ON DIVISORS OF MODULAR FORMS

KATHRIN BRINGMANN, BEN KANE, STEFFEN LÖBRICH, KEN ONO, AND LARRY ROLEN

In celebration of Don Zagier's 65th birthday.

Abstract. The *denominator formula* for the Monster Lie algebra is the product expansion for the modular function $J(z) - J(\tau)$ given in terms of the Hecke system of $\mathrm{SL}_2(\mathbb{Z})$-modular functions $j_n(\tau)$. It is prominent in Zagier's seminal paper on traces of singular moduli, and in the Duncan-Frenkel work on Moonshine. The formula is equivalent to the description of the generating function for the $j_n(z)$ as a weight 2 modular form with a pole at z. Although these results rely on the fact that $X_0(1)$ has genus 0, here we obtain a generalization, framed in terms of polar harmonic Maass forms, for all of the $X_0(N)$ modular curves. We use these functions to study divisors of modular forms.

1. Introduction and statement of results

As usual, let $J(\tau)$ be the $\mathrm{SL}_2(\mathbb{Z})$ Hauptmodul defined by

$$J(\tau) = \sum_{n=-1}^{\infty} c(n) e^{2\pi i n\tau} := \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = e^{-2\pi i\tau} + 196884 e^{2\pi i\tau} + \cdots ,$$

where $E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}$ is the weight $k \in 2\mathbb{N}$ Eisenstein series, $\sigma_\ell(n) := \sum_{d|\ell} d^\ell$, B_k is the kth Bernoulli number, and $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$. By Moonshine (for example, see [14]), $J(\tau)$ is the McKay-Thompson series for the identity (i.e., its coefficients are the graded dimensions of the Monster module $V^\natural$). Moonshine also offers the striking infinite product

$$J(z) - J(\tau) = e^{-2\pi i z} \prod_{m>0,\ n\in\mathbb{Z}} \left(1 - e^{2\pi i m z} e^{2\pi i n\tau}\right)^{c(mn)},$$

the *denominator formula* for the Monster Lie algebra. Here we let $\tau, z \in \mathbb{H}$. This formula is equivalent to the following identity of Asai, Kaneko, and Ninomiya (see Theorem 3 of [2])

$$H_z(\tau) := \sum_{n=0}^{\infty} j_n(z) e^{2\pi i n\tau} = \frac{E_4(\tau)^2 E_6(\tau)}{\Delta(\tau)} \frac{1}{J(\tau) - J(z)} = -\frac{1}{2\pi i}\frac{J'(\tau)}{J(\tau) - J(z)}. \tag{1.1}$$

The functions $j_n(\tau)$ form a Hecke system. Namely, if we let $j_0(\tau) := 1$ and $j_1(\tau) := J(\tau)$, then the others are obtained by applying the normalized Hecke operator $T(n)$

$$j_n(\tau) := j_1(\tau) \mid T(n). \tag{1.2}$$

Remark. The functions $H_z(\tau)$ and $j_n(\tau)$ played central roles in Zagier's [20] seminal paper on traces of singular moduli and the Duncan-Frenkel work [13] on the Moonshine Tower. Carnahan [10] has obtained similar denominator formulas for completely replicable modular functions.

2010 *Mathematics Subject Classification.* 11F03, 11F37, 11F30.

The first and third author are supported by the Deutsche Forschungsgemeinschaft (DFG) Grant No. BR 4082/3-1. The second author was supported by grant project numbers 27300314, 17302515, and 17316416 of the Research Grants Council. The fourth author thanks the support of the NSF and the Asa Griggs Candler Fund.

If $z \in \mathbb{H}$, then $H_z(\tau)$ is a weight 2 meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a single pole (modulo $\mathrm{SL}_2(\mathbb{Z})$) at the point z. Using these functions, the *divisor modular form* of a normalized weight k meromorphic modular form $f(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ was defined in [9] as[1]

$$f^{\mathrm{div}}(\tau) := \sum_{z \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} e_z \mathrm{ord}_z(f) H_z(\tau), \tag{1.3}$$

where $e_z := 2/\#\mathrm{Stab}_z\,(\mathrm{SL}_2(\mathbb{Z}))$ With $\Theta := \frac{1}{2\pi i}\frac{d}{d\tau}$, Theorem 1 of [9] asserts that

$$f^{\mathrm{div}}(\tau) = -\frac{\Theta(f(\tau))}{f(\tau)} + \frac{kE_2(\tau)}{12}. \tag{1.4}$$

Although these results rely on the fact that $X_0(1)$ has genus 0, there is a natural extension for congruence subgroups. This extension requires polar harmonic Maass forms, which are harmonic Maass forms with poles in the upper half-plane (see [5] for details). Here we consider the modular curves $X_0(N)$. For $n \in \mathbb{N}$, we define a Hecke system of $\Gamma_0(N)$ harmonic Maass functions $j_{N,n}(\tau)$ in Section 3 which generalize the $j_n(\tau)$.

In Section 2 we construct weight 2 polar harmonic Maass forms $H^*_{N,z}(\tau)$ which generalize the $H_z(\tau)$. We have two cases for the $H^*_{N,z}(\tau)$, according to whether $z \in \mathbb{H}$ or z is a cusp, which we consider separately. The following theorem summarizes the essential properties of these functions when $z \in \mathbb{H}$.

