Day-to-day modal choice with a Pareto improvement or zero-sum revenue scheme

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Abstract. We investigate the day-to-day modal choice of commuters in a bi-modal transportation system comprising both private transport and public transit. On each day, commuters adjust their modal choice, based on the previous day's perceived travel cost and intraday toll or subsidy of each mode, to minimize their perceived travel cost. Meanwhile, the transportation authority sets the number of bus runs and the tolls or subsidies of two modes on each day, based on the previous day's modal choice of commuters, to simultaneously reduce the daily total actual travel cost of the transportation system and achieve a Pareto improvement or zero-sum revenue target at a stationary state. The evolution process of the modal choice of commuters, associated with the strategy adjustment process of the authority, is formulated as a dynamical system model. We analyze several properties of the dynamical system with respect to its stationary point and evolutionary trajectory. Moreover, we introduce new concepts of Pareto improvement and zero-sum revenue in a day-to-day dynamic setting and propose the two targets' implementations in either a prior or a posterior form. We show that, although commuters have different perceived travel costs for using the same travel mode, the authority need not know the probability distribution of perceived travel costs of commuters to achieve the Pareto improvement target. Finally, we give a set of numerical examples to show the properties of the model and the implementation of the toll or subsidy schemes.

Keywords: dynamical system; modal choice; stochastic user equilibrium (SUE); Pareto improvement; zero-sum revenue

1. Introduction

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In most cities around the world, multiple transport modes exist to provide substitutable transportation services for people, e.g., people can choose either private cars or public transit to travel from an origin to a destination. Thus, there is a problem, i.e., how to plan and manage a congested transport system with multiple modes so as to improve its system performance, which can be evaluated by travel time, transport safety, fuel consumption, environmental pollution, and so on. This paper considers a bi-modal transportation system comprising both private transport and public transit, in which travelers adjust their individual travel mode choice from day to day. This study is important because it gives new insights into the management of a transport system with multiple modes from the dynamic and evolutionary viewpoint.

Existing studies on the multi-modal problem extend mainly along two lines. Along the first line, researchers follow with interest the distribution of traffic flows among travel modes at a static state, i.e., which mode travelers choose (e.g., Cantarella, 1997; Arnott and Yan, 2000; Ahn, 2009; Li et al., 2012; Tirachini and Hensher, 2012; David and Foucart, 2014; Zhang et al., 2014). The distribution of flows among travel modes is generally evaluated under static user equilibrium (UE) or system optimum (SO). Along the other line, researchers focus on not only the distribution of traffic flows among travel modes but also the within-day dynamics of traffic flows, i.e., how and when people travel (e.g., Huang, 2000, 2002; Kraus, 2003; Huang et al., 2007; Gonzales and Daganzo, 2012; van der Weijde et al., 2013; van den Berg and Verhoef, 2014; Wu and Huang, 2014; Gonzales, 2015; Tian and Huang, 2015; Liu et al., 2016). The distribution of flows along the temporal dimension and among travel modes is generally evaluated under dynamic UE or SO. The within-day dynamics of traffic flows is generally formulated by the bottleneck model proposed by Vickrey (1969).

The above two categories of studies only show an end result rather than the choice adjustment process of traffic states. It is well known that, in realistic traffic systems, commuters adjust their travel modes or routes from day to day with their experiences or information provided by an advanced traffic information system (ATIS), and the resultant traffic flows evolve over days before reaching an equilibrium state. On one hand, the exploration of the day-to-day dynamics is useful for better understanding various processes of forming traffic jam and better using an ATIS from either a theoretical or practical standpoint. On the other hand, the exploration of the day-to-day dynamics opens up another avenue for improving total travelers' utility (e.g., decreasing traffic congestion or total travel cost) in traffic systems.

So far, there are a number of studies focusing on the day-to-day evolution of traffic flows in a single-modal system, e.g., Smith (1984), Friesz et al. (1994), Cantarella and Cascetta (1995), Zhang and Nagurney (1996), Watling (1999), Yang and Zhang (2009), He et al. (2010), Smith and Mounce (2011), He and Liu (2012), Cantarella and Watling (2016), He and Peeta (2016), Hazelton and Parry (2016), Rambha and Boyles (2016), Wang et al. (2016), Wei et al. (2016), Xiao and Lo (2016), Xiao et al., (2016), Guo, et al. (2017), and Han et al., (2017). Readers may refer to Watling and Cantarella (2013, 2015) for a comprehensive review of the day-to-day dynamics of traffic flows. Recently, Cantarella et al. (2015), Li and Yang (2016), Liu and Geroliminis (2017), and Liu et al. (2017) concerned the day-to-day dynamics of traffic assignment in bi-modal or multi-modal transport systems.

In this paper, we also focus on the day-to-day modal choice of commuters in a transportation system comprising both private transport and public transit. Despite that Cantarella (1997), Cantarella et al. (2015), Li and Yang (2016), Liu and Geroliminis (2017), Liu et al. (2017), and our study all focus on the subject of bi-modal or multi-modal choice, the specific issues involved in these works are essentially different. Cantarella (1997) proposed a fixed-point formulation of multi-mode equilibrium assignment with elastic demand and analyzed the properties of the equilibrium, which can be considered to be the stationary state of a day-to-day dynamical system. They further established the theoretical conditions of the existence and uniqueness of equilibrium for stationary multi-mode systems. Cantarella et al. (2015) proposed a dynamical system of formulating the joint adjustment of modal choice and transit operation from day to day in a bi-modal transport system. In the system, the frequency of bus runs is prefixed to meet the demand with all the buses available or is daily updated to meet the demand with the minimum number of buses needed to avoid oversaturation. They also showed the non-uniqueness of equilibrium by a numerical example. Li and Yang (2016) proposed a dynamical system model of formulating travelers' day-to-day modal choice in a bi-modal transportation system with responsive transit services. In their model, the frequency of bus runs is adjusted from period to period so that a given target profit of the transit operator is achieved at a stationary state. Liu and Geroliminis (2017) modeled and controlled a multi-region and multi-modal transportation system, in which the travelers adjust their mode choices from day to day and also the within-day traffic dynamics evolve over days. They developed an adaptive mechanism to update parking pricing from period to period so as to improve the system's efficiency. Liu et al. (2017) modeled the joint evolution of travelers' departure time and mode choices in a bi-modal transportation system by considering the impact of user inertia. They also analyzed the dynamic interactions between transport users and the traffic information provider.

Different from Cantarella (1997), Cantarella et al. (2015), Li and Yang (2016), Liu and Geroliminis (2017), and Liu et al. (2017), we concern the issue of Pareto improvement or zero-sum revenue associated with day-to-day modal choice, introduce new concepts of Pareto improvement and zero-sum revenue in a day-to-day dynamic setting, and propose the two targets' implementations in either a prior or a posterior form. We propose a control scheme implemented from the government's standpoint. The transportation authority adjusts not only the number of bus runs but also the tolls/subsidies of both car and bus users from day to day so as to simultaneously reduce the daily total actual travel cost of the transportation system and achieve either a Pareto improvement or zero-sum revenue target. In this way, both society and individual commuters become better off under the implementation of the scheme.

The tolls or subsidies herein imply that all commuters using a travel mode may be charged on some days and may be subsidized on other days. The utilities of some commuters may decrease under the implementation of the control (and redistribution) scheme. With proper subsidies to them, their utility losses can be compensated.

The Pareto improvement refers to that the net travel cost of each commuter is reduced under the implementation of the control scheme, compared with the initial state, i.e., each commuter becomes better off. Such a Pareto-improvement can make the control scheme as a policy more acceptable to commuters because everyone is a winner under such a policy (Guo and Yang, 2010). Some researchers, e.g., Daganzo (1995), Eliasson (2001), Song et al. (2009), Guo and Yang (2010), and Lawphongpanich and Yin (2010), proposed their toll or subsidy schemes (or pricing and revenue refund schemes) implemented at the steady state to realize a Pareto-improvement target. Different from these toll or subsidy schemes, our toll or subsidy scheme is implemented in the day-to-day adjustment process of traffic flows and makes traffic flows to get to a stationary state with Pareto-improvement.

To the best of our knowledge, we give the first implementation of Pareto improvement and zero-sum revenue in the context of day-to-day dynamics. This implementation scheme is not a straightforward extension of the Pareto improvement in the static UE traffic assignment and the result could not simply be derived in a quite straightforward manner. When a toll or subsidy scheme (or pricing and revenue refund scheme) for realizing a Pareto-improvement target, e.g., the one by Daganzo (1995), Eliasson (2001), Song et al. (2009), Guo and Yang (2010), and Lawphongpanich and Yin (2010), is introduced into a traffic system at an equilibrium state, the scheme, as a controlled input, will perturb the traffic system from its equilibrium state and make the flow distribution fall into a disequilibrium state. In this case, commuters will adjust their travel modes from day to day for reducing their travel costs and the flow distribution will then evolve over time before reaching a new equilibrium state. As a result, the scheme may become ineffective for achieving Pareto improvement at the new equilibrium state.

We give an example to show that a static Pareto-improvement toll/subsidy scheme can be ineffective for achieving Pareto improvement when it is implemented in the day-to-day adjustment process of traffic flow. Consider an origin-destination (OD) pair connected by two parallel links. Let x_1 and x_2 be the flows on links 1 and 2, respectively, and $\mathbf{x}=(x_1,x_2)$ be the corresponding vector. The demand between the OD pair is 6 and then $x_1+x_2=6$. The travel costs of links 1 and 2 are respectively governed by

$$c_1(\mathbf{x})=x_1+x_1x_2 \quad \text{and} \quad c_2(\mathbf{x})=x_2+1.$$

Let p_1 and p_2 be the tolls/subsidies of links 1 and 2, respectively. The day-to-day dynamics of link flows is formulated using the proportional swap rule in Smith (1984), i.e.,

$$x_1^{(n+1)}=x_1^{(n)}+\rho\left(x_2^{(n)}\left[c_2\left(\mathbf{x}^{(n)}\right)+p_2-c_1\left(\mathbf{x}^{(n)}\right)-p_1\right]_+-x_1^{(n)}\left[c_1\left(\mathbf{x}^{(n)}\right)+p_1-c_2\left(\mathbf{x}^{(n)}\right)-p_2\right]_+\right)$$

and

$$x_2^{(n+1)}=x_2^{(n)}+\rho\left(x_1^{(n)}\left[c_1\left(\mathbf{x}^{(n)}\right)+p_1-c_2\left(\mathbf{x}^{(n)}\right)-p_2\right]_+-x_2^{(n)}\left[c_2\left(\mathbf{x}^{(n)}\right)+p_2-c_1\left(\mathbf{x}^{(n)}\right)-p_1\right]_+\right),$$

where $x_i^{(n)}$ is the flow on link i at iteration n, ρ (>0) is an adjustment parameter, and the operation $[a]_+$ takes the maximum one between a and 0.

The initial flow vector is set to $\mathbf{x}^{(1)}=(1,5)$. Obviously, when there are no tolls/subsidies (i.e., both p_1 and p_2 are zero), $\mathbf{x}^{(1)}$ is the unique UE point and the travel costs $c_1\left(\mathbf{x}^{(1)}\right)$ and $c_2\left(\mathbf{x}^{(1)}\right)$ of links 1 and 2 at this point are the same (i.e., both equal 6). Under the static toll/subsidy scheme $(p_1,p_2)=(-4,4)$, $\mathbf{x}^*=(6,0)$ is a UE point and the travel costs $c_1\left(\mathbf{x}^*\right)$ and $c_2\left(\mathbf{x}^*\right)$ of links 1 and 2 at this point are 6 and 1, respectively. It immediately follows that

$$c_1\left(\mathbf{x}^*\right)+p_1=2<6=c_1\left(\mathbf{x}^{(1)}\right)=c_2\left(\mathbf{x}^{(1)}\right).$$

That is to say, the static toll/subsidy scheme $(p_1,p_2)=(-4,4)$ is a Pareto improvement toll/subsidy scheme. Figure 1 shows the adjustment process of link flows from the initial point $\mathbf{x}^{(1)}=(1,5)$ under the static toll/subsidy scheme $(p_1,p_2)=(-4,4)$ and the adjustment parameter $\rho=0.03$ when the proportional swap rule is applied. It can be seen that the link flows adjust to a UE point $\mathbf{x}^{(50)}=(3,3)$. At the UE state, it holds that

$$c_1\left(\mathbf{x}^{(50)}\right)+p_1=8>6=c_1\left(\mathbf{x}^{(1)}\right) \quad \text{and} \quad c_2\left(\mathbf{x}^{(50)}\right)+p_2=8>6=c_2\left(\mathbf{x}^{(1)}\right).$$

Namely, under the static toll/subsidy scheme $(p_1,p_2)=(-4,4)$, the generalized travel costs of all users at the final state become higher than those at the initial state. This indicates that a static Pareto improvement scheme may not work when it is applied to a day-to-day setting.

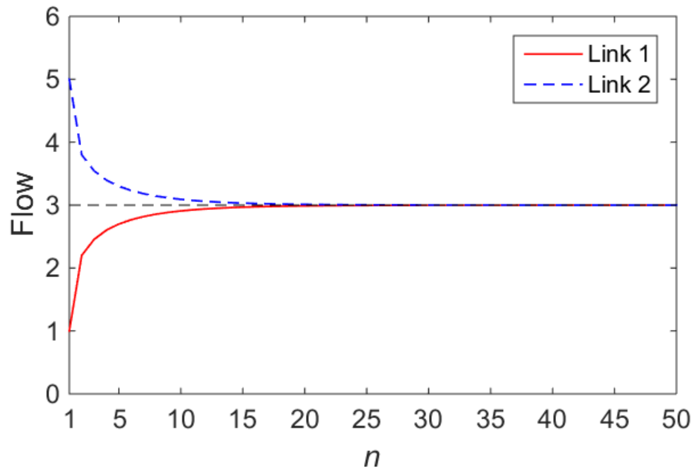


Figure 1. The evolutionary trajectories of link flows from the initial point $\mathbf{x}^{(1)}=(1,5)$ under the static toll/subsidy scheme $(p_1,p_2)=(-4,4)$ and the adjustment parameter $\rho=0.03$ when the proportional swap rule is applied.

Therefore, it is essential for us to solve a problem, namely, how to implement a toll or subsidy scheme for a Pareto-improvement target in the day-to-day adjustment process of traffic flows. The solution to the problem can narrow the gap between Pareto-improvement schemes' theory and their application to a real traffic system.

The zero-sum revenue means that, on each day, some commuters are charged; at the same time, some commuters are subsidized so that the total revenue of the transportation authority from tolls and subsidies is zero. In this way, commuters may abandon the idea that the toll charge is just like another tax collected by the government and may accept the control scheme more easily.

The contribution of this paper can be summarized as follows. On one hand, this paper introduces new concepts of Pareto improvement and zero-sum revenue in a day-to-day dynamic setting and also proposes a new control scheme to a bi-modal day-to-day dynamical system so as to simultaneously reduce the daily total actual travel cost of the system and achieve either a Pareto improvement target or a zero-sum revenue target. In this way, both society and individual commuters are better off after the scheme is implemented. On the other hand, it provides new managerial insights for a transport system with multiple modes from the dynamic and evolutionary viewpoint. It indicates that a static Pareto improvement scheme may fail to achieve Pareto improvement when it is applied to a day-to-day setting. Thus, it is essential to examine static control schemes in the context of day-to-day dynamics. When a control scheme in a static setting is extended to a day-to-day dynamic setting, the implementation of the scheme can be in either a prior or a posterior form. Under the dynamic implementation, the transport system is gradually guided to a stationary state with either Pareto improvement or zero-sum revenue.

