Robust Stability of Uncertain Linear Systems with Input and Output Quantization and Packet Loss ⋆

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Abstract

This paper investigates the robust stability of uncertain discrete-time linear systems subject to input and output quantization and packet loss. First, a necessary and sufficient condition in terms of LMIs is proposed for the quadratic stability of the closed-loop system with double quantization and norm bounded uncertainty in the plant. Moreover, it is shown that the proposed condition can be exploited to derive the coarsest logarithmic quantization density under which the uncertain plant can be quadratically stabilized via quantized state feedback. Second, a new class of Lyapunov function which depends on the quantization errors in a multilinear way is developed to obtain less conservative results. Lastly, the case with input and output packet-loss channels is investigated.

Key words: Robust stability, Input and output quantization, Packet-loss channels, Uncertain systems, LMIs.

1 Introduction

Motivated by finite network resource, quantized feedback control has been one of the most popular research trends in the field of networked control systems (see, e.g., Zhang et al. (2013)). It naturally becomes significant that how the quantization error will influence the stability and performance of the feedback systems. In the meanwhile, a great deal of effort has gone into establishing the minimum feedback information needed to stabilize a open-loop unstable system. Perhaps the most important results in recent years on the quantized feedback control should be traced back to Elia & Mitter (2001) where the logarithmic quantization was proposed and shown to be the coarsest quantizer to quadratically stabilize discrete-time linear time-invariant systems. The logarithmic quantizer was further investigated by Fu & Xie (2005) in which the sector bound approach was exploited to relate the design problem for quantized feedback control to the optimal H_∞ control problem. Besides, the quantized feedback control problem has been studied in different scenarios. For instance, Gu & Qiu (2014) put forward the polar logarithmic quantization for multi-input systems; Gu et al. (2015) studied the mean-square stabilization for networked control systems with both fading channels and logarithmic quantization; (Coutinho et al. 2010), Xia et al. (2013) considered feedback control systems with input and output quantization. On the other hand, packet loss is also a widely studied topic as one of the main communication constraints, see, e.g., Rich & Elia (2015). Among the works considering both the effect of quantization and packet loss, one should mention Ishido et al. (2011) which investigated the digital channel subject to packet loss and finite-level quantization, Tsumura et al. (2009) which analyzed the tradeoffs between the coarsest quantizer, packet-loss rate and the instability of the plant.

More recently, another research aspect that researchers have started to deal with is the effect of plant uncertainty. See, e.g., Su & Chesi (2017*a*) which considered robust stability of uncertain system over fading channels, Fu & Xie (2010) where sufficient condition was proposed for robust stabilization for linear uncertain systems via logarithmic quantized feedback, Liu et al. (2015) which studied the stability analysis of continuous-time uncertain system with dynamic quantization and communication delays, Hayakawa et al. (2009) in which adaptive quantized control was designed for nonlinear uncertain system, Kang & Ishii (2015) which considered coarsest quantization for a class of finite-order uncertain autoregressive plant.

In this paper, we first consider the model of double quantization as studied in Coutinho et al. (2010) with the plant affected by unstructured uncertainty and then further integrate the effect of input and output packet

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loss. Specifically, the controller output and the plant output are transmitted through input and output packet-loss channels respectively after being quantized via two independent logarithmic quantizers. First, a necessary and sufficient condition in terms of LMIs is proposed for the quadratic stability of the closed-loop system with double quantization and norm bounded uncertainty in the plant. Moreover, it is shown that the proposed condition can be exploited to derive the coarsest logarithmic quantization density under which the uncertain plant can be robustly quadratically stabilized via quantized state feedback. Second, a new class of Lyapunov function which depends on the quantization errors in a multilinear way is developed to obtain less conservative result. Lastly, a sufficient condition is established to ensure the robust stability in the mean square sense for the uncertain closed-loop systems with input and output quantization and packet-loss channels. A conference version of this paper (without Section 4 and part of Section 5) is reported in Su & Chesi (2017*b*).

2 Quadratic stability of uncertain systems with input and output quantization

In this section, we focus on the robust quadratic stability of uncertain systems with input and output quantization. Let us first consider the single-input single-output plant affected by uncertainty described as

$$\begin{cases} x_p(k+1) = (A+A_1)x_p(k) + (B+B_1)u(k) \\ y(k) = Cx_p(k) \end{cases} \quad (1)$$

where $x_p(k) \in \mathbb{R}^n$ is the plant state, $u(k) \in \mathbb{R}$ is the plant input and $y(k) \in \mathbb{R}$ is the plant output, $(A,\ B)$ is the nominal system and the time-varying uncertainty $(A_1,\ B_1)$ is assumed to be norm bounded satisfying

$$[A_1\ B_1] = HF(k)[E_1\ E_2],\ F(k)F(k)^T \le I. \quad (2)$$

The controller is assumed to be dynamic, described as

$$\begin{cases} x_c(k+1) = A_c x_c(k) + B_c v(k) \\ w(k) = C_c x_c(k) + D_c v(k) \end{cases} \quad (3)$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state, $v(k) \in \mathbb{R}$ is the controller input and $w(k) \in \mathbb{R}$ is the controller output.

Following the works Elia & Mitter (2001) and Fu & Xie (2005), we utilize the logarithmic quantization defined as

$$Q(v) = \begin{cases} \rho^i & \text{if } \frac{1}{1+\delta}\rho^i < v \le \frac{1}{1-\delta}\rho^i \\ & v > 0,\ i = \pm 1, \pm 2, \dots \\ 0 & \text{if } v = 0 \\ -Q(-v) & \text{if } v < 0 \end{cases} \quad (4)$$

where $0 < \rho < 1$ is the quantization density and $\delta = \frac{1-\rho}{1+\rho}$. It is assumed that the output of the plant $y(k)$ is quantized before being sent to the input of the controller $v(k)$ and the the output of the controller $w(k)$ is quantized before being sent to the input of the plant $u(k)$. The two quantizers are modeled as

$$v(k) = Q_1(y(k)),\ u(k) = Q_2(w(k)) \quad (5)$$

where $Q_1(\cdot)$ and $Q_2(\cdot)$ are static logarithmic quantizers with quantization density ρ_1 and ρ_2.

