (2015), and references therein. In the actuarial context, many authors including Promislow & Young (2005), Bayraktar & Young (2008), Azcue & Muler (2013), Bäuerle & Bayraktar (2014) and Bayraktar & Zhang (2015) adopted the objective of minimizing probability of ruin to carry out various optimality studies. However, in real financial markets, investors would rather prefer maintaining the values of their surplus processes at or above a certain positive level such as a fixed proportion of its maximum value to date. In this regard, researchers are motivated to study the optimization problem of minimizing the so-called probability of drawdown, i.e., the probability that the value of the surplus process drops below some fixed proportion of its maximum value to date. Recently, Angoshtari et al.(2016a) and Han et al. (2017) investigated the minimum drawdown probability problems over an infinite-time horizon, and showed that the optimal strategy which minimizes the probability of ruin also minimizes the probability of drawdown if drawdown does not happen. Besides, Chen et al. (2015) and Angoshtari et al. (2016b) both studied a lifetime investment problem aiming at minimizing the risk of drawdown occurrence. They found that the optimal strategy for a random (or finite) maturity setting such as lifetime drawdown is very different from that of the corresponding ruin problem. Other earlier works related to drawndown can be found in Grossman & Zhou (1993), Cvitanić & Karatzas (1995), and Elie & Touzi (2008).

In this paper, under the criterion of minimizing the probability of drawdown, we investigate the optimal proportional reinsurance problem for the diffusion approximation to the model of Wang & Yuen (2005) with thinning dependence. When the surplus follows the risk process with thinning dependence, the special method in Bäuerle & Bayraktar (2014) does not apply. Therefore, following the analysis of Chen et al.(2015) and Angoshtari et al.(2016a,b), we apply the technique of stochastic control theory to tackle the optimal problem. Furthermore, the problem becomes more challenging if we require the reinsurance proportion to lie in the interval $[0, 1]$. By some nonstandard analytical analysis, we obtain explicit expressions for the optimal reinsurance strategy and the corresponding minimum probability of drawdown for both of the expected value principle and the variance premium principle. We find that the optimal strategies under the two different premium principles depend not only on the safety loading but also on the claim-size distribution and the claim-number process. Under the variance premium principle, we show that when the same safety loading applies to all classes, a simple expression for the optimal strategy can be derived even though the reinsurance premium formula seems more complex than the one with different safety loadings. Interestingly, we work out an optimal retention level which holds for all classes, and falls into the interval $[0, 1]$. Moreover, we can conclude that the optimal strategy for the drawdown problem coincides with the one for the ruin problem if drawdown does not happen.

The rest of the paper is organized as follows. In Section 2, the model and the optimization problem are presented. Under the expected value principle, explicit expressions for the optimal strategies and the corresponding minimum probabilities of drawdown are derived in Sections 3. Optimal results under the variance premium principle are given in Section 4. In Section 5, we present some numerical examples which show the impact of some model parameters on the optimal results. Finally, we conclude the paper in Section 6.

2. Model and problem formulation

In this section, we first introduce the thinning model proposed in Wang & Yuen (2005). Suppose that an insurance company has n dependent classes of insurance business, such as life insurance, motor insurance, and so on. Stochastic sources related to claim occurrences of the n classes are classified into l groups. Assume that each event in the kth ($k = 1, 2, \cdots, l$) group may cause a claim in the jth ($j = 1, 2, \cdots, n$) class with probability p_{kj}, and that there exists at least some k for each j such that $p_{kj} > 0$. With this set-up, we denote by $N^k(t)$ the number of events from the kth group occurred up to time t. Let $N_{kj}(t)$ be the number of claims of the jth class up to time t generated from the events in group k. Then the claim-number process $N_j(t)$ for class j ($j = 1, 2, \ldots, n$) takes the form

$$N_j(t) = N_{1j}(t) + N_{2j}(t) + \cdots + N_{lj}(t).$$

Moreover, it is natural to assume that $N^1(t), N^2(t), \ldots, N^l(t)$ are independent Poisson processes with parameters $\lambda_1, \lambda_2, \ldots, \lambda_l$, respectively, and that $N_{kj}(t)$ is a homogenous Poisson process with intensity $\lambda_k p_{kj}$, i.e., $N_{kj}(t)$ is the $p_{kj}-$thinning of $N^k(t)$. It is further assumed that the two vectors of claim-number processes, $(N^k(t), N_{kj}(t), \ldots, N_{kn}(t))$ and $(N^{k'}(t), N_{k'j}(t), \ldots, N_{k'n}(t))$ are independent for $k \neq k'$, and that $N_{k1}(t), N_{k2}, \ldots, N_{kn}(t)$ are conditionally independent given $N^k(t)$ for each k ($k = 1, 2, \ldots, l$). We label these the partial independence assumptions on the claim-number processes.

Let $Y_j(i)$ be the claim-size random variable for the ith claim in the jth class. Then, the total amount of claims from the jth class up to time t can be expressed as

$$S_j(t) = \sum_{i=1}^{N_j(t)} Y_j(i).$$

Therefore, the aggregate claims process of the company is given by

$$S(t) = \sum_{j=1}^{n} S_j(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} Y_j(i),$$

where $\{Y_j(i); i = 1, 2, \ldots\}$ are a sequence of independent and identically distributed positive random variables having common distribution F_j with mean μ_j and variance σ_j^2 for each j. As usual, we assume that the n sequences $\{Y_1(i); i = 1, 2, \ldots\}, \ldots, \{Y_n(i); i = 1, 2, \ldots\}$ are mutually independent and are independent of all claim-number processes.

Define the surplus process $R(t)$ by

$$R(t) = u + ct - S(t),$$

where u is the initial surplus, and c is the premium rate. Moreover, we allow the insurance company to continuously reinsure a fraction of its claim with the retention level $q_j(\cdot) \in [0, 1]$ for each risk $Y_j(i)$ in class j ($j = 1, 2, \ldots, n$), and the reinsurance premium rate at time t is $\delta(q_t)$ with $q_t = (q_{1t}(\cdot), q_{2t}(\cdot), \ldots, q_{nt}(\cdot)) \in [0, 1]^n$. Furthermore, the insurer is allowed to invest all its surplus in a risk free asset (bond or bank account) with interest rate r. Let $U(t)$ denote the associated surplus process, i.e., $U(t)$ is the surplus of the insurer at time t under the strategy q. This process then evolves as

$$dU(t) = [rU(t) + (c - \delta(q_t))]dt - dS^q(t), \tag{1}$$

with the initial surplus $U(0) = u$, where

$$S^q(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} q_j Y_j(i).$$

It follows from Wang & Yuen (2005) that $S^q(t)$ is statistically equivalent to a compound Poisson process

$$\tilde{S}^q(t) = \sum_{i=1}^{N_t^{\tilde{Y}}} \tilde{Y}_i,$$

where $N_t^{\tilde{Y}}$ is a Possion process with intensity

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_l,$$

and $\{\tilde{Y}_i;\ i = 1, 2, \ldots\}$ are independent with common distribution $F_{\tilde{Y}}$ having moment generating function

$$M_{\tilde{Y}}(r) = \frac{1}{\lambda} \sum_{j=1}^{l} \prod_{k=1}^{n} (M_k(q_k r) p_{jk} + 1 - p_{jk}),$$

with $M_k(r)$ being the moment generating function of distribution F_k for $k = 1, 2, \ldots, n$. Furthermore, following the derivations of Yuen & Wang (2002), one can show that

$$ES^q(t) = E\tilde{S}^q(t) = \sum_{j=1}^{n} \mu_j q_j \sum_{k=1}^{l} \lambda_k p_{kj} t,$$

and

$$VarS^q(t) = Var\tilde{S}^q(t) = \sum_{j=1}^{n} q_j^2(\mu_j^2 + \sigma_j^2) \sum_{k=1}^{l} \lambda_k p_{kj} t + \sum_{j=1}^{l} \sum_{k=1}^{n} \sum_{i \neq k}^{n} \mu_i \mu_k q_i q_k \lambda_j p_{ji} p_{jk} t.$$

Let B_t be a standard Brownian motion. Then it follows from Grandell (1991) that the Brownian motion risk model given by

$$\hat{S}^q(t) = a(q)t - b(q)B_t,$$

with

$$a(q) = \sum_{j=1}^{n} \mu_j q_j \sum_{k=1}^{l} \lambda_k p_{kj},$$

and

$$b^2(q) = \sum_{j=1}^{n} q_j^2(\mu_j^2 + \sigma_j^2) \sum_{k=1}^{l} \lambda_k p_{kj} + \sum_{j=1}^{l} \sum_{k=1}^{n} \sum_{i \neq k}^{n} \mu_i \mu_k q_i q_k \lambda_j p_{ji} p_{jk} \tag{2}$$

can be treated as a diffusion approximation to the compound Poisson process $S^q(t)$. Replacing $S^q(t)$ of (1) by $\hat{S}^q(t)$, one can obtain the following surplus process

$$d\hat{U}(t) = [r\hat{U}(t) + c - \delta(q_t) - a(q_t)]dt + b(q_t)dB_t, \tag{3}$$

with $\hat{U}(0) = u$.

Define the maximum surplus value $M(t)$ at time t by

$$M(t) = \max\left\{ \sup_{0 \le s \le t} \hat{U}(s),\ M(0) \right\},$$

where $M(0) = m > 0$. Note that we allow the surplus process to have a financial past and that m

is no less than the initial surplus u by definition. Here the term 'drawdown' means that the value of the surplus process reaches $\alpha \in [0,1)$ times its maximum value. Define the corresponding hitting time by

$$\tau_\alpha = \inf\{t \geq 0 : \hat{U}(t) \leq \alpha M(t)\}.$$

It is easy to see that we are in the case of minimizing the probability of ruin for the fixed ruin level 0 if $\alpha = 0$. Besides, if the value of the investment fund is no less than

$$u_s = \frac{\delta(0) + a(0) - c}{r}, \tag{4}$$

which is the safe level defined in Angoshtari et al. (2016a), then the insurer can transfer all the risk, and hence the surplus value will never decrease, i.e., drawdown cannot occur in this case.

In the following definition, we give the admissible set of q.

Definition 2.1: *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a complete filtration $\mathcal{F}_t$ which is generated by $\hat{U}(s)$ $(0 \leq s \leq t)$. A strategy $q = (q_1(\cdot), q_2(\cdot), \ldots, q_n(\cdot))$ is said to be admissible if the following conditions are satisfied:*
(a) $q = (q_1(\cdot), q_2(\cdot), \ldots, q_n(\cdot))$ is $(\mathcal{F}_t)_{t\geq 0}$ progressively measurable;
(b) $q_i(\cdot) \in [0,1]$ for $i = 1, 2, \ldots, n$;
(c) Equation (3) for $\hat{U}(t)$ has a unique strong solution.
The set of all admissible strategies is denoted by $\mathcal{D}$.

Denote the minimum probability of drawdown by $\phi(u,m)$ which depends on the initial surplus u and the maximum (past) value m. Specifically, ϕ is the minimum probability of $\tau_\alpha < \infty$. Thus, the objective function can be written as

$$J^q(u,m) = \mathbf{P}^{u,m}(\tau_\alpha < \infty) = \mathbf{E}^{u,m}(\mathbf{1}_{\{\tau_\alpha < \infty\}}),$$

where $\mathbf{P}^{u,m}$ and $\mathbf{E}^{u,m}$ denote the probability and expectation, respectively, conditional on $\hat{U}(0) = u$ and $M(0) = m$; and the corresponding value function is given by

$$\phi(u,m) = \inf_{q \in \mathcal{D}} J^q(u,m).$$

3. Optimal results under expected value principle

In this section, we consider the optimization problem for the risk model (3) under the expected value principle.

Recall the safe level u_s of (4). In the case of $m \geq u_s$, we can see that if $\hat{U}(0) = u \geq u_s$, then drawdown is impossible; and if $\hat{U}(0) = u \leq \alpha m$, then drawdown has occurred and the game is over. Thus, we assume that $\hat{U}(0) = u \in [\alpha m, u_s]$. If $\hat{U}(0) = u \leq u_s$, either $\hat{U}(t) < u_s$ almost surely for all $t \geq 0$ or $\hat{U}(t) = u_s$ for some $t > 0$. Since $m \geq u_s$, $M(t) = m$ holds almost surely for all $t \geq 0$. Therefore, avoiding drawdown is equivalent to avoiding ruin with a (fixed) ruin level of αm. On the other hand, in the case of $m \leq u_s$, $M(t)$ can be larger than m, i.e., the level that we set is not necessarily a fixed one. Based on the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, we obtain the optimal proportional reinsurance strategy and the minimum probability of drawdown for both cases.