The control scheme may have realism issues and it may not be directly applicable to a realistic traffic system. On one hand, for policymakers, the scheme would be difficult to implement as it requires marginal travel cost information that is difficult to observe (Yang et al., 2010). On the other hand, if the number of bus runs and the tolls or subsidies of both car and bus users were adjusted frequently from day to day, travelers would perceive that the scheme is an unstable policy and would certainly have concerns about trusting the scheme (Higgins, 1994; Chu, 1999). However, a day in the model can be regarded as a period of covering a number of days (or even several months), the tolls or subsidies on a day are regarded as the tolls or subsidies on each day in a period (the tolls or subsidies remain unchanged on all days in a period), and the traffic flows on each day are regarded as the traffic flows on the first day of each period. In this way, the model formulates the dynamic evolution of traffic flows under the implementation of a control scheme, in which the tolls or subsidies are adjusted from period to period, which is not so frequent compared to from day to day. Thus, the model should provide an avenue of designing a period-to-period control scheme that is acceptable to travelers. The effectiveness of the period-to-period scheme for driving a traffic system to get to a state with a lower system travel cost and Pareto improvement (or zero-sum revenue) is evaluated through observing whether the tolls or subsidies in each period and the traffic flows on the first day in each period satisfy the conditions in subsequent Theorems 2 to 4 and Propositions 1 to 4.

The model associated with the scheme is potentially applicable to (*i*) evaluate whether a multi-modal system can approach a state with minimum system travel cost and Pareto improvement (or zero-sum revenue) and (*ii*) understand how commuters adjust their modal choice from day to day under the scheme.

The remainder of this paper is organized as follows. In the next section, we present the dynamical system model for formulating the day-to-day evolution of the numbers of car users and bus runs and the tolls or subsidies of car users and bus users so that the system gets to a state with a lower total actual travel cost compared with the initial state. In Section 3, some properties of the dynamical system with respect to its stationary points and evolutionary trajectory are analyzed. In Section 4, we further formulate the dynamic implementation of tolls or subsidies of car users and bus users so that the system gets to a stationary state with Pareto-improvement or zero-sum revenue. Some numerical examples are given to show the properties of the dynamical system and the implementation effects of the toll or subsidy schemes in Section 5. Finally, some remarks and conclusions are provided in Section 6.

2. System description

2.1. Notations and assumptions

On every morning, a fixed number d (>0) of commuters travel from an origin to a destination. The origin-destination (OD) pair is connected by a congested highway running in parallel to an exclusive transit (i.e., bus) line. The two modes are separated, i.e., the highway is used only by cars and the transit line is used only by buses. Commuters have to make a discrete choice between using private transport (auto) or public transit (bus). For simplicity, the occupancy of each private car is assumed to be one.

The number of auto users is denoted by x (≥ 0). Then, the number of public transport users is $d-x$ (≥ 0). y (≥ 0) represents the number of bus runs or bus departures from the origin. A bus has a strict capacity of s (>0) passengers. y is related to x by $sy \geq d-x$, i.e., the number of public transport users is not more than the maximum number of commuters who can be transported by all these operating buses on a day. The feasible set of the numbers of auto users and bus runs is expressed as

$$\Omega \equiv \{(x,y) \mid x+sy-d \geq 0, x \geq 0, d-x \geq 0, y \geq 0\}. \tag{1}$$

Obviously, the set is nonempty and convex.

The notations $t_a(x)$ and $t_b(y)$ respectively represent the average travel time costs of private car and public transport, including both free flow travel cost and congestion cost (occurring on the road), and they are positive. The functions t_a and t_b are twice continuously differentiable with respect to x and y, respectively. Moreover, it is supposed that

$$t_a'(x) > 0, \quad t_b'(y) > 0, \tag{2}$$

$$t_a''(x) \geq 0, \text{ and } t_b''(y) \geq 0. \tag{3}$$

Condition (2) indicates that a higher number of car users (bus runs) generate more congestion for car users (bus users) because there are more cars (buses) on the highway (transit line). Hence, the functions t_a and t_b are increasing in x and y, respectively. In some studies, e.g., Wu and Huang (2014) and Zhang et al. (2014), it is assumed that the travel time cost of public transport is constant. However, the assumption is based on a precondition, i.e., the number of bus runs on the transit line is relatively less than the capacity of the transit line. When there are many bus runs on the transit line, congestion also exists on the transit line. In this work, we adopt an assumption that $t_b'(y)$ is sufficiently small for $y \le d/s$. This assumption means that the travel time cost of public transport almost remains unchanged when the number of bus runs on the transit line is not more than the certain value d/s.

The notation $w(y)$ (≥ 0) is the average waiting time cost of passengers at a public transit stop. The function w is twice continuously differentiable in y. Moreover, it is supposed that

$$w'(y) < 0 \text{ and } w''(y) \ge 0, \tag{4}$$

i.e., the waiting time cost declines and also the decline rate decreases when the number of bus runs increases.

The notation $g(z)$ (≥ 0) denotes the average in-vehicle congestion cost of passengers in bus carriage, where $z = sy - (d - x) \ge 0$, and it reflects the discomfort generated by body congestion or crowdedness in carriage, which has a significant effect on the choices of passengers between transit service and other travel modes (Huang, 2000, 2002; Huang et al., 2007; Li et al., 2012; van den Berg and Verhoef, 2014; Wu and Huang, 2014). The function g is twice continuously differentiable in z. It is assumed that

$$g'(z) < 0 \text{ and } g''(z) \ge 0, \tag{5}$$

i.e., the in-vehicle congestion cost decreases and the decrease rate descends when the free space in bus carriage increases.

The notations p_a and p_b respectively stand for the toll charges from (or financial subsidies to) each auto and bus user, respectively. A positive (negative) p_a-value or p_b-value represents a toll charge (financial subsidy). All these costs and charges, mentioned above, are measured in monetary units.

Commuters have imperfect information regarding travel costs and they choose travel modes according to the perceived generalized travel costs of the two modes. The perceived generalized travel costs c_a and c_b of private car and public transit are respectively formulated as

$$c_a = t_a(x) + p_a + \xi_a \text{ and } c_b = t_b(y) + w(y) + g(z) + p_b + \xi_b, \tag{6}$$

where the terms $t_a(x) + p_a$ and $t_b(y) + w(y) + g(z) + p_b$ are the actual generalized travel costs of private car and public transit, respectively, and ξ_a and ξ_b are two random error terms. The random terms prescribe the differences among travel costs perceived by different commuters. The tolls/subsidies are explicitly presented in the defined perceived travel cost function (6) and this implies that the random terms ξ_a and ξ_b are independent of tolls/subsidies.

Let $\Delta(x, y, p_a, p_b)$ be the difference between the actual generalized travel costs of auto users and public transport users, i.e.,

$$\Delta(x, y, p_a, p_b) = t_a(x) + p_a - t_b(y) - w(y) - g(z) - p_b . \tag{7}$$

Let ε_a and ε_b be two realizations of the two random variables ξ_a and ξ_b, respectively; let $\xi = \xi_b - \xi_a$ be the difference between the two random terms ξ_b and ξ_a, and $\varepsilon = \varepsilon_b - \varepsilon_a$ be a realization of the random variable ξ. It is assumed that ξ is continuously distributed with a probability density function $\varepsilon \mapsto f(\varepsilon)$ over the support $(-\infty, +\infty)$. Moreover, $f(\varepsilon) > 0$ holds for any ε. The probability P_a that the private transport mode is chosen is then governed by

$$P_a = \Pr(c_a < c_b) = \Pr\left(\xi > \Delta(x, y, p_a, p_b)\right) = \int_{\Delta(x,y,p_a,p_b)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon . \tag{8}$$

At the stochastic user equilibrium (SUE) state, no commuter can reduce his/her perceived travel cost by unilaterally altering his/her travel mode (Daganzo and Sheffi, 1977; Sheffi, 1985). The SUE condition can be characterized by the following equation

$$x = dP_a = d\int_{\Delta(x,y,p_a,p_b)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon . \tag{9}$$

The above equation means that these commuters with a difference ε more than (less than) $\Delta(x, y, p_a, p_b)$ choose to travel by private car (public transit) at the SUE state.

By the equilibrium definition, the difference ε can be explained as the preference of commuters for the use of private transport relative to public transit (David and Foucart, 2014; Li and Yang, 2016). The distribution of the random variable ξ is a result of the coincidence of plenty of factors, which affect commuters' modal choice, e.g., travel time, comfort in the carriage, waiting time, commuter income, and so on.

2.2. Formulation of optimizing total actual travel cost

The optimization problem of minimizing the total actual travel cost of the system is formulated as

$$\min_{(x,y)\in\Omega} V(x, y) = xt_a(x) + (d - x)\left(t_b(y) + w(y)\right). \tag{10}$$

Tolls and subsidies are transferred between the government and all commuters in the system, and hence the total actual travel cost of the system does not include tolls and subsidies.

Let $\left(x^*, y^*\right)$ be an optimal solution to the above optimization problem and it then satisfies the following Karush–Kuhn–Tucker (KKT) conditions:

$$t_a(x)+xt_a'(x)-t_b(y)-w(y)=\lambda_1+\lambda_2-\lambda_3, \tag{11}$$

$$\left(t_b'(y)+w'(y)\right)(d-x)=s\lambda_1+\lambda_4, \tag{12}$$

$$\lambda_1 \geq 0,\ \ \lambda_1(x+sy-d)=0, \tag{13}$$

$$\lambda_2 \geq 0,\ \ \lambda_2 x=0, \tag{14}$$

$$\lambda_3 \geq 0,\ \ \lambda_3(d-x)=0, \tag{15}$$

$$\lambda_4 \geq 0, \text{ and } \ \lambda_4 y=0, \tag{16}$$

where λ_1, λ_2, λ_3, and λ_4 are the Lagrange multipliers associated with the four constraints in condition (1), respectively.

If the optimal solution $\left(x^*, y^*\right)$ is in the interior of the set Ω, then the Lagrange multipliers λ_1, λ_2, λ_3, and λ_4 are all zero and the optimal solution $\left(x^*, y^*\right)$ satisfies the following conditions:

$$t_a(x)+xt_a'(x)-t_b(y)-w(y)=0 \ \text{ and} \tag{17}$$

$$\left(t_b'(y)+w'(y)\right)(d-x)=0. \tag{18}$$

2.3. Dynamical system model

Based on the above notations, the day-to-day adjustment of the number of auto users is formulated as

$$x^{(n+1)}=\max\left\{(1-\delta)x^{(n)}+\delta d\int_{\Delta\left(x^{(n)},y^{(n)},p_a^{(n+1)},p_b^{(n+1)}\right)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon,\ \ d-sy^{(n+1)}\right\}, \tag{19}$$

for $n=0,\ 1,\ 2,\cdots$. The superscript (n) refers to the nth day, e.g., $x^{(n)}$ represents the number of auto users on day n. The operation $\max\{a,b\}$ takes the maximum one between a and b. The adjustment parameter $\delta \in (0,1]$.

Formula (19) describes the following dynamic modal choice behaviour of commuters. On each day, a portion of commuters do not change their previous day's choice. The other portion of commuters reconsider their travel modes based on the perceived travel costs of the two modes on the previous day and the transport tolls/subsidies on that day, and they choose to commute by car with the probability

$$\Pr\left(t_a\left(x^{(n)}\right)+p_a^{(n+1)}+\xi_a<t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+p_b^{(n+1)}+\xi_b\right), \tag{20}$$

i.e., $\Pr\left(\xi>\Delta\left(x^{(n)},y^{(n)},p_a^{(n+1)},p_b^{(n+1)}\right)\right)$. Thus, the number of commuters using car is computed as

$$(1-\delta)x^{(n)}+\delta d\int_{\Delta\left(x^{(n)},y^{(n)},p_a^{(n+1)},p_b^{(n+1)}\right)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon\,. \tag{21}$$

Of course, the number of commuters using their own car is not less than $d-sy^{(n+1)}$ due to the capacity constraint of buses.

The transportation authority sets the number of bus runs and the tolls or subsidies of two modes from day to day to simultaneously reduce the daily total actual travel cost of the transportation system and achieve a Pareto improvement target or a zero-sum revenue target at a stationary state. The number of bus runs on day $n+1$ is computed by the previous day's information as follows:

$$y^{(n+1)}=\max\left\{y^{(n)}-\theta\left(t_b'\left(y^{(n)}\right)+w'\left(y^{(n)}\right)\right)\left(d-x^{(n)}\right),0\right\}, \tag{22}$$

for $n=0,\ 1,\ 2,\ \cdots$. The adjustment parameter $\theta>0$.

The difference $p_a^{(n+1)}-p_b^{(n+1)}$ between the tolls or subsidies of two modes on day $n+1$ is governed by

$$p_a^{(n+1)}-p_b^{(n+1)}=x^{(n)}t_a'\left(x^{(n)}\right)+h^{(n)}+g\left(z^{(n)}\right), \tag{23}$$

for $n=0,1,2,\cdots$, where $h^{(n)}$ satisfies

$$x^{(n)}=d\int_{h^{(n)}}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon\,. \tag{24}$$

Equation (23) only formulates the difference between the tolls or subsidies of two modes on each day and does not show how to set specific tolls or subsidies of two modes. However, this guarantees that the trajectory of variable (x,y) evolves in a descent direction of the objective function V of the optimization problem (10). To make the tolls or subsidies of two modes on each day satisfy equation (23), after the toll or subsidy of bus user (auto user) is determined, the toll or subsidy of auto user (bus user) is the toll or subsidy of bus user (auto user) plus (minus) the term on the right hand side of equation (23).

To sum up, formulae (19), (22), and (23) form a dynamical system model with state variables (x,y,p_a,p_b). As shown by formula (23), p_a-p_b can be formulated in terms of variables (x,y). Thus, the state variables of the dynamical system can be further reduced to (x,y).

3. System properties

In this section, we show several properties of the dynamical system model (19), (22), and (23).

First, it is obvious that, if $\left(x^{(0)},y^{(0)}\right)\in\Omega$, then $\left(x^{(n)},y^{(n)}\right)\in\Omega$, for $n=1,\ 2,\cdots$. Namely, if the initial point is in the feasible set Ω, then the trajectory of the dynamical system model must be in the set Ω.

Let Ω° denote the set of all interior points of the set Ω. In subsequent analyses, it is assumed that the initial point $\left(x^{(0)}, y^{(0)}\right)$ belongs to the set Ω°. Moreover, the adjustment parameters δ and θ in formulae (19) and (22) are sufficiently small, i.e., the adjustment of the trajectory of the dynamical system is gradual and fine. Then, when the trajectory of the dynamical system approaches to the boundary line $x+sy-d=0$ to a certain extent, the subsequent trajectory will not further approach to the boundary line. In fact, when $\left(x^{(n)}, y^{(n)}\right)$ approaches to the boundary line to a certain extent, $x^{(n)}$ can be either equal to d or less than d. As can be seen from formula (22), in the case of $x^{(n)}=d$, $y^{(n+1)}$ will remain unchanged. In the case of $x^{(n)}<d$, because that $t_b'(y)$ is positive and sufficiently small for $y \le d/s$ and $w'(y)$ is negative, the term $\left(t_b'\left(y^{(n)}\right)+w'\left(y^{(n)}\right)\right)\left(d-x^{(n)}\right)$ in formula (22) becomes negative and $y^{(n+1)}$ will increase. Therefore, it is impossible that the subsequent trajectory further approaches to the boundary line. As a result, formulae (19) and (22) can be further written as

$$x^{(n+1)}=(1-\delta)x^{(n)}+\delta d\int_{\Delta\left(x^{(n)},y^{(n)},p_a^{(n+1)},p_b^{(n+1)}\right)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon \quad \text{and} \tag{25}$$

$$y^{(n+1)}=y^{(n)}-\theta\left(t_b'\left(y^{(n)}\right)+w'\left(y^{(n)}\right)\right)\left(d-x^{(n)}\right). \tag{26}$$

Second, the following theorem shows the relation between the stationary state of the dynamical system and conditions (17) and (18).

