A note on joint occupation times of spectrally negative Lévy risk processes with tax

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Abstract

In this paper we consider the joint Laplace transform of occupation times over disjoint intervals for spectrally negative Lévy processes with a general loss-carry-forward taxation structure. This tax structure was first introduced by Albrecher and Hipp in their paper in 2007. We obtain representations of the joint Laplace transforms in terms of scale functions and the Lévy measure associated with the driven spectrally negative Lévy processes. Two numerical examples, i.e. a Brownian motion with drift and a compound Poisson model, are provided at the end of this paper and explicit results are presented with discussions.

Key words: Occupation time; Spectrally negative Lévy process; Loss-carry-forward taxation; Brownian Motion with drift; Compound Poisson process

1 Introduction

Lévy processes are stochastic processes with independent and stationary increments. Spectrally negative Lévy processes (SNLPs) are Lévy processes with no upward jumps, which find many applications in risk theory, mathematical finance and branching processes. In the recent literature of risk theory and mathematical finance, there have been increasing interests in studying the Laplace transforms of occupation times for Lévy processes. For

*Supported in part by National Natural Science Foundation of China (No. 11601197) and Program for New Century Excellent Talents in Fujian Province University (No. Z0210103).

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general SNLPs, Laplace transforms of occupation times were studied in Landriault et al. (2011) and Loeffen et al. (2014), by adopting different approximation schemes. Quite recently, the joint Laplace transforms of occupation times under Lévy processes have been attracting much research attention. For instance, Li and Zhou (2013) considered joint occupation times under general time-homogeneous diffusion processes. Further more, Li and Zhou (2014) derived the joint Laplace transforms of occupation times over disjoint intervals under SNLPs, by adopting a fairly new approach. Some recent papers considering SNLPs include Yin and Yuen (2014) and Li et al. (2017).

The so-called loss-carry-forward taxation system (in a simplified version) was first introduced into a compound Poisson process with drift by Albrecher and Hipp (2007). Meanwhile, Kyprianou and Zhou (2009) introduced a very general taxation structure into the Lévy framework. Results regarding stochastic processes with loss-carry-forward taxation can be found in Wang and Hu (2012), Wang et al. (2011), Ming et al. (2010), Albrecher et al. (2008) and the references therein.

This paper aims to study the impact of a loss-carry-forward taxation system on the joint Laplace transforms of occupation times in SNLPs. It is motivated by the increasing role of occupation times on managing risks in risk theory. For $0 < a < b$, the occupation times of the surplus process being in intervals $(0, a)$ and (a, b) prior to ruin can be used to evaluate the performance of an insurance portfolio as well as monitoring the time an insurer's surplus remaining at critically low levels, which may help to measure the solvency risk. By incorporating taxes into the suplus models will enable us to better examine the above mentioned risks in a more real-life related environment. The obtained new occupation-time functionals are of much interest on both theoretical and practical aspects.

2 Preliminary identities for SNLPs

In this section, we shall provide some preliminary scale function related results for SNLPs. Then, we shall present our model, i.e., an SNLP imbedded with a general loss-carry-forward taxation system. Some existing fluctuational and distributional identities on our taxed model will also be given in this section.

2.1 SNLPs without tax

Let process $X = \{X_t; t \geq 0\}$ be an SNLP defined on a filtered probability space $(\Omega, \{\mathcal{F}_t; t \geq 0\}, P)$. We exclude the case of X being the negative of a subordinator. Denote by $\mathbb{P}_x$ the probability law of X given $X_0 = x$, and by $\mathbb{E}_x$ its corresponding expectation operator. The Laplace exponent of X is defined as

$$\psi(\theta) = \log \mathbb{E}_x[e^{\theta(X_1 - x)}],$$

which is finite at least for $\theta \in [0, \infty)$, where it is strictly convex and infinitely differentiable. The scale functions $\{W^{(q)}; q \geq 0\}$ of X are defined such that for each $q \geq 0$, $W^{(q)} : [0, \infty) \to$

$[0,\infty)$ is the unique strictly increasing and continuous function with a Laplace transform satisfying $\int_0^\infty e^{-\lambda x}W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\lambda)-q}$, $\lambda > \Phi(q)$, where $\Phi(q)$ is the larger solution of the equation $\psi(\lambda) = q$ (there are at most two). Let $W^{(q)\prime}(x)$ be its density and we let $W^{(q)}(x) = 0$ for $x < 0$.

Define the first up-crossing and down-crossing times of X as follows,

$$T_b^+ = \inf\{t \geq 0;\, X_t \geq b\}, \quad T_a^- = \inf\{t \geq 0;\, X_t < a\}$$

with the convention that $\inf \phi = \infty$. In addition, define for $q \geq 0$,

$$Z^{(q)}(x) = \begin{cases} 1 + q\int_0^x W^{(q)}(y)dy, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

According to Kyprianou (2014) we know that, for $x \leq b$,

$$\mathbb{E}_x\left[e^{-qT_b^+}; T_b^+ < T_0^-\right] = \frac{W^{(q)}(x)}{W^{(q)}(b)}, \quad \mathbb{E}_x\left[e^{-qT_0^-}; T_0^- < T_b^+\right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(b)}Z^{(q)}(b).$$

Furthermore, it has been verified by Li and Zhou (2014) that, for $q_1, q_2 \geq 0$, $a, x \in [0,b]$ and $b \in (0,\infty)$, we have,

$$\begin{aligned} g_1(x,b) &:= \mathbb{E}_x\Big[\exp\Big\{-q_1\int_0^{T_0^-}\mathbf{1}_{(0,a)}(X_s)ds - q_2\int_0^{T_0^-}\mathbf{1}_{(a,b)}(X_s)ds\Big\}; T_0^- < T_b^+\Big] \\ &= Z_a^{(q_1,q_2)}(x) - \frac{W_a^{(q_1,q_2)}(x)Z_a^{(q_1,q_2)}(b)}{W_a^{(q_1,q_2)}(b)}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} g_2(x,b) &:= \mathbb{E}_x\Big[\exp\Big\{-q_1\int_0^{T_b^+}\mathbf{1}_{(0,a)}(X_s)ds - q_2\int_0^{T_b^+}\mathbf{1}_{(a,b)}(X_s)ds\Big\}; T_0^- > T_b^+\Big] \\ &= \frac{W_a^{(q_1,q_2)}(x)}{W_a^{(q_1,q_2)}(b)}. \end{aligned} \tag{2.2}$$