Let $x(k) = [x_p(k)^T\ x_c(k)^T]^T$ be the state of the closed-loop system. Comprising the plant, the controller and the quantizers, such a closed-loop system is given by

$$\begin{aligned} x(k+1) &= \begin{pmatrix} x_p(k+1) \\ x_c(k+1) \end{pmatrix} = \begin{pmatrix} (A+A_1)x_p(k) \\ A_c x_c(k) \end{pmatrix} \\ &+ \begin{pmatrix} (B+B_1)Q_2(C_c x_c(k) + D_c Q_1(Cx_p(k))) \\ B_c Q_1(Cx_p(k)) \end{pmatrix}. \end{aligned} \quad (6)$$

When there is no uncertainty in the plant, i.e., $A_1 = 0$ and $B_1 = 0$, it is shown in Theorem 2 of Coutinho et al. (2010) that the closed-loop system (6) is quadratically stable if and only if there exists $P > 0$ such that

$$\begin{gathered} \bar{A}(\Delta_1, \Delta_2)^T P \bar{A}(\Delta_1, \Delta_2) - P < 0 \\ \forall |\Delta_1| \le \delta_1, |\Delta_2| \le \delta_2 \end{gathered} \quad (7)$$

where

$$\bar{A}(\Delta_1, \Delta_2) = \begin{pmatrix} A + B(1+\Delta_2)D_c(1+\Delta_1)C & B(1+\Delta_2)C_c \\ B_c(1+\Delta_1)C & A_c \end{pmatrix}. \quad (8)$$

Lemma 1 *(Amato et al. (1996), Garofalo et al. (1993)) Consider the matrix-valued function $M(p) : \mathcal{P} \to \mathbb{R}^{n\times n}$, where $p \in \mathcal{P} \subset \mathbb{R}^q$ and the set $\mathcal{P}$ is a hyper-box, i.e., $\mathcal{P} := [\underline{p_1}, \overline{p_1}] \times [\underline{p_2}, \overline{p_2}] \times \cdots \times [\underline{p_q}, \overline{p_q}]$. Let us assume*

$$M(p) = \frac{N(p)}{d(p)}, \quad (9)$$

with $N(\cdot)$ a multi-affine matrix-valued function of p, $d(\cdot)$ a multi-affine polynomial of p and $d(p) \neq 0$ for all $p \in \mathcal{P}$. Then $M(p) > 0, \forall p \in \mathcal{P}$ if and only if $M(p_{(i)}) > 0, i = 1, \dots, 2^q$ where $p_{(i)}$ is the i-th vertex of $\mathcal{P}$.

Therefore, it is necessary and sufficient to check the quadratic stability of an uncertain system depending multi-affinely on uncertain parameters constrained into a hyper-box on the vertices of the hyper-box under the same Lyapunov function $v(x(k)) = x(k)^T P x(k)$.

Next, let us take the uncertainty (A_1, B_1) into consideration. By treating the quantization errors as sector bounded time-varying uncertainties, let us define the auxiliary system for (6) as

$$\begin{cases} x(k+1) = \hat{A}(\Delta_1(k), \Delta_2(k))x(k) \\ \hat{A}(\Delta_1, \Delta_2) = \begin{pmatrix} A + A_1 & 0 \\ 0 & A_c \end{pmatrix} + \\ \begin{pmatrix} (B+B_1)(1+\Delta_2)([0\ C_c] + D_c(1+\Delta_1)[C\ 0]) \\ B_c(1+\Delta_1)[C\ 0] \end{pmatrix} \\ \forall |\Delta_1(k)| \le \delta_1, |\Delta_2(k)| \le \delta_2. \end{cases} \quad (10)$$

Before proceeding to our main result, let us report the following result (see, e.g., Xie (1996)).

Lemma 2 *Given real matrices $\mathcal{S} = \mathcal{S}^T, \mathcal{U}, \mathcal{V}$ with appropriate dimension, then*

$$\mathcal{S} + \mathcal{U}F(k)\mathcal{V} + \mathcal{V}^T F(k)^T \mathcal{U}^T > 0 \quad (11)$$

holds for all $F(k)$ satisfying $F(k)F(k)^T \leq I$ if and only if there exists a scalar $\sigma > 0$ such that

$$\mathcal{S} - \sigma\mathcal{U}\mathcal{U}^T - \sigma^{-1}\mathcal{V}^T\mathcal{V} > 0. \quad (12)$$

Theorem 3 *The closed-loop system (6) is robustly quadratically stable if and only if there exist $Q > 0$ and a scalar $\sigma(\Delta_1, \Delta_2) > 0$* [1] *such that*

$$\begin{pmatrix} Q & Q\bar{A}(\Delta_1,\Delta_2)^T & Q\bar{E}(\Delta_1,\Delta_2)^T \\ * & Q - \sigma(\Delta_1,\Delta_2)\bar{H}\bar{H}^T & 0 \\ * & * & \sigma(\Delta_1,\Delta_2)I \end{pmatrix} > 0 \quad (13)$$
$$\forall \Delta_1 \in \{-\delta_1, \delta_1\}, \Delta_2 \in \{-\delta_2, \delta_2\}$$

where $\bar{A}(\Delta_1, \Delta_2)$ is defined in (8), and

$$\begin{cases} \bar{E}(\Delta_1,\Delta_2) = \\ \quad \left(E_1 + E_2(1+\Delta_2)D_c(1+\Delta_1)C \;\; E_2(1+\Delta_2)C_c\right) \\ \bar{H} = \left(H^T \;\; 0\right)^T. \end{cases} \quad (14)$$