Theorem 1. $\left(x^{(n)}, y^{(n)}\right)\in\Omega$ is a stationary point of the dynamical system model (23), (25), and (26) if and only if it satisfies conditions (17) and (18).

Theorem 1 is proved in Appendix A.1. The theorem indicates that all the stationary points of the dynamical system model (19), (22), and (23) out of the boundary line $x+sy-d=0$ are identical to the solutions of equations (17) and (18). The dynamical system model also has a stationary point $(x,y)=(d,0)$; however, the stationary point may not satisfy conditions (17) and (18). The stationary point is located at the boundary line $x+sy-d=0$, and hence when the trajectory of the dynamical system approaches to the stationary point $(x,y)=(d,0)$ to a certain extent, the subsequent trajectory will not further approach to the stationary point.

Third, in some existing literatures, e.g., Huang et al. (2007), Ahn (2009), Li et al. (2012), Tirachini and Hensher (2012), David and Foucart (2014), and Cantarella et al. (2015), it is assumed that two modes share a road. Thus, the travel cost of one mode is affected by the flow of the other mode, i.e., there exist interactions between two modes. In other some literatures, e.g., Arnott and Yan (2000), Huang (2000, 2002), Kraus (2003), Gonzales and Daganzo (2012), van der Weijde et al. (2013), van den Berg and Verhoef (2014), Wu and Huang (2014), Zhang et al. (2014), Gonzales (2015), Tian and Huang (2015), Li and Yang (2016), and Liu et al. (2016), it is assumed that two modes are separated. However, it does not mean that there are no interactions between two modes. For our model, despite that two modes are separated, the travel cost of public transport is affected by both the flows of public transport and private car due to involving in-vehicle congestion cost (see the definition of in-vehicle congestion cost $g(z)$ and formula (6)). It may lead to multiple stationary points (or equilibrium points) of the system.

The Hessian matrix of the objective function V is written as

$$\nabla^2 V(x,y) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{27}$$

where

$$H_{11} = 2t_a'(x) + x t_a''(x), \tag{28}$$

$$H_{12} = H_{21} = -t_b'(y) - w'(y), \text{ and} \tag{29}$$

$$H_{22} = \left(t_b''(y) + w''(y)\right)(d - x). \tag{30}$$

For each point of satisfying conditions (17) and (18) (or each stationary point out of the boundary line $x + sy - d = 0$), $H_{11} > 0$ and $H_{22} \geq 0$ since $t_a'(x)$ is positive and $t_a''(x)$, $t_b''(y)$, and $w''(y)$ are nonnegative. As a result, at least an eigenvalue of the Hessian matrix $\nabla^2 V(x,y)$ is positive. Thus, the set of stationary points out of the boundary line $x + sy - d = 0$ comprise globally and locally minimum points and saddle points of the optimization problem (10).

For a stationary point out of the boundary line $x + sy - d = 0$, when $H_{11}H_{22} - H_{12}H_{21} > 0$, both eigenvalues are positive, and hence the stationary point is a globally or locally minimum point of the optimization problem (10) and also the stationary point is an isolated point (Bertsekas, 2003; Horn and Johnson, 2013). When $H_{11}H_{22} - H_{12}H_{21} = 0$, an eigenvalue is positive and the other one is zero, and hence the stationary point is a globally or locally minimum point and the stationary point may be connected with other stationary points, i.e., some stationary points may form a connected set. A connected set is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other. When $H_{11}H_{22} - H_{12}H_{21} < 0$, an eigenvalue is positive and the other one is negative, and hence the stationary point is a saddle point and also is an isolated point.

Fourth, let the set of globally and locally minimum points of the optimization problem (10) be $S_{\min}$ and the set of saddle points of the optimization problem (10) be S_{sad}. The following theorem shows that the trajectory of the dynamical system model (23), (25), and (26) from any initial point in the set $\Omega^{\circ} \setminus (S_{\min} \bigcup S_{\text{sad}})$ converges to a point in the set $S_{\min} \bigcup S_{\text{sad}}$.

Theorem 2. The trajectory of the dynamical system model (23), (25), and (26) from any initial point $\left(x^{(0)}, y^{(0)}\right)$ in the set $\Omega^{\circ} \setminus (S_{\min} \bigcup S_{\text{sad}})$ converges to a point in the set $S_{\min} \bigcup S_{\text{sad}}$.

Theorem 2 is proved in Appendix A.2. This theorem indicates that, under the implementation of the control scheme, the daily total actual travel cost of the transportation system will decline over days before reaching the final state. The declining degree is determined by the initial state (i.e., the numbers of car users and bus runs on day 0). If the initial state is in the attraction domain of a stationary point, then the trajectory converges to the stationary point. The attraction domain of a stationary point defines the collection of all states that will evolve towards the stationary point over time (Bie and Lo, 2010). If the dynamical system (23), (25), and (26) has a unique stationary point and also $H_{11}H_{22} - H_{12}H_{21} > 0$ at the stationary point, then the stationary point is the globally minimum point of the optimization problem (10).

It is worth mentioning that a saddle point, as a stationary point of the dynamical system, is unstable, because giving the saddle point a small perturbation can lead to the departure of the trajectory from that point. For an isolated minimum point, as a stationary point, after giving it a small perturbation, the trajectory will return to that point; and hence the minimum point is stable.

Theorem 2 implies that, if the tolls or subsidies of two modes on each day satisfy condition (23), then the trajectory of the dynamical system will converge to a stationary point. Thus, all the toll or subsidy schemes given in subsequent Propositions 1 to 4 can drive the system to reach a stationary state.

4. Specific implementation of toll or subsidy schemes

Equation (23) only formulates the difference between the tolls or subsidies of two modes on each day and does not give the setting of specific tolls or subsidies of two modes. Under the intervention of the toll or subsidy scheme satisfying formula (23), it can be guaranteed that the daily total actual travel cost of the transportation system decreases, as stated by Theorem 2. To further guarantee that a Pareto improvement target or a zero-sum revenue target is achieved at a stationary state, we propose a specific implementation of the toll or subsidy scheme in this section.

To achieve the Pareto improvement or zero-sum revenue target, the implementation of tolls or subsidies can be formulated as either a prior form or a posterior form. In subsequent four implementations, it is assumed that the tolls or subsidies $\left(p_a^{(n)}, p_b^{(n)}\right)$ on each day n satisfy equation (23).

4.1. Prior Pareto-improvement

An implementation of the toll or subsidy scheme is called a prior Pareto-improvement implementation on day $n+1$, if the tolls or subsidies $\left(p_a^{(n+1)}, p_b^{(n+1)}\right)$ on day $n+1$ satisfies that, when $x^{(0)} \geq x^{(n)}$,

$$t_a\left(x^{(0)}\right)+\varepsilon_a > t_a\left(x^{(n)}\right)+\varepsilon_a + p_a^{(n+1)}, \text{ for } \varepsilon \in \left(h^{(n)}, +\infty\right), \tag{31}$$

$$t_a\left(x^{(0)}\right)+\varepsilon_a > t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\varepsilon_b + p_b^{(n+1)}, \text{ for } \varepsilon \in \left(h^{(0)}, h^{(n)}\right), \text{ and} \tag{32}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b > t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\varepsilon_b + p_b^{(n+1)}, \text{ for } \varepsilon \in \left(-\infty, h^{(0)}\right); \tag{33}$$

when $x^{(0)} < x^{(n)}$,

$$t_a\left(x^{(0)}\right)+\varepsilon_a > t_a\left(x^{(n)}\right)+\varepsilon_a + p_a^{(n+1)}, \text{ for } \varepsilon \in \left(h^{(0)}, +\infty\right), \tag{34}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b > t_a\left(x^{(n)}\right)+\varepsilon_a + p_a^{(n+1)}, \text{ for } \varepsilon \in \left(h^{(n)}, h^{(0)}\right), \text{ and} \tag{35}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b > t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\varepsilon_b + p_b^{(n+1)}, \text{ for } \varepsilon \in \left(-\infty, h^{(n)}\right). \tag{36}$$

It follows from formula (24) that, when $x^{(0)} \geq x^{(n)}$, $h^{(0)} \leq h^{(n)}$ holds. Hence, the commuters with the difference $\varepsilon \in \left(h^{(n)}, +\infty\right)$ choose to travel by auto on both days 0 and n; the commuters with $\varepsilon \in \left(h^{(0)}, h^{(n)}\right)$ use their own auto on day 0 and use the bus mode on day n; the commuters with $\varepsilon \in \left(-\infty, h^{(0)}\right)$ use the bus mode on both days 0 and n. When $x^{(0)} < x^{(n)}$, $h^{(0)} > h^{(n)}$ holds. Hence, the commuters with $\varepsilon \in \left(h^{(0)}, +\infty\right)$ choose auto on both days 0 and n; the commuters with $\varepsilon \in \left(h^{(n)}, h^{(0)}\right)$ use the bus mode on day 0 and use their own auto on day n; the commuters with $\varepsilon \in \left(-\infty, h^{(n)}\right)$ use the bus mode on both days 0 and n. Thus, in the above definition, the tolls or subsidies are formulated in two different cases: $x^{(0)} \geq x^{(n)}$ and $x^{(0)} < x^{(n)}$. Nevertheless, in both cases, the above definition indicates that the perceived travel cost savings of commuter on day $n+1$ are evaluated by the perceived travel costs on the previous day n and the tolls or subsidies on day $n+1$. Under the implementation of the tolls or subsidies on day $n+1$, the perceived travel cost of each commuter, based on the previous day's perceived travel costs and the tolls or subsidies on the current day $n+1$, is less than his or her initial perceived travel cost. This implies that at a stationary state, the perceived travel cost of each commuter on each day is less than his or her initial perceived travel cost and each commuter becomes better off.

For the prior Pareto-improvement implementation, we have the following property.

Theorem 3. If the system has entered a stationary state on day $\bar{n}$ and also the tolls or subsidies $\left(p_a^{(\bar{n}+1)}, p_b^{(\bar{n}+1)}\right)$ on day $\bar{n}+1$ satisfy

$$t_a\left(x^{(0)}\right) > t_a\left(x^{(\bar{n})}\right) + p_a^{(\bar{n}+1)} \quad \text{and} \tag{37}$$

$$t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) > t_b\left(y^{(\bar{n})}\right) + w\left(y^{(\bar{n})}\right) + g\left(z^{(\bar{n})}\right) + p_b^{(\bar{n}+1)}, \tag{38}$$

then the implementation of the toll or subsidy scheme is a prior Pareto-improvement implementation on day $\bar{n}+1$, i.e., inequalities (31) to (36) hold.

Theorem 3 is proved in Appendix A.3. Inequalities (37) and (38) imply that the stationary point is not equal to the initial point. In this section, we only concern whether the Pareto-improvement target can be achieved under the implementation of the toll or subsidy scheme. When the system has non-unique stationary points, the toll or subsidy scheme drives the trajectory of the dynamical system to reach a stationary point with Pareto-improvement. It is difficult to know the probability distribution of perceived travel costs of commuters for the two modes in reality. Theorem 3 provides a method for determining the toll or subsidy scheme, without knowing the probability distribution information, so that each commuter is better off at a stationary state, i.e., a Pareto-improvement target is achieved at the stationary state. By Theorem 3, a prior Pareto-improvement implementation can be formulated as follows.

Proposition 1. The tolls or subsidies $\left(p_a^{(n+1)}, p_b^{(n+1)}\right)$ on day $n+1$ are governed by

$$p_a^{(n+1)} = t_a\left(x^{(0)}\right) - t_a\left(x^{(n)}\right) - \kappa_a^{(n)} \quad \text{and} \tag{39}$$

$$p_b^{(n+1)} = t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - t_b\left(y^{(n)}\right) - w\left(y^{(n)}\right) - g\left(z^{(n)}\right) - \kappa_b^{(n)}, \tag{40}$$

for $n = 0,\ 1,\ 2, \cdots$, where the parameters $\kappa_a^{(n)}$ and $\kappa_b^{(n)}$ are positive and also satisfy

$$\kappa_b^{(n)} - \kappa_a^{(n)} = t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - t_b\left(y^{(n)}\right) - w\left(y^{(n)}\right)$$
$$-t_a\left(x^{(0)}\right) + t_a\left(x^{(n)}\right) + x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)}. \tag{41}$$

Then, the toll or subsidy scheme (39) and (40) satisfies not only conditions (37) and (38) but also formula (23) on each day.

Proposition 1 is proved in Appendix A.4. Under the implementation of the toll or subsidy scheme (39) and (40), the daily total actual travel cost of the transportation system decreases, at the same time, a Pareto improvement target is achieved at a stationary state. The values of the parameters $\kappa_a^{(n)}$ and $\kappa_b^{(n)}$ can be set easily. For example, the minimum one between them is set as a positive number and the other one is computed by formula (41).

We then analyze the perceived travel cost savings of all commuters at a stationary state under the prior Pareto-improvement implementation (39) and (40). Suppose that the system has entered a stationary state on day $\bar{n}$. When $x^{(0)} \geq x^{(\bar{n})}$, each commuter with $\varepsilon \in \left(h^{(\bar{n})}, +\infty\right)$ chooses to use his/her auto on both days 0 and $\bar{n}$, and hence his or her perceived travel cost saving on day $\bar{n}$ is

$$\left(t_a\left(x^{(0)}\right) + \varepsilon_a\right) - \left(t_a\left(x^{(\bar{n})}\right) + \varepsilon_a + p_a^{(\bar{n})}\right) = \kappa_a^{(\bar{n})}. \tag{42}$$

The right hand side of equality (42) is generated by substituting formula (39) into the left hand side of equality (42) and simplifying the resultant expression. Each commuter with $\varepsilon \in \left(-\infty, h^{(0)}\right)$ chooses to use the bus mode on both days 0 and $\bar{n}$, and hence his or her perceived travel cost saving is

$$\left(t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) + \varepsilon_b\right) - \left(t_b\left(y^{(\bar{n})}\right) + w\left(y^{(\bar{n})}\right) + g\left(z^{(\bar{n})}\right) + \varepsilon_b + p_b^{(\bar{n})}\right) = \kappa_b^{(\bar{n})}. \tag{43}$$

Similarly, the right hand side of the above equality is deduced from formula (40). Each commuter with $\varepsilon \in \left(h^{(0)}, h^{(\bar{n})}\right)$ chooses to use his/her auto on day 0 and use the bus mode on day $\bar{n}$, and hence his or her perceived travel cost saving is

$$\left(t_a\left(x^{(0)}\right) + \varepsilon_a\right) - \left(t_b\left(y^{(\bar{n})}\right) + w\left(y^{(\bar{n})}\right) + g\left(z^{(\bar{n})}\right) + \varepsilon_b + p_b^{(\bar{n})}\right)$$
$$= t_a\left(x^{(0)}\right) - t_a\left(x^{(\bar{n})}\right) - p_a^{(\bar{n})} + h^{(\bar{n})} - \varepsilon > t_a\left(x^{(0)}\right) - t_a\left(x^{(\bar{n})}\right) - p_a^{(\bar{n})} = \kappa_a^{(\bar{n})}. \tag{44}$$

The first equality in expression (44) is obtained by relation (A.17). Expressions (42) to (44) indicate that the perceived travel cost savings of these commuters, who travel by private auto on both days 0 and $\bar{n}$, are equal and are less than the perceived travel cost savings of these commuters changing their travel modes from private auto on day 0 to public transit on day $\bar{n}$. Moreover, for these commuters changing their travel modes, a smaller difference ε corresponds to a higher perceived travel cost saving. The perceived travel cost savings of these commuters travelling by public transit on both days 0 and $\bar{n}$ are also equal.