Theorem 1.1. *If $z \in \mathbb{H}$, then $H^*_{N,z}(\tau)$ is a weight 2 polar harmonic Maass form on $\Gamma_0(N)$ which vanishes at all cusps and has a single simple pole at z. Moreover, the following are true:*

(1) *If $z \in \mathbb{H}$ and $\mathrm{Im}(\tau) > \max\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\}$, then we have that*

$$H^*_{N,z}(\tau) = \frac{3}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)} + \sum_{n=1}^{\infty} j_{N,n}(z) e^{2\pi i n\tau}.$$

(2) *For $\gcd(N,n) = 1$, we have $j_{N,n}(\tau) = j_{N,1}(\tau) \mid T(n)$.*

(3) *For $n \mid N$, we have $j_{N,n}(\tau) = j_{\frac{N}{n},1}(n\tau)$.*

(4) *As $n \to \infty$, we have*

$$j_{N,n}(\tau) = \sum_{\substack{\lambda \in \Lambda_\tau \\ \lambda \le n}} \sum_{(c,d) \in S_\lambda} e\left(-\frac{n}{\lambda} r_\tau(c,d)\right) e^{\frac{2\pi n \mathrm{Im}(\tau)}{\lambda}} + O_\tau(n) \tag{1.5}$$

for some real numbers $r_\tau(c,d)$ (see (3.2)), Λ_τ a lattice in $\mathbb{R}$ (see (3.3)), and S_λ the set of solutions to $Q_\tau(c,d) = \lambda$ for a certain positive-definite binary quadratic form Q_τ (see (3.4)).

Five Remarks.

(1) In Theorem 1.1 (1), the inequality on $\mathrm{Im}(\tau)$ is required for convergence.

(2) For $N = 1$, we have that $H^*_{1,z}(\tau) = H_z(\tau) - E^*_2(\tau)$, where $E^*_2(\tau) := -\frac{3}{\pi \mathrm{Im}(\tau)} + E_2(\tau)$ is the usual weight 2 nonholomorphic Eisenstein series, and we have that $j_{1,n}(\tau) = j_n(\tau) + 24\sigma_1(n)$.

(3) The sums (1.5) were introduced by Hardy and Ramanujan [15] (see also [3, 4]) to study the Fourier coefficients of $1/E_6$. Their formulas have been generalized [7, 8] to negative weight meromorphic modular forms. Theorem 1.1 (4) extends these results to weight 0 where the series are not absolutely convergent.

[1]Note that this summation does not include the cusp $i\infty$.

(4) Theorem 1.1 (4) gives asymptotics for $j_{N,n}(z)$ in the n-aspect. If $\mathrm{Im}(z) \geq \mathrm{Im}\,(Mz)$ for all $M \in \Gamma_0(N)$, then

$$j_{N,n}(z) = e^{-2\pi i n z} + \sum_{\substack{c\geq 1\\ N|c}} \sum_{\substack{d\in\mathbb{Z}\\ \gcd(c,d)=1\\ |cz+d|^2=1}} e\left(n\frac{d-a}{c}\right) e^{2\pi i n\overline{z}} + O_z(n) \tag{1.6}$$

as $n \to \infty$.

The second case we consider are those $H^*_{N,\rho}(\tau)$ where ρ is a cusp of $X_0(N)$. These functions are compatible with the $H^*_{N,z}(\tau)$ considered in Theorem 1.1. More precisely, since $z \mapsto H^*_{N,z}(\tau)$ is continuous (even harmonic) and $\Gamma_0(N)$-invariant, it follows that

$$H^*_{N,\rho}(\tau) := \lim_{z\to\rho} H^*_{N,z}(\tau) \tag{1.7}$$

is well-defined and only depends on the equivalence class of ρ. The next result summarizes these functions' properties. We use the Kloosterman sums $K_{i\infty,\rho}(0,-n;c)$ of (2.4) and the weight 2 harmonic Eisenstein series $E^*_{2,N,\rho}(\tau)$ for $\Gamma_0(N)$ defined in Section 2. These have constant term 1 at ρ and vanish at all other cusps.

Theorem 1.2. *We have that $H^*_{N,\rho}(\tau) = -E^*_{2,N,\rho}(\tau)$. Moreover, the following are true:*

(1) *We have*

$$\begin{aligned} H^*_{N,\rho}(\tau) &= \frac{3}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)} - \delta_{\rho,\infty} + \sum_{n=1}^{\infty} j_{N,n}(\rho) e^{2\pi i n\tau}, \qquad \textit{with}\\ j_{N,n}(\rho) &:= \lim_{\tau\to\rho} j_{N,n}(\tau) = \frac{4\pi^2 n}{\ell_\rho} \sum_{\substack{c\geq 1\\ N|c}} \frac{K_{i\infty,\rho}(0,-n;c)}{c^2}, \end{aligned}$$

where ℓ_ρ denotes the cusp width of ρ and $\delta_{\rho,\infty} := 1$ if $\rho = i\infty$ and 0 otherwise.

(2) *For $\gcd(N,n) = 1$, we have $j_{N,n}(\rho) = \lim_{\tau\to\rho} j_{N,1}(\tau) \mid T(n)$.*

(3) *For $n \mid N$, we have $j_{N,n}(\rho) = \lim_{\tau\to\rho} j_{\frac{N}{n},1}(n\tau)$.*

Two Remarks.

(1) Recall that the Fourier expansion in Theorem 1.1 (1) is not valid as $z \to i\infty$.

(2) The $j_{N,n}(\rho)$ are divisor sums, which we leave to the interested reader to verify. From a generalization of the Weil bound (3.9) one can obtain $j_{N,n}(\rho) = O(n^{\frac{3}{2}})$.