Here, for $q_1, q_2 \geq 0$ and $0 \leq a \leq x$,

$$\begin{aligned} W_a^{(q_1,q_2)}(x) &:= W^{(q_1)}(x) - (q_1 - q_2)\int_a^x W^{(q_1)}(y)W^{(q_2)}(x-y)dy, \\ Z_a^{(q_1,q_2)}(x) &:= Z^{(q_1)}(x) - (q_1 - q_2)\int_a^x Z^{(q_1)}(y)W^{(q_2)}(x-y)dy. \end{aligned}$$

2.2 SNLPs with tax

Following the ideas in Albrecher and Hipp (2007) and Kyprianou and Zhou (2009), we are interested in how the loss-carry-forward tax payments will affect the quantitative behavior of the driven SNLP. Assume that the cumulative tax payments by time t are given by

$$\int_0^t \gamma(S_w^X)\mathrm{d}S_w^X,$$

where $\gamma : [0,\infty) \to [0,1)$ is a measurable function such that $\int_0^\infty (1-\gamma(s))ds = \infty$, $\{S_t^X := \sup_{0\le w\le t} X_w; t \ge 0\}$ is the running maximum process of X, and X is the surplus process without tax. The net aggregate surplus process is then given by

$$U_t = X_t - \int_0^t \gamma(S_w^X)\mathrm{d}S_w^X. \tag{2.3}$$

Define the first up-crossing and down-crossing times of $\{U_t; t \ge 0\}$, respectively, as,

$$\tau_b^+ = \inf\{t \ge 0;\, U_t \ge b\}, \qquad \tau_a^- = \inf\{t \ge 0;\, U_t < a\}.$$

We further define $\rho^+_{S^U_{\tau_a^-}} = \inf\{t > \tau_a^-;\, U_t \ge S^U_{\tau_a^-}\}$ with $S_t^U := \sup_{0\le w\le t} U_w$. Actually, $\rho^+_{S^U_{\tau_a^-}}$ is the first time such that the driven SNLP X is taxed again after the first down-crossing time τ_a^-. For the purpose of simplification, we shall define two auxiliary functions. The first one is

$$\xi_x^{(q)}(y;a) := \exp\Big\{ -\int_x^y \frac{W^{(q)\prime}(w-a)}{W^{(q)}(w-a)(1-\gamma(\bar\gamma^{-1}(w)))}dw\Big\}$$

with its first derivative w.r.t. y in the form of $\xi_x^{(q)\prime}(y;a) = -\frac{\xi_x^{(q)}(y)}{1-\gamma(\bar\gamma^{-1}(y))}\frac{W^{(q)\prime}(y-a)}{W^{(q)}(y-a)}$, where $\bar\gamma^{-1}(\cdot)$ denotes the inverse function of $\bar\gamma(s) = x + \int_x^s (1-\gamma(y))dy$. Let $\xi_x^{(q)}(y) \hat{=} \xi_x^{(q)}(y;0)$. As mentioned in Kyprianou and Zhou (2009), when $\gamma \in (0,1)$ is a constant and $a = 0$, we have $\bar\gamma(s) = s(1-\gamma) + \gamma x$ and $\xi_x^{(q)}(y) = \Big(\frac{W^{(q)}(x)}{W^{(q)}(y)}\Big)^{1/(1-\gamma)}$. The second auxiliary function is $\zeta_x^{(q)}(y) = W^{(q)\prime}(x-y) - \frac{W^{(q)\prime}(x)}{W^{(q)}(x)}W^{(q)}(x-y)$, which has $\zeta_x^{(q)\prime}(0) = \frac{W^{(q)\prime}(x)^2}{W^{(q)}(x)} - W^{(q)\prime\prime}(x)$.

Theorems 1.1 and 1.3 in Kyprianou and Zhou (2009) give that:

- for $a \le x \le b$,
$$\mathbb{E}_x[e^{-q\tau_b^+}; \tau_b^+ < \tau_a^-] = \xi_x^{(q)}(b;a); \tag{2.4}$$
- for $z > 0, \theta - a \ge y \ge 0$ and $\theta \ge x \ge a$,
$$\begin{aligned}&\mathbb{E}_x\Big[e^{-q\tau_a^-}; S^U_{\tau_a^-} \in d\theta, U_{\tau_a^- -} \in a+dy, -U_{\tau_a^-} \in -a+dz\Big]\\ =&\frac{\xi_x^{(q)}(\theta;a)}{1-\gamma(\bar\gamma^{-1}(\theta))}\Big[\upsilon(y+dz)\mathbf{1}_{\{y<\theta-a\}}\zeta^{(q)}_{\theta-a}(y)dy + W^{(q)}(0+)\upsilon(\theta-a+dz)\delta_{\theta-a}(dy)\Big]d\theta;\end{aligned} \tag{2.5}$$
- and for $\theta \ge x \ge a$,
$$\mathbb{E}_x[e^{-q\tau_a^-}; S^U_{\tau_a^-} \in d\theta, U_{\tau_a^-} = a] = \frac{\sigma^2\xi_x^{(q)}(\theta;a)}{2[1-\gamma(\bar\gamma^{-1}(\theta))]}\zeta^{(q)\prime}_{\theta-a}(0)\,d\theta, \tag{2.6}$$

  where σ is the Gaussian coefficient in the Lévy-Itô decomposition of X, υ denotes the Lévy measure of $-X$ and $\delta_{\theta-a}(dy)$ is the Dirac measure which assigns unit mass to the point $\theta - a$.

3 Main results

First of all, we shall define the joint Laplace transforms of the occupation times of the disjoint sets $(0,a)$ and (a,b) for the process given in (2.3) prior to its two-sided exit from the set $[0,b]$, which are the two primary objects in this paper:

$$f_1(x) := \mathbb{E}_x\Big[\exp\Big\{-q_1\int_0^{\tau_0^-}\mathbf{1}_{(0,a)}(U_s)ds - q_2\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(U_s)ds\Big\};\tau_0^- < \tau_b^+\Big],$$

$$f_2(x) := \mathbb{E}_x\Big[\exp\Big\{-q_1\int_0^{\tau_b^+}\mathbf{1}_{(0,a)}(U_s)ds - q_2\int_0^{\tau_b^+}\mathbf{1}_{(a,b)}(U_s)ds\Big\};\tau_0^- > \tau_b^+\Big].$$

Let $h_0(x) = \mathbb{E}_x[e^{-q\tau_0^-};\tau_0^- < \infty]$. To calculate $f_1(x)$ and $f_2(x)$, we need the following lemma.