When $x^{(0)} < x^{(\bar{n})}$, the perceived travel cost saving of each commuter with $\varepsilon \in \left(h^{(0)}, +\infty\right)$ on day $\bar{n}$ is

$$\left(t_a\left(x^{(0)}\right)+\varepsilon_a\right)-\left(t_a\left(x^{(\bar{n})}\right)+\varepsilon_a+p_a^{(\bar{n})}\right)=\kappa_a^{(\bar{n})}, \tag{45}$$

the perceived travel cost saving of each commuter with $\varepsilon \in \left(-\infty, h^{(\bar{n})}\right)$ is

$$\left(t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b\right)-\left(t_b\left(y^{(\bar{n})}\right)+w\left(y^{(\bar{n})}\right)+g\left(z^{(\bar{n})}\right)+\varepsilon_b+p_b^{(\bar{n})}\right)=\kappa_b^{(\bar{n})}, \tag{46}$$

and the perceived travel cost saving of each commuter with $\varepsilon \in \left(h^{(\bar{n})}, h^{(0)}\right)$ is

$$\begin{aligned}&\left(t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b\right)-\left(t_a\left(x^{(\bar{n})}\right)+\varepsilon_a+p_a^{(\bar{n})}\right)\\&\quad=t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)-t_b\left(y^{(\bar{n})}\right)-w\left(y^{(\bar{n})}\right)-g\left(z^{(\bar{n})}\right)-p_b^{(\bar{n})}-h^{(\bar{n})}+\varepsilon\\&\quad>t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)-t_b\left(y^{(\bar{n})}\right)-w\left(y^{(\bar{n})}\right)-g\left(z^{(\bar{n})}\right)-p_b^{(\bar{n})}=\kappa_b^{(\bar{n})}.\end{aligned} \tag{47}$$

The first equality in expression (47) is obtained by relation (A.17). Expressions (45) to (47) indicate that the perceived travel cost savings of these commuters travelling by car on both days 0 and $\bar{n}$ are equal. The perceived travel cost savings of these commuters, who travel by bus on both days 0 and $\bar{n}$, are also equal and are less than the perceived travel cost savings of these commuters changing their travel modes from bus on day 0 to car on day $\bar{n}$. Moreover, for these commuters changing their travel modes, a larger difference ε corresponds to a higher perceived travel cost saving. It can be seen from formulae (42) to (47) that the parameter $\kappa_a^{(\bar{n})}$ represents the target perceived travel cost saving of each commuter using his/her car on both days 0 and $\bar{n}$, and $\kappa_b^{(\bar{n})}$ represents the target perceived travel cost saving of each commuter using the bus mode on both days 0 and $\bar{n}$.

4.2. Posterior Pareto-improvement

An implementation of the toll or subsidy scheme is called a posterior Pareto-improvement implementation on day n, if the tolls or subsidies $\left(p_a^{(n)}, p_b^{(n)}\right)$ on day n satisfies that, when $x^{(0)} \geq x^{(n)}$,

$$t_a\left(x^{(0)}\right)+\varepsilon_a > t_a\left(x^{(n)}\right)+\varepsilon_a+p_a^{(n)}-\mu_a^{(n+1)}, \text{ for } \varepsilon \in \left(h^{(n)}, +\infty\right), \tag{48}$$

$$t_a\left(x^{(0)}\right)+\varepsilon_a > t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\varepsilon_b+p_b^{(n)}-\mu_b^{(n+1)}, \text{ for } \varepsilon \in \left(h^{(0)}, h^{(n)}\right), \text{ and} \tag{49}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b > t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\varepsilon_b+p_b^{(n)}-\mu_b^{(n+1)},$$

$$\text{for } \varepsilon\in\left(-\infty,h^{(0)}\right); \tag{50}$$

when $x^{(0)} < x^{(n)}$,

$$t_a\left(x^{(0)}\right)+\varepsilon_a > t_a\left(x^{(n)}\right)+\varepsilon_a+p_a^{(n)}-\mu_a^{(n+1)}, \text{ for } \varepsilon\in\left(h^{(0)},+\infty\right), \tag{51}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b > t_a\left(x^{(n)}\right)+\varepsilon_a+p_a^{(n)}-\mu_a^{(n+1)}, \text{ for } \varepsilon\in\left(h^{(n)},h^{(0)}\right), \text{ and} \tag{52}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b > t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\varepsilon_b+p_b^{(n)}-\mu_b^{(n+1)},$$

$$\text{for } \varepsilon\in\left(-\infty,h^{(n)}\right), \tag{53}$$

where $\mu_a^{(n+1)}$ is the (n+1)th day's financial subsidy to (or toll charge from) each commuter, who uses his/her auto on the previous day n, and $\mu_b^{(n+1)}$ is the (n+1)th day's financial subsidy to (or toll charge from) each commuter, who uses public transit on the previous day n. Both of them are functions of $\left(p_a^{(n)},p_b^{(n)}\right)$.

The above definition indicates that each commuter is charged or subsidized twice on each day n, dependent on his or her modal choice on that day and the previous day $n-1$. The subsidies or tolls on day n $\left(\mu_a^{(n)},\mu_b^{(n)}\right)$ dependent on the previous day' choice are used to compensate his or her loss or to cut down his or her excessive benefit on the previous day $n-1$ due to the implementation of the control scheme. On each day n, the perceived travel cost of each commuter, based the travel costs on that day and the tolls or subsidies on both that day and the next day $n+1$, is less than his or her initial perceived travel cost. As a result, at a stationary state, the perceived travel cost of each commuter on each day is less than his or her initial perceived travel cost and each commuter becomes better off.

Similar to Theorem 3, the following theorem can also be obtained.

Theorem 4. If the system has entered a stationary state on day $\overline{n}$ and also $\left(p_a^{(n)},p_b^{(n)}\right)$ and $\left(\mu_a^{(n+1)},\mu_b^{(n+1)}\right)$ satisfy

$$t_a\left(x^{(0)}\right) > t_a\left(x^{(\overline{n})}\right)+p_a^{(\overline{n})}-\min\left\{\mu_a^{(\overline{n}+1)},\mu_b^{(\overline{n}+1)}\right\} \text{ and} \tag{54}$$

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right) > t_b\left(y^{(\overline{n})}\right)+w\left(y^{(\overline{n})}\right)+g\left(z^{(\overline{n})}\right)+p_b^{(\overline{n})}-\min\left\{\mu_a^{(\overline{n}+1)},\mu_b^{(\overline{n}+1)}\right\}, \tag{55}$$

then the implementation of the toll or subsidy scheme is a posterior Pareto-improvement implementation on day $\overline{n}$, i.e., inequalities (48) to (53) hold.

The proof of Theorem 4 is similar to that of Theorem 3 and hence it is omitted here. By

Theorem 4 and formula (23), a posterior Pareto-improvement implementation can be formulated as follows.

Proposition 2. The tolls or subsidies $\left(p_a^{(n+1)}, p_b^{(n+1)}\right)$ on day $n+1$ and $\left(\mu_a^{(n+2)}, \mu_b^{(n+2)}\right)$ on day $n+2$ are formulated as

$$p_a^{(n+1)} = x^{(n)} t_a'\left(x^{(n)}\right), \tag{56}$$

$$p_b^{(n+1)} = -h^{(n)} - g\left(z^{(n)}\right), \text{ and} \tag{57}$$

$$\mu_a^{(n+2)} = \mu_b^{(n+2)} = \max\left\{\tau_a^{(n+1)}, \tau_b^{(n+1)}\right\} + \bar{\kappa}^{(n+1)}, \tag{58}$$

for $n = 0,\ 1,\ 2, \cdots$, where the parameters $\tau_a^{(n+1)}$ and $\tau_b^{(n+1)}$ are given by

$$\tau_a^{(n+1)} = t_a\left(x^{(n+1)}\right) + p_a^{(n+1)} - t_a\left(x^{(0)}\right) \text{ and} \tag{59}$$

$$\tau_b^{(n+1)} = t_b\left(y^{(n+1)}\right) + w\left(y^{(n+1)}\right) + g\left(z^{(n+1)}\right) + p_b^{(n+1)} - t_b\left(y^{(0)}\right) - w\left(y^{(0)}\right) - g\left(z^{(0)}\right), \tag{60}$$

and the parameter $\bar{\kappa}^{(n+1)}$ is positive. Then, the toll or subsidy scheme (56) to (58) satisfies conditions (54) and (55) and formula (23) on each day.

Proposition 2 is proved in Appendix A.5. Thus, under the implementation of the toll or subsidy scheme, not only the daily total actual travel cost of the transportation system decreases, but also a Pareto improvement target is achieved at a stationary state. Meanwhile, the transportation authority need not know commuters' perceived travel cost information to implement the toll or subsidy scheme.

We then analyze the perceived travel cost savings of all commuters at a stationary state under the posterior Pareto-improvement implementation (56) to (58). Assume that the system has entered a stationary state on day $\bar{n}$. When $x^{(0)} \geq x^{(\bar{n})}$, the perceived travel cost saving of each commuter with $\varepsilon \in \left(h^{(\bar{n})}, +\infty\right)$ on day $\bar{n}$ (which includes the subsidy $\mu_a^{(\bar{n}+1)}$) is

$$\left(t_a\left(x^{(0)}\right) + \varepsilon_a\right) - \left(t_a\left(x^{(\bar{n})}\right) + \varepsilon_a + p_a^{(\bar{n})} - \mu_a^{(\bar{n}+1)}\right) = \max\left\{\tau_a^{(\bar{n})}, \tau_b^{(\bar{n})}\right\} - \tau_a^{(\bar{n})} + \bar{\kappa}^{(\bar{n})}. \tag{61}$$

The above equality is generated by substituting formulae (58) and (59) into the left hand side of the equality. The perceived travel cost saving of each commuter with $\varepsilon \in \left(-\infty, h^{(0)}\right)$ is

$$\left(t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) + \varepsilon_b\right) - \left(t_b\left(y^{(\bar{n})}\right) + w\left(y^{(\bar{n})}\right) + g\left(z^{(\bar{n})}\right) + \varepsilon_b + p_b^{(\bar{n})} - \mu_b^{(\bar{n}+1)}\right)$$

$$= \max\left\{\tau_a^{(\bar{n})}, \tau_b^{(\bar{n})}\right\} - \tau_b^{(\bar{n})} + \bar{\kappa}^{(\bar{n})}. \tag{62}$$

Similarly, the above equality is obtained by substituting formulae (58) and (60) into the left hand side of the equality. The perceived travel cost saving of each commuter with

$\varepsilon \in \left(h^{(0)}, h^{(\bar{n})}\right)$ is

$$\left(t_a\left(x^{(0)}\right)+\varepsilon_a\right)-\left(t_b\left(y^{(\bar{n})}\right)+w\left(y^{(\bar{n})}\right)+g\left(z^{(\bar{n})}\right)+\varepsilon_b+p_b^{(\bar{n})}-\mu_b^{(\bar{n}+1)}\right)$$

$$=t_a\left(x^{(0)}\right)-t_a\left(x^{(\bar{n})}\right)-p_a^{(\bar{n})}+\mu_b^{(\bar{n}+1)}+h^{(\bar{n})}-\varepsilon>t_a\left(x^{(0)}\right)-t_a\left(x^{(\bar{n})}\right)-p_a^{(\bar{n})}+\mu_b^{(\bar{n}+1)}$$

$$=\max\left\{\tau_a^{(\bar{n})},\tau_b^{(\bar{n})}\right\}-\tau_a^{(\bar{n})}+\bar{\kappa}^{(\bar{n})}. \tag{63}$$

The first equality in expression (63) is obtained by relation (A.17). Expressions (61) to (63) indicate that the perceived travel cost savings of these commuters, who travel by car on both days 0 and $\bar{n}$, are equal and are less than the perceived travel cost savings of these commuters changing their travel modes from car on day 0 to bus on day $\bar{n}$. Moreover, for these commuters changing their travel modes, a smaller difference ε corresponds to a higher perceived travel cost saving. The perceived travel cost savings of these commuters travelling by bus on both days 0 and $\bar{n}$ are also equal.

When $x^{(0)}<x^{(\bar{n})}$, the perceived travel cost saving of each commuter with $\varepsilon \in \left(h^{(0)}, +\infty\right)$ on day $\bar{n}$ is

$$\left(t_a\left(x^{(0)}\right)+\varepsilon_a\right)-\left(t_a\left(x^{(\bar{n})}\right)+\varepsilon_a+p_a^{(\bar{n})}-\mu_a^{(\bar{n}+1)}\right)=\max\left\{\tau_a^{(\bar{n})},\tau_b^{(\bar{n})}\right\}-\tau_a^{(\bar{n})}+\bar{\kappa}^{(\bar{n})}, \tag{64}$$

the perceived travel cost saving of each commuter with $\varepsilon \in \left(-\infty, h^{(\bar{n})}\right)$ is

$$\left(t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b\right)-\left(t_b\left(y^{(\bar{n})}\right)+w\left(y^{(\bar{n})}\right)+g\left(z^{(\bar{n})}\right)+\varepsilon_b+p_b^{(\bar{n})}-\mu_b^{(\bar{n}+1)}\right)$$

$$=\max\left\{\tau_a^{(\bar{n})},\tau_b^{(\bar{n})}\right\}-\tau_b^{(\bar{n})}+\bar{\kappa}^{(\bar{n})}, \tag{65}$$

and the perceived travel cost saving of each commuter with $\varepsilon \in \left(h^{(\bar{n})}, h^{(0)}\right)$ is

$$\left(t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)+\varepsilon_b\right)-\left(t_a\left(x^{(\bar{n})}\right)+\varepsilon_a+p_a^{(\bar{n})}-\mu_a^{(\bar{n}+1)}\right)$$

$$=t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)-t_b\left(y^{(\bar{n})}\right)-w\left(y^{(\bar{n})}\right)-g\left(z^{(\bar{n})}\right)-p_b^{(\bar{n})}+\mu_a^{(\bar{n}+1)}-h^{(\bar{n})}+\varepsilon$$

$$>t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)-t_b\left(y^{(\bar{n})}\right)-w\left(y^{(\bar{n})}\right)-g\left(z^{(\bar{n})}\right)-p_b^{(\bar{n})}+\mu_a^{(\bar{n}+1)}$$

$$=\max\left\{\tau_a^{(\bar{n})},\tau_b^{(\bar{n})}\right\}-\tau_b^{(\bar{n})}+\bar{\kappa}^{(\bar{n})}. \tag{66}$$

The first equality in expression (66) is obtained by relation (A.17). Expressions (64) to (66) indicate that the perceived travel cost savings of the commuters travelling by car on both days 0 and $\bar{n}$ are equal. The perceived travel cost savings of the commuters, who travel by bus on both days 0 and $\bar{n}$, are also equal and are less than the perceived travel cost savings of these commuters changing their travel modes from bus on day 0 to car on day $\bar{n}$. Moreover, for the commuters changing their travel modes, a larger difference ε corresponds to a higher perceived travel cost saving. It can be seen from formulae (61) to (66) that the parameter $\bar{\kappa}^{(n)}$ represents the minimum perceived travel cost saving of commuters.

Comparing the prior and posterior implementations for Pareto improvement, we find that each one has its superiority and inferiority. First, in the prior implementation, each commuter is charged or subsidized once on each day; however, in the posterior implementation, each commuter is charged or subsidized twice on each day. The cost of the authority for the posterior implementation is higher than that for the prior implementation.