We turn to the task of extending (1.4) to generic $\Gamma_0(N)$. Suppose that f is a weight k meromorphic modular form on $\Gamma_0(N)$. In analogy with (1.3), we define the *divisor polar harmonic Maass form*

$$f^{\mathrm{div}}(\tau) := \sum_{z\in X_0(N)} e_{N,z}\mathrm{ord}_z(f) H^*_{N,z}(\tau), \tag{1.8}$$

where $e_{N,z} := 2/\#\mathrm{Stab}_z(\Gamma_0(N))$ and $e_{N,\rho} := 1$ when ρ is a cusp. Generalizing (1.4), we show the following.

Theorem 1.3. *If $S_2(\Gamma_0(N))$ denotes the space of weight 2 cusp forms on $\Gamma_0(N)$, then*

$$f^{\mathrm{div}}(\tau) \equiv \frac{k}{4\pi\,\mathrm{Im}(\tau)} - \frac{\Theta(f(\tau))}{f(\tau)} \pmod{S_2(\Gamma_0(N))}.$$

Three Remarks.
(1) The coefficient of $1/\operatorname{Im}(\tau)$ in $H^*_{N,z}(\tau)$ is independent of z. By the valence formula, summing over every element of $X_0(N)$ in the definition of $f^{\mathrm{div}}(\tau)$ multiplies this constant by $\frac{k}{12}\left[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)\right]$, giving the nonholomorphic correction term on the right-hand side of Theorem 1.3.

(2) At first glance, definitions (1.3) and (1.8) might appear different for $N=1$. Indeed, $H^*_{1,z}(\tau)=H_z(\tau)-E_2^*(\tau)$, and the sum in (1.8) includes the cusp $i\infty$ whereas (1.3) omits it. The quasimodular Eisenstein series $E_2(\tau)$ in (1.4) and the valence formula guarantee that they coincide.

(3) The formula in Theorem 1.3 has already been obtained by Choi using a regularized inner product due to Petersson, but without relating the Fourier coefficients of f^{div} to the polar harmonic Maass forms $H^*_{N,z}$ (see Theorem 1.4 of [11]).

Theorem 1.3 can be used to numerically compute divisors of meromorphic modular forms $f(\tau)$, which, in general, is a difficult task (for example, see [12]). The series $-\frac{\Theta(f(\tau))}{f(\tau)}$ is the logarithmic derivative of $f(\tau)$, and this fact converts the points $z\in\mathbb{H}$ in the divisor of $f(\tau)$ into simple poles. These can be identified by the asymptotic properties of the coefficients of $H^*_{N,z}(\tau)$ given in Theorem 1.1. This follows from Theorem 1.3 and the fact that coefficients of cusp forms satisfy Deligne's bound. In the case of the modular functions $j(\tau)-\alpha$, where $\alpha\in\mathbb{C}$, this has been carried out recently by Alwaise [1]. The method is based on the following immediate corollary to Theorems 1.1–1.3.

Corollary 1.4. *Suppose that $f(\tau)$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ whose divisor is not supported at cusps. Let y_1 be the largest imaginary part of any points in the divisor of $f(\tau)$ lying in $\mathbb{H}$. Then if $-\frac{\Theta(f(\tau))}{f(\tau)}=:\sum_{n\gg-\infty}a(n)q^n$ $(q=e^{2\pi i\tau})$, we have that*

$$y_1=\limsup_{n\to\infty}\frac{\log|a(n)|}{2\pi n}.$$

Two Remarks.
(1) We require lim sup in Corollary 1.4 because the $a(n)$ can vanish on arithmetic progressions.

(2) It would be interesting to develop a practical algorithm for numerically computing modular form divisors. The idea would be to carefully peel away poles of $f^{\mathrm{div}}(\tau)$ in descending order until one is left with a linear combination of functions $E^*_{N,\rho}(\tau)$.

Example 1. For the Eisenstein series $E_4(\tau)$, we have

$$-\frac{\Theta(E_4(\tau))}{E_4(\tau)}=-240q+53280q^2-12288960q^3+2835808320q^4-654403831200q^5+\cdots.$$

The sequence $\{b(n)\}_{n\geq1}=\{\log|a(n)|/(2\pi n)\}_{n\geq1}$ converges rapidly. Indeed, $b(2)=0.866066794\ldots$, and $b(10)=0.866025404\ldots$ matches the first 16 digits of the limiting value. The divisor of $E_4(\tau)$ is supported on a zero at $\omega:=(-1+\sqrt{-3})/2$. By (1.6), since ω lies on the unit circle (implying that the second term on the right-hand side of (1.6) appears) for large n, $a(n)$ should very nearly be $\frac{1}{3}\left(e^{-2\pi in\omega}+2e^{2\pi in\overline{\omega}}\right)=e^{-2\pi in\omega}$, which is very easily seen numerically.