***Lemma* 1.** *(Two-sided exit problem) For any $q>0$ and $x<b$, we have,*

$$\mathbb{E}_x\Big[e^{-q\tau_0^-};\tau_0^- < \tau_b^+\Big] = h_0(x) - \xi_x^{(q)}(b)h_0(b), \tag{3.1}$$

where, for $x \geq 0$,

$$h_0(x) = \int_x^\infty \frac{\xi_x^{(q)}(\theta)}{1-\gamma(\bar{\gamma}^{-1}(\theta))}\Big[\int_0^{\theta-}\zeta_\theta^{(q)}(y)\upsilon((y,\infty))dy + W^{(q)}(0+)\upsilon((\theta,\infty))\Big]d\theta$$
$$+\frac{\sigma^2}{2}\int_x^\infty \frac{\xi_x^{(q)}(\theta)}{1-\gamma(\bar{\gamma}^{-1}(\theta))}\zeta_\theta^{(q)\prime}(0)\,d\theta. \tag{3.2}$$

Proof. The proof of this lemma is straightforward so it is omitted.

Remark. When $\gamma \in (0,1)$ is a constant and $a=0$, we have $\xi_x^{(q)}(\theta) = \Big(\frac{W^{(q)}(x)}{W^{(q)}(\theta)}\Big)^{1/(1-\gamma)}$ and

$$h_0(x) = \frac{1}{1-\gamma}\int_x^\infty \Big(\frac{W^{(q)}(x)}{W^{(q)}(\theta)}\Big)^{\frac{1}{1-\gamma}}\Big[\int_0^{\theta-}\zeta_\theta^{(q)}(y)\upsilon((y,\infty))dy + W^{(q)}(0+)\upsilon((\theta,\infty))\Big]d\theta$$
$$+\frac{\sigma^2}{2(1-\gamma)}\int_x^\infty \Big(\frac{W^{(q)}(x)}{W^{(q)}(\theta)}\Big)^{\frac{1}{1-\gamma}}\zeta_\theta^{(q)\prime}(0)\,d\theta. \tag{3.3}$$

Within the rest of this section we shall present our main results.

***Theorem* 1.** *(1) For $0 \leq x \leq a$, we have,*

$$f_1(x) = \mathbb{E}_x[e^{-q_1\tau_0^-};\tau_0^- < \tau_a^+] + \mathbb{E}_x[e^{-q_1\tau_a^+};\tau_a^+ < \tau_0^-]f_1(a), \tag{3.4}$$
$$f_2(x) = \mathbb{E}_x[e^{-q_1\tau_a^+};\tau_a^+ < \tau_0^-]f_2(a), \tag{3.5}$$

with $\mathbb{E}_x[e^{-q_1\tau_a^+};\tau_a^+ < \tau_0^-]$ and $\mathbb{E}_x[e^{-q_1\tau_0^-};\tau_0^- < \tau_a^+]$ given by (2.4) and (3.1), respectively.

(2) For $a \leq x \leq b$, we have,

$$f_1(x) = \int_x^b h_2(y;a)\exp\Big\{-\int_x^y h_1(w;a)dw\Big\}dy, \tag{3.6}$$

$$f_2(x) = \exp\Big\{-\int_x^b h_1(y;a)dy\Big\}. \tag{3.7}$$

Here,

$$\begin{aligned}
h_1(x;a) \;=\; & \frac{1}{1-\gamma(\bar{\gamma}^{-1}(x))}\left[\frac{W^{(q_2)\prime}(x-a)}{W^{(q_2)}(x-a)} - \frac{\sigma^2}{2} g_2(a,x)\zeta_{x-a}^{(q_2)\prime}(0)\right. \\
& \left. - \int_0^a g_2(a-z,x)\left(\int_0^{(x-a)-} \zeta_{x-a}^{(q_2)}(y)\upsilon(y+dz)dy + W^{(q_2)}(0+)\upsilon(x-a+dz)\right)\right], \\
h_2(x;a) \;=\; & \frac{1}{1-\gamma(\bar{\gamma}^{-1}(x))}\left[\frac{\sigma^2}{2} g_1(a,x)\zeta_{x-a}^{(q_2)\prime}(0)\right. \\
& \left. + \int_0^\infty g_1(a-z,x)\left(\int_0^{(x-a)-} \zeta_{x-a}^{(q_2)}(y)\upsilon(y+dz)dy + W^{(q_2)}(0+)\upsilon(x-a+dz)\right)\right],
\end{aligned}$$

with $g_1(\cdot,\cdot)$ and $g_2(\cdot,\cdot)$ being the two binary functionals given by (2.1) and (2.2).

Proof. The proof is given in Appendix.

Remark. When $\gamma(x) \equiv 0$, the net aggregate surplus process U agrees with the SNLP X, and so Theorem 1 provides some new expressions of the Laplace transforms of the joint occupation times, which are different in form from the results given in Theorem 3.1 of Li and Zhou (2014). The equivalence can be shown in special cases, eg Brownian motion with drift and compound Poisson processes, but not in general.

4 Two Special Cases

In this section, we shall examine two special cases of our model, i.e. a Brownian Motion with drift and a compound Poisson model. Explicit results are derived in respect of the joint Laplace transforms of the occupation times discussed in previous section.

Example 1. Let $X_t = x + \mu t + \sigma B_t$ ($\mu \neq 0,\ \sigma > 0$) be a Brownian motion with drift. It is worth mentioning that its associated Lévy measure is identically zero. One can verify that it has the scale function

$$W^{(q)}(x) = \kappa(q)\big[e^{\lambda_1(q)x} - e^{\lambda_2(q)x}\big], \qquad x \geq 0, \tag{4.1}$$

where $\kappa(q) \hat{=} (2q\sigma^2+\mu^2)^{-\frac{1}{2}}$, $\lambda_1(q) \hat{=} \frac{-\mu+\sqrt{2q\sigma^2+\mu^2}}{\sigma^2}$ and $\lambda_2(q) \hat{=} \frac{-\mu-\sqrt{2q\sigma^2+\mu^2}}{\sigma^2}$. This is a well-known result and can be found in many references including Kuznetsov et al. (2012). Hence, by the aforementioned definition of the function $Z^{(q)}(\cdot)$, we have, for $x, q \geq 0$,

$$Z^{(q)}(x) \;=\; 1 - q\kappa(q)\Big[\frac{1}{\lambda_1(q)} - \frac{1}{\lambda_2(q)}\Big] + q\kappa(q)\Big[\frac{e^{\lambda_1(q)x}}{\lambda_1(q)} - \frac{e^{\lambda_2(q)x}}{\lambda_2(q)}\Big].$$

