Comparative Higher-Order Risk Aversion and Higher-Order Prudence

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In this paper, we define the nth-degree utility premium as the pain associated with facing the passage from a more favorable risk to a less favorable risk, where the risk increase is specified by the notion of more nth-degree risk. We further define the nth-degree prudence utility premium as the increase in pain when the individual suffers a sure loss. We show that the nth-degree utility premium, normalized by the $(n-1)$th derivative of the utility function evaluated at the initial wealth, can explain comparative risk aversion of higher orders. On the other hand, the nth-degree prudence utility premium, normalized by the nth derivative of the utility function evaluated at the initial wealth, can explain comparative prudence of higher orders.

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1. Introduction

In an expected utility framework, Eeckhoudt and Schlesinger (2009) re-examine the concept of utility premium, originally introduced by Friedman and Savage (1948) seventy years ago, and explain its relevance to decision making under uncertainty.[1] The utility premium of Friedman and Savage (1948) is defined as the degree of "pain" associated with a given risk introduction or risk increase, where pain is measured by means of the reduction in expected utility. Since the utility premium depends on the unit to which the utility level is referred, such a measure varies when the utility function undergoes a positive linear transformation. To facilitate interpersonal comparisons of the utility premium, Crainich and Eeckhoudt (2008) propose to normalize the utility premium by the marginal utility (see also Li and Liu, 2014). Indeed, Huang and Stapleton (2015) have recently shown that the utility premium due to the introduction of a zero-mean risk, normalized by the marginal utility evaluated at the initial wealth, is related to the Arrow-Pratt measure of absolute risk aversion in exactly the same way as the risk premium of Pratt (1964), i.e., the greater the Arrow-Pratt measure of absolute risk aversion, the greater the utility premium normalized by the marginal utility.

In this paper, we follow Courbage et al. (2017) to define the nth-degree utility premium as the pain associated with facing the passage from a more favorable risk to a less favorable risk, where the risk increase is specified by the notion of more nth-degree risk in the sense of Ekern (1980). In order to conduct interpersonal comparisons, we normalize the nth-degree utility premium by the $(n-1)$th derivative of the utility function evaluated at the initial wealth, where the latter can be interpreted as the marginal utility of the $(n-1)$th moment of a zero-mean risk. Hence, our normalized nth-degree utility premium is measured in terms of an increase (when n is even) or a decrease (when n is odd) in the $(n-1)$th moment of a zero-mean risk (Li and Liu, 2014). Such normalization is also employed by Jindapon and Neilson (2007) in a comparative statics problem wherein effort can make risk improvements

[1] Eeckhoudt and Schlesinger (2006) use the utility premium of Friedman and Savage (1948) to formulate a unified approach that offers reasonable meanings of the signs of successive derivatives of utility functions.

at a utility cost by means of simple nth-degree risk changes that satisfy a single-crossing property. Focusing on simple nth-degree risk increases that include the introduction of a zero-mean risk as a special case, we show that the comparative risk aversion result of Huang and Stapleton (2015) naturally extends to higher orders in that individuals who are more higher-order Arrow-Pratt risk averse than a reference individual have larger normalized nth-degree utility premiums. When nth-degree risk increases are used, we show that the stronger measure of risk aversion *à la* Ross (1981) is called for to warrant the comparative risk aversion result in this general case.

Kimball (1990, 1993) refers to prudence as preferences for bearing zero-mean risks in the wealthier states of nature. Following Courbage et al. (2017), we define the nth-degree prudence utility premium as the increase in pain associated with facing the passage from a more favorable risk to a less favorable risk when the individual suffers a sure loss, where the risk increase is specified by the notion of more nth-degree risk. To facilitate interpersonal comparisons, we normalize the nth-degree prudence utility premium by the nth derivative of the utility function evaluated at the initial wealth. As such, the normalized nth-degree prudence utility premium is measured in terms of an increase (when n is odd) or a decrease (when n is even) in the nth moment of a zero-mean risk (Li and Liu, 2014). Huang and Stapleton (2015) show that the normalized second-degree prudence utility premium due to the introduction of a zero-mean risk is positively related to the measure of absolute prudence (Kimball, 1990, 1993). We show further that their result of comparative prudence extends to higher orders in an analogous manner as their result of comparative risk aversion.