Proof. First, let us observe that, from Theorem 2 in Coutinho et al. (2010), the system (6) is robustly quadratically stable if and only if the auxiliary system (10) is robustly stable with a Lyapunov function $v(x(k)) = x(k)^T P x(k)$ for all $|\Delta_1(k)| \leq \delta_1, |\Delta_2(k)| \leq \delta_2$ and all $[A_1 \; B_1]$ defined in (2). Next, let us observe that

$$\hat{A}(\Delta_1,\Delta_2) = \bar{A}(\Delta_1,\Delta_2) + \hat{A}_1(\Delta_1,\Delta_2)$$

where $\hat{A}_1(\Delta_1,\Delta_2) = \bar{H}F(k)\bar{E}(\Delta_1,\Delta_2)$. It follows that the system (10) is robustly quadratically stable if and only if there exists $Q > 0$ such that

$$\begin{pmatrix} Q & Q[\bar{A}(\Delta_1,\Delta_2) + \hat{A}_1(\Delta_1,\Delta_2)]^T \\ * & Q \end{pmatrix} > 0 \quad (15)$$
$$\forall |\Delta_1| \leq \delta_1, |\Delta_2| \leq \delta_2, \forall F(k)F(k)^T \leq I.$$

Since the above matrix depends on (Δ_1, Δ_2) bilinearly, by exploiting Lemma 1, we have that the condition in (15) can be equivalently rewritten as

$$\begin{pmatrix} Q & Q[\bar{A}(\Delta_1,\Delta_2) + \hat{A}_1(\Delta_1,\Delta_2)]^T \\ * & Q \end{pmatrix} > 0$$
$$\forall \Delta_1 \in \{-\delta_1,\delta_1\}, \Delta_2 \in \{-\delta_2,\delta_2\}, \forall F(k)F(k)^T \leq I. \quad (16)$$

Next, let us denote

$$\mathcal{S}(\Delta_1,\Delta_2) = \begin{pmatrix} Q & Q\bar{A}(\Delta_1,\Delta_2)^T \\ * & Q \end{pmatrix}, \mathcal{U} = \begin{pmatrix} 0 \\ \bar{H} \end{pmatrix}$$
$$\mathcal{V}(\Delta_1,\Delta_2) = \left(\bar{E}(\Delta_1,\Delta_2)Q \;\; 0\right),$$

one has that

$$\begin{pmatrix} Q & Q[\bar{A}(\Delta_1,\Delta_2) + \hat{A}_1(\Delta_1,\Delta_2)]^T \\ * & Q \end{pmatrix}$$
$$= \mathcal{S}(\Delta_1,\Delta_2) + \mathcal{U}F(k)\mathcal{V}(\Delta_1,\Delta_2) + \mathcal{V}(\Delta_1,\Delta_2)^T F(k)^T \mathcal{U}^T.$$

Let us further denote

$$\begin{cases} \mathcal{S}_1 = \mathcal{S}(\delta_1,\delta_2) \\ \mathcal{S}_2 = \mathcal{S}(-\delta_1,\delta_2) \\ \mathcal{S}_3 = \mathcal{S}(\delta_1,-\delta_2) \\ \mathcal{S}_4 = \mathcal{S}(-\delta_1,-\delta_2) \end{cases} \text{and} \begin{cases} \mathcal{V}_1 = \mathcal{S}(\delta_1,\delta_2) \\ \mathcal{V}_2 = \mathcal{S}(-\delta_1,\delta_2) \\ \mathcal{V}_3 = \mathcal{S}(\delta_1,-\delta_2) \\ \mathcal{V}_4 = \mathcal{S}(-\delta_1,-\delta_2). \end{cases}$$

Then one can rewrite the condition in (16) as

$$\mathcal{S}_i + \mathcal{U}F(k)\mathcal{V}_i + \mathcal{V}_i^T F(k)^T \mathcal{U}^T > 0, \; i = 1,2,3,4$$

for all admissible $F(k)$ defined in (2). Based on Lemma 2 together with the Schur complement lemma, it can be verified that the condition in (16) holds if and only if there exist $\sigma_i > 0, \; i = 1,2,3,4$ satisfying

$$\begin{pmatrix} \mathcal{S}_i - \sigma_i\mathcal{U}\mathcal{U}^T & \mathcal{V}_i^T \\ * & \sigma_i I \end{pmatrix} > 0, \; i = 1,2,3,4$$

which is equivalent to the condition (13). It can be noted that the scalar function $\sigma(\Delta_1, \Delta_2)$ in (13) amounts to the four scalar variables $\sigma_i, i = 1,2,3,4$ corresponding to each vertex of $[-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$. □

Remark 4 *Theorem 3 proposes a necessary and sufficient condition in terms of LMIs for the robust quadratic stability of the closed-loop system (6). As mentioned in Fu & Xie (2010) and Kang & Ishii (2015), it is difficult to deal with the quadratic stabilization for quantized feedback uncertain systems since there are two blocks of uncertainties as a result of the uncertain plant and quantization error. It should be noticed that the system (6) involves three blocks of uncertainties, the quadratic stability analysis of which cannot be fully investigated with the existing approaches including the scaled H_∞ analysis method. When considering only one quantizer, i.e., $\Delta_1 = 0$, the necessary and sufficient condition proposed in Theorem 3 can be led to derive the coarsest logarithmic quantization density δ_{sup} under which the uncertain system (1) can be quadratically stabilized via quantized state feedback by bisection algorithm. This is shown in the following corollary, see also Example 12 for illustration.*