Remark 3.1: *Note that when $m \geq u_s$, the level of drawdown is not changing, then the problem is essentially minimizing the probability of ruin. Therefore, in Angoshtari et al. (2016a) and Han et.al (2017), the optimal results in the case of $m \geq u_s$ are derived by maximizing the ratio of the drift of the diffusion to its volatility squared, i.e., the method used in Bäuerle and Bayraktar (2014). However, for n-dimensional control variables (especially when $n \geq 3$), it is more direct to derive the optimal results using the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation.*

As u and m respectively indicate the initial surplus and the maximum (past) value, we only need to consider the function f on the domain $\mathcal{O} := \{(u, m) \in (R^+)^2 : \alpha m \leq u \leq \min(m, u_s)\}$. Let $C^{2,1}$ denote the space of $f(u, m)$ such that f and its partial derivatives f_u, f_{uu}, f_m are continuous on $\mathcal{O}$. It follows from standard arguments that if the value function $\phi(u, m) \in C^{2,1}$, then ϕ satisfies the following Hamilton-Jacobi-Bellman(HJB) equation

$$\inf_{q \in \mathcal{D}} \mathcal{A}^q \phi(u, m) = 0,$$

where

$$\mathcal{A}^q \phi(u, m) = [ru + c - \delta(q) - a(q)]\, \phi_u + \frac{1}{2} b^2(q) \phi_{uu}. \tag{5}$$

Applying the method of Angoshtari et al. (2016a), we obtain the following verification theorem.

Theorem 3.1: *(Verification Theorem) Suppose that* $h : \mathcal{O} \to R$ *is a bounded continuous function satisfying the following conditions:*

(i) $h(\cdot, m) \in \mathcal{C}^2\left(\alpha m, \min(m, u_s)\right)$ *is a non-increasing convex function,*

(ii) $h(u, \cdot)$ *is continuously differentiable, except possibly at* u_s*,*

(iii) $h_m(m, m) \geq 0 \; if \; m \leq u_s$*,*

(iv) $h(\alpha m, m) = 1$*,*

(v) $h(u_s, m) = 0$*,*

(vi) $\mathcal{A}^q h \geq 0$ *for* $q \in \mathcal{D}$*.*

Then $h(u, m) \leq \phi(u, m)$ *on* $\mathcal{O}$*. Furthermore, suppose that the function* h *satisfies all the conditions, and that Conditions (iii) and (vi) hold with equality for some admissible strategy* q^* *defined in feedback form* $(q_1^*(\hat{U}(t)), q_2^*(\hat{U}(t)), \ldots, q_n^*(\hat{U}(t)))$*. Then we have* $h(u, m) = \phi(u, m)$ *on* $\mathcal{O}$*, and* $(q_1^*(u), q_2^*(u), \ldots, q_n^*(u))$ *is the optimal reinsurance strategy.*

We now consider the following boundary-value problems and try to find a solution at which a certain function is minimized according to Theorem 3.1.

$$\begin{cases} (ru + c)h_u + \min_q \left[-(\delta(q) + a(q))h_u + \frac{1}{2} b^2(q) h_{uu} \right] = 0, \\ h(\alpha m, m) = 1, \qquad h(u_s, m) = 0, \end{cases} \tag{6}$$

for $\alpha m \leq u \leq u_s \leq m$; and

$$\begin{cases} (ru + c)h_u + \min_q \left[-(\delta(q) + a(q))h_u + \frac{1}{2} b^2(q) h_{uu} \right] = 0, \\ h(\alpha m, m) = 1, \qquad h(u_s, u_s) = 0, \\ h_m(m, m) = 0, \end{cases} \tag{7}$$

for $\alpha m \leq u \leq m \leq u_s$. Notice that once we derive the ratio of $\frac{h_u}{h_{uu}}$ under the optimal strategy, solutions to the problems can be obtained easily through some calculations. Therefore, we devote ourselves to the study of $\frac{h_u}{h_{uu}}$ in different cases.

For notational convenience, we denote

$$\hat{f}(q) = \left[c - \delta(q) - a(q) \right] h_u + \frac{1}{2} b^2(q) h_{uu}.$$

Before investigating the optimal problems of (6) and (7), we give the following lemma.

Lemma 3.1: *For a given scalar d_1, constant vector $\boldsymbol{e}_1$ and an arbitrary positive definite matrix $\boldsymbol{A}_1$, if $f_1(q)$ is a quadratic function of $q = (q_1(u), q_1(u), \ldots, q_n(u))$ in the form*

$$f_1(q) = d_1 + q\boldsymbol{e}_1 + \frac{1}{2} q \boldsymbol{A}_1 q^T,$$

then the minimizer of $f_1(q)$ is given by

$$q = -(\boldsymbol{A}_1^{-1}\boldsymbol{e}_1)^T,$$

where the superscripts '-1' and 'T' denote the inverse of a matrix and the transpose of a matrix or vector, respectively.

When the reinsurance premium is calculated according to the expected value principle, the insurance premium rate is

$$c = \sum_{j=1}^{n} (1+\theta_j)\mu_j \sum_{k=1}^{l} \lambda_k p_{kj},$$

and the reinsurance premium rate is

$$\delta(q) = \sum_{j=1}^{n} (1+\eta_j)\mu_j(1-q_j) \sum_{k=1}^{l} \lambda_k p_{kj},$$

where θ_j $(j = 1, 2, \ldots, n)$ and η_j $(j = 1, 2, \ldots, n)$ are the insurer's and reinsurer's safety loadings of the n classes of the insurance business, respectively. Without loss of generality, we assume that $\eta_j > \theta_j$ $(j = 1, 2, \ldots, n)$. Then we have

$$c - \delta(q) - a(q) = \sum_{j=1}^{n} (\theta_j - \eta_j)\mu_j \sum_{k=1}^{l} \lambda_k p_{kj} + \sum_{j=1}^{n} \eta_j \mu_j q_j \sum_{k=1}^{l} \lambda_k p_{kj},$$

and thus

$$\hat{f}(q) = \sum_{j=1}^{n} (\theta_j - \eta_j)\mu_j \sum_{k=1}^{l} \lambda_k p_{kj} h_u - q\mathbf{C}h_u + \frac{1}{2} q\mathbf{A}q^T h_{uu},$$

where the matrix

$$\mathbf{A} = \begin{pmatrix} (\mu_1^2+\sigma_1^2)\sum_{k=1}^l \lambda_k p_{k1} & \sum_{j=1}^l \mu_2\mu_1\lambda_j p_{j2}p_{j1} & \cdots & \sum_{j=1}^l \mu_n\mu_1\lambda_j p_{jn}p_{j1} \\ \sum_{j=1}^l \mu_1\mu_2\lambda_j p_{j1}p_{j2} & (\mu_2^2+\sigma_2^2)\sum_{k=1}^l \lambda_k p_{k2} & \cdots & \sum_{j=1}^l \mu_n\mu_2\lambda_j p_{jn}p_{j2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^l \mu_1\mu_n\lambda_j p_{j1}p_{jn} & \sum_{j=1}^l \mu_2\mu_n\lambda_j p_{j2}p_{jn} & \cdots & (\mu_n^2+\sigma_n^2)\sum_{k=1}^l \lambda_k p_{kn} \end{pmatrix}, \tag{8}$$

and the vector

$$\mathbf{C} = \begin{pmatrix} -\eta_1\mu_1\sum_{k=1}^l \lambda_k p_{k1} \\ -\eta_2\mu_2\sum_{k=1}^l \lambda_k p_{k2} \\ \vdots \\ -\eta_n\mu_n\sum_{k=1}^l \lambda_k p_{kn} \end{pmatrix}.$$

Define

$$\begin{cases} a_i = (\mu_i^2+\sigma_i^2)\sum_{k=1}^l \lambda_k p_{ki}, & i = 1, 2, \ldots, n, \\ b_{ik} = b_{ki} = \sum_{j=1}^l \mu_k\mu_i\lambda_j p_{jk}p_{ji}, & i, k = 1, 2, \ldots, n, \ i \neq k, \end{cases} \tag{9}$$

and

$$c_i = -\eta_i\mu_i\sum_{k=1}^l \lambda_k p_{ki}, \qquad i = 1, 2, \ldots, n. \tag{10}$$

Assumption 1: *We assume that* $\boldsymbol{A}$ *defined in* (8) *is positive definite.*

Under Assumption 1, the matrix $\mathbf{A}h_{uu}$ is also positive definite since $h_{uu} > 0$. It follows from Lemma 3.1 that the minimizer $\hat{q} = (\hat{q}_1(u), \hat{q}_1(u), \ldots, \hat{q}_n(u))$ of $\hat{f}(q)$ is given by

$$\hat{q} = (\mathbf{A}^{-1}\mathbf{C})^T \frac{h_u}{h_{uu}}. \tag{11}$$

Note that we cannot make sure whether or not the reinsurance strategy $\hat{q} = (\hat{q}_1(u), \hat{q}_1(u), \ldots, \hat{q}_n(u))$ in (11) belongs to the interval $[0,1]^n$. Before investigating the optimal reinsurance strategy, we present Lemma 3.2 which plays a key role in the following discussion.

Lemma 3.2: *A continuous and strictly convex function* $\psi(x) : R^n \to R$ *is defined on a closed convex set* Ω*. If the stationary point is located in* $R^n \backslash \Omega$*, then the minimum value is on* $\partial\Omega$*, i.e.,* *the boundary of Ω.*

In the following subsections, we restrict our attention to solve the optimization problem for the thinning model with n dependent classes of insurance business.

3.1. ***The case of*** $n = 2$

When $n = 2$, the minimizer of $\hat{f}(\hat{q}_1(u), \hat{q}_2(u))$ has the form

$$\begin{cases} \hat{q}_1(u) = \dfrac{a_2 c_1 - b_{12} c_2}{a_1 a_2 - b_{12}^2} \cdot \dfrac{h_u}{h_{uu}}, \\ \\ \hat{q}_2(u) = \dfrac{a_1 c_2 - b_{12} c_1}{a_1 a_2 - b_{12}^2} \cdot \dfrac{h_u}{h_{uu}}, \end{cases} \tag{12}$$

where $b_{12},\ a_i$ *and* c_i $(i = 1, 2)$ are given by (9) and (10), respectively. After some calculations, the following lemma can be obtained.