Second, in the prior implementation, the perceived travel cost saving of each commuter on each day is evaluated by the previous day's perceived travel cost information. Moreover, after the prior implementation on a day during a non-stationary state, the perceived generalized travel costs of some commuters on that day (including the perceived travel costs and the tolls or subsidies on that day) may increase. In the posterior implementation, the perceived generalized travel cost of each commuter on a day comprises not only the perceived travel costs and the tolls or subsidies on that day but also the tolls or subsidies on the next day. It can be guaranteed that each commuter is better off on each day during a non-stationary state under the posterior implementation.

Third, the prior implementation may make the perceived travel cost savings of commuters using a mode relatively higher than those of commuters using the other mode, because the perceived travel cost savings of commuters using each mode are affected by the toll or subsidy of that mode, and at the same time, the tolls or subsidies $p_a^{(n)}$ and $p_b^{(n)}$ of both modes need to satisfy constraint (23). However, under the posterior implementation, the difference between the perceived travel cost savings of commuters using different modes can be relatively smaller, because the perceived travel cost savings of car and bus users are determined by the net tolls or subsidies $p_a^{(n)} - \mu_a^{(n+1)}$ and $p_b^{(n)} - \mu_b^{(n+1)}$, which can take any values despite that $p_a^{(n)}$ and $p_b^{(n)}$ need satisfy constraint (23). Thus, more and better toll or subsidy patterns can be achieved by the posterior implementation compared with the prior implementation.

4.3. Prior zero-sum revenue

A prior zero-sum revenue implementation of the toll or subsidy scheme on day n is defined as follows. On day n, all of the commuters using a mode (any one of the two modes) are charged; however, all of the commuters using the other mode are subsidized. The total amount of tolls equals the total amount of subsidies based on the numbers of car users and bus runs on the previous day, i.e., the revenue of the authority from tolls and subsidies is zero based on the numbers of car users and bus runs on the previous day. According to the above definition, a prior zero-sum revenue implementation can be formulated as follows.

Proposition 3. The tolls or subsidies $\left(p_a^{(n+1)}, p_b^{(n+1)}\right)$ on day $n+1$ is governed by

$$p_a^{(n+1)} = \frac{1}{d}\left(d - x^{(n)}\right)\left(x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right)\right) \text{ and} \tag{67}$$

$$p_b^{(n+1)} = -\frac{1}{d} x^{(n)}\left(x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right)\right). \tag{68}$$

for $n = 0, 1, 2, \cdots$. Then, formulae (67) and (68) satisfy condition (23) and the zero-sum condition $p_a^{(n+1)} x^{(n)} + p_b^{(n+1)}\left(d - x^{(n)}\right) = 0$.

By substituting formulae (67) and (68) into the left hand side of both condition (23) and the zero-sum condition, the conclusion of Proposition 3 immediately follows. Under the implementation of the toll or subsidy scheme (67) and (68), not only the daily total actual travel cost of the transportation system decreases, but also the daily revenue of the authority from tolls and subsidies is zero at a stationary state.

4.4. Posterior zero-sum revenue

A posterior zero-sum revenue implementation of the toll or subsidy scheme on day n is defined as follows. On day n, only all of the commuters using a mode (any one of the two modes) are charged and at the same time, all their tolls collected on the day n are averagely distributed to each of all the commuters on the next day $n+1$. As a result, the revenue of the authority from tolls on each day is equal to the refund from the authority on the next day. Based on the above definition, a posterior zero-sum revenue implementation is proposed as follows.

Proposition 4. A posterior zero-sum revenue implementation is formulated in two different cases. If

$$x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right) \geq 0,$$

then

$$p_a^{(n+1)} = x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right) - \frac{R^{(n)}}{d} \text{ and} \tag{69}$$

$$p_b^{(n+1)} = -\frac{R^{(n)}}{d}, \tag{70}$$

where

$$R^{(n+1)} = \left(x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right)\right) x^{(n+1)}; \tag{71}$$

else

$$p_a^{(n+1)} = -\frac{R^{(n)}}{d} \text{ and} \tag{72}$$

$$p_b^{(n+1)} = -x^{(n)} t_a'\left(x^{(n)}\right) - h^{(n)} - g\left(z^{(n)}\right) - \frac{R^{(n)}}{d}, \tag{73}$$

where

$$R^{(n+1)} = -\left(x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right)\right)\left(d - x^{(n+1)}\right), \tag{74}$$

for $n = 0,\ 1,\ 2, \cdots$. Then, under the toll or subsidy scheme (69) to (74), conditions (23) and $p_a^{(n+1)} x^{(n+1)} + p_b^{(n+1)}\left(d - x^{(n+1)}\right) = R^{(n+1)} - R^{(n)}$ are satisfied.

By substituting formulae (69) to (74) into the left hand side of the two conditions, the conclusion of Proposition 4 can be easily verified. Here, $R^{(n)}$ is the toll revenue of the authority on day n (excluding the subsidies on day n). There is no toll charge for both the two modes on day 0, and hence $R^{(0)} = 0$. The toll or subsidy scheme (69) to (74) makes the daily total actual travel cost of the transportation system decrease and, simultaneously, the daily revenue of the authority from tolls and subsidies be zero at a stationary state.

The difference and connection between the two zero-sum revenue implementations lie in the following two aspects. On one hand, under the prior zero-sum revenue implementation, the tolls and subsidies on each day are simultaneously computed based on the previous day's numbers of car users and bus runs, and hence the transportation authority is required to set aside a budget for the implementation to compensate the difference between all the subsidies and all the tolls during a non-stationary state. However, under the posterior zero-sum revenue implementation, the subsidies to users on each day are offset by all the charges on the previous day, and hence the authority need not set aside a budget for this implementation.

On the other hand, if the system under the posterior zero-sum revenue implementation (69) to (74) has got to a stationary state on day $\bar{n}-1$, then the tolls or subsidies $p_a^{(\bar{n}+1)}$ and $p_b^{(\bar{n}+1)}$ on day $\bar{n}+1$ can be further formulated as

$$p_a^{(\bar{n}+1)} = \frac{1}{d}\left(d - x^{(n)}\right)\left(x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right)\right) \text{ and} \tag{75}$$

$$p_b^{(\bar{n}+1)} = -\frac{1}{d} x^{(n)}\left(x^{(n)} t_a'\left(x^{(n)}\right) + h^{(n)} + g\left(z^{(n)}\right)\right). \tag{76}$$

By comparing formulae (75) and (67) and comparing (76) and (68), it is concluded that the stationary tolls or subsidies under the posterior zero-sum revenue implementation (69) to (74) are equal to those under the prior zero-sum revenue implementation (67) and (68) if the stationary numbers of car users and bus runs under the former implementation are equal to those under the latter implementation.

5. Numerical examples

In this section, we give six numerical examples to show the properties of the dynamical system model and the application of the toll or subsidy schemes proposed in this paper. In all these examples, the total number of commuters is $d = 6000$. The capacity of each bus is $s = 50$ passengers. The average travel time cost of private cars is formulated as

$$t_a(x) = 0.08 \times \left(\frac{x}{1000}\right)^4 + 8. \tag{77}$$

The average travel time cost associated with bus is expressed as

$$t_b(y) = 0.6 \times \left(\frac{y}{200}\right)^4 + 8.2. \tag{78}$$

The average waiting time cost of bus users at a bus stop is governed by

$$w(y) = \frac{1000}{4y+1}. \tag{79}$$

The average in-vehicle congestion cost of bus users is given by

$$g(z) = \frac{1000}{20z+1}, \tag{80}$$

where $z = sy - (d - x)$.

The random variable ξ follows a bimodal distribution and its density function is formulated as

$$f(\varepsilon) = \frac{1}{2\sqrt{2\pi}\sigma}\left(\exp\left(-\frac{(\varepsilon-\mu_1)^2}{2\sigma^2}\right) + \exp\left(-\frac{(\varepsilon-\mu_2)^2}{2\sigma^2}\right)\right), \quad \varepsilon \in (-\infty, +\infty), \tag{81}$$

where the parameters $\mu_1 = -3$, $\mu_2 = 6$, and $\sigma = 3$. The bimodal distribution is a mixture of two different unimodal distributions and it prescribes that two groups of commuters show obvious preference to the two modes, respectively. For example, some commuters have no private cars, and hence they have to choose to travel by public transit. Some commuters, who have private cars, may always choose to travel by private auto mode due to inertia or aversion to in-vehicle congestion in bus carriage. Moreover, the two parameters μ_1 and μ_2 satisfy $|\mu_1| < \mu_2$. This means that commuters prefer to use private transport. The adjustment parameter δ in formula (19) takes 0.1. The adjustment parameter θ in formula (22) is 0.1. The following depicts the six examples.

First, we investigate the evolutionary process of the numbers of car users and bus runs under the implementation of the toll or subsidy scheme (23). The numbers $\left(x^{(0)}, y^{(0)}\right)$ of auto users and bus runs on day 0 are near the boundary of the feasible region

$$\bar{\Omega} \equiv \{(x, y) \mid x + sy - d \geq 0, d \geq x \geq 0, 300 \geq y \geq 0\}. \tag{82}$$

Figure 2 shows the evolutionary trajectories of the numbers (x, y) of auto users and bus runs in the feasible region $\bar{\Omega}$, formulated by the dynamical system (19), (22), and (23). We can see that these trajectories adjust to a unique stationary point $(1491.26, 175.44)$. Meanwhile, we record the evolutionary trajectories of the total actual travel cost $V(x, y)$ of the system under the implementation of the toll or subsidy scheme, as depicted in Figure 3. It is seen that the total actual travel cost is reduced to 57509.29 when the number of days increases. Thus, the scheme makes the numbers of auto users and bus runs evolve to a globally minimum point.

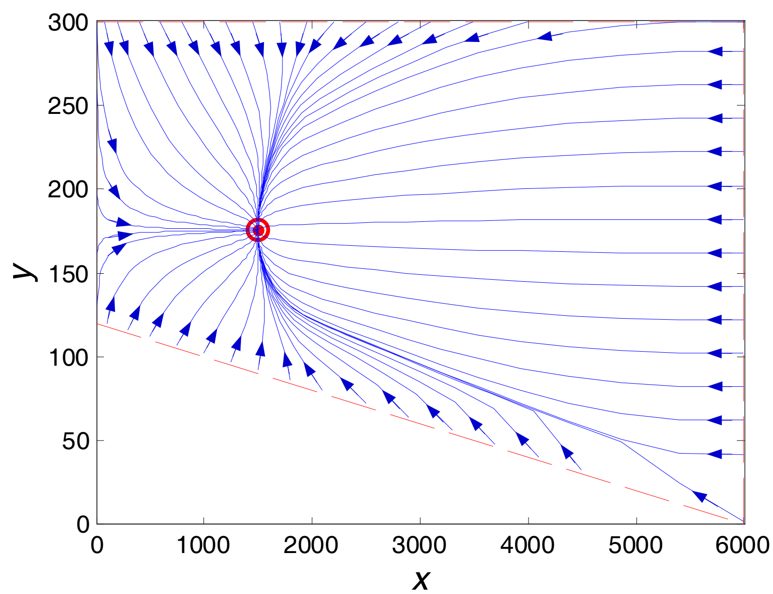


Figure 2. The evolutionary trajectories of the numbers (x, y) of auto users and bus runs in the feasible region $\bar{\Omega}$, formulated by the dynamical system (19), (22), and (23).

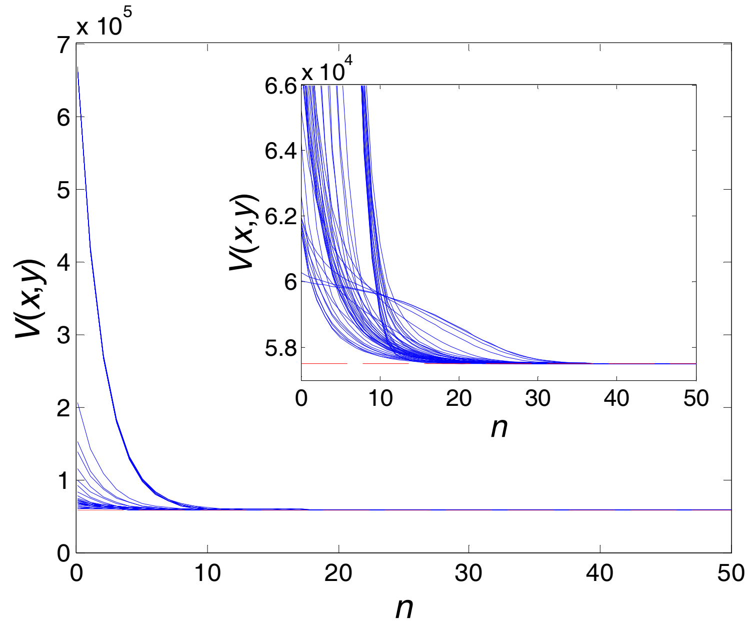


Figure 3. The evolutionary trajectories of the total actual travel cost $V(x, y)$ of the system over day n under the implementation of the toll or subsidy scheme.

Second, we examine the prior Pareto-improvement implementation introduced in Section 4.1. Figure 4 shows the evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (39) and (40), when the minimum target perceived travel cost saving $\min\left\{\kappa_a^{(n)}, \kappa_b^{(n)}\right\}=1$, i.e., the minimum one between $\kappa_a^{(n)}$ and $\kappa_b^{(n)}$ takes 1 and the other one is computed by formula (41), and the initial numbers of car users and bus runs $\left(x^{(0)}, y^{(0)}\right)=(605,300)$, $(2405,300)$, $(3305,300)$, and $(3005,61.9)$. Despite that the numbers of car users and bus runs adjust to the same stationary point $(x, y)=(1491.26,175.44)$ from different initial points, the trajectories for the four different initial points evolve to four different points $(p_a, p_b)=(-1.38,-9.01)$, $(1.28,-6.35)$, $(8.15,0.52)$, and $(5.13,-2.50)$. This illustrates that stationary prior Pareto-improvement tolling schemes are not unique and the stationary toll/subsidy scheme achieved depends on the initial solution $\left(x^{(0)}, y^{(0)}\right)$. Moreover, when the initial number of car users is less than (more than) the number of car users at the stationary state, car users are subsidized (charged) to encourage more commuters to use cars (bus).

Let $U^{(n)}$ be the revenue of the authority from tolls and subsidies on day n and it is formulated as

$$U^{(n)}=p_a^{(n)} x^{(n)}+p_b^{(n)}\left(d-x^{(n)}\right). \tag{83}$$

As shown in Figure 5, the trajectories of U for the four different initial points respectively adjust to the stationary values of -42693.27, -26699.20, 14512.41, and -3617.73 when the number of days increases. This indicates that the initial distribution of traffic flows affects the revenue of the authority from tolls and subsidies on each day and the revenue of the authority can be either positive or negative. The prior implementation can simultaneously achieve a Pareto improvement and let the authority make a profit at the stationary state, depending on the initial distribution of traffic flows.

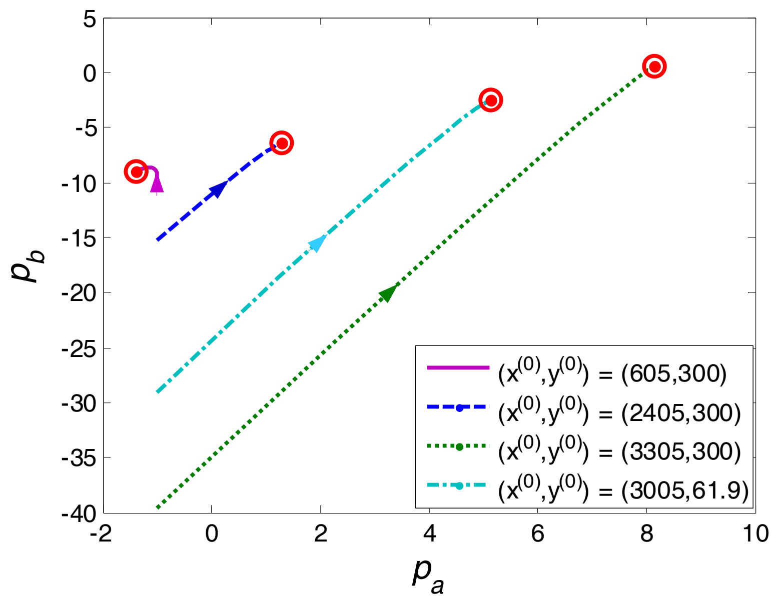


Figure 4. The evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (39) and (40).