Corollary 5 *The coarsest logarithmic quantization density under which the uncertain system (1)-(2) can be robustly quadratically stabilized via quantized state*

[1] Since (Δ_1, Δ_2) takes value only at the vertices of $[-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$ in (13), $\sigma(\Delta_1, \Delta_2)$ amounts to 4 scalar variables corresponding to the 4 vertices, i.e., the variable σ in the LMI (13) is allowed to vary with different vertex.

feedback is given by $\rho_{inf} = \frac{1-\delta_{sup}}{1+\delta_{sup}}$ *, where*

$$\delta_{sup} = \sup_{Q,Y,\sigma(\Delta)} \delta \ s.t.$$

$$\begin{pmatrix} Q & QA^T + (1+\Delta)YB^T & QE_1^T + (1+\Delta)YE_2^T \\ * & Q - \sigma(\Delta)HH^T & 0 \\ * & * & \sigma(\Delta)I \end{pmatrix} > 0, \quad \forall \Delta \in \{-\delta, \delta\} \tag{17}$$

and the corresponding linear feedback gain is given by $K^* = (Y^*)^T (Q^*)^{-1}$.

Remark 6 *The results proposed in this section and the following section can be extended to multiple-input multiple-output systems by quantizing each components of the vector input and vector output independently. Moreover, such results remain applicable when there are more than two quantizers concatenated in a closed loop as long as the quantizers are independent, which is potential since it is often required to apply a decentralized structure for the quantizers in networked control systems.*

3 Multilinear parameter-dependent quadratic Lyapunov function

As proposed in Gao & Chen (2008), a quantization dependent Lyapunov function will provide less conservative condition for the robust stability of the quantized system. Here we define the following Lyapunov function:

$$v(x(k)) = x(k)^T P(\Delta_1(k), \Delta_2(k)) x(k) \tag{18}$$

where

$$\begin{aligned} &P(\Delta_1(k), \Delta_2(k)) \\ &= P_0 + P_1\Delta_1(k) + P_2\Delta_2(k) + P_3\Delta_1(k)\Delta_2(k) \end{aligned} \tag{19}$$

where P_0, P_1, P_2, P_3 are symmetric matrices. We call (18)-(19) multilinear parameter-dependent quadratic Lyapunov function. As the name suggests, this class of Lyapunov function is allowed to depend on the quantization errors $(\Delta_1(k), \Delta_2(k))$ in a multilinear way. To be more specific, $P(\Delta_1(k), \Delta_2(k))$ is a bilinear matrix function w.r.t $(\Delta_1(k), \Delta_2(k))$. It should be noted that $P_i, i = 1, 2, 3$ do not necessarily have to be positive definite. For simplicity, let $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ denote (Δ_1, Δ_2) and (Δ_1', Δ_2'), respectively.

Theorem 7 *The closed-loop system* (6) *is asymptotically stable if there exist* P_0, P_1, P_2, P_3 *and a scalar* $\mu(\mathbf{\Delta}, \mathbf{\Delta}') > 0$ *such that*

$$\begin{pmatrix} P(\mathbf{\Delta}) & \bar{A}(\mathbf{\Delta})^T P(\mathbf{\Delta}') & \mu(\mathbf{\Delta}, \mathbf{\Delta}')\bar{E}(\mathbf{\Delta})^T & 0 \\ * & P(\mathbf{\Delta}') & 0 & P(\mathbf{\Delta}')\bar{H} \\ * & * & \mu(\mathbf{\Delta}, \mathbf{\Delta}')I & 0 \\ * & * & * & \mu(\mathbf{\Delta}, \mathbf{\Delta}')I \end{pmatrix} > 0$$
$$\forall \Delta_1, \Delta_1' \in \{-\delta_1, \delta_1\} \ \forall \Delta_2, \Delta_2' \in \{-\delta_2, \delta_2\} \tag{20}$$

where $\bar{A}, \bar{E}, \bar{H}$ *are defined in* (8) *and* (14).

Proof. Suppose there exist P_0, P_1, P_2, P_3 and $\mu(\mathbf{\Delta}, \mathbf{\Delta}') > 0$ such that the condition (20) holds. By replacing $\mu(\mathbf{\Delta}, \mathbf{\Delta}')$ with $\sigma(\mathbf{\Delta}, \mathbf{\Delta}')^{-1}$ and Schur complement lemma, one has that

$$\begin{pmatrix} P(\mathbf{\Delta}) & \bar{A}(\mathbf{\Delta})^T P(\mathbf{\Delta}') & \bar{E}(\mathbf{\Delta})^T \\ * & P(\mathbf{\Delta}') - \sigma(\mathbf{\Delta}, \mathbf{\Delta}')P(\mathbf{\Delta}')\bar{H}\bar{H}^T P(\mathbf{\Delta}') & 0 \\ * & * & \sigma(\mathbf{\Delta}, \mathbf{\Delta}')I \end{pmatrix}$$
$$> 0, \ \ \forall \Delta_1, \Delta_1' \in \{-\delta_1, \delta_1\} \ \forall \Delta_2, \Delta_2' \in \{-\delta_2, \delta_2\}.$$

By similar reasoning in the proof of Theorem 3 based on Lemma 2, it can be implicated that the above condition is equivalent to that

$$\begin{pmatrix} P(\mathbf{\Delta}) & (\bar{A}(\mathbf{\Delta}) + \hat{A}_1(\mathbf{\Delta}))^T P(\mathbf{\Delta}') \\ * & P(\mathbf{\Delta}') \end{pmatrix} > 0$$
$$\forall \Delta_1, \Delta_1' \in \{-\delta_1, \delta_1\} \ \forall \Delta_2, \Delta_2' \in \{-\delta_2, \delta_2\}$$

for all $F(k)F(k)^T \leq I$ where $\hat{A}_1(\mathbf{\Delta}) = \bar{H}F(k)\bar{E}(\mathbf{\Delta})$.