In addition, for $q_1, q_2 \geq 0$ and $0 \leq a \leq x$, by some algebraic manipulations we have

$$W_a^{(q_1,q_2)}(x) = l_{12}e^{\lambda_1(q_2)x} + l_{22}e^{\lambda_2(q_2)x},$$

where

$$l_{12} = \frac{[\kappa(q_1)+\kappa(q_2)]e^{\lambda_1(q_1)a} - [\kappa(q_1)-\kappa(q_2)]e^{\lambda_2(q_1)a}}{2e^{\lambda_1(q_2)a}},$$
$$l_{22} = \frac{[\kappa(q_1)-\kappa(q_2)]e^{\lambda_1(q_1)a} - [\kappa(q_1)+\kappa(q_2)]e^{\lambda_2(q_1)a}}{2e^{\lambda_2(q_2)a}}.$$

Define

$$l_{11} = \frac{[\kappa(q_1)+\kappa(q_2)]e^{\lambda_1(q_1)a} + [\kappa(q_1)-\kappa(q_2)]e^{\lambda_2(q_1)a}}{2e^{\lambda_1(q_2)a}},$$
$$l_{21} = \frac{[\kappa(q_1)-\kappa(q_2)]e^{\lambda_1(q_1)a} + [\kappa(q_1)+\kappa(q_2)]e^{\lambda_2(q_1)a}}{2e^{\lambda_2(q_2)a}}.$$

Then for $x \geq a \geq 0$, $Z_a^{(q_1,q_2)}(x) = k_{12}e^{\lambda_1(q_2)x} + k_{22}e^{\lambda_2(q_2)x}$, where $k_{12} = \frac{\mu}{2}l_{12} + \frac{1}{2\kappa(q_1)}l_{11}$ and $k_{22} = \frac{\mu}{2}l_{22} + \frac{1}{2\kappa(q_1)}l_{21}$. By (2.1) and (2.2), it can be verified that, for $q_1, q_2 \geq 0$ and $a, x \in [0, b]$ with $b \in (0, \infty)$,

$$g_1(x,b) = \sum_{i=1}^{2} k_{i2}e^{\lambda_i(q_2)x} - \Big(\sum_{i=1}^{2} k_{i2}e^{\lambda_i(q_2)b}\Big)\frac{\sum_{i=1}^{2} l_{i2}e^{\lambda_i(q_2)x}}{\sum_{i=1}^{2} l_{i2}e^{\lambda_i(q_2)b}} \tag{4.2}$$

and

$$g_2(x,b) = \frac{l_{12}e^{\lambda_1(q_2)x} + l_{22}e^{\lambda_2(q_2)x}}{l_{12}e^{\lambda_1(q_2)b} + l_{22}e^{\lambda_2(q_2)b}}. \tag{4.3}$$

Finally, combining (4.1), (4.2) and (4.3) we have

$$h_1(x;a) = \frac{\lambda_1(q_2)e^{\lambda_1(q_2)(x-a)} - \lambda_2(q_2)e^{\lambda_2(q_2)(x-a)} - 2(\frac{\mu^2}{\sigma^2}+2q_2)\kappa(q_2)e^{-\frac{2\mu(x-a)}{\sigma^2}}g_2(a,x)}{[1-\gamma(\bar{\gamma}^{-1}(x))][e^{\lambda_1(q_2)(x-a)} - e^{\lambda_2(q_2)(x-a)}]}$$

and

$$h_2(x;a) = \frac{2(\frac{\mu^2}{\sigma^2}+2q_2)\kappa(q_2)e^{-\frac{2\mu(x-a)}{\sigma^2}}g_1(a,x)}{[1-\gamma(\bar{\gamma}^{-1}(x))][e^{\lambda_1(q_2)(x-a)} - e^{\lambda_2(q_2)(x-a)}]}.$$

Also, by (3.2) we have,

$$h_0(x) = \int_x^\infty \frac{\exp\big\{-\int_x^\theta \frac{\lambda_1(q)e^{\lambda_1(q)w}-\lambda_2(q)e^{\lambda_2(q)w}}{\left(e^{\lambda_1(q)w}-e^{\lambda_2(q)w}\right)(1-\gamma(\bar{\gamma}^{-1}(w)))}dw\big\}}{1-\gamma(\bar{\gamma}^{-1}(\theta))} \frac{2(\frac{\mu^2}{\sigma^2}+2q)\kappa(q)e^{-\frac{2\mu\theta}{\sigma^2}}}{e^{\lambda_1(q)\theta}-e^{\lambda_2(q)\theta}}d\theta.$$

Particularly, for $\gamma(\cdot) \equiv \gamma \in (0,1)$, (3.3) yields

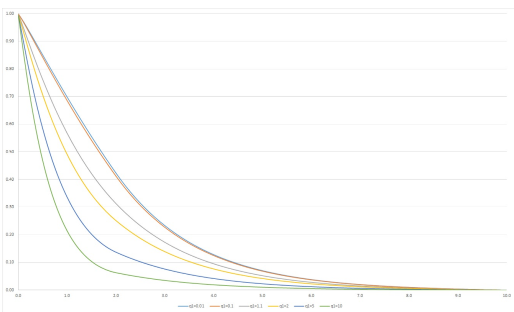
$$h_0(x) = \frac{1}{1-\gamma}\int_x^\infty \left(\frac{e^{\lambda_1(q)x}-e^{\lambda_2(q)x}}{e^{\lambda_1(q)\theta}-e^{\lambda_2(q)\theta}}\right)^{\frac{1}{1-\gamma}} \frac{2(\frac{\mu^2}{\sigma^2}+2q)\kappa(q)e^{-\frac{2\mu\theta}{\sigma^2}}}{e^{\lambda_1(q)\theta}-e^{\lambda_2(q)\theta}}\,d\theta.$$

Having obtained the explicit expressions of $h_0(x)$, $h_1(x)$ and $h_2(x)$, by Theorem 1 we can calculate $f_1(x)$ and $f_2(x)$, which are the main objects of this paper.

