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ON HILBERT'S INEQUALITIES WITH ALTERNATING SIGNS

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Abstract. Some new Hilbert type inequalities with alternating signs are established. These also generalize some existing results of Hilbert type inequalities in the literature.

1. Introduction

The well-known Hilbert's double-series inequality can be stated as follows (see [1, p. 253]).

THEOREM. *If* $p > 1$, $q = p/(p-1)$ *and* $\sum_{m=1}^{\infty} a_m < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant \frac{1}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q},$$

unless the sequence $\{a_m\}$ *or* $\{b_n\}$ *is null.*

Hilbert's inequality were studied extensively and numerous variants, generalizations, and extensions appeared in the literature [2–14] and the references cited therein. The research for reverse Hilbert inequalities were published in [15–17] et al. In particular, Pachpatte [18] established some new inequalities similar to the Hilbert's inequality. The main purpose of this paper is to establish some new Hilbert type inequalities with alternating signs.

2. Main results

The following inequality involving series of nonnegative terms was established in [18].

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THEOREM A. *Let $p \geqslant 1$, $q \geqslant 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1,\ldots,k$ and $n = 1,\ldots,r$, where k,r are natural numbers. Let $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \leqslant C(p,q,k,r)\Big(\sum_{m=1}^{k}\big(k-m+1\big)\big(a_m A_m^{p-1}\big)^2\Big)^{1/2} \times\Big(\sum_{n=1}^{r}\big(r-n+1\big)\big(b_n B_n^{q-1}\big)^2\Big)^{1/2}, \tag{2.1}$$

where

$$C(p,q,k,r) = \frac{1}{2}pq(kr)^{1/2}.$$

In this paper, we first establish the following Hilbert type inequality with alternating signs.

THEOREM 2.1. *Let $p \geqslant 1$, $q \geqslant 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$, $\alpha > 1$ and $\{a_m\}$ and $\{b_n\}$ be two positive non-increasing sequences of real numbers defined for $m = 1,\ldots,k$ and $n = 1,\ldots,r$, where k,r are natural numbers. If*

$$\bar{A}_m = \sum_{s=1}^{m}(-1)^{s+1}a_s \ \ and\ \ \bar{B}_n = \sum_{t=1}^{n}(-1)^{t+1}b_t,$$

then

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\alpha\beta\bar{A}_m^p\bar{B}_n^q}{\alpha M+\beta N} \leqslant \bar{C}(p,q,k,r,\alpha,\beta)\left(\sum_{m=1}^{K}(K-m+1)\left((a_{2m-1}-a_{2m})\tilde{A}_m^{p-1}\right)^{\alpha}\right)^{1/\alpha} \times\left(\sum_{n=1}^{R}(R-n+1)\left((b_{2n-1}-b_{2n})\tilde{B}_n^{q-1}\right)^{\beta}\right)^{1/\beta}, \tag{2.2}$$

where

$$\bar{C}(p,q,k,r,\alpha,\beta) = pqK^{1/\beta}R^{1/\alpha},$$

and

$$\tilde{A}_m = \sum_{k=1}^{m}(a_{2k-1}-a_{2k}), \ \ and\ \ \tilde{B}_n = \sum_{r=1}^{n}(b_{2r-1}-b_{2r}),$$

and for any positive integer z (lowercase), its capital Z denotes $z/2$ if z is even, and $(z+1)/2$ if z is odd.

Proof. Note that any sum with alternating signs can be written in the forms

$$\sum_{s=1}^{m}(-1)^{s+1}a_s = \sum_{s=1}^{M}(a_{2s-1}-a_{2s}) =: \bar{A}_m,$$

and

$$\sum_{t=1}^{n}(-1)^{t+1}b_t = \sum_{t=1}^{N}(b_{2t-1}-b_{2t}) =: \bar{B}_n,$$

and we assume that $a_{m+1}=0$ and $b_{n+1}=0$. By using the following inequality (see e.g. [19])

$$\left(\sum_{m=1}^{n} z_m\right)^{\alpha} \leqslant \alpha \sum_{m=1}^{n} z_m \left(\sum_{k=1}^{m} z_k\right)^{\alpha-1},$$

where $\alpha \geqslant 1$ is a constant and $z_m \geqslant 0$, it easily follows that

$$\bar{A}_m^p \leqslant p \sum_{s=1}^{M} (a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}, \tag{2.3}$$

and

$$\bar{B}_n^q \leqslant q \sum_{t=1}^{N} (b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}. \tag{2.4}$$