The rest of the paper is organized as follows. In the next section, we introduce the concept of nth-degree utility premium. In Section 3, we use the normalized nth-degree utility premium to explain comparative higher-order risk aversion. In Section 4, we use the normalized nth-degree prudence utility premium to explain comparative higher-order prudence. The final section offers some concluding remarks.

2. The nth-degree utility premium

Consider an individual who has initial wealth, x, that is subject to a zero-mean shock, $\tilde{\varepsilon}$. The individual possesses a von Neumann-Morgenstern utility function, $u(x)$. We say that $u(x)$ exhibits nth-degree risk aversion if $(-1)^{n+1}u^{(n)}(x) \geq 0$ for all x, where $u^{(n)}(x) = \mathrm{d}^n u(x)/\mathrm{d}x^n$ is the nth derivative of $u(x)$.[2]

Let $F(\varepsilon)$ and $G(\varepsilon)$ be the cumulative distribution functions (CDFs) of two zero-mean random variables, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, respectively, over support $[\underline{\varepsilon}, \overline{\varepsilon}]$, where $\underline{\varepsilon} < 0 < \overline{\varepsilon}$, $F(\underline{\varepsilon}) = G(\underline{\varepsilon}) = 0$, and $F(\overline{\varepsilon}) = G(\overline{\varepsilon}) = 1$. Denote $F_1(\varepsilon) = F(\varepsilon)$ and $F_k(\varepsilon) = \int_{\underline{\varepsilon}}^{\varepsilon} F_{k-1}(y)\mathrm{d}y$ for all $k = 2, ..., n$, where $n \geq 2$. Similar notation applies to $G(\varepsilon)$. We say that $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$ in the sense of Ekern (1980) if $F_k(\overline{\varepsilon}) = G_k(\overline{\varepsilon})$ for all $k = 1, ..., n$ and $F_n(\varepsilon) \geq G_n(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$. More nth-degree risk is a special case of nth-order stochastic dominance in that the first $n-1$ moments of $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ are the same.[3]

Denote $\mathrm{E}[\cdot]$ as the expectations operator. The concept of nth-degree risk aversion and that of more nth-degree risk are closely related, as stated in the following lemma, where a formal proof can be found in Ekern (1980).

Lemma 1. For any $n \geq 2$ and any two zero-mean random variables, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, all individuals who exhibit nth-degree risk aversion prefer $\tilde{\varepsilon}_2$ to $\tilde{\varepsilon}_1$, i.e., $\mathrm{E}[u(x+\tilde{\varepsilon}_2)] \geq \mathrm{E}[u(x+\tilde{\varepsilon}_1)]$ for all x, if, and only if, $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$.

Following Courbage et al. (2017), we define the nth-degree utility premium as follows:

$$w_A^u(x; \tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = \mathrm{E}[u(x+\tilde{\varepsilon}_2)] - \mathrm{E}[u(x+\tilde{\varepsilon}_1)], \tag{1}$$

where $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$. As is evident from Eq. (1), $w_A^u(x; \tilde{\varepsilon}_1, \tilde{\varepsilon}_2)$ measures the "pain" associated with facing the passage from the more favorable risk, $\tilde{\varepsilon}_2$, to

[2] Throughout the paper, we use the notation, $f^{(k)}(x) = \mathrm{d}^k f(x)/\mathrm{d}x^k$, to denote the kth derivative of the function, $f(x)$. For the first, second, and third derivatives of $f(x)$, we use the usual notation, $f'(x)$, $f''(x)$, and $f'''(x)$, respectively.

[3] When $n = 2$, more second-degree risk refers to mean-preserving spreads in the sense of Rothschild and Stiglitz (1970). When $n = 3$, more third-degree risk is equivalent to an increase in downside risk *à la* Menezes et al. (1980), which moves risk from right to left while keeping the mean and variance constant.

the less favorable risk, $\tilde{\varepsilon}_1$. It follows from Lemma 1 that $w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2) \geq 0$ for all x and for all $u(x)$ that exhibits nth-degree risk aversion.

3. Comparative higher-order risk aversion

Let $u(x)$ and $v(x)$ be two von Neumann-Morgenstern utility functions. We refer to the individual with $u(x)$ as individual u and the one with $v(x)$ as individual v. In order to conduct interpersonal comparisons, we normalize the nth-degree utility premium, $w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)$, by the $(n-1)$th derivative of $u(x)$ evaluated at the initial wealth, i.e., by $(-1)^n u^{(n-1)}(x)$.