Lemma 3.3: *For $n = 2$, the following two statements hold:*

(i) $a_1 a_2 - b_{12}^2 > 0$*;*

(ii) Inequalities $a_2 c_1 - b_{12} c_2 > 0$ *and* $a_1 c_2 - b_{12} c_1 > 0$ *cannot hold true at the same time.*

***Proof*:** It follows from (9) that

$$\begin{aligned} a_1 a_2 - b_{12}^2 &= \left(\mu_1^2 + \sigma_1^2\right) \textstyle\sum_{k=1}^{l} \lambda_k p_{k1} \cdot \left(\mu_2^2 + \sigma_2^2\right) \sum_{k=1}^{l} \lambda_k p_{k2} - \left(\sum_{j=1}^{l} \mu_1 \mu_2 \lambda_j p_{j1} p_{j2}\right)^2 \\ &\geq \left(\mu_1^2 + \sigma_1^2\right)\left(\mu_2^2 + \sigma_2^2\right) \textstyle\sum_{k=1}^{l} \lambda_k p_{k1}^2 \sum_{k=1}^{l} \lambda_k p_{k2}^2 - \left(\sum_{j=1}^{l} \mu_1 \mu_2 \lambda_j p_{j1} p_{j2}\right)^2 \\ &\geq \left(\mu_1^2 + \sigma_1^2\right)\left(\mu_2^2 + \sigma_2^2\right) (\textstyle\sum_{k=1}^{l} \lambda_k p_{k1} p_{k2})^2 - \left(\sum_{j=1}^{l} \mu_1 \mu_2 \lambda_j p_{j1} p_{j2}\right)^2 > 0. \end{aligned}$$

The second inequality above comes from the Hölder inequality. Moreover, if $a_2 c_1 - b_{12} c_2 > 0$, i.e, $c_1 > \frac{b_{12} c_2}{a_2}$, then

$$\begin{aligned} a_1 c_2 - b_{12} c_1 &< a_1 c_2 - b_{12} \cdot \frac{b_{12} c_2}{a_2} \\ &= \frac{c_2}{a_2}(a_1 a_2 - b_{12}^2) < 0, \end{aligned}$$

because $c_i < 0$ $(i = 1, 2)$. Along the same lines, one can show that if $a_1 c_2 - b_{12} c_1 > 0$, then we have $a_2 c_1 < b_{12} c_2$. □

If Condition (i) of Theorem 3.1 holds, then we must have $\frac{h_u}{h_{uu}} < 0$. Therefore, to find the optimal strategies in $\mathcal{D}$, we need to examine the optimal problem in the following three cases.

$$\begin{cases} \mathit{Case}\ 1: & a_2c_1 - b_{12}c_2 \le 0 \ \ \mathit{and} \ \ a_1c_2 - b_{12}c_1 \le 0 \qquad (i.e., \hat{q}_1(u) \ge 0, \hat{q}_2(u) \ge 0), \\ \mathit{Case}\ 2: & a_2c_1 - b_{12}c_2 \ge 0 \ \ \mathit{and} \ \ a_1c_2 - b_{12}c_1 \le 0 \qquad (i.e., \hat{q}_1(u) \le 0, \hat{q}_2(u) \ge 0), \\ \mathit{Case}\ 3: & a_2c_1 - b_{12}c_2 \le 0 \ \ \mathit{and} \ \ a_1c_2 - b_{12}c_1 \ge 0 \qquad (i.e., \hat{q}_1(u) \ge 0, \hat{q}_2(u) \le 0). \end{cases}$$

We first discuss Case 1: $a_2c_1 - b_{12}c_2 \le 0$ and $a_1c_2 - b_{12}c_1 \le 0$. In this case, $\hat{q}_1(u) \ge 0$ and $\hat{q}_2(u) \ge 0$. If $0 \le \hat{q}_1(u) \le 1$ and $0 \le \hat{q}_2(u) \le 1$ hold, then $q_1^*(u) = \hat{q}_1(u)$, $q_2^*(u) = \hat{q}_2(u)$. Inserting $(q_1^*(u), q_2^*(u)) = (\hat{q}_1(u), \hat{q}_2(u))$ into (5) and putting $\mathcal{A}^q h(u,m) = 0$, we obtain

$$\frac{1}{\xi_{11}(u)} = \frac{h_u}{h_{uu}} = \frac{2(ru + \triangle_1)(a_1a_2 - b_{12}^2)^2}{\triangle_2}, \tag{13}$$

where

$$\begin{cases} \triangle_1 = \sum_{j=1}^{2}(\theta_j - \eta_j)\mu_j \sum_{k=1}^{l} \lambda_k p_{kj} < 0, \\ \triangle_2 = a_1a_2^2c_1^2 + a_1^2a_2c_2^2 + 2b_{12}^3c_1c_2 - a_1b_{12}^2c_2^2 - a_2b_{12}^2c_1^2 - 2a_1a_2b_{12}c_1c_2. \end{cases}$$

Lemma 3.4: *In Case 1, inequality $\triangle_2 > 0$ holds.*

***Proof*:** It follows from the form of $\triangle_2$ that

$$\begin{aligned} \triangle_2 &= a_2c_1^2(a_1a_2 - b_{12}^2) + a_1c_2^2(a_1a_2 - b_{12}^2) - 2c_1c_2b_{12}(a_1a_2 - b_{12}^2) \\ &= (a_1a_2 - b_{12}^2)(a_1c_2^2 + a_2c_1^2 - 2c_1c_2b_{12}) \\ &\ge (a_1a_2 - b_{12}^2)(2c_1c_2\sqrt{a_1a_2} - 2c_1c_2b_{12}) \ > 0, \end{aligned}$$

where the last inequality is due to Lemma 3.3. □

Substituting $\frac{h_u}{h_{uu}}$ of (13) back into (12), we obtain

$$\begin{cases} \hat{q}_1(u) = \dfrac{2[ru + \Delta_1](a_1a_2 - b_{12}^2)(a_2c_1 - b_{12}c_2)}{\Delta_2}, \\ \hat{q}_2(u) = \dfrac{2[ru + \Delta_1](a_1a_2 - b_{12}^2)(a_1c_2 - b_{12}c_1)}{\Delta_2}. \end{cases} \tag{14}$$

Let

$$\begin{cases} u_1 = \dfrac{1}{r}\left[\dfrac{\Delta_2}{2(a_1a_2-b_{12}^2)(a_2c_1-b_{12}c_2)} - \Delta_1\right], \\[2ex] u_2 = \dfrac{1}{r}\left[\dfrac{\Delta_2}{2(a_1a_2-b_{12}^2)(a_1c_2-b_{12}c_1)} - \Delta_1\right], \end{cases}$$

it is easy to see that $\hat{q}_1(u_1) = 1$ and $\hat{q}_2(u_2) = 1$.

For simplicity, we assume that $u_1 < u_2$ as similar results can be obtained for $u_1 > u_2$. It follows from Lemma 3.4 that $\hat{q}_1(u)$ and $\hat{q}_2(u)$ are decreasing functions in u. Thus, when $u_2 \le u \le u_s$, we have $0 \le \hat{q}_1(u) < 1$ and $0 \le \hat{q}_2(u) \le 1$, and hence $q_1^*(u) = \hat{q}_1(u)$ and $q_2^*(u) = \hat{q}_2(u)$. On the other hand, when $u < u_2$, we have $\hat{q}_2(u) > 1$. So, we have to choose $q_2^*(u) = 1$, and derive the corresponding

$$\widetilde{q}_1(u) = \frac{c_1}{a_1}\frac{h_u}{h_{uu}} - \frac{b_{12}}{a_1}.$$

Therefore, if $0 \le \widetilde{q}_1(u) \le 1$, we have $q_1^*(u) = \widetilde{q}_1(u)$ and $q_2^*(u) = 1$. Substituting them into (5) and letting $\mathcal{A}^q h(u,m) = 0$ yield

$$\frac{1}{\xi_{12}(u)} = \frac{h_u}{h_{uu}} = -\frac{a_1\left(ru+\Delta_1-c_2+\frac{b_{12}c_1}{a_1}\right) - a_1\sqrt{\left(ru+\Delta_1-c_2+\frac{b_{12}c_1}{a_1}\right)^2 + \frac{c_1^2(a_1a_2-b_{12}^2)}{a_1^2}}}{c_1^2}. \tag{15}$$

Then it is easy to show that

$$\widetilde{q}_1(u) = \frac{\left(ru+\Delta_1-c_2+\frac{b_{12}c_1}{a_1}\right) - \sqrt{\left(ru+\Delta_1-c_2+\frac{b_{12}c_1}{a_1}\right)^2 + \frac{c_1^2(a_1a_2-b_{12}^2)}{a_1^2}}}{c_1} - \frac{b_{12}}{a_1}. \tag{16}$$

Note that $\widetilde{q}_1(u)$ is also a decreasing function in u. Let

$$\widetilde{u}_1 = \frac{1}{r}\left[\frac{a_1c_1 - a_2c_1 + 2b_{12}c_2 + 2a_1c_2}{2(a_1+b_{12})} - \Delta_1\right].$$

Then we have $\widetilde{q}_1(\widetilde{u}_1) = 1$. After some tedious calculations, we show in Appendix B that $\widetilde{u}_1 < u_2$ under the assumption of $u_1 < u_2$. Therefore, we can come to the conclusion that when $\widetilde{u}_1 \le u < u_2$, we have $0 \le \widetilde{q}_1(u) \le 1$, and thus $q_1^*(u) = \widetilde{q}_1(u)$. Finally, when $u < \widetilde{u}_1$, we have to choose $q_1^*(u) = 1$ and $q_2^*(u) = 1$. Inserting $(q_1^*(u), q_2^*(u)) = (1,1)$ into (5) yields

$$\frac{1}{\xi_{13}(u)} = \frac{h_u}{h_{uu}} = -\frac{a_1^2 + a_2^2 + 2b_{12}}{2(ru+\Delta_1-c_1-c_2)}. \tag{17}$$

To summarize, we give the optimal reinsurance strategy and the corresponding minimum probability of drawdown for the case of $m \geq u_s$ with $a_2c_1 - b_{12}c_2 \leq 0$ and $a_1c_2 - b_{12}c_1 \leq 0$ in the following theorem.

Theorem 3.2: *Suppose that $a_2c_1 - b_{12}c_2 \leq 0$ and $a_1c_2 - b_{12}c_1 \leq 0$. Let $\xi_{11}(u)$, $\xi_{12}(u)$ and $\xi_{13}(u)$ be given in* (13), (15) *and* (17), *respectively; $\hat{q}_i$ $(i = 1, 2)$ and $\widetilde{q}_1$ be given in* (14) *and* (16), *respectively; and g_{1i} $(i = 1, 2, 3)$ be given in Appendix A.1. If $u_s \leq m$, then the minimum probability of drawdown for the surplus process* (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - \dfrac{g_{11}(u,m)}{g_{13}(u_s,m)}, & \alpha m \leq u < \max(\alpha m, \widetilde{u}_1), \\ 1 - \dfrac{g_{12}(u,m)}{g_{13}(u_s,m)}, & \max(\alpha m, \widetilde{u}_1) \leq u < \max(\alpha m, u_2), \\ 1 - \dfrac{g_{13}(u,m)}{g_{13}(u_s,m)}, & \max(\alpha m, u_2) \leq u \leq u_s, \end{cases}$$

for any $u \in [\alpha m, u_s]$, and the corresponding optimal reinsurance strategy is

$$(q_1^*, q_2^*) = \begin{cases} (1,1)\,, & \alpha m \leq u < \max(\alpha m, \widetilde{u}_1), \\ (\widetilde{q}_1(u), 1), & \max(\alpha m, \widetilde{u}_1) \leq u < \max(\alpha m, u_2), \\ (\hat{q}_1(u), \hat{q}_2(u)), & \max(\alpha m, u_2) \leq u \leq u_s. \end{cases} \tag{18}$$

***Proof*:** Because h in (6) satisfies the differential equation as well as the boundary conditions, taking the integral of h_u over $[\alpha m, u]$ yields

$$h(u,m) = 1 + d_1 \int_{\alpha m}^{u} \exp\left\{ \int_{\alpha m}^{y} \xi(w) dw \right\} dy.$$

Therefore, when $\max(\alpha m, u_2) \leq u \leq u_s$, we have

$$h(u,m) = 1 + d_1 \cdot g_{13}(u,m),$$

where

$$d_1 = -\frac{1}{g_{13}(u_s,m)}.$$

It follows from the continuity of h that the results for the other two cases, i.e., $\max(\alpha m, \widetilde{u}_1) \leq u <$

$\max(\alpha m, u_2)$ and $\alpha m \le u < \max(\alpha m, \widetilde{u}_1)$, can be obtained along the same lines. By Appendix C, it is straightforward to show that h satisfies Conditions (i), (ii), (iv), (v) and (vi) of Theorem 3.1. Condition (iii) is moot because $m \ge u_s$. Thus, we have $\phi = h$, and (q_1^*, q_2^*) given by (18) is the optimal reinsurance strategy. $\square$

In the next theorem, the optimal results for the case of $m \le u_s$ with $a_2c_1 - b_{12}c_2 \le 0$ and $a_1c_2 - b_{12}c_1 \le 0$ are presented.