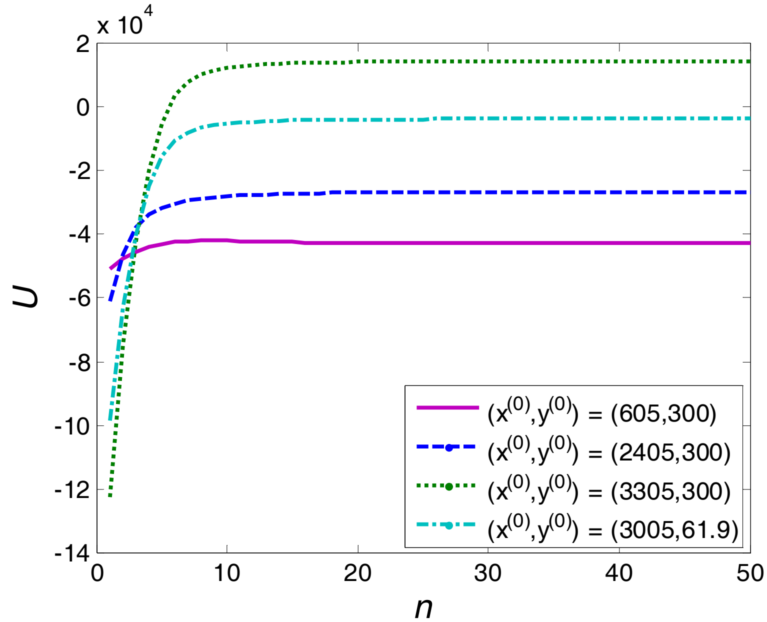


Figure 5. The evolutionary trajectories of the revenue U of the authority from tolls and subsidies over day n under the implementation of scheme (39) and (40).

Third, we study the posterior Pareto-improvement implementation stated in Section 4.2. Figure 6 displays the evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (56) to (58), when the minimum perceived travel cost saving parameter $\bar{\kappa}^{(n)} = 1$ in formula (58) and the initial numbers of car users and bus runs $\left(x^{(0)}, y^{(0)}\right) = (605, 300)$, $(2405, 300)$, $(3305, 300)$, and $(3005, 61.9)$, which are the same as those under the prior implementation. Different from the case under the prior implementation, the trajectories for the four different initial points evolve to the same stationary point $(1.58, -6.04)$ under the posterior implementation. Figure 7 shows the evolutionary trajectories of the subsidy or toll μ_a (which equals μ_b according to (58)) of auto users (or bus users) over day *n* for the four initial points. As can be seen, the trajectories adjust to the stationary points of 2.97, 0.30, -6.57, and -3.55, respectively. The phenomena shown in Figures 6 and 7 indicate that the intraday tolls or subsidies (p_a, p_b) and the next day's subsidies or tolls (μ_a, μ_b) take different actions for the two control targets. The intraday tolls or subsidies (p_a, p_b) can be unchanged for different initial points and are used to make the total actual travel cost of the system minimum. The next day's subsidies or tolls (μ_a, μ_b) are used to compensate commuters' losses or to cut down their excessive benefits due to the setting of target perceived travel cost saving in the current day and (μ_a, μ_b) can take different values for different initial points.

Let $U^{(n)}$ be the revenue of the authority from tolls and subsidies on day n and it is formulated as

$$U^{(n)} = \left(p_a^{(n)} - \mu_a^{(n+1)}\right)x^{(n)} + \left(p_b^{(n)} - \mu_b^{(n+1)}\right)\left(d - x^{(n)}\right). \tag{84}$$

Figure 8 depicts the evolutionary trajectories of U for the four different initial points with the number of days n. The trajectories evolve to the stationary values of -42693.27, -26699.20, 14512.41, and -3617.73, respectively, which are the same as those in Figure 5. By comparing Figures 5 and 8, it is also seen that their trajectories with the same initial point have similar evolutionary trends and also evolve to the same stationary value, although the two Pareto-improvement implementations are different.

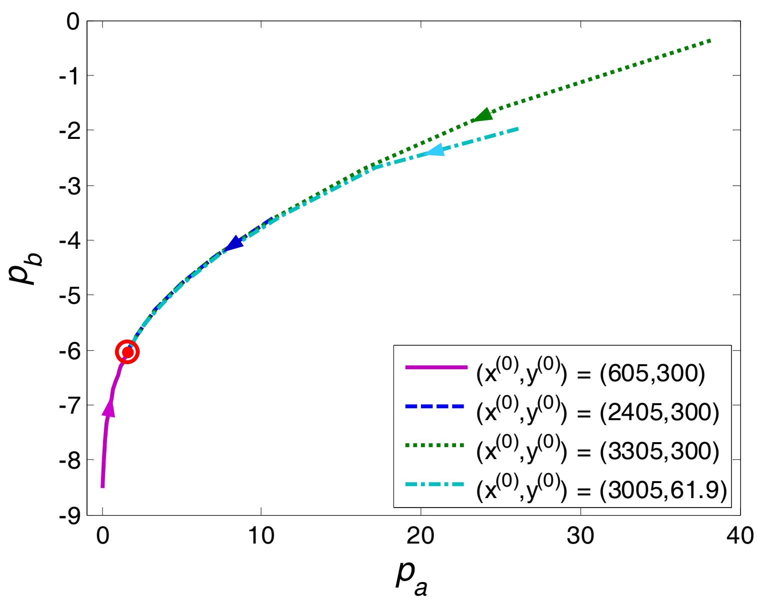


Figure 6. The evolutionary trajectories of the tolls or subsidies p_a and p_b of auto and bus users under the implementation of scheme (56) to (58).

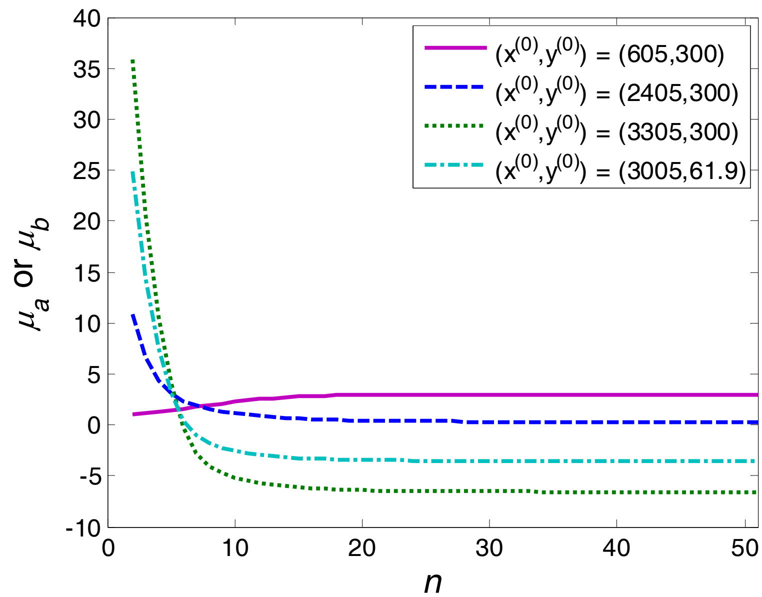


Figure 7. The evolutionary trajectories of the subsidy or toll μ_a (or μ_b) of auto users (or bus users) over day n under the implementation of scheme (56) to (58).

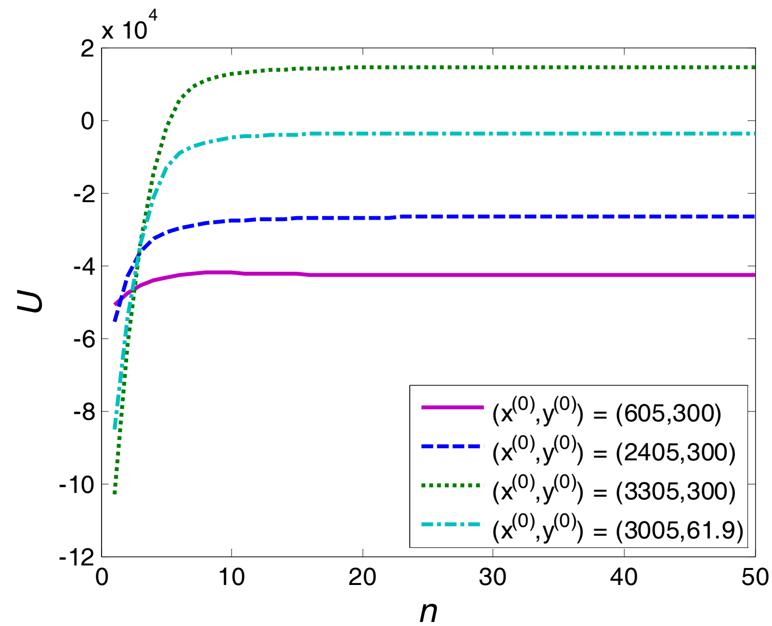


Figure 8. The evolutionary trajectories of the revenue U of the authority from tolls and subsidies over day n under the implementation of scheme (56) to (58).

Fourth, we investigate the prior zero-sum revenue implementation depicted in Section 4.3. Figure 9 shows the evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (67) and (68), when the initial numbers of car users and bus runs $\left(x^{(0)}, y^{(0)}\right) = (605,300)$, $(2405,300)$, $(3305,300)$, and $(3005,61.9)$. The trajectories of (p_a, p_b) for the four different initial points evolve to the same stationary point, i.e., $(5.73,-1.90)$. When the initial number of car users is less than (more than) the number of car users at the stationary state, the trajectory approaches to (the trajectories approach to) the stationary point in a direction, along which the toll p_a of auto users decreases and the subsidy p_b of bus users increases (decreases). Figure 10 shows the evolutionary trajectories of the revenue U of the authority from tolls and subsidies on each day, formulated by formula (83) for the four different initial points. We can see that the revenues adjust to zero when the number of days increases. During non-stationary states, the revenues can be either positive or negative. In particular, the revenue on each day is always positive for an initial point of $(605,300)$ and is always negative for the other three initial points. Thus, the authority may require setting aside a budget to compensate the difference between the subsidy expenditures and the toll incomes on some days during the implementation.

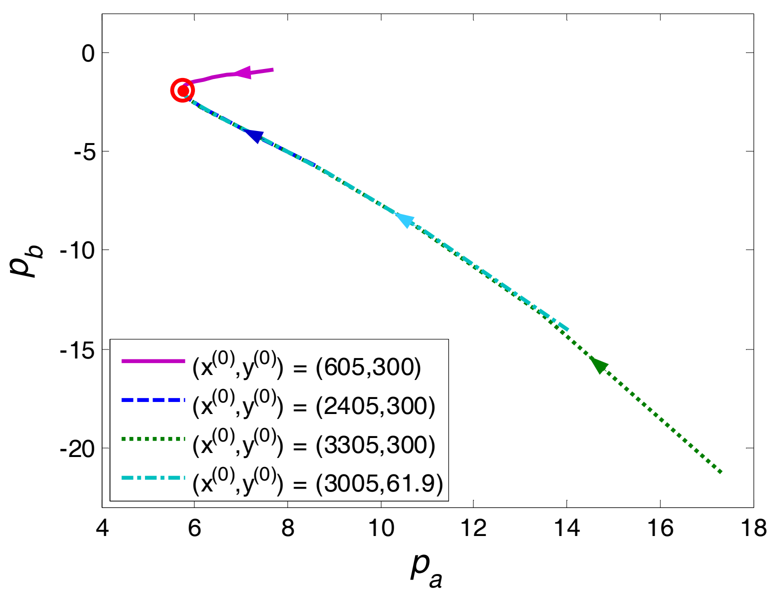


Figure 9. The evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (67) and (68).

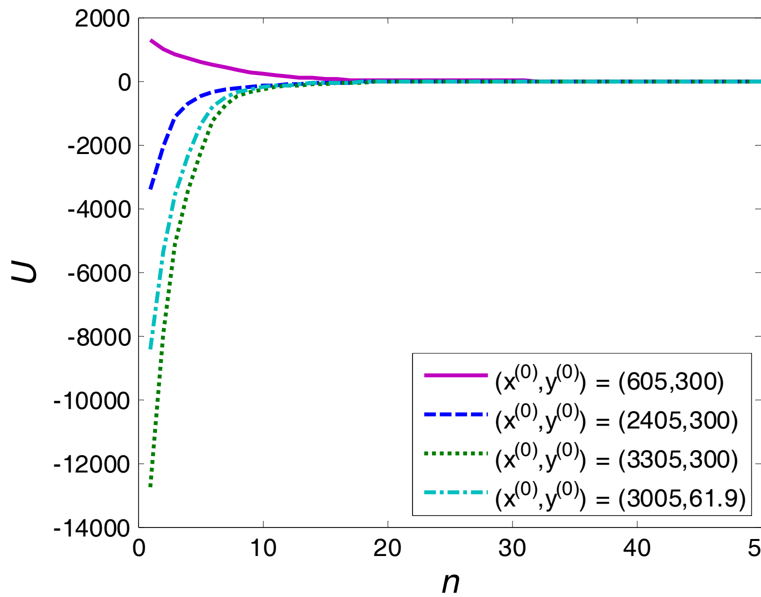


Figure 10. The evolutionary trajectories of the revenue U of the authority from tolls and subsidies over day n under the implementation of scheme (67) and (68).

Fifth, we study the posterior zero-sum revenue implementation described in Section 4.4. Figure 11 displays the evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (69) to (74), when the initial numbers of car users and bus runs $\left(x^{(0)}, y^{(0)}\right) = (605,300)$, $(2405,300)$, $(3305,300)$, and $(3005,61.9)$. The trajectories for the four different initial points evolve to the same stationary point $(5.73,-1.90)$. By comparing Figures 9 and 11, it is seen that the tolls or subsidies (p_a, p_b) at the stationary state are equal and also the trajectories with the same initial point approach to the stationary point in the same direction though the two zero-sum revenue implementations are different.

Figure 12 shows the evolutionary trajectories of the revenue U of the authority from tolls and subsidies over day, based on formula (83), for the four different initial points. We can see that all of the trajectories evolve to zero when the number of days n increases. During non-stationary states, the revenues can be either positive or negative for a certain initial point. However, this does not mean that the authority needs to set aside a budget to compensate the difference between the subsidy expenditures and the toll incomes for the operation because all of the subsidies to users on each day are offset by the charges on the previous day under the posterior zero-sum revenue implementation.