From (2.3) and (2.4), using Hölder's inequality and Young's inequality $c^{1/\alpha}d^{1/\beta} \leqslant \frac{c}{\alpha}+\frac{d}{\beta}$ (for c, d non-negative reals), we obtain

$$\begin{aligned}
\bar{A}_m^p\bar{B}_n^q &\leqslant pq\left(\sum_{s=1}^{M}(a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}\right)\left(\sum_{t=1}^{N}(b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}\right)\\
&\leqslant pqM^{1/\beta}N^{1/\alpha}\left(\sum_{s=1}^{M}\left((a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}\right)^{\alpha}\right)^{1/\alpha}\left(\sum_{t=1}^{N}\left((b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}\right)^{\beta}\right)^{1/\beta}\\
&\leqslant pq\frac{\alpha M+\beta N}{\alpha\beta}\left(\sum_{s=1}^{M}\left((a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}\right)^{\alpha}\right)^{1/\alpha}\left(\sum_{t=1}^{N}\left((b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}\right)^{\beta}\right)^{1/\beta}.
\end{aligned} \tag{2.5}$$

Dividing both sides of (2.5) by $\frac{\alpha M+\beta N}{\alpha\beta}$ and summing up over N from 1 to R first, and then summing up over M from 1 to K, and using again Hölder's inequality and then interchanging the order of summation, we obtain

$$\begin{aligned}
\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\alpha\beta\bar{A}_m^p\bar{B}_n^q}{\alpha M+\beta N} &\leqslant pq\sum_{M=1}^{K}\left(\left(\sum_{s=1}^{M}\left((a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}\right)^{\alpha}\right)^{1/\alpha}\right)\\
&\quad\times\sum_{N=1}^{R}\left(\left(\sum_{t=1}^{N}\left((b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}\right)^{\beta}\right)^{1/\beta}\right)\\
&\leqslant pqK^{1/\beta}R^{1/\alpha}\left(\sum_{M=1}^{K}\left(\sum_{s=1}^{M}\left((a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}\right)^{\alpha}\right)\right)^{1/\alpha}\\
&\quad\times\left(\sum_{N=1}^{R}\left(\sum_{t=1}^{N}\left((b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}\right)^{\beta}\right)\right)^{1/\beta}
\end{aligned}$$

$$= \bar{C}(p,q,k,r,\alpha,\beta)\left(\sum_{s=1}^{K}\left((a_{2s-1}-a_{2s})\tilde{A}_s^{p-1}\right)^{\alpha}\left(\sum_{M=s}^{K}1\right)\right)^{1/\alpha}$$

$$\times\left(\sum_{t=1}^{R}\left((b_{2t-1}-b_{2t})\tilde{B}_t^{q-1}\right)^{\beta}\left(\sum_{N=t}^{R}1\right)\right)^{1/\beta}$$

$$= \bar{C}(p,q,k,r,\alpha,\beta)\left(\sum_{m=1}^{K}\left((a_{2m-1}-a_{2m})\tilde{A}_m^{p-1}\right)^{\alpha}(K-m+1)\right)^{1/\alpha}$$

$$\times\left(\sum_{n=1}^{R}\left((b_{2n-1}-b_{2n})\tilde{B}_n^{q-1}\right)^{\beta}(R-n+1)\right)^{1/\beta}.$$

This completes the proof. $\square$

REMARK 2.1. Taking $\alpha=\beta=2$ in (2.2), we have

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\overline{A}_m^p\overline{B}_n^q}{M+N}\leqslant \tilde{C}(p,q,k,r)\left(\sum_{m=1}^{K}(K-m+1)\left((a_{2m-1}-a_{2m})\tilde{A}_m^{p-1}\right)^2\right)^{1/2}$$
$$\times\left(\sum_{n=1}^{R}(R-n+1)\left((b_{2n-1}-b_{2n})\tilde{B}_n^{q-1}\right)^2\right)^{1/2}, \tag{2.6}$$

where

$$\tilde{C}(p,q,k,r)=\frac{1}{2}pqK^{1/2}R^{1/2}.$$

This is a new Hilbert type inequality with alternating signs which is different from inequality (2.1) in Theorem A.