Theorem 3.3: *Suppose that $a_2c_1 - b_{12}c_2 \le 0$ and $a_1c_2 - b_{12}c_1 \le 0$. Let $\xi_{11}(u)$, $\xi_{12}(u)$ and $\xi_{13}(u)$ be given in* (13), (15) *and* (17), *respectively; $\hat{q}_i$ $(i = 1, 2)$ and $\widetilde{q}_1$ be given in* (14) *and* (16), *respectively; and g_{1i}, f_{1i} $(i = 1, 2, 3)$ be given in Appendix A.1. For $m \le u_s$,*

(i) if $\max(\alpha m, u_2) \le m \le u_s$, the minimum probability of drawdown for the surplus process (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - k_{13}(m) \cdot \dfrac{g_{11}(u,m)}{g_{13}(u_s,u_s)}, & \alpha m \le u < \max(\alpha m, \widetilde{u}_1), \\ 1 - k_{13}(m) \cdot \dfrac{g_{12}(u,m)}{g_{13}(u_s,u_s)}, & \max(\alpha m, \widetilde{u}_1) \le u < \max(\alpha m, u_2), \\ 1 - k_{13}(m) \cdot \dfrac{g_{13}(u,m)}{g_{13}(u_s,u_s)}, & \max(\alpha m, u_2) \le u \le m \le u_s, \end{cases} \tag{19}$$

for any $u \in [\alpha m, m]$, where

$$k_{13}(m) = \exp\left\{\int_m^{u_s} -f_{13}(y)dy\right\};$$

(ii) if $\max(\alpha m, \widetilde{u}_1) \le m < \max(\alpha m, u_2)$, the minimum probability of drawdown for the surplus process (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - k_{12}(m) \cdot \dfrac{g_{11}(u,m)}{g_{13}(u_s,u_s)}, & \alpha m \le u < \max(\alpha m, \widetilde{u}_1), \\ 1 - k_{12}(m) \cdot \dfrac{g_{12}(u,m)}{g_{13}(u_s,u_s)}, & \max(\alpha m, \widetilde{u}_1) \le u \le m \le u_2, \end{cases} \tag{20}$$

for any $u \in [\alpha m, m]$, where

$$k_{12}(m) = \exp\left\{\left(\int_m^{u_2} -f_{12}(y) - \int_{u_2}^{u_s} f_{13}(y)\right) dy\right\};$$

(iii) if $\alpha m \le m < \max(\alpha m, \widetilde{u}_1)$, the minimum probability of drawdown for the surplus process (3)

is given by

$$\phi(u,m) = 1 - k_{11}(m) \cdot \frac{g_{11}(u,m)}{g_{13}(u_s,u_s)}, \tag{21}$$

for any $u \in [\alpha m, m]$*, where*

$$k_{11}(m) = \exp\left\{ \left(\int_m^{\widetilde{u}_1} -f_{11}(y) - \int_{\widetilde{u}_1}^{u_2} f_{12}(y) - \int_{u_2}^{u_s} f_{13}(y) \right) dy \right\}.$$

Also, the corresponding optimal reinsurance strategy has the form

$$(q_1^*, q_2^*) = \begin{cases} (1,1)\,, & \alpha m \le u \le m < \max(\alpha m, \widetilde{u}_1), \\ \\ (\widetilde{q}_1(u), 1), & \max(\alpha m, \widetilde{u}_1) \le u \le m < \max(\alpha m, u_2), \\ \\ (\hat{q}_1(u), \hat{q}_2(u)), & \max(\alpha m, u_2) \le u \le m \le u_s. \end{cases} \tag{22}$$

***Proof*:** We present the proof for the case of $m \in [\max(\alpha m, u_2), u_s]$ only. The proofs for $m \in [\max(\alpha m, \widetilde{u}_1), \max(\alpha m, u_2))$ and $m \in [\alpha m, \max(\alpha m, \widetilde{u}_1))$ can be derived similarly. When $\max(\alpha m, u_2) \le u \le m \le u_s$, the general solution to (7) has the form

$$h(u,m) = 1 + d_1(m) \cdot g_{13}(u,m).$$

Using the condition of $h_m(m,m) = 0$, one can show that

$$d_1(m) = -\frac{1}{g_{13}(u_s,u_s)} \exp\left\{ \int_m^{u_s} -f_{13}(y)dy \right\}$$

with f_{13} given in Appendix A.1. Along the same lines, we can derive the results for the other two cases shown in (19).

It is not difficult to see that h satisfies Conditions (iv), (v) and (vi) of Theorem 3.1. Besides, in Appendix C, we prove that $h(u,m)$ is a non-increasing convex function in u but an increasing function in m. Then the only item remaining to show is that the expressions given in (19), (20) and (21) as well as their derivatives with respect to u and m are continuous at $u = \widetilde{u}_1$, $u = u_2$, $m = \widetilde{u}_1$, $m = u_2$ and $m = u_s$. The proof is similar to Han et al. (2017), so we omit the details here. Thus, h also satisfies Conditions (i), (ii) and (iii). Therefore, we have $\phi = h$ with the optimal reinsurance strategy (q_1^*, q_2^*) given in (22). □

Remark 3.2: *Note that the relationship between αm and $\widetilde{u}_1$ (u_2) is uncertain. Since we are only interested in $u \in [\alpha m, u_s]$, $\max(\alpha m, \widetilde{u}_1)$ and $\max(\alpha m, u_2)$ are used in the expressions for the optimal results which depend on the values of α and m.*

We now switch our attention to Case 2: $a_2c_1 - b_{12}c_2 \geq 0$ *and* $a_1c_2 - b_{12}c_1 \leq 0$. In this case, $\hat{q}_1(u) \leq 0$ and $\hat{q}_2(u) \geq 0$, and thus we have to choose $q_1^*(u) = 0$ based on which we obtain

$$\bar{q}_2(u) = \frac{c_2}{a_2}\frac{h_u}{h_{uu}} > 0.$$

If $0 \leq \bar{q}_2(u) \leq 1$, we get $q_2^*(u) = \bar{q}_2(u)$, and

$$\frac{1}{\xi_{21}(u)} = \frac{h_u}{h_{uu}} = \frac{2a_2(ru + \Delta_1)}{c_2^2}. \tag{23}$$

Thus, we have

$$\bar{q}_2(u) = -\frac{2(\Delta_1 + ru)}{a_2\eta_2}. \tag{24}$$

Let

$$u_2' = \frac{1}{r}\left(\frac{c_2}{2} - \Delta_1 + c_1\right). \tag{25}$$

It is not difficult to see that $\bar{q}_2(u_2') = 1$. In particular, when $u_2' \leq u \leq u_s$, we have $0 \leq \bar{q}_2(u) \leq 1$. However, when $u \leq u_2'$, we have to choose $q_1^*(u) = 0$ and $q_2^*(u) = 1$. It follows that

$$\frac{1}{\xi_{22}(u)} = \frac{h_u}{h_{uu}} = -\frac{a_2}{2(ru + \Delta_1 - c_2)}. \tag{26}$$

Theorem 3.4: *Suppose that $a_2c_1 - b_{12}c_2 \geq 0$ and $a_1c_2 - b_{12}c_1 \leq 0$. Let $\xi_{21}(u)$ and $\xi_{22}(u)$ be given in* (23) *and* (26)*, respectively; $\bar{q}_2$ and u_2' be given in* (24) *and* (25)*, respectively; and g_{2i}, f_{2i} ($i = 1, 2$) be given in Appendix A.2. If $u_s \leq m$, then for any $u \in [\alpha m, u_s]$, the minimum probability of drawdown for the surplus process* (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - \dfrac{g_{21}(u,m)}{g_{22}(u_s,m)}, & \alpha m \leq u < \max(\alpha m, u_2'), \\[2ex] 1 - \dfrac{g_{22}(u,m)}{g_{22}(u_s,m)}, & \max(\alpha m, u_2') \leq u \leq u_s. \end{cases}$$

For $m \leq u_s$, (i) if $\max(\alpha m, u_2') \leq m \leq u_s$, then for any $u \in [\alpha m, m]$, the minimum probability of *drawdown for the surplus process* (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - k_{22}(m) \cdot \dfrac{g_{21}(u,m)}{g_{22}(u_s,u_s)}, & \alpha m \le u < \max(\alpha m, u_2'), \\ \\ 1 - k_{22}(m) \cdot \dfrac{g_{22}(u,m)}{g_{22}(u_s,u_s)}, & \max(\alpha m, u_2') \le u \le m \le u_s, \end{cases}$$

where

$$k_{22}(m) = \exp\left\{\int_m^{u_s} -f_{22}(y)dy\right\};$$

(ii) if $\alpha m \le m < \max(\alpha m, u_2')$*, then for any* $u \in [\alpha m, m]$*, the minimum probability of drawdown for the surplus process* (3) *is given by*

$$\phi(u,m) = 1 - k_{21}(m) \cdot \frac{g_{21}(u,m)}{g_{22}(u_s,u_s)},$$

where

$$k_{21}(m) = \exp\left\{\left(\int_m^{u_2'} -f_{21}(y) - \int_{u_2'}^{u_s} f_{22}(y)\right) dy\right\}.$$

Also, the corresponding optimal reinsurance strategy has the form

$$(q_1^*, q_2^*) = \begin{cases} (0,1), & \alpha m \le u < \min(\max(\alpha m, u_2'), m), \\ \\ (0, \bar{q}_2(u)), & \max(\alpha m, u_2') \le u \le \min(m, u_s). \end{cases}$$

***Proof*:** Since one can derive the results by using arguments similar to those in the proof of Theorem 3.2 and Theorem 3.3, we omit the details here. □

In Case 3: $a_2c_1 - b_{12}c_2 \le 0$ *and* $a_1c_2 - b_{12}c_1 \ge 0$, we have $\hat{q}_1(u) \ge 0$ and $\hat{q}_2(u) \le 0$. Following the derivations in Case 2, we can get the following result.

Theorem 3.5: *Suppose that* $a_2c_1 - b_{12}c_2 \le 0$ *and* $a_1c_2 - b_{12}c_1 \ge 0$*. Let* $\xi_{31}(u)$*,* $\xi_{32}(u)$*,* u_1' *and* g_{3i} $(i = 1,2)$ *be given in Appendix A.3. If* $u_s \le m$*, then for any* $u \in [\alpha m, u_s]$*, the minimum probability of drawdown for the surplus process* (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - \dfrac{g_{31}(u,m)}{g_{32}(u_s,m)}, & \alpha m \le u < \max(\alpha m, u_1'), \\ \\ 1 - \dfrac{g_{32}(u,m)}{g_{32}(u_s,m)}, & \max(\alpha m, u_1') \le u \le u_s. \end{cases}$$

For $m \le u_s$, (i) if $\max(\alpha m, u_1') \le m \le u_s$, then for any $u \in [\alpha m, m]$, the minimum probability of drawdown for the surplus process (3) *is given by*

$$\phi(u,m) = \begin{cases} 1 - k_{32}(m) \cdot \dfrac{g_{31}(u,m)}{g_{32}(u_s,u_s)}, & \alpha m \le u < \max(\alpha m, u_1'), \\ \\ 1 - k_{32}(m) \cdot \dfrac{g_{32}(u,m)}{g_{32}(u_s,u_s)}, & \max(\alpha m, u_1') \le u \le m \le u_s, \end{cases}$$

where

$$k_{32}(m) = \exp\left\{ \int_m^{u_s} -f_{32}(y)dy \right\};$$

(ii) if $\alpha m \le m < \max(\alpha m, u_1')$, then for any $u \in [\alpha m, m]$, the minimum probability of drawdown for the surplus process (3) *is given by*

$$\phi(u,m) = 1 - k_{31}(m) \cdot \frac{g_{31}(u,m)}{g_{32}(u_s,u_s)},$$

where

$$k_{31}(m) = \exp\left\{ \left(\int_m^{u_1'} -f_{31}(y) - \int_{u_1'}^{u_s} f_{32}(y) \right) dy \right\}.$$

Finally, the corresponding optimal reinsurance strategy has the form

$$(q_1^*, q_2^*) = \begin{cases} (1,0)\,, & \alpha m \le u < \min(\max(\alpha m, u_1'), m), \\ \\ (\bar{q}_1(u), 0), & \max(\alpha m, u_1') \le u \le \min(m, u_s), \end{cases}$$

where

$$\bar{q}_1(u) = -\frac{2(\Delta_1 + ru)}{a_1 \eta_1}.$$

Remark 3.3: *Note that if we set the two reinsurance safety loadings equal, i.e., $\eta_1 = \eta_2$, it is not difficult to show that both $a_2 c_1 - b_{12} c_2 \ge 0$ and $a_1 c_2 - b_{12} c_1 \ge 0$ never hold, i.e., we have only Case 1 left. In particular, when the reinsurance safety loadings of the two classes differ greatly, we can guarantee that c_1 is larger or smaller than both $\frac{b_{12} c_2}{a_2}$ and $\frac{a_1 c_2}{b_{12}}$, and thus Case 2 or Case 3 holds. Example 5.2 also illustrates this property for $n = 3$.*

Remark 3.4: *Note that when $m \ge u_s$, avoiding drawdown is equivalent to avoiding ruin with a (fixed) ruin level of αm. From the expressions for the optimal reinsurance policy in each case, we see that, for a specific level of net surplus u_0, the optimal drawdown policy, in some sense, follows the* *optimal ruin policy until drawdown happens. In fact, as was mentioned in Remark 3.2 of Angoshtari et al. (2016a), we can also conclude that the same reinsurance strategy minimizes the expectation of any function that is non-increasing in the minimum surplus value and non-decreasing in the maximum surplus value, if the differential equation remains the same. The changes only happen in the boundary conditions.*

3.2. The case of $n \geq 3$

In this subsection, we investigate the optimization problem for the thinning model with more than two ($n \geq 3$) dependent classes of insurance business. For $n \geq 3$, the feasible region Ω is $[0,1]^n$. Recall the minimizer of $\hat{q}$ of (11). Two possible scenarios are as follows:

- if for $i = 1, 2, \ldots, n$, $\hat{q}_i(u) \in [0,1]$, then $q_i^*(u) = \hat{q}_i(u)$, and the minimum probability of drawdown is $\phi^{\hat{q}}(u, m)$;
- if for some i, $\hat{q}_i(u) \notin [0,1]$, Lemma 3.2 implies that the minimizer is on $\partial\Omega$. Under the definition of admissible control in $\mathcal{D}$, the feasible region is convex polyhedron. For $q = (q_1(u), q_2(u), \ldots, q_n(u))$, we consider $q_j(u)$ ($j = 1, 2, \ldots, n$) taking a value of 0 or 1, and hence there are $2n$ combinations of optimal problems whose dimensions are $n - 1$. Therefore, the corresponding drawdown probability of the original problem is the one which is the minimum of these $2n$ optimal results. In these $2n$ optimal problems, if some minimizers are out of the feasible region, then repeat the steps above to get the minimizer of the problem.