Let us observe that the above matrix depend multiaffinely on $(\Delta_1, \Delta_2, \Delta_1', \Delta_2')$. It follows by exploiting Lemma 1 that

$$\begin{pmatrix} P(\Delta_1, \Delta_2) & \hat{A}(\Delta_1, \Delta_2)^T P(\Delta_1', \Delta_2') \\ * & P(\Delta_1', \Delta_2') \end{pmatrix} > 0$$
$$\forall |\Delta_1| \leq \delta_1, |\Delta_2| \leq \delta_2, \ \forall |\Delta_1'| \leq \delta_1, |\Delta_2'| \leq \delta_2$$

for all $F(k)F(k)^T \leq I$. Consequently, it can be implied that for all nonzero $x(k)$,

$$\begin{aligned} &\nabla v(x(k)) \\ &= v(x(k+1)) - v(x(k)) \\ &= x(k)^T \hat{A}(\Delta_1(k), \Delta_2(k))^T P(\Delta_1(k+1), \Delta_2(k+1)) \times \\ &\hat{A}(\Delta_1(k), \Delta_2(k))x(k) - x(k)^T P(\Delta_1(k), \Delta_2(k))x(k) < 0 \end{aligned}$$

for all $|\Delta_1| \leq \delta_1, |\Delta_2| \leq \delta_2$ and all $F(k)F(k)^T \leq I$.

Thus, the uncertain system (10) is asymptotically stable, so is the closed-loop system (6). □

It can be verified that the condition (20) will be reduced to (13) when the common quadratic Lyapunov function is adopted, i.e., $P_1 = P_2 = P_3 = 0$. Thus, Theorem 7 provides a less conservative condition than (13) for the asympotical stability of the closed-loop system (6). See Example 13 for demonstration.

4 Packet-loss channels

In this section, we further consider the scenario where the communication channels are affected by packet loss, described by Figure 1.

To be specific, the quantized output of the controller and the plant are transmitted over the packet-loss channel $\xi_1(k)$ and $\xi_2(k)$, respectively. The packet loss experienced by the input channel and output channel is characterized via two independent Bernoulli processes with

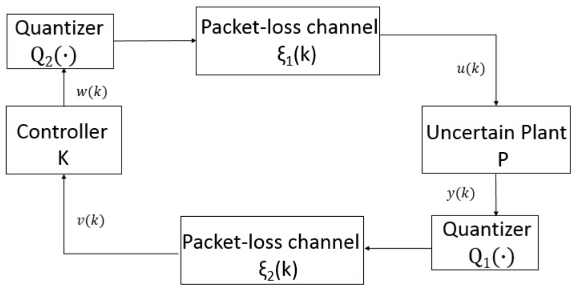

Fig. 1. The uncertain closed-loop system with input and output quantization over packet-loss channels

probability $1-\tau_1$ and $1-\tau_2$, described by

$$\Pr(\xi_i(k)=j)=\begin{cases}1-\tau_i, & j=0,\\ \tau_i, & j=1,\end{cases}\quad 0<\tau_i\le 1,\ i=1,2. \tag{21}$$

To further take the packet loss into consideration, the closed-loop quantized system over packet-loss channels as shown in Figure 1 is represented as

$$x(k+1)=\begin{pmatrix}x_p(k+1)\\ x_c(k+1)\end{pmatrix}=\begin{pmatrix}(A+A_1)x_p(k)\\ A_cx_c(k)\end{pmatrix}+\begin{pmatrix}(B+B_1)\xi_1(k)Q_2(C_cx_c(k)+D_c\xi_2(k)Q_1(Cx_p(k)))\\ B_c\xi_2(k)Q_1(Cx_p(k))\end{pmatrix} \tag{22}$$

and the auxiliary system obtained by treating the quantization errors as sector-bounded uncertainties is given by

$$\begin{cases}x(k+1)=\tilde{A}(\Delta_1(k),\Delta_2(k),\xi_1(k),\xi_2(k))x(k)\\ \quad\tilde{A}(\Delta_1,\Delta_2,\xi_1,\xi_2)\\ =\begin{pmatrix}A+A_1 & 0\\ 0 & A_c\end{pmatrix}+\begin{pmatrix}(B+B_1)\xi_1(k) & 0\\ 0 & B_c\xi_2(k)\end{pmatrix}\\ \times\begin{pmatrix}(1+\Delta_2)D_c\xi_2(k)(1+\Delta_1)C & (1+\Delta_2)C_c\\ (1+\Delta_1)C & 0\end{pmatrix}\\ \forall|\Delta_1(k)|\le\delta_1,|\Delta_2(k)|\le\delta_2.\end{cases} \tag{23}$$

To this end, let us introduce the stability in the mean square sense to tackle the stochastic uncertainties used to model the packet-loss channels. Define

$$X(k)=\mathcal{E}(x(k)x(k)'). \tag{24}$$

Definition 8 *(Elia (2005)) The closed-loop system* (22) *is said to be robustly stable in the mean square sense if* $X(k)$ *is well-defined for all* $k\ge 0$ *and*

$$\lim_{k\to\infty}X(k)=0\ \forall x(0)\in\mathbb{R}^n \tag{25}$$

for all admissible $F(k)$ *defined in* (2).