REMARK 2.2. By using Cauchy inequality on the right side of (2.6) twice, we obtain the following interesting Hilbert inequality with alternating signs:

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\overline{A}_m^p\overline{B}_n^q}{M+N}\leqslant \tilde{C}(p,q,k,r)\left(\sum_{m=1}^{k}(-1)^{m+1}a_m\cdot\sum_{m=1}^{K}(K-m+1)^{1/2}\tilde{A}_m^{p-1}\right)^{1/2}$$
$$\times\left(\sum_{n=1}^{r}(-1)^{n+1}a_n\cdot\sum_{n=1}^{R}(R-n+1)^{1/2}\tilde{B}_n^{q-1}\right)^{1/2},$$

where

$$\tilde{C}(p,q,k,r)=\frac{1}{2}pqK^{1/2}R^{1/2}.$$

The following inequality involving series of nonnegative terms was also established in [18].

THEOREM B. *Let* $\{a_m\}$, $\{b_n\}$, A_m, B_n *be defined as in Theorem A. Let* $\{p_m\}$ *and* $\{q_n\}$ *be positive sequences for* $m = 1,\dots,k$ *and* $n = 1,\dots,r$, *where* k,r *are natural numbers. Define* $P_m = \sum_{s=1}^{m} p_s$ *and* $Q_n = \sum_{t=1}^{n} q_t$. *Let* ϕ *and* ψ *be real-valued, nonnegative, convex, submultiplicative functions defined on* $\mathbb{R}_+ = [0,+\infty)$. *Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n} \leqslant M(k,r)\left(\sum_{m=1}^{k}\big(k-m+1\big)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{1/2} \times\left(\sum_{n=1}^{r}\big(r-n+1\big)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{1/2}, \tag{2.7}$$

where

$$M(k,r) = \frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)^{1/2}.$$

Next, we establish another Hilbert type inequality with alternating signs.

THEOREM 2.2. *Let* $\{a_m\}$, $\{b_n\}$, $\bar{A}_m$, $\bar{B}_n$, α, β, M, N, K, R *be defined as in Theorem 2.1. Let* $\{p_m\}$ *and* $\{q_n\}$ *be positive non-increasing sequences for* $m = 1,\dots,k$ *and* $n = 1,\dots,r$, *where* k,r *are natural numbers. Define*

$$\bar{P}_m = \sum_{s=1}^{m}(-1)^{s+1}p_s \ \text{ and } \ \bar{Q}_n = \sum_{t=1}^{n}(-1)^{t+1}q_t.$$

Let ϕ *and* ψ *be real-valued, nonnegative, convex, sub-multiplicative functions defined on* $\mathbb{R}_+ = [0,+\infty)$. *Then*

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\alpha\beta\phi(\bar{A}_m)\psi(\bar{B}_n)}{\alpha M+\beta N} \leqslant \bar{M}(k,r,\alpha,\beta)\left(\sum_{m=1}^{K}(K-m+1)\left((p_{2m-1}-p_{2m})\phi\left(\frac{a_{2m-1}-a_{2m}}{p_{2m-1}-p_{2m}}\right)\right)^{\alpha}\right)^{1/\alpha} \times\left(\sum_{n=1}^{R}(R-n+1)\left((q_{2n-1}-q_{2n})\psi\left(\frac{b_{2n-1}-b_{2n}}{q_{2n-1}-q_{2n}}\right)\right)^{\beta}\right)^{1/\beta}, \tag{2.8}$$

where

$$\bar{M}(k,r,\alpha,\beta) = \left(\sum_{M=1}^{K}\left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\right)^{\beta}\right)^{1/\beta}\left(\sum_{N=1}^{R}\left(\frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\right)^{\alpha}\right)^{1/\alpha}.$$

Proof. By the hypotheses, Jensen's inequality and Hölder's inequality, we obtain

$$\begin{aligned}\phi(\bar{A}_m) &= \phi\left(\frac{\bar{P}_m\sum_{s=1}^{m}(-1)^{s+1}a_s}{\sum_{s=1}^{m}(-1)^{s+1}p_s}\right)\\ &= \phi\left(\frac{\bar{P}_m\sum_{s=1}^{M}(p_{2s-1}-p_{2s})\dfrac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}}{\sum_{s=1}^{M}(p_{2s-1}-p_{2s})}\right)\\ &\leqslant \frac{\phi(\bar{P}_m)}{\bar{P}_m}\sum_{s=1}^{M}(p_{2s-1}-p_{2s})\phi\left(\frac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}\right). \end{aligned} \tag{2.9}$$

Similarly

$$\psi(\bar{B}_n) \leqslant \frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\sum_{t=1}^{N}(q_{2t-1}-q_{2t})\psi\left(\frac{b_{2t-1}-b_{2t}}{q_{2t-1}-q_{2t}}\right). \tag{2.10}$$