To illustrate how the optimal results can be obtained in the two scenarios, we take $n = 3$ as an example. To keep things simple, we constrain the reinsurance proportion in the interval $[0, \infty)$. For $q_i(u) \in [0,1]$, the insurer has a proportional reinsurance cover. On the other hand, the case with $q_i(u) \in (1, \infty)$ may be thought of as acquiring new business. Therefore, the feasible region Ω is $[0, \infty)^3$.

In the first scenario with $n = 3$, $q_i^*(u) = \hat{q}_i(u)$ for $i = 1, 2, 3$. Inserting the optimal strategy back into (5) and letting $\mathcal{A}^q h(u, m) = 0$, it can be shown that

$$\frac{h_u}{h_{uu}} = \frac{2(ru + \bar{\triangle}_1)}{\mathbf{C}^T (\mathbf{A}^{-1})^T \mathbf{C}},$$

where

$$\bar{\triangle}_1 = \sum_{j=1}^{n} (\theta_j - \eta_j)\mu_j \sum_{k=1}^{l} \lambda_k p_{kj}.$$

Thus,

$$q^* = \frac{2(ru + \bar{\triangle}_1)(\mathbf{A}^{-1}\mathbf{C})^T}{\mathbf{C}^T(\mathbf{A}^{-1})^T\mathbf{C}},$$

and the corresponding minimum probability of drawdown is $\phi^{\hat{q}}(u,m)$.

In the second scenario with $n = 3$, $\hat{q}_i(u) \notin [0,\infty)$ for some i. Parallel to the analysis for $n = 2$, we consider the boundary of Ω which is formed by the following three faces:

$$\begin{aligned}
\mathcal{D}_1 &= (q_1(u), q_2(u), q_3(u))|q_1(u) = 0, q_2(u) \geq 0, q_3(u) \geq 0), \\
\mathcal{D}_2 &= (q_1(u), q_2(u), q_3(u))|q_1(u) \geq 0, q_2(u) = 0, q_3(u) \geq 0), \\
\mathcal{D}_3 &= (q_1(u), q_2(u), q_3(u))|q_1(u) \geq 0, q_2(u) \geq 0, q_3(u) = 0).
\end{aligned}$$

Therefore, we need to investigate the optimal reinsurance strategy in these three faces. The following steps show how the optimal reinsurance strategy is derived:

S1. Let $q_1(u) = 0$. Differentiating $\mathcal{A}^q\phi(u,m)$ with respect to $q_i(u)$ $(i = 2,3)$ and setting $\frac{\mathcal{A}^q\phi(u,m)}{q_i(u)} = 0$, we have

$$\begin{cases}
\hat{q}_2^{1_0}(u) = \dfrac{a_3c_2 - b_{23}c_3}{a_2a_3 - b_{23}^2} \cdot \dfrac{h_u}{h_{uu}}, \\[2ex]
\hat{q}_3^{1_0}(u) = \dfrac{a_2c_3 - b_{23}c_2}{a_2a_3 - b_{23}^2} \cdot \dfrac{h_u}{h_{uu}}.
\end{cases}$$

If $\hat{q}_i^{1_0}(u) \in [0,\infty)$ $(i = 2,3)$, then the minimizer of function $\phi(u,m)$ in $\mathcal{D}_1$ is $q^{*1_0} = (0, \hat{q}_2^{1_0}(u), \hat{q}_3^{1_0}(u))$. We denote the corresponding probability of drawdown by $\phi_{1_0}(u,m)$. If for some i, $\hat{q}_i^{1_0}(u) \notin [0,\infty)$, we need to find the minimizers on the boundary of $\mathcal{D}_1$, i.e.,

$$\mathcal{D}_{11} = \{(0,0,q_3(u))|q_3(u) \geq 0\}, \qquad \mathcal{D}_{12} = \{(0,q_2(u),0)|q_2(u) \geq 0\}.$$

Let $q_1(u) = q_2(u) = 0$. Differentiating $\mathcal{A}^q\phi(u,m)$ with respect to q_3 yields

$$\hat{q}_3^{1_02_0}(u) = \frac{c_3}{a_3} \cdot \frac{h_u}{h_{uu}} \geq 0.$$

Along the same lines, we have

$$\hat{q}_2^{1_0 3_0}(u) = \frac{c_2}{a_2} \cdot \frac{h_u}{h_{uu}} \geq 0.$$

Under the strategy $(0, 0, \hat{q}_3^{1_0 2_0}(u))$ and $(0, \hat{q}_2^{1_0 3_0}(u), 0)$, we denote the corresponding probability of drawdown by $\phi_{1_0 2_0}(u, m)$ and $\phi_{1_0 3_0}(u, m)$, respectively. It follows that

$$\phi_{1_0}(u, m) = \min\left\{\phi_{1_0 2_0}(u, m), \phi_{1_0 3_0}(u, m)\right\},$$

and

$$q^{*1_0} = \begin{cases} (0, 0, \hat{q}_3^{1_0 2_0}(u)), & if \quad \phi_{1_0 2_0}(u, m) < \phi_{1_0 3_0}(u, m), \\ \\ (0, \hat{q}_2^{1_0 3_0}(u), 0), & if \quad \phi_{1_0 2_0}(u, m) \geq \phi_{1_0 3_0}(u, m). \end{cases}$$

S2. Mimicking the steps in **S1**, one can find the minimizer q^{*i_0} and the corresponding minimum probability of drawdown $\phi_{i_0}(u, m)$ in $\mathcal{D}_i$ $(i = 2, 3)$, respectively.

S3. It follows from the results in **S1** and **S2** that the minimum probability of drawdown in Ω is

$$\phi(u, m) = \min\{\phi_{1_0},\ \phi_{2_0},\ \phi_{3_0}\};$$

and the corresponding optimal reinsurance proportional strategy is

$$q^* = \begin{cases} \hat{q}^{*1_0}, & if \quad \phi(u, m) = \phi_{1_0}(u, m), \\ \\ \hat{q}^{*2_0}, & if \quad \phi(u, m) = \phi_{2_0}(u, m), \\ \\ \hat{q}^{*3_0}, & if \quad \phi(u, m) = \phi_{3_0}(u, m). \end{cases}$$

Remark 3.5: *Suppose that $l = n+1$, $p_{lj} = 1$ for $j = 1, 2, \ldots, n$, $p_{ij} = 0$ $(i \neq j)$ for $i, j = 1, 2, \ldots, n$, and $p_{ii} = 1$ for $i = 1, 2, \ldots, n$. Then the resulting risk model becomes the risk model with common shock studied in Yuen et al.(2015) in which the optimal proportional reinsurance problem under the criterion of maximizing the expected utility of terminal wealth was examined. Under the model of Yuen et al. (2015), Han et.al (2017) investigated the optimal proportional reinsurance problem with the objective of minimizing the probability of drawdown. With $n = 2$, one can verify that the* *optimal results given in Theorems 3.2-3.5 coincide with those in Han et al. (2017) .*

4. Optimal results under variance premium principle

In this section, we discuss the problems given by (6) and (7) under the variance premium principle based on which the insurance premium rate has the form

$$c=\sum_{j=1}^{n}\mu_j\sum_{k=1}^{l}\lambda_k p_{kj}+\sum_{j=1}^{n}\bar{\theta}_j(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj},$$

and the reinsurance premium rate can be expressed as

$$\delta(q)=\sum_{j=1}^{n}\mu_j(1-q_j)\sum_{k=1}^{l}\lambda_k p_{kj}+\sum_{j=1}^{n}\bar{\eta}_j(1-q_j)^2(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj}, \tag{27}$$

where $\bar{\theta}_j$ $(j=1,2,\ldots,n)$ and $\bar{\eta}_j$ $(j=1,2,\ldots,n)$ are the insurer's and reinsurer's safety loadings of the n classes of the insurance business, respectively. Again, we assume that $\bar{\eta}_j>\bar{\theta}_j$ $(j=1,2,\ldots,n)$. Thus, we obtain

$$c-\delta(q)-a(q)=\sum_{j=1}^{n}(\bar{\theta}_j-\bar{\eta}_j)(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj}+\sum_{j=1}^{n}\bar{\eta}_j(2q_j-q_j^2)(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj}.$$

Then it follows that

$$\begin{aligned}\hat{f}(q)&=\textstyle\sum_{j=1}^{n}(\bar{\theta}_j-\bar{\eta}_j)(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj}h_u+2\mathbf{D}qh_u-\frac{1}{2}q\mathbf{B}q^Th_u+\frac{1}{2}q\mathbf{A}q^Th_{uu}\\&=\textstyle\sum_{j=1}^{n}(\bar{\theta}_j-\bar{\eta}_j)(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj}h_u+2\mathbf{D}qh_u+\frac{1}{2}q(\mathbf{A}h_{uu}-\mathbf{B}h_u)q^T,\end{aligned}$$

where the matrix

$$\mathbf{B}=\begin{pmatrix}2\eta_1(\mu_1^2+\sigma_1^2)\sum_{k=1}^{l}\lambda_k p_{k1} & 0 & \cdots & 0\\ 0 & 2\eta_2(\mu_2^2+\sigma_2^2)\sum_{k=1}^{l}\lambda_k p_{k2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 2\eta_n(\mu_n^2+\sigma_n^2)\sum_{k=1}^{l}\lambda_k p_{kn}\end{pmatrix},$$

and the vector

$$\mathbf{D} = \begin{pmatrix} \eta_1(\mu_1^2+\sigma_1^2)\sum_{k=1}^{l}\lambda_k p_{k1} \\ \eta_2(\mu_2^2+\sigma_2^2)\sum_{k=1}^{l}\lambda_k p_{k2} \\ \vdots \\ \eta_n(\mu_n^2+\sigma_n^2)\sum_{k=1}^{l}\lambda_k p_{kn} \end{pmatrix}.$$

Obviously, the matrix $\mathbf{B}$ is positive definite. Under Assumption 1, we know that the matrix $\mathbf{A}$ is positive definite, and hence the matrix $\mathbf{A}h_{uu} - \mathbf{B}h_u$ is also positive definite. Therefore, it follows from Lemma 3.1 that

$$\hat{q} = 2\mathbf{D}^T(\mathbf{B} - \mathbf{A}\frac{h_{uu}}{h_u})^{-1}.$$

Note that once the ratio of $\frac{h_u}{h_{uu}}$ is derived, one can carry out the analysis presented in Section 3 to study the problem. However, even though we can show that $\frac{h_u}{h_{uu}}$ is the solution to the equation

$$2\mathbf{D}^T(\mathbf{B} - \mathbf{A}\frac{h_{uu}}{h_u})^{-1}\mathbf{D} = -(ru + c - \delta(0)),$$

for $q^* = \hat{q}$, explicit expression for $\frac{h_u}{h_{uu}}$ cannot be obtained easily.