Huang and Stapleton (2015) show that $w_A^v(x;\tilde{\varepsilon},0)/v'(x) \geq w_A^u(x;\tilde{\varepsilon},0)/u'(x)$ if, and only if, the Arrow-Pratt measure of absolute risk aversion for $v(x)$ is uniformly larger than that for $u(x)$, i.e., $-v''(x)/v'(x) \geq -u''(x)/u'(x)$ for all x. In other words, Huang and Stapleton (2015) compare the normalized second-degree utility premium of individual v with that of individual u, where $\tilde{\varepsilon}_1 = \tilde{\varepsilon}$ is any zero-mean random variable and $\tilde{\varepsilon}_2 \equiv 0$. In this case, we have $G(\varepsilon) = 0$ for all $\varepsilon \in [\underline{\varepsilon}, 0)$ and $G(\varepsilon) = 1$ for all $\varepsilon \in [0, \overline{\varepsilon}]$ so that $F(\varepsilon) \geq G(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, 0)$ and $F(\varepsilon) \leq G(\varepsilon)$ for all $\varepsilon \in [0, \overline{\varepsilon}]$.

Following Jindapon and Neilson (2007), we say that $\tilde{\varepsilon}_1$ has more simple nth-degree risk than $\tilde{\varepsilon}_2$ if $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$, and $F(\varepsilon)$ and $G(\varepsilon)$ satisfy a single-crossing property in that $F_{n-1}(\varepsilon) \geq G_{n-1}(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, 0)$, and $F_{n-1}(\varepsilon) \leq G_{n-1}(\varepsilon)$ for all $\varepsilon \in [0, \overline{\varepsilon}]$. A zero-mean random variable as such has more simple second-degree risk than zero, which is the risk transition considered by Huang and Stapleton (2015).

Following Liu and Meyer (2013), we say that individual v is more (n/m)th-degree Arrow-Pratt risk averse than individual u if

$$\frac{(-1)^{n+1}v^{(n)}(x)}{(-1)^{m+1}v^{(m)}(x)} \geq \frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(x)}, \tag{2}$$

for all x, where $n > m \geq 1$. In our first proposition, we extend the result of comparative risk aversion obtained by Huang and Stapleton (2015) to higher orders.

Proposition 1. For any $n \geq 2$ and any two utility functions, $u(x)$ and $v(x)$, that exhibit $(n-1)$th- and nth-degree risk aversion, the normalized nth-degree utility premium of individual v is larger than that of individual u, i.e.,

$$\frac{w_A^v(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n v^{(n-1)}(x)} \geq \frac{w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n u^{(n-1)}(x)}, \tag{3}$$

for all x and for all zero-mean random variables, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, such that $\tilde{\varepsilon}_1$ has more simple nth-degree risk than $\tilde{\varepsilon}_2$ if, and only if, individual v is more $(n/n-1)$th-degree Arrow-Pratt risk averse than individual u.

Proof. To prove the sufficiency part, we note that

$$\begin{aligned}&\frac{w_A^v(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n v^{(n-1)}(x)} - \frac{w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n u^{(n-1)}(x)}\\ &= \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\frac{v^{(n-1)}(x+\varepsilon)}{v^{(n-1)}(x)} - \frac{u^{(n-1)}(x+\varepsilon)}{u^{(n-1)}(x)}\right][F_{n-1}(\varepsilon) - G_{n-1}(\varepsilon)]\mathrm{d}\varepsilon,\end{aligned} \tag{4}$$

where we have applied integration by parts and $F_k(\overline{\varepsilon}) = G_k(\overline{\varepsilon})$ for all $k = 1, ..., n-1$. As shown by Jindapon and Neilson (2007), Eq. (2) with $m = n-1$ implies that

$$\frac{v^{(n-1)}(x+\varepsilon)}{v^{(n-1)}(x)} \geq (\leq)\ \frac{u^{(n-1)}(x+\varepsilon)}{u^{(n-1)}(x)}, \tag{5}$$

for all $\varepsilon \leq (\geq)\ 0$. Since $F_{n-1}(\varepsilon) \geq G_{n-1}(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, 0]$ and $F_{n-1}(\varepsilon) \leq G_{n-1}(\varepsilon)$ for all $\varepsilon \in (0, \overline{\varepsilon}]$, it follows from Eq. (5) that the right-hand side of Eq. (4) is non-negative. As such, Eq. (3) holds.