Example 2. We consider $f(\tau):=E_4(2\tau)+\frac{\eta^{16}(2\tau)}{\eta^8(\tau)}$, where $\eta(\tau)$ is Dedekind's eta-function. By the valence formula for $\Gamma_0(2)$, it has a single zero, say z_0, in $X_0(2)$. We find that

$$-\frac{\Theta(f(\tau))}{f(\tau)}=-q-495q^2+659q^3+113233q^4-261211q^5+\cdots.$$

After the first 3000 terms the sequence $\log|a(n)|/(2\pi n)$ stabilizes and offers $\operatorname{Im}(z_0)\approx0.4357$. As $f(\tau)$ has real coefficients and there is only one zero, $-\overline{z}_0$ must be $\Gamma_0(2)$-equivalent to z_0. We choose the fundamental domain

$$\left\{z \in \mathbb{H}\colon -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \text{ and } \forall M \in \Gamma_0(2)\colon \left(\mathrm{Im}\,(Mz) \geq \mathrm{Im}(z) \text{ and } \mathrm{Im}\,(Mz) > \mathrm{Im}(z) \text{ if } \mathrm{Re}(z) < 0\right)\right\}.$$

Thus, either $\mathrm{Re}(z) \in \{0, \frac{1}{2}\}$, or z lies on the arc $|2z-1| = 1$. The first two cases are easily excluded by the sign patterns of $a(n)$, and the zero on the arc is easily approximated as $z_0 \approx 0.2547 + 0.4357i$.

This paper is organized as follows. In Section 2 we construct the weight 2 polar harmonic Maass forms $H^*_{N,z}(\tau)$. In Section 3 we relate their Fourier coefficients to the values of the weight 0 weak Maass forms at $\tau = z$, proving Theorems 1.1, 1.2, and 1.3.

2. Weight 2 Polar Harmonic Maass forms

2.1. The $H^*_{N,z}(\tau)$ when $z \in \mathbb{H}$. Define for $z, \tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$

$$P_{N,s}(\tau, z) := \sum_{M \in \Gamma_0(N)} \frac{\varphi_s\left(M\tau, z\right)}{j\left(M, \tau\right)^2 \left|j\left(M, \tau\right)\right|^{2s}} \tag{2.1}$$

with $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) := c\tau + d$ and

$$\varphi_s(\tau, z) := \left(\mathrm{Im}(z)\right)^{1+s} (\tau - z)^{-1} (\tau - \overline{z})^{-1} \left|\tau - \overline{z}\right|^{-2s}.$$

These functions were introduced and investigated in the z-variable in [6], where it was shown that these are *polar harmonic Maass forms*. These functions are allowed to have poles in the upper half plane instead of only at the cusps. In this paper, we are interested in properties of $P_{N,s}(\tau, z)$ as functions of τ. A direct calculation shows that for $L \in \Gamma_0(N)$

$$P_{N,s}\left(L\tau, z\right) = j\left(L, \tau\right)^2 \left|j\left(L, \tau\right)\right|^{2s} P_{N,s}(\tau, z).$$

In [6] it was shown, by a lengthy calculation, that the function $P_{N,s}(\tau, z)$ has an analytic continuation to $s = 0$, which we denote by $\mathrm{Im}(z)\Psi_{2,N}(\tau, z)$. Let $\mathcal{H}_k(\Gamma_0(N))$ be the space of weight k polar harmonic Maass forms with respect to $\Gamma_0(N)$. Lemma 4.4 of [6] then states that $z \mapsto \mathrm{Im}(z)\Psi_{2,N}(\tau, z) \in \mathcal{H}_0(\Gamma_0(N))$. In the τ variable, these functions are also polar harmonic Maass forms, as the next proposition shows. For this, for $w \in \mathbb{C}$, let $e(w) := e^{2\pi i w}$, and

$$K(m, n; c) := \sum_{\substack{a,d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{md + na}{c}\right).$$

Moreover, I_k and J_k denote the usual I- and J-Bessel functions. The following proposition can be obtained by a careful inspection of the proof of Theorem 3.1 of [6].

Proposition 2.1. *We have that* $\tau \mapsto \mathrm{Im}(z)\Psi_{2,N}(\tau, z) \in \mathcal{H}_2(\Gamma_0(N))$. *For* $\mathrm{Im}(\tau) > \max\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\}$, *its Fourier expansion (in τ) has the form*

$$\begin{aligned}
\mathrm{Im}(z)\Psi_{2,N}(\tau, z) = &-\frac{6}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)} - 2\pi \sum_{m \geq 1} \left(e^{-2\pi i m z} - e^{-2\pi i m \overline{z}}\right) e^{2\pi i m \tau} \\
&- 4\pi^2 \sum_{m \geq 1} \sum_{\substack{n, c \geq 1 \\ N | c}} \sqrt{\frac{m}{n}} \frac{K(m, -n; c)}{c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i n z} e^{2\pi i m \tau} \\
&- 4\pi^2 \sum_{m \geq 1} \sum_{\substack{n, c \geq 1 \\ N | c}} \sqrt{\frac{m}{n}} \frac{K(m, n; c)}{c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i n \overline{z}} e^{2\pi i m \tau} - 8\pi^3 \sum_{m \geq 1} m \sum_{\substack{c \geq 1 \\ N | c}} \frac{K(m, 0; c)}{c^2} e^{2\pi i m \tau}.
\end{aligned}$$

We then set

$$H^*_{N,z}(\tau) := -\frac{\operatorname{Im}(z)}{2\pi}\Psi_{2,N}(\tau, z). \tag{2.2}$$

Remark. We have, as $\tau \to z$,

$$H^*_{N,z}(\tau) = \frac{1}{2\pi i e_{N,z}}\frac{1}{\tau - z} + O(1) \tag{2.3}$$

with $e_{N,z}$ as defined after (1.8).