By (2.9) and (2.10) and using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\phi(\bar{A}_m)\psi(\bar{B}_n) &\leqslant \left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\sum_{s=1}^{M}(p_{2s-1}-p_{2s})\phi\left(\frac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}\right)\right)\\ &\quad\times\left(\frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\sum_{t=1}^{N}(q_{2t-1}-q_{2t})\psi\left(\frac{b_{2t-1}-b_{2t}}{q_{2t-1}-q_{2t}}\right)\right)\\ &\leqslant \frac{\alpha M+\beta N}{\alpha\beta}\left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\left(\sum_{s=1}^{M}\left((p_{2s-1}-p_{2s})\phi\left(\frac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}\right)\right)^{\alpha}\right)^{1/\alpha}\right)\\ &\quad\times\left(\frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\left(\sum_{t=1}^{N}\left((q_{2t-1}-q_{2t})\psi\left(\frac{b_{2t-1}-b_{2t}}{q_{2t-1}-q_{2t}}\right)\right)^{\beta}\right)^{1/\beta}\right). \end{aligned} \tag{2.11}$$

Dividing both sides of (2.11) by $\frac{\alpha M+\beta N}{\alpha\beta}$ and then taking sum over N from 1 to R first and then sum over M from 1 to K, and using again Hölder's inequality and then interchanging the order of summation, we obtain

$$\begin{aligned}&\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\alpha\beta\phi(\bar{A}_m)\psi(\bar{B}_n)}{\alpha M+\beta N}\\ &\leqslant \sum_{M=1}^{K}\left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\left(\sum_{s=1}^{M}\left((p_{2s-1}-p_{2s})\phi\left(\frac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}\right)\right)^{\alpha}\right)^{1/\alpha}\right)\\ &\quad\times\sum_{N=1}^{R}\left(\frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\left(\sum_{t=1}^{N}\left((q_{2t-1}-q_{2t})\psi\left(\frac{b_{2t-1}-b_{2t}}{q_{2t-1}-q_{2t}}\right)\right)^{\beta}\right)^{1/\beta}\right)\end{aligned}$$

$$\leqslant \left(\sum_{M=1}^{K}\left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\right)^{\beta}\right)^{1/\beta}\left(\sum_{M=1}^{K}\sum_{s=1}^{M}\left((p_{2s-1}-p_{2s})\phi\left(\frac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}\right)\right)^{\alpha}\right)^{1/\alpha}$$

$$\times\left(\sum_{N=1}^{R}\left(\frac{\phi(\bar{Q}_n)}{\bar{Q}_n}\right)^{\alpha}\right)^{1/\alpha}\left(\sum_{N=1}^{R}\sum_{t=1}^{N}\left((q_{2t-1}-q_{2t})\psi\left(\frac{b_{2t-1}-ba_{2t}}{q_{2t-1}-q_{2t}}\right)\right)^{\beta}\right)^{1/\beta}$$

$$=\bar{M}(k,r,\alpha,\beta)\left(\sum_{s=1}^{K}\left((p_{2s-1}-p_{2s})\phi\left(\frac{a_{2s-1}-a_{2s}}{p_{2s-1}-p_{2s}}\right)\right)^{\alpha}\sum_{M=s}^{K}1\right)^{1/\alpha}$$

$$\times\left(\sum_{t=1}^{R}\left((q_{2t-1}-q_{2t})\psi\left(\frac{b_{2t-1}-ba_{2t}}{q_{2t-1}-q_{2t}}\right)\right)^{\beta}\sum_{N=t}^{R}1\right)^{1/\beta}$$

$$=\bar{M}(k,r,\alpha,\beta)\left(\sum_{m=1}^{K}(K-m+1)\left((p_{2m-1}-p_{2m})\phi\left(\frac{a_{2m-1}-a_{2m}}{p_{2m-1}-p_{2m}}\right)\right)^{\alpha}\right)^{1/\alpha}$$

$$\times\left(\sum_{n=1}^{R}(R-n+1)\left((q_{2n-1}-q_{2n})\psi\left(\frac{b_{2n-1}-b_{2n}}{q_{2n-1}-q_{2n}}\right)\right)^{\beta}\right)^{1/\beta}.$$