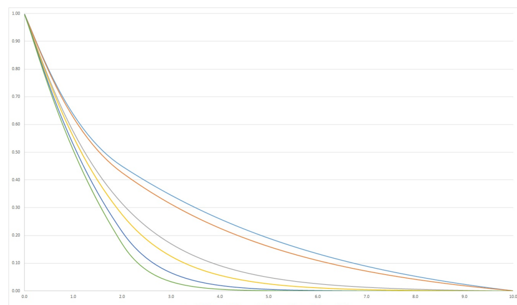
Let $\mu = 1, \sigma = 3, a = 2, b = 10$ and $\gamma(x) \equiv 0.1$. Using the above results we calculate $f_1(x)$ and $f_2(x)$ numerically with the software *Mathematica*. To better demonstrate the results, we let x vary from 0 to b with a step of 0.1 for different choices of (q_1, q_2) values. The results are summarised in Figures 1-2.

- The patterns of $f_1(x), 0 \leq x \leq b$, in Figure 1 show that when x is either too close to 0 or too close to b, the occupation times of the Brownian Motion for intervals $(0, a)$ and (a, b) are all very small. This is because the former case leads to an early down-crossing of level 0 and the latter one causes a very quick up-crossing of the boundary b, i.e. $f_1(0) = 1$ and $f_1(b) = 0$. A trend that $f_1(x)$ decreases when q_1, q_2 increase is also evident. Also, Figure 1 demonstrates a relationship that when $x < a$, changing the value of q_1 has much bigger impact on the shape of $f_1(x)$ than changing the value of q_2 and vice versa when $x \geq a$.

- Figure 2 shows the shape of $f_2(x)$, $0 \leq x \leq b$, for the same choices of q_1 and q_2. Interestingly, changing values of q_1 nearly has no impact on $f_2(x)$, with only minor effect when x is small. It is because under the definition of $f_2(x)$, when $\tau_0^- > \tau_b^+$, the Brownian motion shall spend most time in interval (a, b) before up-crossing b. On the contrary, different values of q_2 generate dramatically different shapes of $f_2(x)$ for the same reason.

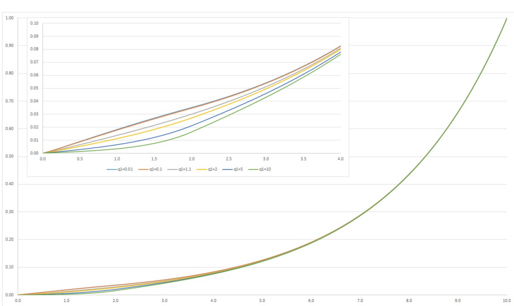


(a) $q_2 = 1$ with various q_1 values

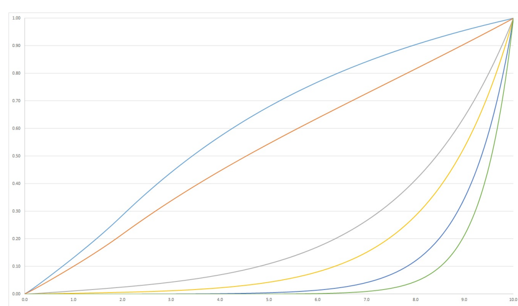


(b) $q_1 = 1$ with various q_2 values

Figure 1: The graph of $f_1(x)$ with $a = 2$ and $b = 10$



(a) $q_2 = 1$ with various q_1 values



(b) $q_1 = 1$ with various q_2 values

Figure 2: The graph of $f_2(x)$ with $a = 2$ and $b = 10$

Example 2. Let $X_t = x + ct - S_t$ with $x \geq 0$, where S_t is a compound Poisson process with rate $\lambda > 0$ and an exponential jump distribution $F(x) = 1 - e^{-\mu x}$, $\mu > 0$. It can be verified that $\upsilon(dx) = \lambda F(dx)$ and it has the scale function

$$W^{(q)}(x) = \frac{A_1(q)}{c} e^{\theta_1(q)x} - \frac{A_2(q)}{c} e^{\theta_2(q)x}, \qquad x \geq 0, \tag{4.4}$$

where $A_1(q) = \frac{\mu+\theta_1(q)}{\theta_1(q)-\theta_2(q)}$, $A_2(q) = \frac{\mu+\theta_2(q)}{\theta_1(q)-\theta_2(q)}$, $\theta_1(q) = \frac{\lambda+q-c\mu+K(q)}{2c}$, $\theta_2(q) = \frac{\lambda+q-c\mu-K(q)}{2c}$ and $K(q) = \sqrt{(c\mu-\lambda-q)^2+4cq\mu}$. So $W^{(q)}(0+) = \frac{1}{c}$. Further, for $x, q \geq 0$,

$$Z^{(q)}(x) = 1 + \frac{qA_1(q)}{c\theta_1(q)}\big(e^{\theta_1(q)x} - 1\big) - \frac{qA_2(q)}{c\theta_2(q)}\big(e^{\theta_2(q)x} - 1\big).$$

In addition, for $q_1, q_2 \geq 0$ and $0 \leq a \leq x$, by some algebraic manipulations we have

$$\begin{aligned}
W_a^{(q_1,q_2)}(x) &= \frac{A_1(q_2)(q_1-q_2)}{c^2}\Big[A_1(q_1)\frac{e^{[\theta_1(q_1)-\theta_1(q_2)]a}}{\theta_1(q_1)-\theta_1(q_2)} - A_2(q_1)\frac{e^{[\theta_2(q_1)-\theta_1(q_2)]a}}{\theta_2(q_1)-\theta_1(q_2)}\Big]e^{\theta_1(q_2)x} \\
&\quad - \frac{A_2(q_2)(q_1-q_2)}{c^2}\Big[A_1(q_1)\frac{e^{[\theta_1(q_1)-\theta_2(q_2)]a}}{\theta_1(q_1)-\theta_2(q_2)} - A_2(q_1)\frac{e^{[\theta_2(q_1)-\theta_2(q_2)]a}}{\theta_2(q_1)-\theta_2(q_2)}\Big]e^{\theta_2(q_2)x} \\
&\hat{=}\ \beta_{12}e^{\theta_1(q_2)x} + \beta_{22}e^{\theta_2(q_2)x}, \\
Z_a^{(q_1,q_2)}(x) &= \frac{q_1(q_1-q_2)A_1(q_2)}{c^2}\Big\{\frac{A_1(q_1)e^{[\theta_1(q_1)-\theta_1(q_2)]a}}{\theta_1(q_1)[\theta_1(q_1)-\theta_1(q_2)]} - \frac{A_2(q_1)e^{[\theta_2(q_1)-\theta_1(q_2)]a}}{\theta_2(q_1)[\theta_2(q_1)-\theta_1(q_2)]}\Big\}e^{\theta_1(q_2)x} \\
&\quad - \frac{q_1(q_1-q_2)A_2(q_2)}{c^2}\Big\{\frac{A_1(q_1)e^{[\theta_1(q_1)-\theta_2(q_2)]a}}{\theta_1(q_1)[\theta_1(q_1)-\theta_2(q_2)]} - \frac{A_2(q_1)e^{[\theta_2(q_1)-\theta_2(q_2)]a}}{\theta_2(q_1)[\theta_2(q_1)-\theta_2(q_2)]}\Big\}e^{\theta_2(q_2)x} \\
&\hat{=}\ \chi_{12}e^{\theta_1(q_2)x} + \chi_{22}e^{\theta_2(q_2)x}.
\end{aligned}$$