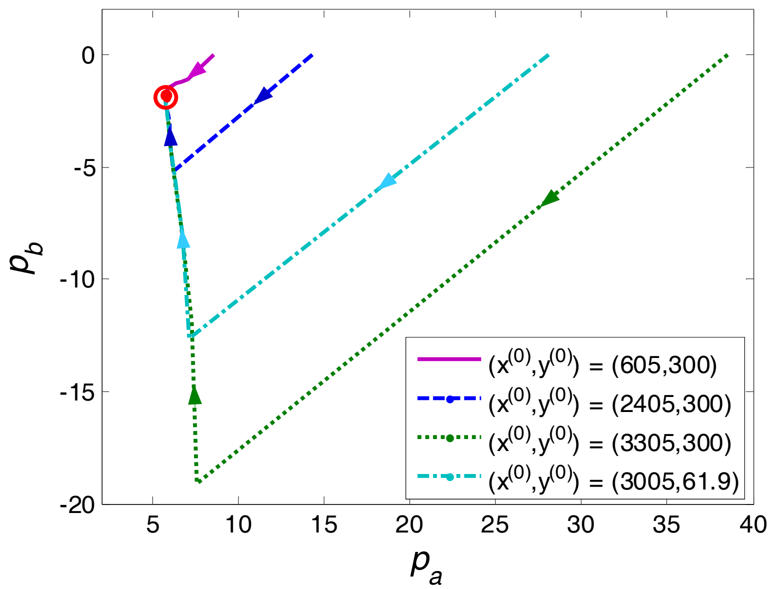


Figure 11. The evolutionary trajectories of the tolls or subsidies (p_a, p_b) of auto and bus users under the implementation of scheme (69) to (74).

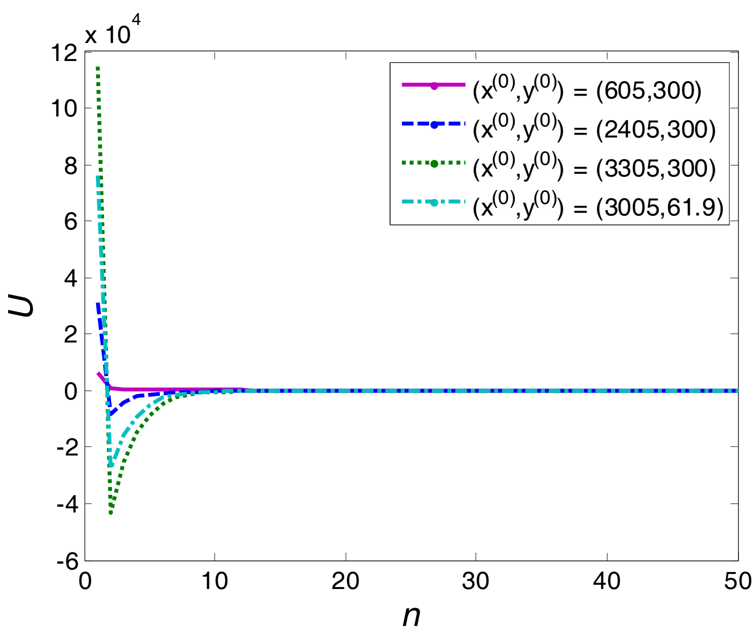


Figure 12. The evolutionary trajectories of the revenue U of the authority from tolls and subsidies over day n under the implementation of scheme (69) to (74).

Sixth, we compare the perceived travel cost savings of commuters under the implementation of the above four schemes. Let $D^{(n)}$ be the minimum perceived travel cost saving of commuter on day n. Similar to the analyses of perceived travel cost saving in

Sections 4.1 and 4.2, it can be obtained that, under the prior Pareto-improvement implementation, the prior zero-sum revenue implementation, and the posterior zero-sum revenue implementation, the minimum perceived travel cost saving of commuter on day n is computed as $D^{(n)} = \min\left\{\tilde{\kappa}_1^{(n)}, \tilde{\kappa}_2^{(n)}, \tilde{\kappa}_3^{(n)}\right\}$, where

$$\tilde{\kappa}_1^{(n)} = t_a\left(x^{(0)}\right) - t_a\left(x^{(n)}\right) - p_a^{(n)}, \tag{85}$$

$$\tilde{\kappa}_2^{(n)} = \begin{cases} t_a\left(x^{(0)}\right) - t_b\left(y^{(n)}\right) - w\left(y^{(n)}\right) - g\left(z^{(n)}\right) - p_b^{(n)} - h^{(n)}, & \text{if } x^{(0)} \geq x^{(n)}, \\ t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - t_a\left(x^{(n)}\right) - p_a^{(n)} + h^{(n)}, & \text{if } x^{(0)} < x^{(n)}, \text{ and} \end{cases} \tag{86}$$

$$\tilde{\kappa}_3^{(n)} = t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - t_b\left(y^{(n)}\right) - w\left(y^{(n)}\right) - g\left(z^{(n)}\right) - p_b^{(n)}. \tag{87}$$

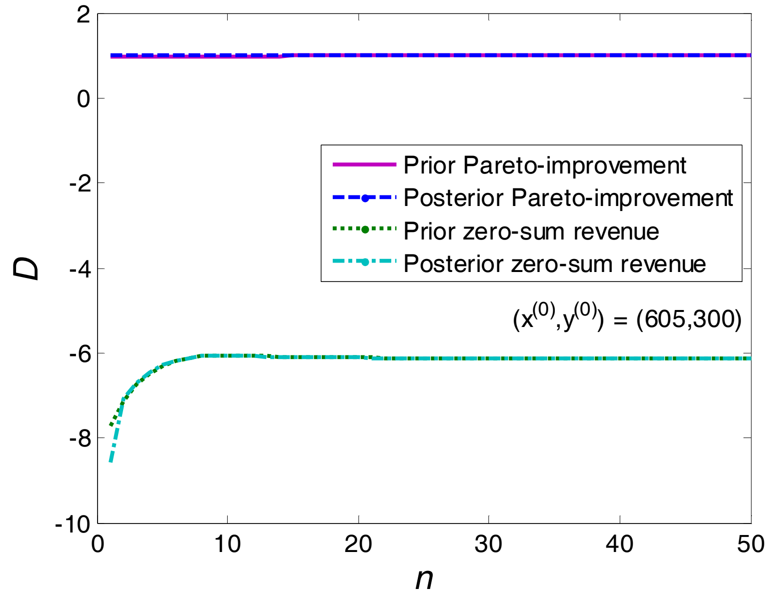
Under the posterior Pareto-improvement implementation, the minimum perceived travel cost saving of commuter on day n is computed as $D^{(n)} = \min\left\{\hat{\kappa}_1^{(n)}, \hat{\kappa}_2^{(n)}, \hat{\kappa}_3^{(n)}\right\}$, where

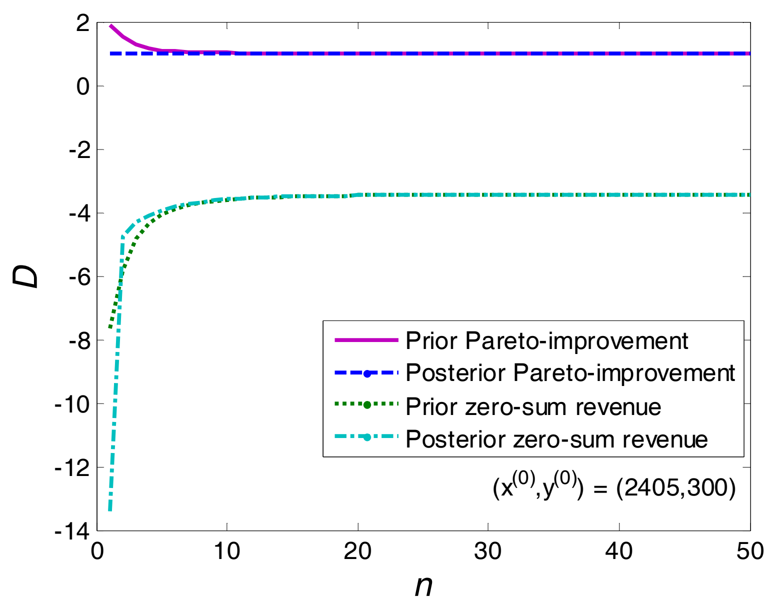
$$\hat{\kappa}_1^{(n)} = t_a\left(x^{(0)}\right) - t_a\left(x^{(n)}\right) - p_a^{(n)} + \mu_a^{(n+1)}, \tag{88}$$

$$\hat{\kappa}_2^{(n)} = \begin{cases} t_a\left(x^{(0)}\right) - t_b\left(y^{(n)}\right) - w\left(y^{(n)}\right) - g\left(z^{(n)}\right) - p_b^{(n)} + \mu_b^{(n+1)} - h^{(n)}, & \text{if } x^{(0)} \geq x^{(n)}, \\ t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - t_a\left(x^{(n)}\right) - p_a^{(n)} + \mu_a^{(n+1)} + h^{(n)}, & \text{if } x^{(0)} < x^{(n)}, \text{ and} \end{cases} \tag{89}$$

$$\hat{\kappa}_3^{(n)} = t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - t_b\left(y^{(n)}\right) - w\left(y^{(n)}\right) - g\left(z^{(n)}\right) - p_b^{(n)} + \mu_b^{(n+1)}. \tag{90}$$

Figure 13 shows the evolutionary trajectories of the minimum perceived travel cost saving D of commuter over day n under the implementations of the four schemes. For all of the four initial points, the trajectories evolve to the same stationary value of 1 and also the minimum perceived travel cost saving on each day during a non-stationary state is always not less than 1 under the two Pareto-improvement implementations, because both the minimum target perceived travel cost saving $\min\left\{\kappa_a^{(n)}, \kappa_b^{(n)}\right\}$ and the minimum perceived travel cost saving parameter $\bar{\kappa}^{(n)}$ are set to 1. That is to say, under the two Pareto-improvement implementations, a Pareto-improvement target is achieved at the stationary state. However, under the two zero-sum revenue implementations, the trajectories evolve to the same stationary value of -6.12 for the initial point $\left(x^{(0)}, y^{(0)}\right) = (605, 300)$, evolve to -3.45 for $\left(x^{(0)}, y^{(0)}\right) = (2405, 300)$, evolve to 3.42 for $\left(x^{(0)}, y^{(0)}\right) = (3305, 300)$, and evolve to 0.40 for $\left(x^{(0)}, y^{(0)}\right) = (3005, 61.9)$. This finding shows that under the two zero-sum revenue implementations, it cannot be guaranteed that a Pareto-implementation is realized at the stationary state since commuters choose their travel modes according to perceived travel costs.


Prior Pareto-improvement
Posterior Pareto-improvement
Prior zero-sum revenue
Posterior zero-sum revenue
(x(0),y(0)) = (605,300)
D
n


Prior Pareto-improvement
Posterior Pareto-improvement
Prior zero-sum revenue
Posterior zero-sum revenue
(x(0),y(0)) = (2405,300)
D
n

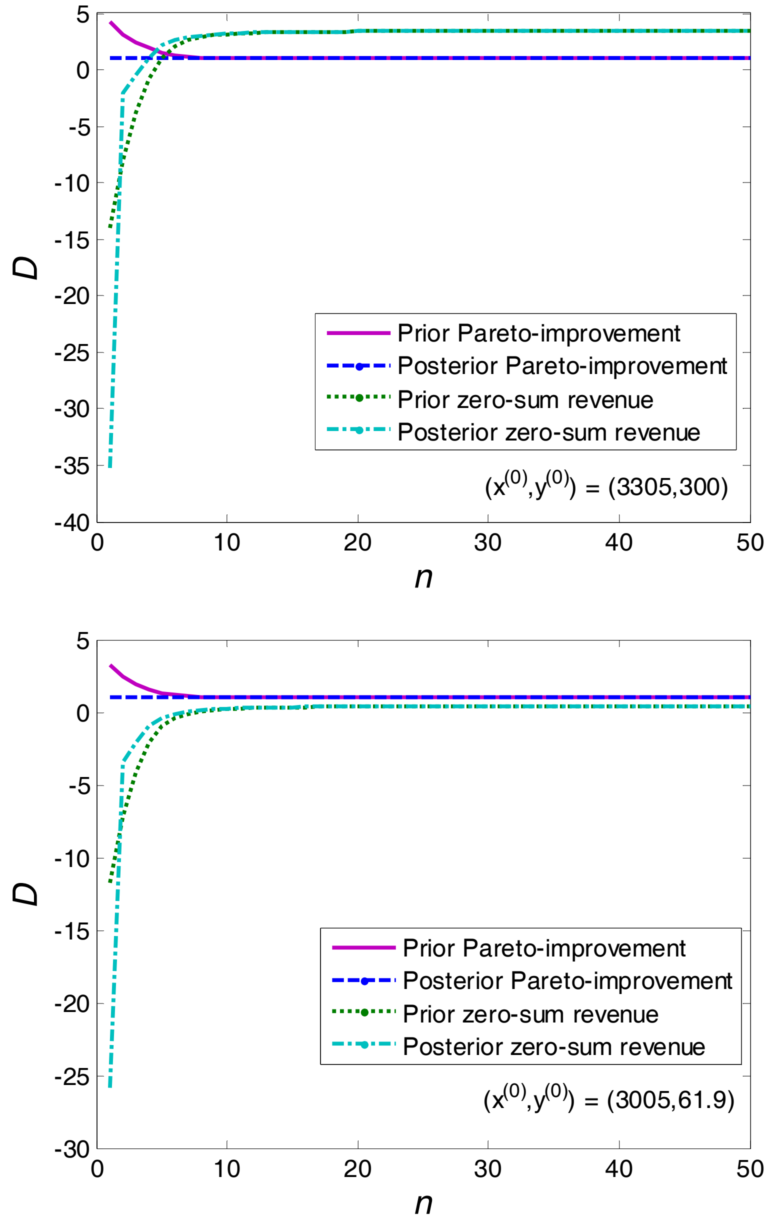


Figure 13. The evolutionary trajectories of the minimum perceived travel cost saving D of commuter over day n under the implementations of the four schemes.

6. Conclusions and remarks

We propose a dynamical system model to formulate the interactive decision-making between the transportation authority and commuters from day to day in a bi-modal system comprising both private transport and public transit. On each day, commuters adjust their modal choice, based on the previous day's perceived travel costs and intraday tolls or subsidies of two modes, to minimize their own perceived travel cost. At the same time, the transportation authority sets the number of bus runs and the tolls or subsidies of two modes, based on the modal choice of commuters on the previous day, to simultaneously reduce the daily total actual travel cost of the transportation system and achieve a Pareto improvement or zero-sum revenue target at a stationary state.

We prove that the set of stationary points of the dynamical system out of the boundary line $x+sy-d=0$ is identical to the set of comprising all the points that satisfy conditions (17) and (18). The system also has a stationary point $(x,y)=(d,0)$ located at the boundary line $x+sy-d=0$; however, the stationary point at the boundary line may not satisfy conditions (17) and (18). The trajectory of the dynamical system from any initial point in the interior of the set Ω will not converge to the stationary point located at the boundary line, but will converge to a stationary point out of the boundary line under certain preconditions. Moreover, a stationary point out of the boundary line can be either a globally/locally minimum point or a saddle point of the optimization problem (10). An isolated saddle point, as a stationary point of the dynamical system, is unstable, and an isolated minimum point, as a stationary point, is stable.

We also prove that with the intervention of the control scheme of the authority, the daily total actual travel cost of the transportation system is reduced. We demonstrate that when a toll or subsidy scheme is implemented in the dynamic evolution process of traffic flows to achieve a Pareto-improvement or zero-sum revenue target, the toll or subsidy scheme can be performed in either a prior pattern or a posterior pattern. We propose both the prior and posterior implementations for the two targets. Moreover, we show that, although commuters have different perceived travel costs for the same travel mode, the authority need not know the probability distribution of perceived travel costs of commuters to achieve the Pareto improvement target.