The proof is complete. □

REMARK 2.3. Taking $\alpha=\beta=2$ in (2.8), we have

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\phi(\bar{A}_m)\psi(\bar{B}_n)}{M+N}$$

$$\leqslant \tilde{M}(k,r)\left(\sum_{m=1}^{K}(K-m+1)\left((p_{2m-1}-p_{2m})\phi\left(\frac{a_{2m-1}-a_{2m}}{p_{2m-1}-p_{2m}}\right)\right)^{2}\right)^{1/2}$$

$$\times\left(\sum_{n=1}^{R}(R-n+1)\left((q_{2n-1}-q_{2n})\psi\left(\frac{b_{2n-1}-b_{2n}}{q_{2n-1}-q_{2n}}\right)\right)^{2}\right)^{1/2}, \tag{2.12}$$

where

$$\tilde{M}(k,r)=\frac{1}{2}\left(\sum_{M=1}^{K}\left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\right)^{2}\right)^{1/2}\left(\sum_{N=1}^{R}\left(\frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\right)^{2}\right)^{1/2}.$$

This is a new Hilbert type inequality with alternating signs which is different from inequality (2.7) in Theorem B.

REMARK 2.4. By using Cauchy inequality on the right side of (2.12) twice, we obtain the following interesting Hilbert inequality with alternating signs:

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\phi(\bar{A}_m)\psi(\bar{B}_n)}{M+N}\leqslant \tilde{M}(k,r)\left(\sum_{m=1}^{k}(-1)^{m+1}p_m\cdot\sum_{m=1}^{K}\phi\left(\frac{a_{2m-1}-a_{2m}}{p_{2m-1}-p_{2m}}\right)(K-m+1)^{1/2}\right)^{1/2}$$
$$\times\left(\sum_{n=1}^{r}(-1)^{n+1}p_n\cdot\sum_{n=1}^{R}\phi\left(\frac{b_{2n-1}-b_{2n}}{q_{2n-1}-p_{2n}}\right)(R-n+1)^{1/2}\right)^{1/2},$$

where

$$\tilde{M}(k,r)=\frac{1}{2}\left(\sum_{M=1}^{K}\left(\frac{\phi(\bar{P}_m)}{\bar{P}_m}\right)^2\right)^{1/2}\left(\sum_{N=1}^{R}\left(\frac{\psi(\bar{Q}_n)}{\bar{Q}_n}\right)^2\right)^{1/2}.$$

The following inequality involving series of nonnegative terms was also established in [18].

THEOREM C. *Let $\{a_m\}$, $\{b_n\}$, $\{p_m\}$, $\{q_n\}$, P_m, Q_n be defined as in Theorem B. Define $A_m=\frac{1}{P_m}\sum_{s=1}^{m}p_sa_s$ and $B_n=\frac{1}{Q_n}\sum_{t=1}^{n}q_tb_t$ for $m=1,\ldots,k$ and $n=1,\ldots,r$, where k,r are natural numbers. Let ϕ and ψ be real-valued, nonnegative, convex functions defined on $\mathbb{R}_+=[0,+\infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{m+n}\leqslant\frac{1}{2}(kr)^{1/2}\left(\sum_{m=1}^{k}(k-m+1)\Big(p_m\phi(a_m)\Big)^2\right)^{1/2}$$
$$\times\left(\sum_{n=1}^{r}(r-n+1)\Big(q_n\psi(b_n)\Big)^2\right)^{1/2}. \tag{2.13}$$

Finally, we establish the following new Hilbert type inequality with alternating signs.

THEOREM 2.3. *Let $\{a_m\}$, $\{b_n\}$, $\{p_m\}$, $\{q_n\}$, $\bar{P}_m$, $\bar{Q}_n$, α, β, M, N, K, R be as defined in Theorem 2.2. Define*

$$\bar{A}_m=\frac{1}{\bar{P}_m}\sum_{s=1}^{m}(-1)^{s+1}p_sa_s \text{ and } \bar{B}_n=\frac{1}{\bar{Q}_n}\sum_{t=1}^{n}(-1)^{t+1}q_tb_t$$

for $m=1,\ldots,k$ and $n=1,\ldots,r$, where k,r are natural numbers. Let ϕ and ψ be real-valued, nonnegative, convex functions defined on $\mathbb{R}_+=[0,+\infty)$. Then

$$\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\alpha\beta\bar{P}_m\bar{Q}_n\phi(\bar{A}_m)\psi(\bar{B}_n)}{\alpha M+\beta N}$$
$$\leqslant\bar{C}(1,1,k,r,\alpha,\beta)\left(\sum_{m=1}^{K}(K-m+1)\left((p_{2m-1}-p_{2m})\phi\left(\frac{p_{2m-1}a_{2m-1}-p_{2m}a_{2m}}{p_{2m-1}-p_{2m}}\right)\right)^{\alpha}\right)^{1/\alpha}$$
$$\times\left(\sum_{n=1}^{R}(R-n+1)\left((q_{2n-1}-q_{2n})\psi\left(\frac{q_{2n-1}b_{2n-1}-q_{2n}b_{2n}}{q_{2n-1}-q_{2n}}\right)\right)^{\beta}\right)^{1/\beta}, \tag{2.14}$$

where $\bar{C}(1,1,k,r,\alpha,\beta)$ *is defined by taking* $p=q=1$ *in Theorem 2.1.*