As a result, for $q_1, q_2 \geq 0$ and $a, x \in [0, b]$ with $b \in (0, \infty)$, we have

$$g_1(x,b) = \sum_{i=1}^{2}\chi_{i2}e^{\theta_i(q_2)x} - \sum_{i=1}^{2}\chi_{i2}e^{\theta_i(q_2)b}\frac{\sum_{i=1}^{2}\beta_{i2}e^{\theta_i(q_2)x}}{\sum_{i=1}^{2}\beta_{i2}e^{\theta_i(q_2)b}}, \tag{4.5}$$

$$g_2(x,b) = \frac{\beta_{12}e^{\theta_1(q_2)x} + \beta_{22}e^{\theta_2(q_2)x}}{\beta_{12}e^{\theta_1(q_2)b} + \beta_{22}e^{\theta_2(q_2)b}}. \tag{4.6}$$

From Theorem 1 and the above results we have

$$\begin{aligned}
h_1(x;a) &= \frac{1}{1-\gamma(\bar{\gamma}^{-1}(x))}\Bigg[\frac{A_1(q_2)\theta_1(q_2)e^{\theta_1(q_2)(x-a)} - A_2(q_2)\theta_2(q_2)e^{\theta_2(q_2)(x-a)}}{A_1(q_2)e^{\theta_1(q_2)(x-a)} - A_2(q_2)e^{\theta_2(q_2)(x-a)}} \\
&\quad - \frac{\lambda\mu}{c}\frac{B(x-a;q_2) + e^{-\mu(x-a)}}{\sum_{i=1}^{2}\beta_{i2}e^{\theta_i(q_2)x}}\sum_{i=1}^{2}\beta_{i2}\frac{e^{\theta_i(q_2)a} - e^{-\mu a}}{\theta_i(q_2)+\mu}\Bigg].
\end{aligned}$$

where

$$B(x;q_2) = \frac{A_1(q_2)A_2(q_2)[\theta_1(q_2)-\theta_2(q_2)]e^{(\theta_1(q_2)+\theta_2(q_2))x}}{A_1(q_2)e^{\theta_1(q_2)x} - A_2(q_2)e^{\theta_2(q_2)x}}\Big[\frac{1-e^{-(\theta_2(q_2)+\mu)x}}{\theta_2(q_2)+\mu} - \frac{1-e^{-(\theta_1(q_2)+\mu)x}}{\theta_1(q_2)+\mu}\Big].$$

Similarly,

$$h_2(x;a) = \frac{1}{1-\gamma(\bar{\gamma}^{-1}(x))}\frac{\lambda\mu}{c}\left(B(x-a;q_2)+e^{-\mu(x-a)}\right)\left[\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\chi_{ij}\left(e^{\theta_i(q_j)a}-e^{-\mu a}\right)}{\theta_i(q_j)+\mu}\right.$$
$$\left.+\frac{\chi_0\left(1-e^{-\mu a}\right)+e^{-\mu a}}{\mu}-\frac{\chi_0+\sum_{i=1}^{2}\sum_{j=1}^{2}\chi_{ij}e^{\theta_i(q_j)x}}{\sum_{i=1}^{2}\sum_{j=1}^{2}\beta_{ij}e^{\theta_i(q_j)x}}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\beta_{ij}\left(e^{\theta_i(q_j)a}-e^{-\mu a}\right)}{\theta_i(q_j)+\mu}\right].$$

In addition, by the expression of $h_0(x)$ we get,

$$h_0(x) = \frac{\lambda}{c}\int_x^{\infty}\frac{\exp\left\{-\int_x^{\theta}\frac{A_1(q)\theta_1(q)e^{\theta_1(q)w}-A_2(q)\theta_2(q)e^{\theta_2(q)w}}{\left(A_1(q)e^{\theta_1(q)w}-A_2(q)e^{\theta_2(q)w}\right)(1-\gamma(\bar{\gamma}^{-1}(w)))}dw\right\}}{1-\gamma(\bar{\gamma}^{-1}(\theta))}\left(B(\theta,q)+e^{-\mu\theta}\right)d\theta.$$

Particularly, when $\gamma(\cdot)\equiv\gamma$,

$$h_0(x) = \frac{\lambda}{c(1-\gamma)}\int_x^{\infty}\left(\frac{A_1(q)e^{\theta_1(q)x}-A_2(q)e^{\theta_2(q)x}}{A_1(q)e^{\theta_1(q)\theta}-A_2(q)e^{\theta_2(q)\theta}}\right)^{\frac{1}{1-\gamma}}\left(B(\theta,q)+e^{-\mu\theta}\right)d\theta.$$

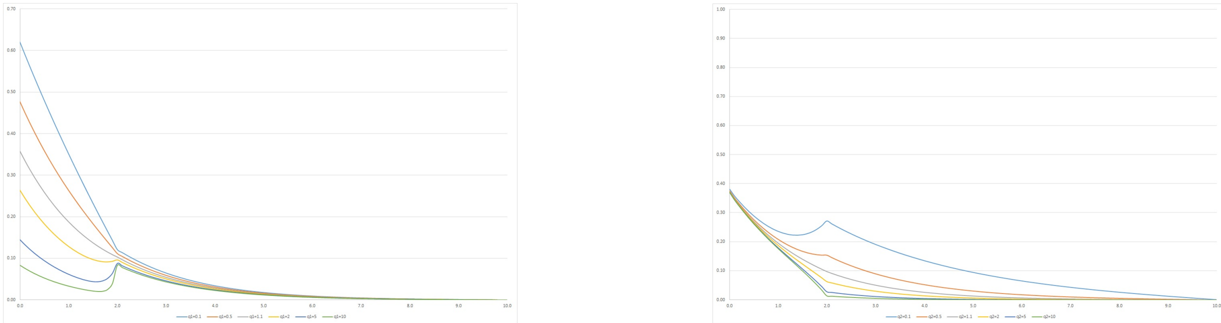
Again, we shall present some numerical results. Let $\lambda = 1, \mu = 1, c = 1.2, a = 2, b = 10$ with a constant tax rate $\gamma = 0.1$. Similar to Example 1, we let initial surplus x to vary from 0 to b with a step of 0.1 for different choices of (q_1, q_2) values. The results are summarised in Figures 3-4.