To prove the necessity part, we suppose the contrary that there exists a point, x, at which Eq. (2) with $m = n-1$ does not hold. By continuity, it must be true that

$$-\frac{v^{(n)}(y)}{v^{(n-1)}(y)} < -\frac{u^{(n)}(y)}{u^{(n-1)}(y)}, \tag{6}$$

for all $y \in [x-a, x+b]$, where a and b are small positive numbers. Construct two CDFs, $F(\varepsilon)$ and $G(\varepsilon)$, over support $[\underline{\varepsilon}, \overline{\varepsilon}]$ such that $F(\varepsilon)$ has more simple nth-degree risk than $G(\varepsilon)$, where $-a < \underline{\varepsilon} < 0 < \overline{\varepsilon} < b$. It follows from Eq. (6) that

$$\frac{v^{(n-1)}(x+\varepsilon)}{v^{(n-1)}(x)} < (>)\ \frac{u^{(n-1)}(x+\varepsilon)}{u^{(n-1)}(x)}, \tag{7}$$

for all $\varepsilon \leq (\geq)\ 0$. Since $F_{n-1}(\varepsilon) \geq G_{n-1}(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, 0]$ and $F_{n-1}(\varepsilon) \leq G_{n-1}(\varepsilon)$ for all $\varepsilon \in (0, \overline{\varepsilon}]$, it follows from Eq. (7) that the right-hand side of Eq. (4) is negative, a contradiction. Hence, Eq. (2) with $m = n-1$ must hold for all x. □

The notion of more simple nth-degree risk imposes a single-crossing property on $F(\varepsilon)$ and $G(\varepsilon)$, which may be unduly restrictive. Following Liu and Meyer (2013), we say that individual v is more (n/m)th-degree Ross risk averse than individual u if

$$\frac{(-1)^{n+1}v^{(n)}(y)}{(-1)^{m+1}v^{(m)}(x)} \geq \frac{(-1)^{n+1}u^{(n)}(y)}{(-1)^{m+1}u^{(m)}(x)}, \tag{8}$$

for all x and y, where $n > m \geq 1$. In the following proposition, we focus on nth-degree risk increases and show that the more $(n/n-1)$th-degree Ross risk aversion plays a pivotal role in comparing risk aversion of higher orders in this general case.

Proposition 2. For any $n \geq 2$ and any two utility functions, $u(x)$ and $v(x)$, that exhibit $(n-1)$th- and nth-degree risk aversion, the normalized nth-degree utility premium of individual v is larger than that of individual u, i.e.,

$$\frac{w_A^v(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n v^{(n-1)}(x)} \geq \frac{w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n u^{(n-1)}(x)}, \tag{9}$$

for all x and for all zero-mean random variables, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, such that $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$ if, and only if, individual v is more $(n/n-1)$th-degree Ross risk averse than individual u.

Proof. The necessity part can be proved by contradiction using similar arguments as those in the proof of Proposition 1. To prove the sufficiency part, we note that

$$\frac{w_A^v(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n v^{(n-1)}(x)} - \frac{w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^n u^{(n-1)}(x)}$$

$$= \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\frac{u^{(n)}(x+\varepsilon)}{u^{(n-1)}(x)} - \frac{v^{(n)}(x+\varepsilon)}{v^{(n-1)}(x)}\right][F_n(\varepsilon) - G_n(\varepsilon)]\mathrm{d}\varepsilon, \tag{10}$$

where we have applied integration by parts and $F_k(\overline{\varepsilon}) = G_k(\overline{\varepsilon})$ for all $k = 1, ..., n$. Since

$F_n(\varepsilon) \geq G_n(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$, it follows from Eq. (8) with $m = n - 1$ that the right-hand side of Eq. (10) is non-negative. As such, Eq. (9) holds. □

Propositions 1 and 2 are readily applicable to the comparative statics problem wherein exerting effort entails a utility cost as considered by Jindapon and Neilson (2007). In this case, individual v, who is more $(n/n-1)$th-degree Arrow-Pratt or Ross risk averse than individual u, optimally exerts more effort than individual u when effort can make risk improvements by means of simple nth-degree risk changes or nth-degree risk changes, respectively.