When the reinsurance safety loadings for all classes are the same, it may be possible to derive explicit expressions for the optimal results. Therefore, in the rest of this section, we focus on investigating the optimization problem with a common reinsurance safety loading.

$$\begin{aligned}\delta(q) = \textstyle\sum_{j=1}^{n}\mu_j(1-q_j)\sum_{k=1}^{l}\lambda_k p_{kj} + \bar{\Lambda}\Big(& \textstyle\sum_{j=1}^{n}(1-q_j)^2(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj} \\ & + \textstyle\sum_{j=1}^{l}\sum_{k=1}^{n}\sum_{i\neq k}^{n}\mu_i\mu_k(1-q_i)(1-q_k)\lambda_j p_{ji}p_{jk}\Big),\end{aligned} \tag{28}$$

where $\bar{\Lambda}$ is the common reinsurance safety loading. Using $b^2(q)$ of (2), we get

$$\begin{aligned}c - \delta(q) - a(q) = \Lambda b^2(\mathbf{1}) - \bar{\Lambda}\Big(& \textstyle\sum_{j=1}^{n}(1-q_j)^2(\mu_j^2+\sigma_j^2)\sum_{k=1}^{l}\lambda_k p_{kj} \\ & + \textstyle\sum_{j=1}^{l}\sum_{k=1}^{n}\sum_{i\neq k}^{n}\mu_i\mu_k(1-q_i)(1-q_k)\lambda_j p_{ji}p_{jk}\Big),\end{aligned}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$, and Λ is the safety loading of the insurer. Without loss of generality, we assume that $\bar{\Lambda} > \Lambda$. Thus, we have

$$
\begin{aligned}
\hat{f}(q) &= (\Lambda - \bar{\Lambda})b^2(\mathbf{1})h_u - \bar{\Lambda}q\mathbf{A}q^T h_u + 2\bar{\Lambda}q\mathbf{D}_1 h_u + \tfrac{1}{2}q\mathbf{A}q^T h_{uu} \\
&= (\Lambda - \bar{\Lambda})b^2(\mathbf{1})h_u + 2\bar{\Lambda}q\mathbf{D}_1 h_u + \tfrac{1}{2}q(\mathbf{A}huu - 2\bar{\Lambda}\mathbf{A}h_u)q^T,
\end{aligned}
$$

where the vector

$$
\mathbf{D}_1 = \begin{pmatrix}
(\mu_1^2+\sigma_1^2)\sum_{k=1}^{l}\lambda_k p_{k1} + \sum_{j=1}^{l}\sum_{k\neq 1}^{n}\mu_k\mu_1\lambda_j p_{jk}p_{j1} \\
(\mu_2^2+\sigma_2^2)\sum_{k=1}^{l}\lambda_k p_{k2} + \sum_{j=1}^{l}\sum_{k\neq 2}^{n}\mu_k\mu_2\lambda_j p_{jk}p_{j2} \\
\vdots \\
(\mu_n^2+\sigma_n^2)\sum_{k=1}^{l}\lambda_k p_{kn} + \sum_{j=1}^{l}\sum_{k\neq n}^{n}\mu_k\mu_n\lambda_j p_{jk}p_{jn}
\end{pmatrix}.
$$

Note that $\mathbf{D}_1 = \mathbf{A}\mathbf{1}^T$. Then, under Assumption 1, it follows from Lemma 3.1 that the minimizer of $\hat{f}(q)$ is given by

$$
\hat{q} = \frac{2\bar{\Lambda}}{2\bar{\Lambda} - \frac{h_{uu}}{h_u}}\mathbf{1}.
$$

For $\frac{h_u}{h_{uu}} < 0$, it is easy to see that $\hat{q} = (\hat{q}_1(u), \hat{q}_1(u), \ldots, \hat{q}_n(u))$ falls into the interval $[0,1]^n$, i.e., $q^* = \hat{q}$.

In the following lemma, we present the form of $\frac{h_{uu}}{h_u}$ under the optimal strategy.

Lemma 4.1: *When $q^* = \hat{q}$, one can show that*

$$
\xi(u) = \frac{h_{uu}}{h_u} = \frac{2ru\bar{\Lambda} + 2\Lambda\bar{\Lambda}b^2(\boldsymbol{1})}{ru + (\Lambda - \bar{\Lambda})b^2(\boldsymbol{1})}. \tag{29}
$$

***Proof*:** When the reinsurance premium is calculated by the variance premium principle with a common safety loading for all classes, we have the corresponding HJB equation

$$
[ru + (\Lambda - \bar{\Lambda})b^2(\mathbf{1})]h_u + \inf_{q\in\mathcal{D}}\left\{2\bar{\Lambda}q\mathbf{D}_1 h_u - \bar{\Lambda}q\mathbf{A}q^T h_u + \frac{1}{2}q\mathbf{A}q^T h_{uu}\right\} = 0.
$$

Instituting $q = q^*$ back into the equation, we have

$$
[ru + (\Lambda - \bar{\Lambda})b^2(\mathbf{1})] + \frac{2\bar{\Lambda}^2\mathbf{1}\mathbf{D}_1}{2\bar{\Lambda} - \frac{h_{uu}}{h_u}} = 0.
$$

Noting that $b^2(\mathbf{1}) = \mathbf{1}\mathbf{D}_1$, one can show that (29) holds. $\square$

In the following theorem, we present the solution to our optimization problem under the variance premium principle.

Theorem 4.1: *Let $\xi(u)$ be given in (29). Then the minimum probability of drawdown on $\mathcal{O} := \{(u,m) \in (R^+)^2 : \alpha m \leq u \leq \min(m, u_s)\}$ is given by*

$$\phi(u,m) = \begin{cases} 1 - \dfrac{g(u,m)}{g(u_s,m)}, & \alpha m \leq u \leq u_s \leq m, \\ \\ 1 - k(m) \cdot \dfrac{g(u,m)}{g(u_s,u_s)}, & \alpha m \leq u \leq m \leq u_s, \end{cases}$$

where

$$g(u,m) = \int_{\alpha m}^{u} \exp\left\{\int_{\alpha m}^{y} \xi(w)dw\right\} dy,$$

and

$$k(m) = \exp\left\{-\int_{m}^{u_s} f(y)dy\right\}$$

with

$$f(y) = \alpha\left[\frac{1}{g(y,y)} + \xi(\alpha y)\right];$$

and the corresponding optimal reinsurance strategy is given by

$$q^* = -\frac{ru + (\Lambda - \bar{\Lambda})b^2(\boldsymbol{1})}{\bar{\Lambda}b^2(\boldsymbol{1})}\boldsymbol{1}. \tag{30}$$

***Proof*:** Following the arguments and steps in Theorem 3.2 and Theorem 3.3, we can derive the solutions to problems (6) and (7), and prove that h satisfies all the conditions stated in Theorem 3.1. Finally, we have $\phi = h$ with the optimal reinsurance strategy q^* given in (30). $\square$

Remark 4.1: *Note that the corresponding safe level u_s equals to*

$$\frac{(\bar{\Lambda} - \Lambda)b^2(\boldsymbol{1})}{r}$$

under the variance premium principle. Then we see that the inequality

$$ru + (\Lambda - \bar{\Lambda})b^2(\boldsymbol{1}) < 0$$

holds for any $u \in [\alpha m, \min(m, u_s)]$*. Besides, it is not difficult to see that*

$$\frac{ru + (\Lambda - \bar{\Lambda})b^2(\boldsymbol{1})}{\bar{\Lambda}b^2(\boldsymbol{1})} = 1 - \frac{ru + \Lambda b^2(\boldsymbol{1})}{\bar{\Lambda}b^2(\boldsymbol{1})} < 1.$$

Therefore, one can show that the optimal strategy q^* *belongs to* $[0, 1]^n$*.*

Remark 4.2: *Even though the reinsurance premium rate given in* (28) *looks more complex than the one given in* (27)*, it leads to a simpler expression for the optimal strategy, which exactly falls into the interval* $[0, 1]^n$ *and perfectly equals to each other. Furthermore, the optimal strategies under the two different premium principles both depend not only on the safety loading but also on the claim-size distribution and the claim-number process. However, comparing with the influence under the expected value principle, the impact of the claim-size and the claim-number process is rather smaller when the reinsurance premium is calculated by the variance premium principle (see Examples 5.4-5.6 for details). This observation can be explained by the expression given in* (30)*, where the impact of the claim size distributions and the counting processes is somehow cancelled out when the reward of risk-free investment is relatively small.*

Remark 4.3: *If* $\alpha = 0$*, then we are in the case of minimizing the probability of ruin for the fixed level 0. Also, it follows from* (4) *that the safe level approaches* ∞ *as* r *tends to 0. Our corresponding optimal results in this case coincide with those in Liang & Yuen (2017) (they studied the same optimization problem with the objective of minimizing ruin probability for the risk model with thinning dependence). Furthermore, as was mentioned in Remark 3.4, we can see from Theorem 4.1 that the optimal strategy in relation to drawdown probability is in some sense equal to the optimal strategy associated with ruin probability. Therefore, in our model, if drawdown has not happened, the optimal strategy in relation to drawdown probability follows the optimal strategy associated with ruin probability not only for the expected value principle but also for the variance premium principle.*

5. Numerical examples

In this section, we provide six examples to show the optimal reinsurance strategy and the effect of different parameters on the optimal results. Examples 5.1$\sim$5.5 is under the expected value principle, while Example 5.6 is under the variance premium principle.

The first example presents the optimal reinsurance strategy with two dependent classes of insurance business and two groups of stochastic sources, i.e., $n = l = 2$, $p_{11} = p_{22} = 1$.

Example 5.1: In this example, we set $u = 5$, $r = 0.05$, $\mu_1 = \mu_2 = 1$, $\sigma_1^2 = 0.75$, $\sigma_2^2 = 0.25$, $\theta_1 = \theta_2 = 0.12$, $\lambda_1 = 4$, $\lambda_2 = 5$, $p_{12} = 0.3$ and $p_{21} = 0.5$. Here we consider two pairs of (η_1, η_2). For (0.22, 0.28), the results are given in Table 5.1. For (0.3, 0.15), the results are shown in Table 5.2.

Table 5.1 Optimal strategy for $(\eta_1 = 0.22, \eta_2 = 0.28)$

i	$\hat{q}_i$	q_i^*
1	0.4084	**0.5228**
2	1.2672	**1**

Table 5.2 Optimal strategy for $(\eta_1 = 0.3, \eta_2 = 0.15)$

i	$\hat{q}_i$	q_i^*
1	1.1326	**1**
2	0.3264	**0.3992**

Example 5.2 involves three classes of insurance business and three groups of stochastic sources, i.e., $n = l = 3$, $p_{11} = p_{22} = p_{33} = 1$. Here $q_i^*(u) \in [0, \infty)$. For $q_i^*(u) \in [0, 1]$, the insurer has a proportional reinsurance cover. For $q_i^*(u) \in (1, \infty)$, it may be thought of as acquiring new business.

Example 5.2: In this example, we set $u = 20$, $r = 0.05$, $\mu_1 = \mu_2 = \mu_3 = 1$, $\sigma_1^2 = 0.49$, $\sigma_2^2 = 0.36$, $\sigma_3^2 = 0.25$, $\theta_1 = \theta_2 = \theta_3 = 0.12$, $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 5$, $p_{12} = p_{13} = p_{23} = 0.3$ and $p_{21} = p_{31} = p_{32} = 0.5$. Again, we consider two triplets of (η_1, η_2, η_2). They are (0.2, 0.25, 0.3) and (0.2, 0.4, 0.35). The results are summarized in Tables 5.3 and 5.4.