Theorem 9 *The closed-loop system* (22) *is robustly stable in the mean square sense if there exist* $Q>0$ *and a scalar* $\sigma(\mathbf{\Delta})>0$ *such that the LMIs in* (26) *hold where*

$$\begin{cases}\kappa_1=\sqrt{(1-\tau_1)(1-\tau_2)}, & \kappa_2=\sqrt{\tau_1\tau_2}\\ \kappa_3=\sqrt{\tau_1(1-\tau_2)}, & \kappa_4=\sqrt{\tau_2(1-\tau_1)}\end{cases}$$

and

$$\begin{cases}\mathcal{A}_1=\begin{pmatrix}A & 0\\ 0 & A_c\end{pmatrix}\\ \mathcal{A}_2=\begin{pmatrix}A+(1+\Delta_1)(1+\Delta_2)BD_cC & (1+\Delta_2)BC_c\\ (1+\Delta_1)B_cC & A_c\end{pmatrix}\\ \mathcal{A}_3=\begin{pmatrix}A & (1+\Delta_2)BC_c\\ 0 & A_c\end{pmatrix},\ \mathcal{A}_4=\begin{pmatrix}A & 0\\ (1+\Delta_1)B_cC & A_c\end{pmatrix}\\ \bar{E}_1=\begin{pmatrix}E_1 & 0\end{pmatrix}\\ \bar{E}_2=\begin{pmatrix}E_1+(1+\Delta_1)(1+\Delta_2)E_2D_cC & (1+\Delta_2)E_2C_c\end{pmatrix}\\ \bar{E}_3=\begin{pmatrix}E_1 & (1+\Delta_2)E_2C_c\end{pmatrix}.\end{cases}$$

Proof. Suppose there exist $Q>0$ and $\sigma(\mathbf{\Delta})>0$ satisfying the condition in (26). First, it can be verified via applying the S-procedure that the condition (26) is equivalent to the condition (27).

Since the matrix in (27) depends on (Δ_1,Δ_2) multi-affinely, it can be followed from Lemma 1 that the matrix inequality in (27) holds for all $|\Delta_1|\le\delta_1,|\Delta_2|\le\delta_2$ and all $F(k)F(k)^T\le I$.

Next, let us pre and post multiply the matrix inequality in (27) with $\mathrm{diag}\{Q^{-1},Q^{-1}\}$ and after applying the Schur complement lemma, one has that (27) is equivalent to

$$P-\kappa_1^2\bar{\mathcal{A}}_1^TP\bar{\mathcal{A}}_1-\kappa_2^2\bar{\mathcal{A}}_2^TP\bar{\mathcal{A}}_2-\kappa_3^2\bar{\mathcal{A}}_3^TP\bar{\mathcal{A}}_3-\kappa_4^2\bar{\mathcal{A}}_4^T\times P\bar{\mathcal{A}}_4>0,\ \forall|\Delta_1|\le\delta_1,|\Delta_2|\le\delta_2,F(k)F(k)^T\le I$$

where $P=Q^{-1}$ and

$$\begin{cases}\bar{\mathcal{A}}_1=\begin{pmatrix}A+A_1 & 0\\ 0 & A_c\end{pmatrix}\\ \bar{\mathcal{A}}_2=\begin{pmatrix}A+A_1 & (1+\Delta_2)(B+B_1)C_c\\ 0 & A_c\end{pmatrix}\\ \quad+\begin{pmatrix}(1+\Delta_1)(1+\Delta_2)(B+B_1)D_cC & 0\\ (1+\Delta_1)B_cC & 0\end{pmatrix}\\ \bar{\mathcal{A}}_3=\begin{pmatrix}A+A_1 & (1+\Delta_2)(B+B_1)C_c\\ 0 & A_c\end{pmatrix}\\ \bar{\mathcal{A}}_4=\begin{pmatrix}A+A_1 & 0\\ (1+\Delta_1)B_cC & A_c\end{pmatrix}.\end{cases}$$

Let us observe that for the auxiliary system (23),

$$\begin{aligned}X(k+1)&=\mathcal{E}(\tilde{A}(\Delta_1(k),\Delta_2(k),\xi_1(k),\xi_2(k))X(k)\\ &\quad\tilde{A}(\Delta_1(k),\Delta_2(k),\xi_1(k),\xi_2(k))^T)\\ &=\kappa_1^2\bar{\mathcal{A}}_1X(k)\bar{\mathcal{A}}_1^T+\kappa_2^2\bar{\mathcal{A}}_2X(k)\bar{\mathcal{A}}_2^T+\\ &\quad\kappa_3^2\bar{\mathcal{A}}_3X(k)\bar{\mathcal{A}}_3^T+\kappa_4^2\bar{\mathcal{A}}_4X(k)\bar{\mathcal{A}}_4^T\end{aligned}$$

$$
\begin{pmatrix}
Q & \kappa_1 Q\mathcal{A}_1^T & \kappa_2 Q\mathcal{A}_2^T & \kappa_3 Q\mathcal{A}_3^T & \kappa_4 Q\mathcal{A}_4^T & \kappa_1 Q\bar{E}_1^T & \kappa_2 Q\bar{E}_2^T & \kappa_3 Q\bar{E}_3^T & \kappa_4 Q\bar{E}_1^T \\
* & Q-\sigma(\boldsymbol{\Delta})\bar{H}\bar{H}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & Q-\sigma(\boldsymbol{\Delta})\bar{H}\bar{H}^T & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & Q-\sigma(\boldsymbol{\Delta})\bar{H}\bar{H}^T & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & Q-\sigma(\boldsymbol{\Delta})\bar{H}\bar{H}^T & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \sigma(\boldsymbol{\Delta})I & 0 & 0 & 0 \\
* & * & * & * & * & * & \sigma(\boldsymbol{\Delta})I & 0 & 0 \\
* & * & * & * & * & * & * & \sigma(\boldsymbol{\Delta})I & 0 \\
* & * & * & * & * & * & * & * & \sigma(\boldsymbol{\Delta})I
\end{pmatrix} > 0 \tag{26}
$$