4. Comparative higher-order prudence

Kimball (1990, 1993) refers to $u'''(x) \geq 0$ as prudence or preferences for bearing zero-mean risks in the wealthier states of nature. The prudence utility premium, introduced by Crainich and Eeckhoudt (2008), measures the increase in pain of facing a zero-mean risk in the presence of a sure loss, $\ell > 0$. Following Courbage et al. (2017), we extend the definition of Crainich and Eeckhoudt (2008) to the nth-degree prudence utility premium as follows:

$$w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2) = w_A^u(x-\ell;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2) - w_A^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2). \tag{11}$$

As is evident from Eq. (11), $w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)$ measures the additional "pain" associated with facing the passage from the more favorable risk, $\tilde{\varepsilon}_2$, to the less favorable risk, $\tilde{\varepsilon}_1$, when the individual suffers a sure loss, $\ell > 0$. It follows from Eq. (11) that $w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2) \geq 0$ if, and only if, $\mathrm{E}[u'(x+\tilde{\varepsilon}_2)] \leq \mathrm{E}[u'(x+\tilde{\varepsilon}_1)]$, which from Lemma 1, given that $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$, is true if, and only if, $(-1)^n u^{(n+1)}(x) \geq 0$.[4] In order to conduct interpersonal comparisons, we normalize the nth-degree prudence utility premium, $w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)$, by the nth derivative of $u(x)$ evaluated at the initial wealth, i.e., by $(-1)^{n+1}u^{(n)}(x)$.

[4] As pointed out by Courbage et al. (2017), $(-1)^n u^{(n+1)}(x) \geq 0$ can be interpreted as "prudence with respect to nth-degree risk increases" or preferences for bearing nth-degree risk increases in the wealthier states of nature. See also Wong (2016, 2017).

Consider two 50-50 lotteries, $[x-\ell+\tilde{\varepsilon}_2; x+\tilde{\varepsilon}_1]$ and $[x-\ell+\tilde{\varepsilon}_1; x+\tilde{\varepsilon}_2]$. The preference of the former over the latter means that an individual prefers bearing the greater nth-degree risk when he is richer, or equivalently he prefers to disaggregate the harm of a greater nth-degree risk and that of lower wealth. This is the case if, and only if,

$$\frac{1}{2}\mathrm{E}[u(x-\ell+\tilde{\varepsilon}_2)]+\frac{1}{2}\mathrm{E}[u(x+\tilde{\varepsilon}_1)] \geq \frac{1}{2}\mathrm{E}[u(x-\ell+\tilde{\varepsilon}_1)]+\frac{1}{2}\mathrm{E}[u(x+\tilde{\varepsilon}_2)]. \tag{12}$$

It is evidence from Eq. (11) that Eq. (12) holds if, and only if, $w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2) \geq 0$.

Huang and Stapleton (2015) show that $-w_P^v(x;\tilde{\varepsilon},0)/v''(x) \geq -w_P^u(x;\tilde{\varepsilon},0)/u''(x)$ if, and only if, the measure of absolute prudence for $v(x)$ is uniformly larger than that for $u(x)$, i.e., $-v'''(x)/v''(x) \geq -u'''(x)/u''(x)$ for all x. In other words, Huang and Stapleton (2015) compare the normalized second-degree prudence utility premium of individual v with that of individual u, where $\tilde{\varepsilon}_1 = \tilde{\varepsilon}$ is any zero-mean random variable and $\tilde{\varepsilon}_2 \equiv 0$ so that $\tilde{\varepsilon}$ has more simple second-degree risk than zero. In the following proposition, we extend their result of comparative prudence to all $n \geq 2$.

Proposition 3. For any $n \geq 2$ and any two utility functions, $u(x)$ and $v(x)$, that exhibit nth- and $(n+1)$th-degree risk aversion, the normalized nth-degree prudence utility premium of individual v is larger than that of individual u, i.e.,

$$\frac{w_P^v(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^{n+1}v^{(n)}(x)} \geq \frac{w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^{n+1}u^{(n)}(x)}, \tag{13}$$

for all x and for all zero-mean random variables, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, such that $\tilde{\varepsilon}_1$ has more simple nth-degree risk than $\tilde{\varepsilon}_2$ if, and only if, individual v is more $(n+1/n)$th-degree Arrow-Pratt risk averse than individual u.