Proof. By the hypotheses, Jensen's inequality, and Hölder's inequality, it is easy to observe that

$$\begin{aligned}\phi(\bar{A}_m) &= \phi\left(\frac{1}{\bar{P}_m}\sum_{s=1}^{m}(-1)^{s+1}p_s a_s\right) = \phi\left(\frac{1}{\bar{P}_m}\sum_{s=1}^{M}(p_{2s-1}a_{2s-1}-p_{2s}a_{2s})\right)\\ &\leqslant \frac{1}{\bar{P}_m}\sum_{s=1}^{M}(p_{2s-1}-p_{2s})\phi\left(\frac{p_{2s-1}a_{2s-1}-p_{2s}a_{2s}}{p_{2s-1}-p_{2s}}\right)\\ &\leqslant \frac{1}{\bar{P}_m}M^{1/\beta}\left(\sum_{s=1}^{M}\left((p_{2s-1}-p_{2s})\phi\left(\frac{p_{2s-1}a_{2s-1}-p_{2s}a_{2s}}{p_{2s-1}-p_{2s}}\right)\right)^{\alpha}\right)^{1/\alpha}.\end{aligned}$$

Similarly

$$\psi(\bar{B}_n) \leqslant \frac{1}{\bar{Q}_n}N^{1/\alpha}\left(\sum_{t=1}^{N}\left((q_{2t-1}-q_{2t})\psi\left(\frac{q_{2t-1}b_{2s-1}-q_{2t}a_{2t}}{q_{2t-1}-q_{2t}}\right)\right)^{\beta}\right)^{1/\beta}.$$

Proceeding now much as in the proof of Theorems 2.1 and 2.2 and with suitable modifications, it is not hard to arrive at the desired inequality (2.14). The details are omitted here. $\square$

REMARK 2.5. Taking $\alpha=\beta=2$ in (2.14), we obtain

$$\begin{aligned}&\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\bar{P}_m\bar{Q}_n\phi(\bar{A}_m)\psi(\bar{B}_n)}{M+N}\\ &\leqslant \frac{1}{2}(KR)^{1/2}\left(\sum_{m=1}^{K}(K-m+1)\left((p_{2m-1}-p_{2m})\phi\left(\frac{p_{2m-1}a_{2m-1}-p_{2m}a_{2m}}{p_{2m-1}-p_{2m}}\right)\right)^2\right)^{1/2}\\ &\quad\times\left(\sum_{n=1}^{R}(R-n+1)\left((q_{2n-1}-q_{2n})\psi\left(\frac{q_{2n-1}b_{2n-1}-q_{2n}b_{2n}}{q_{2n-1}-q_{2n}}\right)\right)^2\right)^{1/2}.\end{aligned}\tag{2.15}$$

REMARK 2.6. By using Cauchy inequality on the right side of (2.15) twice, we obtain the following interesting Hilbert inequality with alternating signs:

$$\begin{aligned}&\sum_{M=1}^{K}\sum_{N=1}^{R}\frac{\bar{P}_m\bar{Q}_n\phi(\bar{A}_m)\psi(\bar{B}_n)}{M+N}\\ &\leqslant \frac{1}{2}(KR)^{1/2}\left(\sum_{m=1}^{k}(-1)^{m+1}p_m\cdot\sum_{m=1}^{K}(K-m+1)^{1/2}\phi\left(\frac{p_{2m-1}a_{2m-1}-p_{2m}a_{2m}}{p_{2m-1}-p_{2m}}\right)\right)^{1/2}\\ &\quad\times\left(\sum_{n=1}^{r}(-1)^{n+1}q_n\cdot\sum_{n=1}^{R}(R-n+1)^{1/2}\psi\left(\frac{q_{2n-1}b_{2n-1}-q_{2n}b_{2n}}{q_{2n-1}qp_{2n}}\right)\right)^{1/2}.\end{aligned}$$

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