Commuters have imperfect information regarding travel costs and they choose travel modes according to perceived travel costs. Thus, the same toll or subsidy has different effects on the mode choice of commuters with different perceived errors. Especially, when some commuters change their travel modes, the perceived errors significantly affect their perceived travel cost savings. For example, when some commuters change their travel modes from private auto on day 0 to public transit on day n, their perceived travel cost saving is

$$t_a\left(x^{(0)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)-g\left(z^{(n)}\right)-p_b^{(n)}-\varepsilon .$$

In some existing studies, e.g., Guo and Yang (2010), they showed that, if the total travel cost of a traffic system is reduced under the implementation of a toll charge scheme, then there exists a refund scheme, which can achieve Pareto improvement. In our paper, we mainly concern the day-to-day implementation of a Pareto improvement scheme rather than the existence of a Pareto improvement scheme. Moreover, the proposed schemes are practical since the authority does not need to know commuters' perceived travel cost information.

In terms of modeling, control scheme, and optimization, our study also differs from the studies of Cantarella et al. (2015) and Li and Yang (2016). Compared with Cantarella et al. (2015), we further consider toll/subsidy scheme from day to day to formulate the interactive adjustment of modal choice, transit operation, and tolls/subsidies. Under the implementation of the transit operation and toll/subsidy scheme, not only the daily total actual travel cost of the transportation system is reduced but also either a Pareto improvement or zero-sum revenue target is achieved to make both society and individual commuters better off. In a word, we mainly concern with a different control scheme that considers the interactive adjustment of modal choice, transit operation and tolls/subsidies in which the scheme allows a redistribution of toll revenue to society and a smaller total actual travel cost compared with the control scheme proposed by Cantarella et al. (2015) and all individuals to be better off after such implementation. Different from the control scheme of Li and Yang (2016), we propose another control scheme implemented from the government's standpoint, not from the private transit operator's standpoint or the target profit level's perspective. The transportation authority can further adjust the tolls or subsidies of both car and bus users from day to day (not just bus frequency) and redistribute toll revenue to society through providing subsidies to travelers from day to day so that the daily total actual travel cost of the transportation system is reduced and all individuals are better off after such implementation.

By numerical examples, we show that the proposed toll or subsidy schemes are effective for reducing the daily total actual travel cost of the transportation system and achieving the Pareto improvement or zero-sum revenue target. When the stationary point of the system out of the boundary line $x + sy - d = 0$ is unique, the control scheme can make the total actual travel cost of the system globally minimum. The initial flow distribution significantly affects both the daily amounts of tolls or subsidies of two modes and the daily revenue of the authority from tolls and subsidies to realize a perceived travel cost saving target for commuters. Under the two zero-sum revenue implementations, it cannot be guaranteed that a Pareto-implementation is realized at a stationary state since commuters choose their travel modes according to perceived travel costs.

Note that while the Pareto conditions do not involve the probability measure, the solutions to the model are dependent on it. On one hand, under the implementation of the Pareto improvement scheme, the perceived travel cost of each commuter is reduced compared with the initial state. The perceived errors of commuters can take any value. Thus, when the probability distributions of the perceived errors change (or the distribution of the random variable ξ changes), the Pareto improvement can be still guaranteed, i.e., the Pareto improvement is independent on the distributions of the perceived errors (or the distribution of ξ). On the other hand, the perceived travel costs of commuters are determined by their perceived errors. Therefore, when the probability distributions of the perceived errors change (or the distribution of the random variable ξ changes), the evolutionary trajectory and stationary state of traffic flows also change. That is to say, the solutions to the model are dependent on the distributions of the perceived errors (or the distribution of ξ).

We assume that bus and car are segregated. In reality, they can share lanes. Hence, one of the future research directions is to consider the shared lane situation, which introduces additional complexity and may lead to a different Pareto improving pricing scheme.

Acknowledgments

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Appendix A. Proofs of theorems and propositions

A.1. Proof of Theorem 1

Proof. We first prove the necessity. $\left(x^{(n)}, y^{(n)}\right) \in \Omega$ is a stationary point of the dynamical system model, i.e., $\left(x^{(n+1)}, y^{(n+1)}\right) = \left(x^{(n)}, y^{(n)}\right)$, and hence it follows from formulae (25) and (26) that

$$x^{(n)} = (1-\delta)x^{(n)} + \delta d \int_{\Delta\left(x^{(n)}, y^{(n)}, p_a^{(n+1)}, p_b^{(n+1)}\right)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon \quad \text{and} \tag{A.1}$$

$$y^{(n)} = y^{(n)} - \theta\left(t_b'\left(y^{(n)}\right) + w'\left(y^{(n)}\right)\right)\left(d - x^{(n)}\right), \tag{A.2}$$

i.e.,

$$x^{(n)} = d \int_{\Delta\left(x^{(n)}, y^{(n)}, p_a^{(n+1)}, p_b^{(n+1)}\right)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon \quad \text{and} \tag{A.3}$$

$$\left(t_b'\left(y^{(n)}\right) + w'\left(y^{(n)}\right)\right)\left(d - x^{(n)}\right) = 0. \tag{A.4}$$

Substituting formulae (7), (23), and (24) into (A.3) generates

$$d\int_{t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)+h^{(n)}}^{h^{(n)}} f(\varepsilon)\mathrm{d}\varepsilon = 0\,. \tag{A.5}$$

For any ε, $f(\varepsilon) > 0$ holds, and hence we have

$$t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)=0\,. \tag{A.6}$$

Equations (A.4) and (A.6) mean that $\left(x^{(n)}, y^{(n)}\right)$ satisfies conditions (17) and (18).

We then prove the sufficiency. If $\left(x^{(n)}, y^{(n)}\right)$ satisfies conditions (17) and (18), then it satisfies conditions (A.1) and (A.2). That is to say, $\left(x^{(n)}, y^{(n)}\right)$ is a stationary point of the dynamical system model. ■

A.2. Proof of Theorem 2

Proof. To prove that the trajectory converges to a point in the set $S_{\min} \cup S_{\text{sad}}$, we need to show that the day-to-day numbers of auto users and bus runs update in a descent direction of the objective function V of the optimization problem (10) if the trajectory is in the set $\Omega \setminus (S_{\min} \cup S_{\text{sad}})$. On one hand, substituting formula (23) into (7) results in

$$\Delta\left(x^{(n)}, y^{(n)}, p_a^{(n+1)}, p_b^{(n+1)}\right)=t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)+h^{(n)}\,. \tag{A.7}$$

Rearranging (25) and substituting formulae (A.7) and (24) into the resultant expression leads to

$$\begin{aligned} x^{(n+1)}-x^{(n)} &= \delta\left(d\int_{\Delta\left(x^{(n)}, y^{(n)}, p_a^{(n+1)}, p_b^{(n+1)}\right)}^{+\infty} f(\varepsilon)\mathrm{d}\varepsilon - x^{(n)}\right) \\ &= -\delta d\int_{h^{(n)}}^{t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)+h^{(n)}} f(\varepsilon)\mathrm{d}\varepsilon\,. \end{aligned} \tag{A.8}$$

In addition, it is obtained that

$$\frac{\partial V\left(x^{(n)}, y^{(n)}\right)}{\partial x^{(n)}}=t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right). \tag{A.9}$$

Combining formulae (A.8) and (A.9) generates

$$\begin{aligned} \frac{\partial V\left(x^{(n)}, y^{(n)}\right)}{\partial x^{(n)}}\left(x^{(n+1)}-x^{(n)}\right) &= -\delta d\left(t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)\right) \\ &\quad \times\int_{h^{(n)}}^{t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)+h^{(n)}} f(\varepsilon)\mathrm{d}\varepsilon\,. \end{aligned} \tag{A.10}$$

On the other hand, it is generated that

$$\frac{\partial V\left(x^{(n)}, y^{(n)}\right)}{\partial y^{(n)}}=\left(t_b'\left(y^{(n)}\right)+w'\left(y^{(n)}\right)\right)\left(d-x^{(n)}\right). \tag{A.11}$$

Thus, it follows from formula (26) that

$$\frac{\partial V\left(x^{(n)},y^{(n)}\right)}{\partial y^{(n)}}\left(y^{(n+1)}-y^{(n)}\right)=-\theta\left(t_b'\left(y^{(n)}\right)+w'\left(y^{(n)}\right)\right)^2\left(d-x^{(n)}\right)^2. \tag{A.12}$$

If $\left(x^{(n)},y^{(n)}\right)\in\Omega\setminus(S_{\min}\cup S_{\text{sad}})$ on day n, then

$$t_a\left(x^{(n)}\right)+x^{(n)}t_a'\left(x^{(n)}\right)-t_b\left(y^{(n)}\right)-w\left(y^{(n)}\right)\neq 0, \tag{A.13}$$

or

$$\left(t_b'\left(y^{(n)}\right)+w'\left(y^{(n)}\right)\right)\left(d-x^{(n)}\right)\neq 0. \tag{A.14}$$

Thus, associating formulae (A.10), (A.12) to (A.14), and the precondition that $f(\varepsilon)>0$ for any ε, it is obtained that

$$\frac{\partial V\left(x^{(n)},y^{(n)}\right)}{\partial x^{(n)}}\left(x^{(n+1)}-x^{(n)}\right)+\frac{\partial V\left(x^{(n)},y^{(n)}\right)}{\partial y^{(n)}}\left(y^{(n+1)}-y^{(n)}\right)<0. \tag{A.15}$$

This indicates that, if $\left(x^{(n)},y^{(n)}\right)\in\Omega\setminus(S_{\min}\cup S_{\text{sad}})$, then $\left(x^{(n)},y^{(n)}\right)$ is updated in a descent direction of the objective function V.

The parameters δ and θ are sufficiently small, and hence it follows that

$$V\left(x^{(n+1)},y^{(n+1)}\right)<V\left(x^{(n)},y^{(n)}\right),\ \ \forall\left(x^{(n)},y^{(n)}\right)\in\Omega\setminus(S_{\min}\cup S_{\text{sad}}). \tag{A.16}$$

That is to say, the value of the function V always decreases before the trajectory of the dynamical system enters the set $S_{\min}\cup S_{\text{sad}}$. Therefore, the trajectory from any initial point $\left(x^{(0)},y^{(0)}\right)$ in the set $\Omega^{\circ}\setminus(S_{\min}\cup S_{\text{sad}})$ will converge to a point in the set $S_{\min}\cup S_{\text{sad}}$. ■

A.3. Proof of Theorem 3

Proof. First, it immediately follows from conditions (37) and (38) that, when $x^{(0)}\geq x^{(\bar{n})}$, inequalities (31) and (33) hold; when $x^{(0)}<x^{(\bar{n})}$, inequalities (34) and (36) hold.

Second, the system has entered the stationary state on day $\bar{n}$. Thus, by formulae (A.3), (7), and (24), it is obtained that

$$h^{(\bar{n})}=t_a\left(x^{(\bar{n})}\right)+p_a^{(\bar{n}+1)}-t_b\left(y^{(\bar{n})}\right)-w\left(y^{(\bar{n})}\right)-g\left(z^{(\bar{n})}\right)-p_b^{(\bar{n}+1)}. \tag{A.17}$$

When $x^{(0)}\geq x^{(\bar{n})}$, rearranging formula (A.17) and substituting the resultant expression into (37) gives

$$\begin{aligned} t_a\left(x^{(0)}\right)&>t_b\left(y^{(\bar{n})}\right)+w\left(y^{(\bar{n})}\right)+g\left(z^{(\bar{n})}\right)+p_b^{(\bar{n}+1)}+h^{(\bar{n})}\\ &>t_b\left(y^{(\bar{n})}\right)+w\left(y^{(\bar{n})}\right)+g\left(z^{(\bar{n})}\right)+p_b^{(\bar{n}+1)}+\varepsilon,\ \text{for}\ \ \varepsilon\in\left(h^{(0)},h^{(\bar{n})}\right). \end{aligned}$$

That is to say, inequality (32) holds.

Similarly, when $x^{(0)}<x^{(\bar{n})}$, rearranging formula (A.17) and substituting the resultant

expression into (38) leads to

$$t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)>t_a\left(x^{(\bar{n})}\right)+p_a^{(\bar{n}+1)}-h^{(\bar{n})}>t_a\left(x^{(\bar{n})}\right)+p_a^{(\bar{n}+1)}-\varepsilon,$$

$$\text{for } \varepsilon\in\left(h^{(\bar{n})},h^{(0)}\right).$$

Namely, inequality (35) holds. Therefore, the implementation of the toll or subsidy scheme is a prior Pareto-improvement implementation on day $\bar{n}+1$. ■

A.4. Proof of Proposition 1

Proof. By formula (39) and the condition $\kappa_a^{(n)}>0$, it follows that

$$t_a\left(x^{(n)}\right)+p_a^{(n+1)}=t_a\left(x^{(0)}\right)-\kappa_a^{(n)}<t_a\left(x^{(0)}\right).$$

Similarly, by formula (40) and the condition $\kappa_b^{(n)}>0$, it follows that

$$t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+p_b^{(n+1)}=t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right)-\kappa_b^{(n)}$$

$$<t_b\left(y^{(0)}\right)+w\left(y^{(0)}\right)+g\left(z^{(0)}\right).$$

Thus, the toll or subsidy scheme (39) and (40) satisfies conditions (37) and (38) on each day. In addition, we have

$$p_a^{(n+1)}-p_b^{(n+1)}=t_a\left(x^{(0)}\right)-t_a\left(x^{(n)}\right)-t_b\left(y^{(0)}\right)-w\left(y^{(0)}\right)$$

$$-g\left(z^{(0)}\right)+t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+\kappa_b^{(n)}-\kappa_a^{(n)}$$

$$=x^{(n)}t_a'\left(x^{(n)}\right)+h^{(n)}+g\left(z^{(n)}\right).$$

The first equality in the above equation is obtained by substituting formulae (39) and (40) into the left hand side of the equality and the second equality is generated by substituting (41) into the left hand side. Therefore, the toll or subsidy scheme also satisfies formula (23) on each day. ■

A.5. Proof of Proposition 2

Proof. By formulae (58) and (59) and the condition $\bar{\kappa}^{(n)}>0$, it follows that

$$t_a\left(x^{(n)}\right)+p_a^{(n)}-\min\left\{\mu_a^{(n+1)},\mu_b^{(n+1)}\right\}=t_a\left(x^{(n)}\right)+p_a^{(n)}-\max\left\{\tau_a^{(n)},\tau_b^{(n)}\right\}-\bar{\kappa}^{(n)}$$

$$\le t_a\left(x^{(n)}\right)+p_a^{(n)}-\tau_a^{(n)}-\bar{\kappa}^{(n)}=t_a\left(x^{(0)}\right)-\bar{\kappa}^{(n)}<t_a\left(x^{(0)}\right).$$

Similarly, by formulae (58) and (60) and the condition $\bar{\kappa}^{(n)}>0$, it is obtained that

$$t_b\left(y^{(n)}\right)+w\left(y^{(n)}\right)+g\left(z^{(n)}\right)+p_b^{(n)}-\min\left\{\mu_a^{(n+1)},\mu_b^{(n+1)}\right\}$$

$$= t_b\left(y^{(n)}\right) + w\left(y^{(n)}\right) + g\left(z^{(n)}\right) + p_b^{(n)} - \max\left\{\tau_a^{(n)}, \tau_b^{(n)}\right\} - \bar{\kappa}^{(n)}$$

$$\leq t_b\left(y^{(n)}\right) + w\left(y^{(n)}\right) + g\left(z^{(n)}\right) + p_b^{(n)} - \tau_b^{(n)} - \bar{\kappa}^{(n)}$$

$$= t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right) - \bar{\kappa}^{(n)} < t_b\left(y^{(0)}\right) + w\left(y^{(0)}\right) + g\left(z^{(0)}\right).$$

Therefore, the toll or subsidy scheme (56) to (58) satisfies conditions (54) and (55) on each day. In addition, it is obvious that the scheme also satisfies formula (23) on each day. ■

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