Table 5.3 Optimal strategy for $(\eta_1 = 0.2, \eta_2 = 0.25, \eta_3 = 0.3)$

i	$\hat{q}_i$	q_i^*
1	0.2525	**0.2525**
2	0.4255	**0.4255**
3	0.9460	**0.9460**

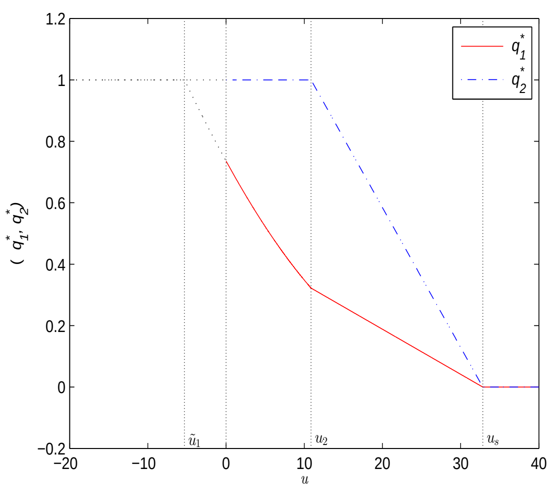
Table 5.4 Optimal strategy for $(\eta_1 = 0.2, \eta_2 = 0.4, \eta_3 = 0.35)$

i	$\hat{q}_i$	q^{*i_0}	$\hat{\phi}_{i0}$	q_i^*
1	-0.1129	(0, 1.3164, 1.0499)	**0.1816**	**0**
2	1.3034	(0.1641, 0, 2.4166)	0.3067	**1.3164**
3	1.1234	(0.2078, 2.1810, 0)	0.2582	**1.0499**

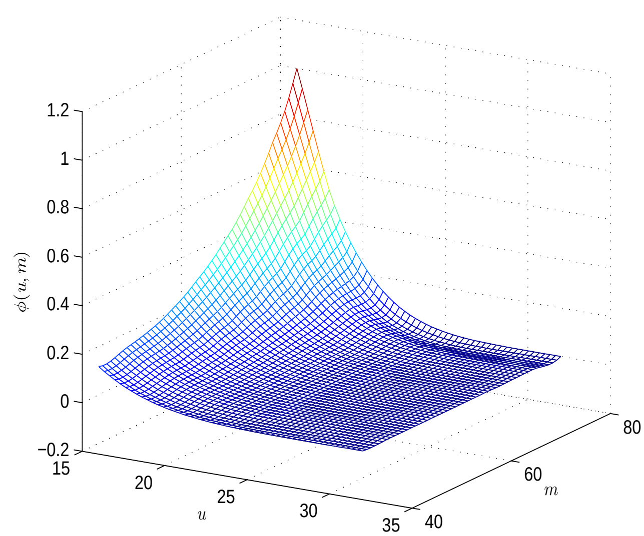
In Table 5.4, since the minimizer $\hat{q} = (-0.1129, 1.3034, 1.1234) \notin [0, \infty)^3$, we investigate q^{*i_0} and the corresponding minimum $\hat{\phi}_{i0}$ $(i = 1, 2, 3)$, respectively. By comparing $\hat{\phi}_{i0}$ for $i = 1, 2, 3$, we get the optimal reinsurance strategy.

In the following examples, with two dependent classes of insurance business and two groups of stochastic sources, we show how the initial surplus u and the maximum (past) value m affect the the optimal reinsurance strategy and its corresponding minimum probability of drawdown. We also present the impact of η_1, η_2, p_{12} and σ_1^2 on the optimal strategy.

Example 5.3: In this example, we set $r = 0.05$, $\alpha = 0.2$, $\mu_1 = \mu_2 = 1$, $\sigma_1^2 = 0.75$, $\sigma_2^2 = 0.25$, $\theta_1 = \theta_2 = 0.12$, $\eta_1 = 0, 22$, $\eta_2 = 0.28$, $\lambda_1 = 4$, $\lambda_2 = 5$, $p_{12} = 0.3$ and $p_{21} = 0.5$. The results are shown in Figure 1.



(a)

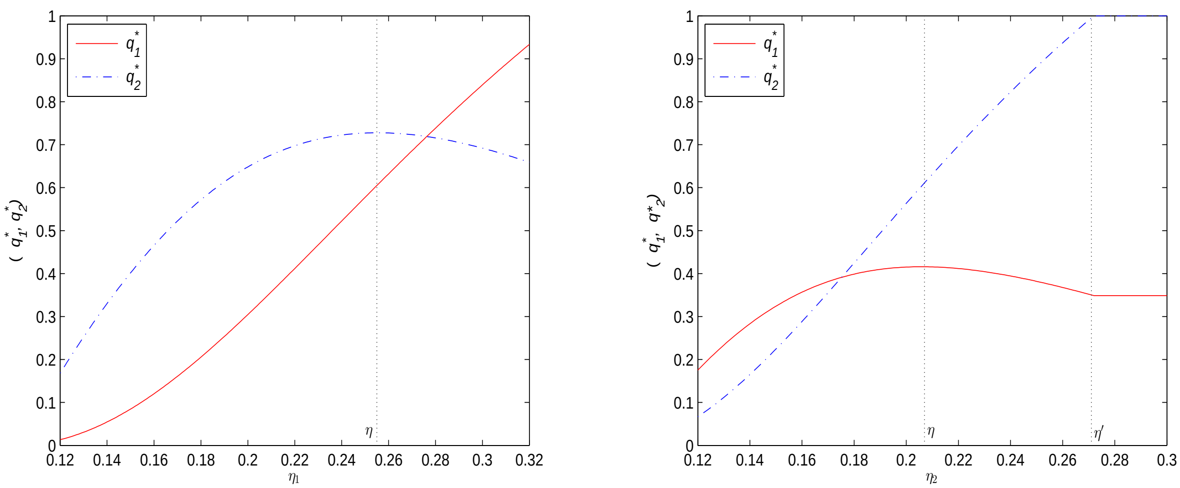


(b)

Figure 1: The influence of u and m on the optimal reinsurance results

We see from Figure 1 that the optimal reinsurance strategy (q_1^*, q_2^*) decreases as u increases. According to (14) and (16), it is not difficult to prove that q_1^* and q_2^* are decreasing functions of u, and independent of m. Meanwhile, the corresponding minimum probability of drawdown $\phi(u,m)$ is a decreasing function of u but an increasing function of m. We give the proof of this property in Appendix C for both cases of $m \geq u_s$ and $m \leq u_s$. These observations are kind of reasonable. When the value of the surplus increases toward u_s, the insurer can transfer all the risk to the reinsurer. As a result, wealth will never decrease, and drawdown cannot happen. On the other hand, the drawdown level increases as the maximum (past) value m increases. This in turn makes drawdown more likely.

Example 5.4: In this example, we set $u = 10$, $r = 0.05$, $\mu_1 = \mu_2 = 1$, $\sigma_1^2 = 0.75$, $\sigma_2^2 = 0.25$, $\theta_1 = \theta_2 = 0.12$, $\lambda_1 = 4$, $\lambda_2 = 5$, $p_{12} = 0.3$ and $p_{21} = 0.5$. The results are shown in Figure 2. We set $\eta_2 = 0.22$ in Figure 2(a), and $\eta_1 = 0.22$ in Figure 2(b).

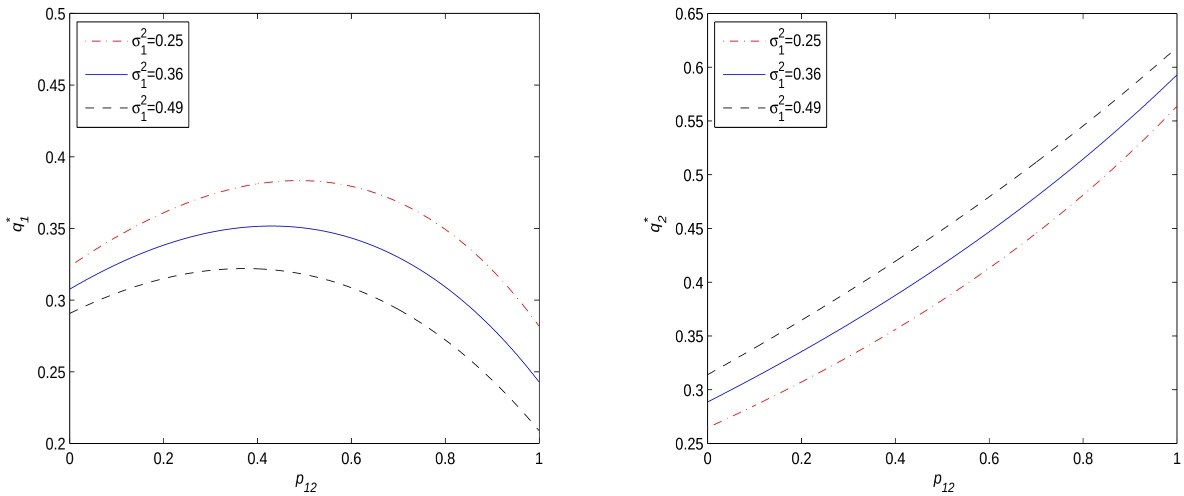


(a) (b)

Figure 2: The influence of η_1 and η_2 on the optimal reinsurance strategy

Figure 2 examines the influence of the reinsurer's safety loadings, i.e., η_1 and η_2 on the optimal reinsurance strategy. It is easy to see that a greater value of η_i $(i = 1, 2)$ yields a greater value of q_i^* $(i = 1, 2)$, which illustrates the intuition that if the reinsurance premium increases, the insurer would rather retain a greater share of each claim by purchasing less reinsurance. We also see that as the value of $\eta_1(\eta_2)$ increases, the retention level of the other class first increases and then decreases after reaching a certain level. When the company keep buying less reinsurance for one class, it eventually needs to reduce the risk of its insurance portfolios by buying a bit more reinsurance for another class. In Figure 2(b), when $\eta_2 > \eta'$, we have $q_2^* = 1$ and a constant q_1^*. This phenomenon can be explained by the expression for q_1^* given in (16), which is independent of η_2.

Example 5.5: In this example, we set $u = 10$, $r = 0.05$, $\mu_1 = \mu_2 = 1$, $\sigma_2^2 = 0.25$, $\theta_1 = \theta_2 = 0.12$, $\eta_1 = \eta_2 = 0.2$, $\lambda_1 = \lambda_2 = 4$ and $p_{21} = 0.5$. The results are shown in Figure 3.



(a) (b)

Figure 3: The influence of p_{12} and σ_1^2 on the optimal reinsurance strategy

Figure 3 shows that a greater value of p_{12} yields a greater value of q_2^*, but the monotonicity does not apply to q_1^*. Besides, we observe that a greater value σ_1^2 yields a smaller value of q_1^* but a greater value of q_2^*. It makes sense because a greater value of σ_1 and p_{12} implies a larger insurance risk in class 1. To reduce the risk, the insurer tends to purchase more reinsurance for class 1 when the value of the σ_1 or p_{12} gets larger. On the other hand, for a fixed σ_2^2, the insurer would like to retain a bit more in class 2 as the risk in class 1 increases.

Example 5.6: In this example, we set $u = 10$, $r = 0.05$, $\mu_1 = \mu_2 = 1$, $\sigma_2^2 = 0.25$, $\Lambda = 0.12$, $\lambda_1 = \lambda_2 = 4$ and $p_{21} = 0.5$. The results are shown in Figure 4.

Figure 4 illustrates the impact of the parameters of $\bar{\Lambda}$, p_{12} and σ_1^2 on the optimal strategy under the variance premium principle. It shows that the strategy q^* increases as $\bar{\Lambda}$ increases, and this monotonicity is similar to the one under the expected value principle. We also observe that a greater value of σ_1^2 and p_{12} yields a greater value of the optimal reinsurance strategy q^*. This phenomenon is in line with (30). Furthermore, comparing with Figure 2 and Figure 3 under the expected value principle, we see from Figure 4 that the impact of p_{12} and σ_1^2 on the optimal strategy is relatively smaller when the reinsurance premium is calculated according to the variance premium principle.