$$\forall\Delta_1 \in \{-\delta_1, \delta_1\}, \Delta_2 \in \{-\delta_2, \delta_2\}$$

$$
\begin{pmatrix}
Q & \kappa_1 Q(\mathcal{A}_1+\bar{H}F(k)\bar{E}_1)^T & \kappa_2 Q(\mathcal{A}_2+\bar{H}F(k)\bar{E}_2)^T & \kappa_3 Q(\mathcal{A}_3+\bar{H}F(k)\bar{E}_3)^T & \kappa_4 Q(\mathcal{A}_4+\bar{H}F(k)\bar{E}_1)^T \\
* & Q & 0 & 0 & 0 \\
* & * & Q & 0 & 0 \\
* & * & * & Q & 0 \\
* & * & * & * & Q
\end{pmatrix} > 0 \tag{27}
$$

$$\forall\Delta_1 \in \{-\delta_1, \delta_1\}, \Delta_2 \in \{-\delta_2, \delta_2\},\ \forall F(k)F(k)^T \le I.$$

where $X(k)$ is defined in (24). Based on a Lyapunov function constructed as $V(X(k)) = \text{tr}(X(k)P)$, we have that for all $|\Delta_1| \le \delta_1, |\Delta_2| \le \delta_2, F(k)F(k)^T \le I$,

$$
\begin{aligned}
V(X(k+1)) &= \text{tr}((\kappa_1^2\bar{\mathcal{A}}_1 X(k)\bar{\mathcal{A}}_1{}^T + \kappa_2^2\bar{\mathcal{A}}_2 X(k)\bar{\mathcal{A}}_2{}^T + \\
&\quad \kappa_3^2\bar{\mathcal{A}}_3 X(k)\bar{\mathcal{A}}_3{}^T + \kappa_4^2\bar{\mathcal{A}}_4 X(k)\bar{\mathcal{A}}_4{}^T)P) \\
&= \text{tr}(X(k))(\kappa_1^2\bar{\mathcal{A}}_1{}^T P\bar{\mathcal{A}}_1 + \kappa_2^2\bar{\mathcal{A}}_2{}^T P\bar{\mathcal{A}}_2 + \\
&\quad \kappa_3^2\bar{\mathcal{A}}_3{}^T P\bar{\mathcal{A}}_3 + \kappa_4^2\bar{\mathcal{A}}_4{}^T P\bar{\mathcal{A}}_4) \\
&< \text{tr}\,(X(k)P) = V(X(k))
\end{aligned}
$$

for nonzero $X(k)$, which indicates that the auxiliary system (23) is stable in the mean square sense. Thus, we can conclude that the closed-loop system (22) is robustly stable in the mean square sense. □

Remark 10 *One can observe that the Lyapunov function used in the proof above, $V(X(k)) = tr(X(k)P)$, is the same with $V(x(k)) = \mathcal{E}(x(k)^T Px(k))$. Thus, based on the definition of stochastic quadratic stability in Tsumura et al. (2009), it can be implicated by reversing the proof that the condition (26) is necessary and sufficient for the stochastic quadratic stability of the system (22).*

When there is no packet loss in the communication channels, i.e., $\tau_1 = \tau_2 = 1$, the Lyapunov function becomes $V(x(k)) = x(k)^T Px(k)$, and thus the condition (26) is reduced to the condition (13). When we consider only one quantizer and one packet-loss channel for the state feedback control of the uncertain system (1), e.g., $\tau_2 = 1$ and $\rho_1 = 1$, by simplifying the result in Theorem 9, the tradeoffs among the quantization, the packet delivery probability and the uncertainty in the plant is shown in the following result.

Corollary 11 *The coarsest logarithmic quantization density under which the uncertain system (1)-(2) with packet-loss input channel can be stochastically quadratically stabilized via quantized state feedback is $\rho_{inf} = \frac{1-\delta_{sup}}{1+\delta_{sup}}$, where δ_{sup} is given in (28) and the corresponding linear feedback gain is given by $K^* = (Y^*)^T (Q^*)^{-1}$.*

The δ_{sup} derived by the LMI problem (28) coincides with the one derived by analytical bounds proposed in Theorem 2.1 in Tsumura et al. (2009) when considering the case without uncertainty in the plant. See Example 14 for more details.

5 Numerical example

Example 12 *The first example is taken from the Example 1 in Fu & Xie (2010). The plant to be considered is described in (1)-(2) with*

$$
\begin{cases}
A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
H = \epsilon \begin{pmatrix} 1 \\ 1 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, E_2 = 1
\end{cases}
$$

where the parameter $\epsilon > 0$ is a measure of the size of uncertainty $[A_1\ B_1]$.

It is assumed that the output of the state feedback controller is quantized by a quantizer $Q(\cdot)$. The problem is to find the minimum required quantization density ρ_{inf} for different size of uncertainty ϵ.