2.2. **The $H^*_{N,z}(\tau)$ for cusps.** We require the Fourier expansion of the functions $H^*_{N,\rho}(\tau)$ defined in (1.7). For any cusp ρ of $\Gamma_0(N)$, denote by ℓ_ρ the cusp width and let M_ρ be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with $\rho = M_\rho i\infty$. For two cusps $\mathfrak{a}, \mathfrak{b}$ of $\Gamma_0(N)$, the generalized Kloosterman sums are

$$K_{\mathfrak{a},\mathfrak{b}}(m, n; c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\infty^{\ell_\mathfrak{a}} \backslash M_\mathfrak{a}^{-1}\Gamma_0(N) M_\mathfrak{b} / \Gamma_\infty^{\ell_\mathfrak{b}}} e\left(\frac{md}{\ell_\mathfrak{b} c} + \frac{na}{\ell_\mathfrak{a} c}\right) \tag{2.4}$$

with $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Note that we have $K_{i\infty,i\infty}(m, n; c) = K(m, n; c)$.

Lemma 2.2 (Lemma 5.4 of [6])**.** *We have*

$$H^*_{N,\rho}(\tau) = \frac{3}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\operatorname{Im}(\tau)} - \delta_{\rho,\infty} + \frac{4\pi^2}{\ell_\rho}\sum_{n\geq 1} n \sum_{c \geq 1}\frac{K_{\rho,i\infty}(n, 0; c)}{c^2} e^{2\pi i n\tau}.$$

The Fourier expansions in Lemma 2.2 yield a relation with the harmonic weight 2 Eisenstein series $E^*_{2,N,\rho}(\tau)$ for $\Gamma_0(N)$. For $\operatorname{Re}(s) > 0$, define

$$E^*_{2,N,\rho,s}(\tau) := \sum_{M \in \Gamma_\rho \backslash \Gamma_0(N)} j\left(M_\rho M, \tau\right)^{-2}\left|j\left(M_\rho M, \tau\right)\right|^{-2s}. \tag{2.5}$$

Using the Hecke trick, it is well-known (cf. Satz 6 of [16]) that $E^*_{2,N,\rho,s}(\tau)$ has an analytic continuation to $s = 0$, denoted by $E^*_{2,N,\rho}(\tau)$. Applying equations (5.3) and (5.4) in Theorem 1 of [19] with $v = 1$, $A_j = M_\rho$, Γ=$\Gamma_0(N)$, and $\mu = 0$ to obtain the Fourier expansion of $E^*_{2,N,\rho}$, we see that

$$H^*_{N,\rho}(\tau) = -E^*_{2,N,\rho}(\tau). \tag{2.6}$$

3. The $j_{N,n}(z)$ and the proofs of Theorems 1.1 and 1.2

3.1. **The functions $j_{N,n}(z)$.** The functions $j_{N,n}(z)$ are constructed as analytic continuations of Niebur's Poincaré series [18]. To be more precise, set for $n \in \mathbb{N}$ and $\operatorname{Re}(s) > 1$

$$F_{N,-n,s}(z) := \sum_{M \in \Gamma_\infty \backslash \Gamma_0(N)} e\left(-n\operatorname{Re}(Mz)\right)\operatorname{Im}(Mz)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\operatorname{Im}(Mz)\right).$$

These functions are *weak Maass forms* of weight 0; instead of being annihilated by Δ_0, they have eigenvalue $s(1-s)$. To obtain an analytic continuation to $s = 1$, one computes the Fourier expansion of $F_{N,-n,s}(z)$ and uses

$$\lim_{s\to 1} y^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n y\right) = y^{\frac{1}{2}} I_{\frac{1}{2}}\left(2\pi n y\right) = \frac{1}{\pi\sqrt{n}}\sinh\left(2\pi n y\right) = \frac{e^{2\pi n y} - e^{-2\pi n y}}{2\pi\sqrt{n}}.$$

Proposition 3.1 (Theorem 1 of [18])**.** *The function $F_{N,-n,s}(z)$ has an analytic continuation $F_{N,-n}(z)$ to $s = 1$, and $F_{N,-n}(z) \in \mathcal{H}_0(\Gamma_0(N))$. It has the Fourier expansion*

$$F_{N,-n}(z) = \frac{e^{-2\pi i n z} - e^{-2\pi i n\overline{z}}}{2\pi\sqrt{n}} + c_N(n, 0) + \sum_{m\geq 1}\left(c_N(n, m) e^{2\pi i m z} + c_N(n, -m) e^{-2\pi i m\overline{z}}\right),$$

where the coefficients are given by

$$c_N(n,m) := \sum_{\substack{c\geq 1\\ N|c}} \frac{K(m,-n;c)}{c} \times \begin{cases} \frac{1}{\sqrt{m}} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) & \text{if } m>0,\\ \frac{2\pi\sqrt{n}}{c} & \text{if } m=0,\\ \frac{1}{\sqrt{|m|}} J_1\left(\frac{4\pi\sqrt{|m|n}}{c}\right) & \text{if } m<0.\end{cases}$$

We then define the functions $j_{N,n}(z)$ by

$$j_{N,n}(z) := 2\pi\sqrt{n} F_{N,-n}(z). \tag{3.1}$$

For $N=1$, we recover the $j_n(z)$ from the introduction up to the constant $2\pi\sqrt{n}c_1(n,0) = 24\sigma_1(n)$.