- The patterns of $f_1(x), 0 \le x \le b$ shown in Figure 3 are quite different from those in Figure 1. Bear in mind that for the compound Poisson model with a positive safety loading, there is a positive probability that the process will never down-cross a when $x = a$. It makes $x = a$ an obvious divisor on the shape of $f_1(x), 0 \le x \le b$.

- In Figure 3(a), when q_2 is fixed, $x = 0$ gives the biggest possibility of $\tau_0^- < \tau_a^+$. The proposed various q_1 values lead to significantly spread-out $f_1(0)$ values. On the contrary, fixing q_1 and changing q_2 shall have minimal impact on $f_1(0)$ which is shown in Figure 3(b). This is because the only component in $f_1(x), 0 \le x < a$, that depends on q_2 is $f_1(a)$.

- When $q_1 < q_2$, the impact of increasing x on $E_x[e^{-q_1\tau_0^-};\tau_0^-<\tau_a^+]$ (negative impact) is bigger than the impact on $E_x[e^{-q_1\tau_a^+};\tau_0^->\tau_a^+]$ (positive impact), so the overall impact is negative and $f_1(x), 0 \le x < a$, displays a clear downward trend. When $q_1 > q_2$, the overall impact of increasing x is initially negative, but when $x \to a$, τ_a^+ decreases faster and so the positive impact over-performs the negative one, which causes the U shape at the end of interval $[0, a)$. These arguments are confirmed by Figure 3.

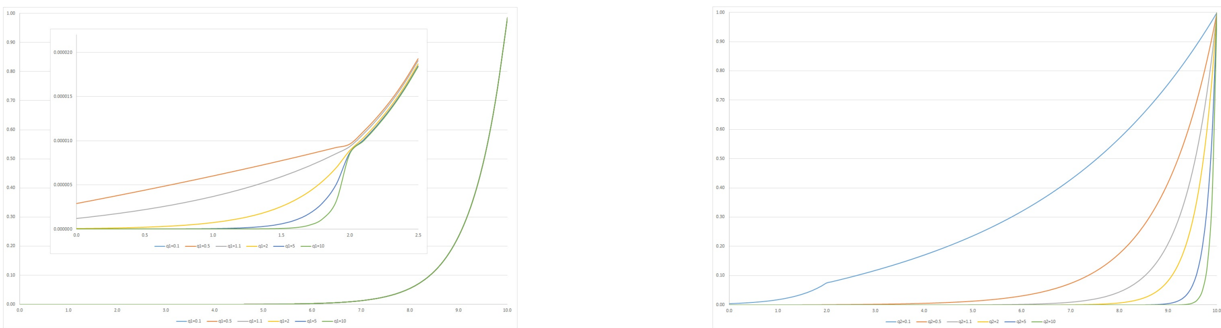
- Figure 4 shows the shape of $f_2(x)$, $0 \le x \le b$, for the same choices of q_1 and q_2. We have similar finds as in Example 1, except the significantly different shapes seen for $x < a$ in Figure 4(a). Again, changing q_1 nearly has no significant impact on $f_2(x)$ but different values of q_2 lead to dramatically different shapes of $f_2(x)$ for $0 \le x \le b$.



(a) $q_2 = 1$ with various q_1 values

(b) $q_1 = 1$ with various q_2 values

Figure 3: The graph of $f_1(x)$ with $a = 2$ and $b = 10$



(a) $q_2 = 1$ with various q_1 values

(b) $q_1 = 1$ with various q_2 values

Figure 4: The graph of $f_2(x)$ with $a = 2$ and $b = 10$

A Appendix

Proof of Theorem 1. For the purpose of convenience, we shall define

$$D_1(t_1, t_2) = \int_{t_1}^{t_2} 1_{(0,a)}(U_s)ds, \qquad D_2(t_1, t_2) = \int_{t_1}^{t_2} 1_{(a,b)}(U_s)ds,$$

which are the total duration of our process U_s in the interval $(0, a)$ and (a, b) between times t_1 and t_2 respectively. Then we have:

(1) For $0 \le x \le a$, we have

$$\begin{aligned}
f_1(x) &= \mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_0^-)-q_2 D_2(0,\tau_0^-)}; \tau_0^- < \tau_a^+\right] + \mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_0^-)-q_2 D_2(0,\tau_0^-)}; \tau_a^+ < \tau_0^- < \tau_b^+\right] \\
&= \mathbb{E}_x\left[e^{-q_1\tau_0^-}; \tau_0^- < \tau_a^+\right] + \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_0^-)-q_2 D_2(0,\tau_0^-)}; \tau_a^+ < \tau_0^- < \tau_b^+ \middle| \mathcal{F}_{\tau_a^+}\right]\right] \\
&= \mathbb{E}_x\left[e^{-q_1\tau_0^-}; \tau_0^- < \tau_a^+\right] + \mathbb{E}_x\left[e^{-q_1\tau_a^+}; \tau_a^+ < \tau_0^-\right] f_1(a), \\
f_2(x) &= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_b^+)-q_2 D_2(0,\tau_b^+)}; \tau_0^- > \tau_b^+ > \tau_a^+ \middle| \mathcal{F}_{\tau_a^+}\right]\right] \\
&= \mathbb{E}_x\left[e^{-q_1\tau_a^+}; \tau_a^+ < \tau_0^-\right] f_2(a).
\end{aligned}$$