Proof. The necessity part can be proved by contradiction using similar arguments as those in the proof of Proposition 1. To prove the sufficiency part, we note first that

$$\frac{\mathrm{E}[v'(z+\tilde{\varepsilon}_1)]-E[v'(z+\tilde{\varepsilon}_2)]}{(-1)^{n+1}v^{(n)}(z)} - \frac{\mathrm{E}[u'(z+\tilde{\varepsilon}_1)]-E[u'(z+\tilde{\varepsilon}_2)]}{(-1)^{n+1}u^{(n)}(z)}$$

$$= \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\frac{v^{(n)}(z+\varepsilon)}{v^{(n)}(z)} - \frac{u^{(n)}(z+\varepsilon)}{u^{(n)}(z)} \right] [F_{n-1}(\varepsilon) - G_{n-1}(\varepsilon)] \mathrm{d}\varepsilon, \tag{14}$$

where we have applied integration by parts and $F_k(\overline{\varepsilon}) = G_k(\overline{\varepsilon})$ for all $k = 1, ..., n-1$. As shown by Jindapon and Neilson (2007), Eq. (2) with $n = n+1$ and $m = n$ implies that

$$\frac{v^{(n)}(z+\varepsilon)}{v^{(n)}(z)} \geq (\leq) \ \frac{u^{(n)}(z+\varepsilon)}{u^{(n)}(z)}, \tag{15}$$

for all $\varepsilon \leq (\geq) \ 0$. Since $F_{n-1}(\varepsilon) \geq G_{n-1}(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, 0]$ and $F_{n-1}(\varepsilon) \leq G_{n-1}(\varepsilon)$ for all $\varepsilon \in (0, \overline{\varepsilon}]$, it follows from Eq. (15) that the right-hand side of Eq. (14) is non-negative. Using Eq. (14), we have

$$\int_{x-\ell}^{z} \frac{u^{(n)}(z)}{v^{(n)}(z)} \{\mathrm{E}[v'(z+\tilde{\varepsilon}_1)] - \mathrm{E}[v'(z+\tilde{\varepsilon}_2)]\} \mathrm{d}z$$

$$\geq \int_{x-\ell}^{x} \{\mathrm{E}[u'(z+\tilde{\varepsilon}_1)] - \mathrm{E}[u'(z+\tilde{\varepsilon}_2)]\} \mathrm{d}z, \tag{16}$$

for $\ell > 0$. From Eq. (15), we have $u^{(n)}(x)/v^{(n)}(x) \geq u^{(n)}(z)/v^{(n)}(z)$ for all $z \in [x-\ell, x]$. Since $-v'(x)$ exhibits nth-degree risk aversion and $F(\varepsilon)$ has more nth-degree risk than $G(\varepsilon)$, Lemma 1 implies that $\mathrm{E}[v'(z+\tilde{\varepsilon}_1)] \geq \mathrm{E}[v'(z+\tilde{\varepsilon}_2)]$ for all z. Hence, we have

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} \int_{x-\ell}^{x} \{\mathrm{E}[v'(z+\tilde{\varepsilon}_1)] - \mathrm{E}[v'(z+\tilde{\varepsilon}_2)]\} \mathrm{d}z$$

$$\geq \int_{x-\ell}^{z} \frac{u^{(n)}(z)}{v^{(n)}(z)} \{\mathrm{E}[v'(z+\tilde{\varepsilon}_1)] - \mathrm{E}[v'(z+\tilde{\varepsilon}_2)]\} \mathrm{d}z. \tag{17}$$

Eqs. (16) and (17) then imply that

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} \{\mathrm{E}[v(x+\tilde{\varepsilon}_1)] - \mathrm{E}[v(x-\ell+\tilde{\varepsilon}_1)] - \mathrm{E}[v(x+\tilde{\varepsilon}_2)] + \mathrm{E}[v(x-\ell+\tilde{\varepsilon}_2)]\}$$

$$\geq \mathrm{E}[u(x+\tilde{\varepsilon}_1)] - \mathrm{E}[u(x-\ell+\tilde{\varepsilon}_1)] - \mathrm{E}[u(x+\tilde{\varepsilon}_2)] + \mathrm{E}[u(x-\ell+\tilde{\varepsilon}_2)]. \tag{18}$$

Rearranging terms of Eq. (18) and using Eq. (11) yields Eq. (13). □

In the following proposition, we focus on nth-degree risk increases and show that the comparative higher-order prudence calls for the use of more $(n+1/n)$th-degree Ross risk aversion in this general case.