3.2. **Proofs of Theorems 1.1 and 1.2.** In order to formally state Theorem 1.1 (4), for an arbitrary solution $a,b\in\mathbb{Z}$ to $ad-bc=1$, we define

$$r_z(c,d) := ac|z|^2 + (ad+bc)\operatorname{Re}(z) + bd, \tag{3.2}$$

$$\Lambda_z := \left\{\alpha^2|z|^2 + \beta\operatorname{Re}(z) + \gamma^2 > 0 : \alpha,\beta,\gamma\in\mathbb{Z}\right\}, \tag{3.3}$$

$$Q_z(c,d) := c^2|z|^2 + 2cd\operatorname{Re}(z) + d^2,$$

$$S_\lambda := \left\{(c,d)\in N\mathbb{Z}\times\mathbb{Z} :\ c\geq 0,\ \gcd(c,d)=1,\ \text{and } Q_z(c,d)=\lambda\right\}. \tag{3.4}$$

Note that although $r_z(c,d)$ is not uniquely determined, $e(-nr_z(c,d)/Q_z(c,d))$ is well-defined.

Proof of Theorem 1.1. (1) For $n\in\mathbb{N}$, inspecting the expansions in Propositions 2.1 and 3.1 yields that $2\pi\sqrt{n}F_{N,-n}(z)$ is the coefficient of $e^{2\pi in\tau}$ in $-\operatorname{Im}(z)\Psi_{2,N}(\tau,z)/(2\pi)$, yielding the claim.
(2) Since $\gcd(N,n)=1$, $T(n)$ commutes with the action of $\Gamma_0(N)$, and so it suffices to show that (by analytic continuation) $f_n(z) = f_1(z) \mid T(n)$, where

$$f_n(z) = f_{n,s}(z) := e\left(-n\operatorname{Re}(z)\right)\left(n\operatorname{Im}(z)\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\operatorname{Im}(z)\right).$$

Let f be a nonholomorphic modular form of weight 0 with Fourier expansion

$$f(z) = \sum_{m\in\mathbb{Z}} a(\operatorname{Im}(z),m)e^{2\pi imz}.$$

Then for $\gcd(n,N)=1$, the action of $T(n)$ on f is given by

$$f(z)\mid T(n) = n\sum_{m\in\mathbb{Z}}\sum_{d|\gcd(m,n)} \frac{a\left(\frac{d^2}{n}\operatorname{Im}(z), \frac{mn}{d^2}\right)}{d} e^{2\pi imz}. \tag{3.5}$$

Write $f_n(z) = f_n^*(\operatorname{Im}(z))e^{-2\pi inz}$ with $f_n^*(y) := (ny)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi ny)e^{-2\pi ny}$. The mth coefficient in (3.5) vanishes unless $m=-n$. Moreover, only $d=n$ contributes, giving

$$f_1(z)\mid T(n) = f_n^*\left(n\operatorname{Im}(z)\right)e^{-2\pi inz} = \left(n\operatorname{Im}(z)\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\operatorname{Im}(z)\right)e^{-2\pi in\operatorname{Re}(z)} = f_n(z).$$

(3) For $n\mid N$, we rewrite

$$\sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} f_n(Mz) = \sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} f_1(nMz).$$

Now, with $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)$, we have $nMz = \frac{anz+bn}{\frac{c}{n}nz+d}$ and $\left(\begin{smallmatrix} a & bn\\ \frac{c}{n} & d\end{smallmatrix}\right)$ runs through $\Gamma_\infty\setminus\Gamma_0(\frac{N}{n})$ if M runs through $\Gamma_\infty\setminus\Gamma_0(N)$, implying the claim for $n\mid N$.

(4) We first rewrite the claimed asymptotic formula in terms of the corresponding points Mz with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)$. Directly plugging in and simplifying yields $r_z(c,d)/Q_z(c,d) = \mathrm{Re}(Mz)$ and $\mathrm{Im}(z)/Q_z(c,d) = \mathrm{Im}(Mz)$, so the claim in Theorem 1.1 (4) is equivalent to

$$j_{N,n}(z) = \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma_0(N) \\ n\,\mathrm{Im}(Mz) \geq \mathrm{Im}(z)}} e^{-2\pi i n Mz} + O_z(n). \tag{3.6}$$

In order to show (3.6), we only expand the Fourier expansion for large c. That is to say, we write

$$\begin{aligned} j_{N,n}(z) = 2 &\sum_{\substack{1 \leq c \leq \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} e(-n\,\mathrm{Re}(Mz)) \sinh(2\pi n\,\mathrm{Im}(Mz)) \\ &+ 2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{m \geq 1} \frac{K(m,-n;c)}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i m z} + 4\pi^2 n \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \frac{K(0,-n;c)}{c^2} \\ &+ 2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{m \geq 1} \frac{K(-m,-n;c)}{\sqrt{m}c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i m \overline{z}}. \end{aligned} \tag{3.7}$$

In order to obtain (3.6), we split the main terms with $n\,\mathrm{Im}(Mz) \geq \mathrm{Im}(z)$ off and rewrite

$$2\sinh(2\pi n\,\mathrm{Im}(Mz)) = e^{2\pi n\,\mathrm{Im}(Mz)} - e^{-2\pi n\,\mathrm{Im}(Mz)}. \tag{3.8}$$

The second term above is obviously bounded. Since

$$\mathrm{Im}(z) \leq n\,\mathrm{Im}(Mz) = \frac{n\,\mathrm{Im}(z)}{c^2\,\mathrm{Im}(z)^2 + (d + c\,\mathrm{Re}(z))^2}$$

implies that $c \leq \sqrt{n}/\mathrm{Im}(z) \ll_z \sqrt{n}$ and $|d| \leq |c\,\mathrm{Re}(z)| + \sqrt{n\,\mathrm{Im}(z)} \ll_z \sqrt{n}$, the contribution to the error from the sum of the second terms in (3.8) yields an error of at most $O_z(n)$.