(2) For $a \le x < b$, we have

$$f_1(x) = \mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_0^-)-q_2 D_2(0,\tau_0^-)}; \tau_a^- < \rho^+_{S^U_{\tau_a^-}} < \tau_0^- < \tau_b^+\right]$$
$$+\mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_0^-)-q_2 D_2(0,\tau_0^-)}; \tau_a^- \le \tau_0^- < \rho^+_{S^U_{\tau_a^-}} < \tau_b^+\right]$$
$$= \mathbb{E}_x\left[e^{-q_2 D_2(0,\tau_a^-)}\mathbb{E}_x\Big[\exp\Big\{-q_1 D_1(\tau_a^-, \rho^+_{S^U_{\tau_a^-}}) - q_2 D_2(\tau_a^-, \rho^+_{S^U_{\tau_a^-}})\Big\}\right.$$
$$\times\mathbb{E}_x\Big[e^{-q_1 D_1(\rho^+_{S^U_{\tau_a^-}},\tau_0^-)-q_2 D_2(\rho^+_{S^U_{\tau_a^-}},\tau_0^-)}; \tau_0^- < \tau_b^+\Big|\mathcal{F}_{\rho^+_{S^U_{\tau_a^-}}}\Big]; \rho^+_{S^U_{\tau_a^-}} < \tau_0^-\Big|\mathcal{F}_{\tau_a^-}\Big]; \left.S^U_{\tau_a^-} < b\right]$$
$$+\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-q_1 D_1(0,\tau_0^-)-q_2 D_2(0,\tau_0^-)}; \tau_a^- \le \tau_0^- < \tau_b^+, \rho^+_{S^U_{\tau_a^-}} > \tau_0^- |\mathcal{F}_{\tau_a^-}\right]\right]$$
$$= \mathbb{E}_x\left[e^{-q_2\tau_a^-}\left(\mathbb{E}_{\tilde{u}}\left[e^{-q_1\int_0^{T^+_{\bar{u}}}\mathbf{1}_{(0,a)}(X_s)ds - q_2\int_0^{T^+_{\bar{u}}}\mathbf{1}_{(a,b)}(X_s)ds}; T^+_{\bar{u}} < T^-_0\right]\Big|_{\substack{\bar{u}=S^U_{\tau_a^-}\\ \tilde{u}=U_{\tau_a^-}}}\right) f_1(S^U_{\tau_a^-}); S^U_{\tau_a^-} < b\right]$$
$$+\mathbb{E}_x\left[e^{-q_2\tau_a^-}\left(\mathbb{E}_{\tilde{u}}\left[e^{-q_1\int_0^{T^-_0}\mathbf{1}_{(0,a)}(X_s)ds - q_2\int_0^{T^-_0}\mathbf{1}_{(a,b)}(X_s)ds}; T^-_0 < T^+_{\bar{u}}\right]\Big|_{\substack{\bar{u}=S^U_{\tau_a^-}\\ \tilde{u}=U_{\tau_a^-}}}\right); S^U_{\tau_a^-} < b\right]$$
$$= \mathbb{E}_x\left[e^{-q_2\tau_a^-} g_2(U_{\tau_a^-}, S^U_{\tau_a^-}) f_1(S^U_{\tau_a^-}); S^U_{\tau_a^-} < b\right] + \mathbb{E}_x\left[e^{-q_2\tau_a^-} g_1(U_{\tau_a^-}, S^U_{\tau_a^-}); S^U_{\tau_a^-} < b\right]. \tag{A.1}$$

The above equalities are based on the fact that $U_{\tau_a^-}$, $S^U_{\tau_a^-}$ and $f_1(S^U_{\tau_a^-})$ are all $\mathcal{F}_{\tau_a^-}$-measurable. Also, the trajectories of the process $\{U_t; t \ge 0\}$ restricted on the time interval $[\tau_a^-, \rho^+_{S^U_{\tau_a^-}}]$ evolve in the same way as those of $\{X_t; t \ge 0\}$ since no taxes are paid during this time period. Further, from (2.5) and (2.6) we have

$$\mathbb{E}_x\left[e^{-q_2\tau_a^-} g_2(U_{\tau_a^-}, S^U_{\tau_a^-}) f_1(S^U_{\tau_a^-}); S^U_{\tau_a^-} < b\right]$$
$$= \int_x^b \frac{\xi_x^{(q_2)}(\theta; a) f_1(\theta)}{1-\gamma(\bar{\gamma}^{-1}(\theta))} d\theta \left\{\int_0^{(\theta-a)-} \zeta^{(q_2)}_{\theta-a}(y) \int_0^a g_2(a-z,\theta)\upsilon(y+dz)dy\right.$$
$$\left.+W^{(q_2)}(0+)\int_0^a g_2(a-z,\theta)\upsilon(\theta-a+dz)\right\} + \int_x^b g_2(a,\theta)\frac{\xi_x^{(q_2)}(\theta; a) f_1(\theta)}{1-\gamma(\bar{\gamma}^{-1}(\theta))}\frac{\sigma^2}{2}\zeta^{(q_2)\prime}_{\theta-a}(0)d\theta.$$

Similarly, the second term in the right-hand side of (A.1) can be re-expressed as,

$$\mathbb{E}_x\left[e^{-q_2\tau_a^-} g_1(U_{\tau_a^-}, S^U_{\tau_a^-}); S^U_{\tau_a^-} < b\right]$$
$$= \int_x^b \int_0^\infty \frac{\xi_x^{(q_2)}(\theta; a) g_1(a-z,\theta)}{1-\gamma(\bar{\gamma}^{-1}(\theta))}\left[\int_0^{(\theta-a)-} \zeta^{(q_2)}_{\theta-a}(y)\upsilon(y+dz)dy + W^{(q_2)}(0+)\upsilon(\theta-a+dz)\right]d\theta$$
$$+\frac{\sigma^2}{2}\int_x^b \frac{g_1(a,\theta)}{1-\gamma(\bar{\gamma}^{-1}(\theta))}\xi_x^{(q_2)}(\theta; a)\zeta^{(q_2)\prime}_{\theta-a}(0)d\theta.$$

Substituting them into (A.1) and differentiating with respect to x yields

$$f_1'(x) = h_1(x; a) f_1(x) - h_2(x; a). \tag{A.2}$$

Using the boundary condition $f_1(b) = 0$ to solve (A.2), we get, for $a \leq x \leq b$,

$$f_1(x) = \int_x^b h_2(y; a) \exp\Big\{ - \int_x^y h_1(w; a) dw \Big\} dy.$$

In a similar manner, we can obtain the closed-form expression for $f_2(x)$, for $a \leq x \leq b$, as given in Theorem 1. This completes the proof of Theorem 1. □

Acknowledgments

The authors are grateful to the anonymous reviewers whose constructive comments have led to substantial improvements of the article.

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