By solving the LMI problem (17), i.e., performing the LMI feasibility test at each step of bisection algorithm,

$$\delta_{sup} = \sup_{Q,Y,\sigma(\Delta)} \delta \text{ s.t.}$$

$$\begin{pmatrix} Q & \sqrt{\tau}(QA^T+(1+\Delta)YB^T) & \sqrt{1-\tau}QA^T & \sqrt{\tau}(QE_1^T+(1+\Delta)YE_2^T) & \sqrt{1-\tau}QE_1^T \\ * & Q-\sigma(\Delta)HH^T & 0 & 0 & 0 \\ * & * & Q-\sigma(\Delta)HH^T & 0 & 0 \\ * & * & * & \sigma(\Delta)I & 0 \\ * & * & * & * & \sigma(\Delta)I \end{pmatrix} > 0, \forall\Delta \in \{-\delta,\delta\} \quad (28)$$

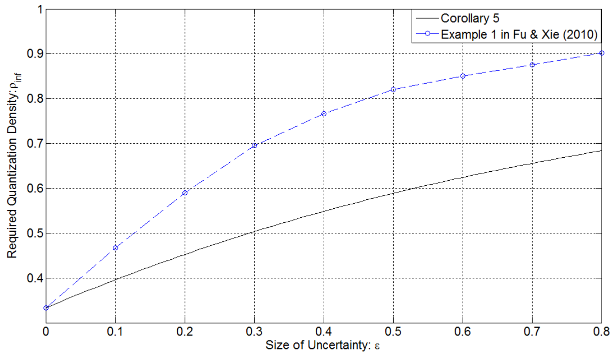

Fig. 2. Minimum required quantization density versus size of uncertainty

we obtained the result plotted in Figure 2. As shown by this figure, the coarsest quantization density ρ_{inf} obtained with Corollary 5 is smaller than the results provided by Fu & Xie (2010) using the H_∞ synthesis techniques.

Example 13 *Let us consider the situation where the plant is described as in* (1)-(3) *with by*

$$\begin{cases} A = \begin{pmatrix} 1 & -0.5 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0.5 & -0.5 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \\ C = \begin{pmatrix} 1 & -1 & 0 & 0.5 \end{pmatrix} \\ H = \begin{pmatrix} 0.25 & 0.5 & 0.25 & 0.5 \end{pmatrix}^T \\ E_1 = \begin{pmatrix} 0.1 & 0 & 0 & 0.1 \end{pmatrix}, \; E_2 = 0.1 \\ A_c = 0.69, B_c = -0.85, C_c = -0.11, D_c = 0.075. \end{cases}$$

When $\delta_1 = 0.05, \delta_2 = 0.2$, the LMIs in (13) *is feasible. Thus, based on Theorem 3, the closed-loop system* (6) *is robustly quadratically stable.*

When $\delta_1 = 0.1, \delta_2 = 0.2$, the LMIs in (13) *is infeasible. Then, we solve the LMIs in* (20) *with $P_3 = 0$, i.e., $P(\Delta_1(k), \Delta_2(k)) = P_0 + P_1\Delta_1(k) + P_2\Delta_2(k)$, and it is feasible. Therefore, the closed-loop system* (6) *is asymptotically stable with linear parameter-dependent quadratic Lyapunov function.*

When $\delta_1 = 0.145, \delta_2 = 0.2$, the LMIs in (13) *is infeasible and* (20) *is also infeasible with $P_3 = 0$. With the multilinear parameter-dependent Lyapunov function defined in* (18)-(19)*, we found that the LMIs in* (20) *is feasible. Thus we can conclude that the closed-loop system* (6) *is asymptotically stable.*

Example 14 *Consider the uncertain plant described as in* (1)-(2) *with*

$$\begin{cases} A = \begin{pmatrix} -1 & 1 & -2 & 1 \\ -0.5 & 1 & -0.5 & 0 \\ 1.5 & -1 & 0 & 0 \\ 1 & -1 & 1 & -0.5 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \\ H = \epsilon \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 0.5 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \end{cases}$$

where the parameter $\epsilon > 0$ is a measure of the size of uncertainty $[A_1 \; B_1]$.

It is assumed that the output of the state feedback controller is quantized and sent to the input of the plant via a packet-loss channel.

With different values of ϵ and τ, we calculate δ_{sup} in (28)*. As shown in Figure 3, on the δ_{sup}-τ plane with $\epsilon = 0$, the derived result based on Corollary 11 coincides with the analytical result proposed in Theorem 2.1 in Tsumura et al. (2009). Moreover, with fixed ϵ, δ_{sup} increases monotonically as τ increases; with fixed τ, δ_{sup} decreases monotonically as ϵ increases. In general, Figure 3 shows the tradeoffs among the size of uncertainty, the probability of packet loss and the quantization density, as described in* (28).

6 Conclusion

In this work, a necessary and sufficient condition in terms of LMIs is proposed to establish the robust quadratic stability for the uncertain closed-loop system with double quantization. Moreover, a new class of Lyapunov function which depends on the quantization errors in a multilinear way is constructed. In the end, robust stability in the mean square sense is considered for the uncertain closed-loop system with input and output quantization and packet-loss channels. Future work will look into the linear dynamic output feedback synthesis based on the stability conditions. Another direction could be considering nonlinear control for the uncertain networked systems.

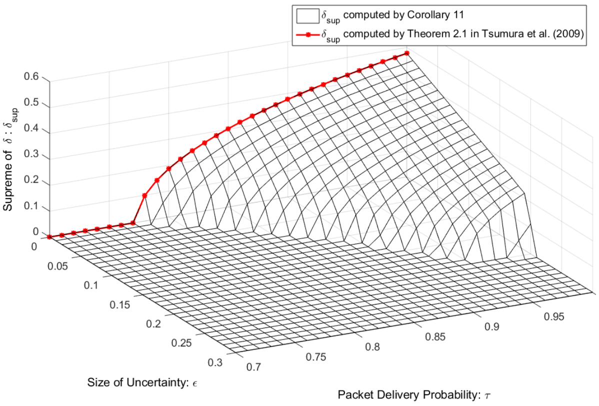

Fig. 3. The relationship among the size of uncertainty ϵ, the probability of packet delivery τ and the supremum of δ

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