Proposition 4. For any $n \geq 2$ and any two utility functions, $u(x)$ and $v(x)$, that exhibit nth- and $(n+1)$th-degree risk aversion, the normalized nth-degree prudence utility premium of individual v is larger than that of individual u, i.e.,

$$\frac{w_P^v(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^{n+1}v^{(n)}(x)} \geq \frac{w_P^u(x;\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)}{(-1)^{n+1}u^{(n)}(x)}, \tag{19}$$

for all x and for all zero-mean random variables, $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$, such that $\tilde{\varepsilon}_1$ has more nth-degree risk than $\tilde{\varepsilon}_2$ if, and only if, individual v is more $(n+1/n)$th-degree Ross risk averse than individual u.

Proof. The necessity part can be proved by contradiction using similar arguments as those in the proof of Proposition 1. To prove the sufficiency part, we note first that

$$\frac{\mathrm{E}[v'(z+\tilde{\varepsilon}_1)]-E[v'(z+\tilde{\varepsilon}_2)]}{(-1)^{n+1}v^{(n)}(x)} - \frac{\mathrm{E}[u'(z+\tilde{\varepsilon}_1)]-E[u'(z+\tilde{\varepsilon}_2)]}{(-1)^{n+1}u^{(n)}(x)}$$

$$= \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\frac{u^{(n+1)}(z+\varepsilon)}{u^{(n)}(x)} - \frac{v^{(n+1)}(z+\varepsilon)}{v^{(n)}(x)}\right][F_n(\varepsilon)-G_n(\varepsilon)]\mathrm{d}\varepsilon, \tag{20}$$

where we have applied integration by parts and $F_k(\overline{\varepsilon}) = G_k(\overline{\varepsilon})$ for all $k = 1, \ldots, n$. Since $F_n(\varepsilon) \geq G_n(\varepsilon)$ for all $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$, it follows from Eq. (8) with $n = n+1$ and $m = n$ that the right-hand side of Eq. (20) is non-negative. Using Eq. (20), we have for $\ell > 0$

$$\int_{x-\ell}^{x} \left\{\frac{\mathrm{E}[v'(z+\tilde{\varepsilon}_1)]-E[v'(z+\tilde{\varepsilon}_2)]}{(-1)^{n+1}v^{(n)}(x)}\right\}\mathrm{d}z \geq \int_{x-\ell}^{x} \left\{\frac{\mathrm{E}[u'(z+\tilde{\varepsilon}_1)]-E[u'(z+\tilde{\varepsilon}_2)]}{(-1)^{n+1}u^{(n)}(x)}\right\}\mathrm{d}z,$$

which is equivalent to

$$\frac{\mathrm{E}[v(x+\tilde{\varepsilon}_1)]-\mathrm{E}[v(x-\ell+\tilde{\varepsilon}_1)]-E[v(w+\tilde{\varepsilon}_2)]+E[v(w-\ell+\tilde{\varepsilon}_2)]}{(-1)^{n+1}v^{(n)}(x)}$$

$$\geq \frac{\mathrm{E}[u(x+\tilde{\varepsilon}_1)]-\mathrm{E}[u(x-\ell+\tilde{\varepsilon}_1)]-E[u(w+\tilde{\varepsilon}_2)]+E[u(w-\ell+\tilde{\varepsilon}_2)]}{(-1)^{n+1}u^{(n)}(x)}. \tag{21}$$

Using Eq. (11), Eq. (21) reduces to Eq. (19). □

5. Conclusion

In this paper, we define the nth-degree utility premium as the pain associated with facing the passage from a more favorable risk to a less favorable risk, where the risk increase is specified by the notion of more nth-degree risk in the sense of Ekern (1980). We further define the nth-degree prudence utility premium as the increase in pain when the individual suffers a sure loss. Using the normalized nth-degree utility premium and the normalized nth-degree prudence utility premium, we show that the results of Huang and Stapleton (2015) regarding comparative risk aversion and comparative prudence naturally extend to higher orders. Our findings as such indicate that the concept of utility premium is a useful tool for research in decision making under uncertainty.

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