For the second, third, and fourth sums in (3.7), we use the Weil bound for Kloosterman sums

$$|K(m,-n;c)| \leq \sqrt{\gcd(m,n,c)}\sigma_0(c)\sqrt{c} \ll \begin{cases} \sqrt{n}c^{\frac{1}{2}+\varepsilon} & \text{if } m = 0, \\ \sqrt{|m|}c^{\frac{1}{2}+\varepsilon} & \text{if } m \neq 0. \end{cases} \tag{3.9}$$

For the third sum in (3.7), this gives

$$2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \frac{K(0,-n;c)}{c^2} \ll n \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} c^{-\frac{3}{2}+\varepsilon} \ll_z n^{\frac{3}{4}+\varepsilon}. \tag{3.10}$$

Next note that for $x \geq 0$ we have $|J_1(x)| \leq I_1(x)$ by their series expansions. Since $x \mapsto \frac{I_1(x)}{x}$ is monotonically increasing and grows at most exponentially, the contribution from the second and fourth terms in (3.7) may be bounded by, using (3.9),

$$\ll \sum_{\substack{c>\frac{\sqrt{n}}{\operatorname{Im}(z)}\\ N|c}} \sum_{m\geq 1} \frac{|K(\pm m,-n;c)|}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi m \operatorname{Im}(z)}$$

$$\ll \sqrt{n} \sum_{\substack{c>\frac{\sqrt{n}}{\operatorname{Im}(z)}\\ N|c}} \sum_{m\geq 1} \frac{|K(\pm m,-n;c)|}{c^2} \frac{I_1\left(4\pi \operatorname{Im}(z)\sqrt{m}\right)}{4\pi \operatorname{Im}(z)\sqrt{m}} e^{-2\pi m \operatorname{Im}(z)} \tag{3.11}$$

$$\ll \sqrt{n} \sum_{m\geq 1} I_1\left(4\pi \operatorname{Im}(z)\sqrt{m}\right) e^{-2\pi m \operatorname{Im}(z)} \ll \sqrt{n}.$$

It remains to bound the terms in the first sum in (3.7) with $|cz+d|^2 > n$. Since each term gives a constant contribution, the terms with $|d| < \sqrt{n} + |c\operatorname{Re}(z)|$ give an error term of at most $O_z(n)$.

We finally assume that $|d| \geq \sqrt{n} + |c\operatorname{Re}(z)|$. Since $x \mapsto \frac{\sinh(x)}{x}$ is monotonically increasing and $|cz+d|^2 > n$, the remaining terms contribute

$$\left| \sum_{\substack{c\leq \frac{\sqrt{n}}{\operatorname{Im}(z)}\\ N|c}} \sum_{\substack{|d|\geq \sqrt{n}+|c\operatorname{Re}(z)|\\ \gcd(c,d)=1,}} e\left(-n\operatorname{Re}\left(Mz\right)\right) \sinh\left(2\pi n \operatorname{Im}\left(Mz\right)\right) \right| \leq \sum_{c\leq \frac{\sqrt{n}}{\operatorname{Im}(z)}} \sum_{|d|\geq\sqrt{n}+|cx|} \sinh\left(\frac{2\pi n\operatorname{Im}(z)}{|cz+d|^2}\right)$$

$$\leq \sum_{c\leq \frac{\sqrt{n}}{\operatorname{Im}(z)}} \sum_{|d|\geq \sqrt{n}} \sinh\left(\frac{2\pi n \operatorname{Im}(z)}{d^2}\right) \leq 2\pi\sqrt{n} \sum_{d\geq\sqrt{n}} \frac{n}{d^2} \frac{\sinh\left(2\pi\operatorname{Im}(z)\right)}{2\pi\operatorname{Im}(z)} = O_z(n),$$

This implies that the terms in the first sum in (3.7) with $|cz+d|^2 > n$ contribute $O_z(n)$. ☐

Remark.

By replacing $c > \sqrt{n}/\operatorname{Im}(z)$ with $c > C$ in (3.10) and (3.11), one finds that the terms decay like $C^{-\frac{1}{2}+\varepsilon}$ times a power of n. For $c \leq C$, the expansions in Proposition 3.1 decay exponentially in m.

Proof of Theorem 1.2. (1) Let K_s denote the usual K-Bessel function. Expanding $F_{N,-n,s}(z)$ at the cusp ρ as in Section 3.4 of [17], we obtain

$$F_{N,-n,s}\left(M_\rho z\right) = \frac{c_{\rho,s}(n,0)}{2s-1}\left(\operatorname{Im}(z)\right)^{1-s} + \sum_{m\in\mathbb{Z}\setminus\{0\}} c_{\rho,s}(n,m) e^{2\pi i m \frac{\operatorname{Re}(z)}{\ell_\rho}} \left(\operatorname{Im}(z)\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi|m|\operatorname{Im}(z)}{\ell_\rho}\right),$$

$$\text{with } c_{\rho,s}(n,m) := \sum_{c\geq 1} K_{i\infty,\rho}(m,-n;c) \times \begin{cases} \frac{2}{c\sqrt{\ell_\rho}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{\ell_\rho c}\right) & \text{if } m>0,\\ \frac{2\pi^s n^{s-\frac{1}{2}}}{\ell_\rho^s c^{2s}\Gamma(s)} & \text{if } m=0,\\ \frac{2}{c\sqrt{\ell_\rho}} J_{2s-1}\left(\frac{4\pi\sqrt{|m|n}}{\ell_\rho c}\right) & \text{if } m<0,\end{cases}$$