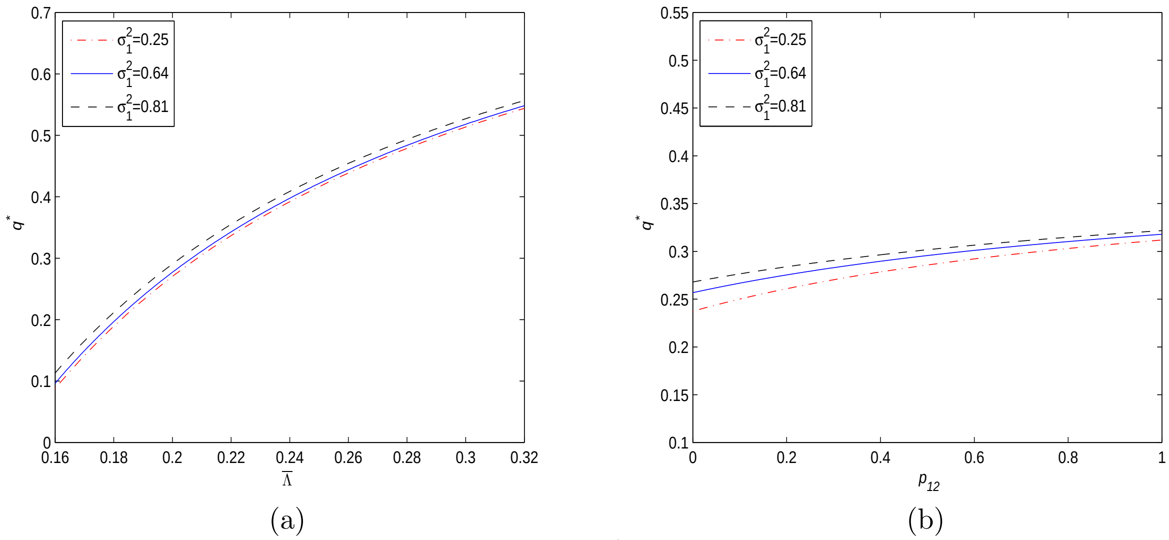


Figure 4: The influence of $\bar{\Lambda}$, p_{12} and σ_1^2 on the optimal reinsurance strategy

6. Conclusion

We first recap the main results of this paper. From an insurer's point of view, we consider the optimal proportional reinsurance problem to minimize the probability of drawdown in a diffusion approximation risk model with thinning-dependence. Using the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, we derive the optimal reinsurance strategy and the corresponding minimized probability of drawdown under the expected value principle and the variance premium principle. Our results show that the optimal reinsurance strategy strongly depends on the value of the initial surplus u, and the expression under the expected value principle is very different from the one under the variance premium principle.

Although the literature on optimal reinsurance is increasing rapidly, there are still many interesting problems that deserve investigation. For further research, one can discuss other types of reinsurance such as excess-of-loss reinsurance or combined reinsurance in the risk model with thinning-dependence. Another interesting research topic is to consider the optimization problem with a more general objective function such as minimizing the expectation of some function that is non-increasing with respect to the minimum surplus value or non-decreasing with respect to the maximum surplus value. Apart from reinsurance, one may consider taking the life time of individual τ_d into consideration so as to investigate the problem of optimal insurance which minimizes the probability of lifetime drawdown.

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Appendix A Auxiliary functions

A.1. The functions g_{1i} and f_{1i} $(i = 1, 2, 3)$ are given by

$$\begin{cases} g_{11}(u,m) = \displaystyle\int_{\alpha m}^{u} \exp\left\{\int_{\alpha m}^{y} \xi_{13}(w)dw\right\} dy, \\ g_{12}(u,m) = \displaystyle\int_{\alpha m}^{\alpha m\vee\widetilde{u}_1} \exp\left\{\int_{\alpha m}^{y} \xi_{13}(w)dw\right\} dy \\ \qquad\qquad + \displaystyle\int_{\alpha m\vee\widetilde{u}_1}^{u} \exp\left\{\left(\int_{\alpha m}^{\alpha m\vee\widetilde{u}_1} \xi_{13}(w) + \int_{\alpha m\vee\widetilde{u}_1}^{y} \xi_{12}(w)\right) dw\right\} dy, \\ g_{13}(u,m) = \displaystyle\int_{\alpha m}^{\alpha m\vee\widetilde{u}_1} \exp\left\{\int_{\alpha m}^{y} \xi_{13}(w)dw\right\} dy \\ \qquad\qquad + \displaystyle\int_{\alpha m\vee\widetilde{u}_1}^{\alpha m\vee u_2} \exp\left\{\left(\int_{\alpha m}^{\alpha m\vee\widetilde{u}_1} \xi_{13}(w) + \int_{\alpha m\vee\widetilde{u}_1}^{y} \xi_{12}(w)\right) dw\right\} dy \\ \qquad\qquad + \displaystyle\int_{\alpha m\vee u_2}^{u} \exp\left\{\left(\int_{\alpha m}^{\alpha m\vee\widetilde{u}_1} \xi_{13}(w) + \int_{\alpha m\vee\widetilde{u}_1}^{\alpha m\vee u_2} \xi_{12}(w) + \int_{\alpha m\vee u_2}^{y} \xi_{11}(w)\right) dw\right\} dy; \end{cases}$$

and

$$f_{1i}(y)=\begin{cases}\alpha\left[\dfrac{1}{g_{1i}(y,y)}+\xi_{11}(\alpha y)\right], & if\ u_2<\alpha m,\\[2ex] \alpha\left[\dfrac{1}{g_{1i}(y,y)}+\xi_{12}(\alpha y)\right], & if\ \widetilde{u}_1\le \alpha m\le u_2,\\[2ex] \alpha\left[\dfrac{1}{g_{1i}(y,y)}+\xi_{13}(\alpha y)\right], & if\ \alpha m<\widetilde{u}_1.\end{cases}$$

A.2. The functions g_{2i} and f_{2i} (i=1,2) are given by

$$\begin{cases} g_{21}(u,m)=\displaystyle\int_{\alpha m}^{u}\exp\left\{\int_{\alpha m}^{y}\xi_{22}(w)dw\right\}dy,\\[2ex] g_{22}(u,m)=\displaystyle\int_{\alpha m}^{\alpha m\vee u_2'}\exp\left\{-2\int_{\alpha m}^{y}\xi_{22}(w)dw\right\}dy\\[2ex] \qquad\qquad +\displaystyle\int_{\alpha m\vee u_2'}^{u}\exp\left\{-2\left(\int_{\alpha m}^{\alpha m\vee u_2'}\xi_{22}(w)+\int_{\alpha m\vee u_2'}^{y}\xi_{21}(w)\right)dw\right\}dy;\end{cases}$$

and

$$f_{2i}(y)=\begin{cases}\alpha\left[\dfrac{1}{g_{2i}(y,y)}+\xi_{21}(\alpha y)\right], & if\ u_2'\le\alpha m,\\[2ex] \alpha\left[\dfrac{1}{g_{2i}(y,y)}+\xi_{22}(\alpha y)\right], & if\ \alpha m<u_2'.\end{cases}$$

A.3. The functions g_{3i} and f_{3i} (i=1,2) are given by

$$\begin{cases} g_{31}(u,m)=\displaystyle\int_{\alpha m}^{u}\exp\left\{\int_{\alpha m}^{y}\xi_{32}(w)dw\right\}dy,\\[2ex] g_{32}(u,m)=\displaystyle\int_{\alpha m}^{\alpha m\vee u_1'}\exp\left\{\int_{\alpha m}^{y}\xi_{32}(w)dw\right\}dy\\[2ex] \qquad\qquad +\displaystyle\int_{\alpha m\vee u_1'}^{u}\exp\left\{\left(\int_{\alpha m}^{\alpha m\vee u_1'}\xi_{32}(w)+\int_{\alpha m\vee u_1'}^{y}\delta_{31}(w)\right)dw\right\}dy,\end{cases}$$

with $\xi_{3i}\ \ (i=1,2)$ and u_1' given by

$$\begin{cases}\xi_{31}(u)=-\dfrac{c_1^2}{2a_1(ru+\Delta_1)},\\[2ex] \xi_{32}(u)=-\dfrac{2(ru+\Delta_1-c_1)}{a_1},\\[2ex] \quad u_1' \ =\dfrac{1}{r}\left(\dfrac{c_1}{2}-\Delta_1+c_2\right);\end{cases}$$

and

$$f_{3i}(y)=\begin{cases}\alpha\left[\dfrac{1}{g_{3i}(y,y)}+\xi_{31}(\alpha y)\right], & if\ u_1'\le \alpha m,\\ \alpha\left[\dfrac{1}{g_{3i}(y,y)}+\xi_{32}(\alpha y)\right], & if\ \alpha m<u_1'.\end{cases}$$

Appendix B Proof of $\widetilde{u}_1 < u_2$ when $u_1 < u_2$

Note that

$$\begin{aligned}u_2-\widetilde{u}_1 &= \frac{1}{r}\left[\frac{\Delta_2}{2(a_1a_2-b_{12}^2)(a_1c_2-b_{12}c_1)}-\frac{a_1c_1-a_2c_1+2b_{12}c_2+2a_1c_2}{2(a_1+b_{12})}\right]\\ &= \frac{\Delta_2(a_1+b_{12})-(a_1c_1-a_2c_1+2b_{12}c_2+2a_1c_2)(a_1a_2-b_{12}^2)(a_1c_2-b_{12}c_1)}{2r(a_1a_2-b_{12}^2)(a_1c_2-b_{12}c_1)(a_1+b_{12})} \qquad \text{(B.1)}\\ &= \frac{(b_{12}^2-a_1a_2)\left(c_2(b_{12}c_2-a_2c_1)+c_1(a_1c_2-c_1b_{12})+(a_1c_2^2-a_2c_1^2)\right)}{2r(a_1a_2-b_{12}^2)(a_1c_2-b_{12}c_1)(a_1+b_{12})}.\end{aligned}$$

Under the assumption of $u_1 < u_2$, we have

$$a_2c_1-c_2b_{12} > a_1c_2-c_1b_{12}.$$

Then it follows from Lemma 3.3 that

$$\begin{aligned}&(b_{12}^2-a_1a_2)\left(c_2(b_{12}c_2-a_2c_1)+c_1(a_1c_2-c_1b_{12})+(a_1c_2^2-a_2c_1^2)\right)\\ &<(b_{12}^2-a_1a_2)(b_{12}c_2^2-a_2c_1c_2-c_1c_2b_{12}+a_1c_2^2)\\ &=c_2(b_{12}^2-a_1a_2)(c_2b_{12}-a_2c_1-c_1b_{12}+a_1c_2)\\ &<0.\end{aligned}$$

In Case 1, we have $a_1c_2-b_{12}c_1\le 0$, and thus the denominator of (B.1) is non-positive. So, we have $\widetilde{u}_1 < u_2$.

Appendix C Proof of monotonicity and convexity of h

If $m \geq u_s$, we have

$$h(u,m) = 1 - \frac{g_{13}(u,m)}{g_{13}(u_s,m)},$$

for $\alpha m < \widetilde{u}_1 < u_2 \leq u \leq u_s$. Differentiating $\phi(u,m)$ with respect to m yields

$$h_m(u,m) = \frac{\alpha\left(g_{13}(u_s,m) - g_{13}(u,m)\right)}{g_{13}^2(u_s,m)} \geq 0.$$

If $m \leq u_s$, we have

$$h(u,m) = 1 - k_{13}(m) \cdot \frac{g_{13}(u,m)}{g_{13}(u_s,u_s)},$$

where

$$k_{13}(m) = \exp\left\{\int_m^{u_s} -f_{13}(y)dy\right\},$$

with

$$f_{13}(y) = \alpha\left[\frac{1}{g_{13}(y,y)} + \xi_{13}(\alpha y)\right],$$

for $\alpha m < \widetilde{u}_1 < u_2 \leq u \leq m \leq u_s$. It follows that

$$\begin{aligned} h_m(u,m) &= -\frac{k_{13}(m)}{g_{13}(u_s,u_s)} \cdot \left[f_{13}(m)g_{13}(u,m) - \alpha\xi_{13}(\alpha m)g_{13}(u,m) - \alpha\right] \\ &= \frac{\alpha \cdot k_{13}(m)}{g_{13}(u_s,u_s)} \cdot \left[1 - \frac{g_{13}(u,m)}{g_{13}(m,m)}\right] \geq 0. \end{aligned}$$

Besides, it is not difficult to see that

$$\frac{\partial g_{13}(u,m)}{\partial u} = \exp\left\{\left(\int_{\alpha m}^{\alpha m \vee \widetilde{u}_1} \xi_{13}(w) + \int_{\alpha m \vee \widetilde{u}_1}^{\alpha m \vee u_2} \xi_{12}(w) + \int_{\alpha m \vee u_2}^{u} \xi_{11}(w)\right) dw\right\} > 0,$$

and

$$\frac{\partial g_{13}^2(u,m)}{\partial u^2} = \frac{\partial g_{13}(u,m)}{\partial u} \cdot \xi_{13}(u) < 0.$$

Thus, we have $h_u < 0$ and $h_{uu} > 0$. Along the same lines, we can get the same results for other cases. Therefore, we conclude that $h(u,m)$ is a non-increasing convex function with respect to the surplus wealth u but a non-decreasing function with respect to the maximum (past) value m.