The right-hand side is analytic at $s=1$, which gives the expansion of $F_{N,-n}(z)$ at ρ. Plugging in $K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}$ and taking the limit $z \to i\infty$, we obtain

$$j_{N,n}(\rho) = 2\pi\sqrt{n} \lim_{s\to 1^+} c_{\rho,s}(n,0) = \frac{4\pi^2 n}{\ell_\rho} \sum_{c\geq 1} \frac{K_{i\infty,\rho}(0,-n;c)}{c^2}. \tag{3.12}$$

We have $K_{i\infty,\rho}(0,-n;c) = K_{\rho,i\infty}(n,0;c)$, since $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ runs through $\Gamma_0(N)M_\rho/\Gamma_\infty^{\ell_\rho}$ iff $-M^{-1} = \left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)$ runs through $\Gamma_\infty^{\ell_\rho}\backslash M_\rho^{-1}\Gamma_0(N)$ in (2.4). Hence (2.6) yields the claim.
Parts (2) and (3) follow by taking limits $\tau \to \rho$ in Theorem 1.1 (2) and (3), respectively. Using the growth in n of $j_{N,n}(\rho)$ from (3.12), these limits may be taken termwise.

□

Proof of Theorem 1.3. We show that the difference of both sides has no poles in $\mathbb{H}$ and decays towards the cusps. We start by considering the points in $\mathbb{H}$. One easily computes that the residue of $-\frac{\Theta(f(\tau))}{f(\tau)}$ at $\tau = z$ equals $\frac{1}{2\pi i}\mathrm{ord}_z(f)$. Using (2.3) gives that the principal part at z agrees. At a cusp ρ one similarly sees that $\frac{\Theta(f(\tau))}{f(\tau)}$ has no pole and its constant term equals $\mathrm{ord}_\rho(f)$. Using that the constant term of $H_{N,z}^*(\tau)$ at ρ is -1 then gives the claim. □

References

[1] E. Alwaise, *An algorithm for numerically inverting the modular j-function*, Res. Numb. Th., accepted for publication.
[2] T. Asai, M. Kaneko, and H. Ninomiya, *Zeros of certain modular functions and an application*, Comm. Math. Univ. Sancti Pauli **46** (1997), 93–101.
[3] B. Berndt, P. Bialek, and A. Yee, *Formulas of Ramanujan for the power series coefficients of certain quotients of Eisenstein series*, Int. Math. Res. Not. **2002** (2002), 1077–1109.
[4] P. Bialek, *Ramanujan's formulas for the coefficients in the power series expansions of certain modular forms*, Ph. D. thesis, University of Illinois at Urbana–Champaign, 1995.
[5] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, *Harmonic Maass forms and mock modular forms: theory and applications*, AMS Colloquium Series, to appear.
[6] K. Bringmann and B. Kane, *A problem of Petersson about weight 0 meromorphic modular forms*, Research in Mathematical Sciences, accepted for publication.
[7] K. Bringmann and B. Kane, *Ramanujan and coefficients of meromorphic modular forms*, J. Math. Pures Appl., accepted for publication.
[8] K. Bringmann and B. Kane, *Ramanujan-like formulas for Fourier coefficients of all meromorphic cusp forms*, submitted for publication.
[9] J. Bruinier, W. Kohnen, and K. Ono, *The arithmetic of the values of modular functions and the divisors of modular forms*, Compositio Math. **130** (2004), 552–566.
[10] S. Carnahan, *Generalized moonshine, II: Borcherds products*, Duke Math. J. **161** (2012), no. 5, 893–950.
[11] D. Choi, *Poincaré series and the divisors of modular forms*, Proc. Amer. Math. Soc. **138** (2010), no 10, 3393–3403
[12] C. Delaunay, *Critical and ramification points of the modular parametrization of an elliptic curve*, J. Théor. Nombres Bordeaux **17** (2005), no. 1, 109–124.
[13] J. Duncan and I. Frenkel, *Rademacher sums, moonshine and gravity*, Comm. Numb. Th. Phys. **5** (2011), 1–128.
[14] J. Duncan, M. Griffin, and K. Ono, *Moonshine*, Res. Math. Sci. **2** (2015), A11.
[15] G. Hardy and S. Ramanujan, *On the coefficients in the expansions of certain modular functions*, Proc. Royal Soc. A **95** (1918), 144–155.
[16] E. Hecke, *Analytische Funktionen und algebraische Zahlen, zweiter Teil*, Abh. Math. Sem. Hamburg Univ. **3** (1924) 213–236.
[17] H. Iwaniec, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematics **53** (2002), ed. 2, American Mathematical Society, Providence, RI; Revista
[18] D. Niebur, *A class of nonanalytic automorphic functions*, Nagoya Math. J. **52** (1973), 133–145 Matemática Iberoamericana, Madrid.
[19] J. Smart, *On modular forms of dimension* -2, J Trans. Amer. Math. Soc. **116** (1965), 86–107.
[20] D. Zagier, *Traces of singular moduli*, Motives, Polylogarithms, and Hodge Theory (Ed. F. Bogomolov and L. Katzarkov), Lect. Ser. **3** Intl. Press, Somerville, 2002, 209–244.P

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: kbringma@math.uni-koeln.de

Department of Mathematics, University of Hong Kong, Pokfluam, Hong Kong
E-mail address: bkane@maths.hku.hk

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: steffen.loebrich@uni-koeln.de

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30022
E-mail address: ono@mathcs.emory.edu

Hamilton Mathematics Institute & School of Mathematics, Trinity College, Dublin 2, Ireland
E-mail address: lrolen@maths.tcd.ie