

OPTIMAL REINSURANCE AND INVESTMENT STRATEGY WITH TWO PIECE UTILITY FUNCTION

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ABSTRACT. This paper studies optimal reinsurance and investment strategies that maximize expected utility of the terminal wealth for an insurer in a stochastic market. The insurer's preference is represented by a two-piece utility function which can be regarded as a generalization of traditional concave utility functions. We employ martingale approach and convex optimization method to transform the dynamic maximization problem into an equivalent static optimization problem. By solving the optimization problem, we derive explicit expressions of the optimal reinsurance and investment strategy and the optimal wealth process.

1. Introduction. Study on continuous-time optimal control problem related to insurance risk management has predominantly centered around expected utility maximization (EUM) for decades. Traditional assumption of EUM is that decision makers are rational and risk averse when facing uncertainty. However, this assumption has been challenged by many researchers, for example, the Allais paradox (see Allais, 1953), the equity premium puzzle (see Mehra and Prescott, 1985) and so on.

A number of alternative preference measures have been proposed to overcome the drawbacks of EUM, such as Lopes' SP/A model, cumulative prospect theory (CPT, see Kahneman and Tversky 1979, 1992) and disappointment theory (DT, see Bell 1985, Loomes and Sugden 1986). CPT has three significant features: existence of preference point, S-shaped utility function and probability distortion. More and more researchers incorporate these new preference measures into optimization problems and pricing principles. The early publications are limited to the single period setting, see, for example, Benarti and Thaler (1995), Lopes and Oden (1999), Shefrin and Statman (2000) and Bernard and Ghossoub (2010). Berkelaar et al. (2004) considers the dynamic portfolio selection problem under a two-piece power utility

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function with loss aversion, where investor takes up a risk-seeking attitude towards loss, and derive optimal investment strategy by employing a convex optimization technique. Jin and Zhou (2008) considers a continuous time portfolio selection model under CPT. They separate the optimization problem into gain part and loss part, use a Choquet integral formulation to deal with the probability distortion and develop a Choquet maximization and minimization technique to solve the problem. Mi and Zhang (2012) investigates an optimal portfolio selection problem assuming a two-piece utility function in an incomplete market.

The optimal reinsurance and investment strategy is an important research topic in insurance and actuarial science. Under EUM setting, the optimal strategy has attracted considerable interest recently. Browne (1995) uses a Brownian motion with drift to model the risk process of the insurer and obtains the corresponding optimal investment strategy. Yang and Zhang (2005) considers the portfolio selection problem under a jump diffusion risk process model. In order to deal with changes in the market environment, Zhang and Siu (2012) investigates the optimal proportional reinsurance and investment strategy under a Markovian regime switching economy. Other related papers include Hipp and Plum (2000, 2003), Schmidli (2001), Liu and Yang (2004), Xu and Yao (2008), Yao et al. (2010). There is also burgeoning research interest in actuarial science under alternative preference measures mentioned above. Tasnakas and Desli (2003) derives a premium principle called generalized expected utility premium principle based on the rank dependent expected utility (RDEU). Sung et al. (2012) studies the optimal insurance policy using a behavioral principle. Chueng et al. (2015a, 2015b) investigate the premium principle and optimal insurance using disappointment theory. He et al. (2015) studies the optimal insurance design using RDEU.

To the best of our knowledge, there has been limited publications incorporating non-concave preferences into the optimal strategy selection for an insurer. Guo (2014) considers the optimal investment problem for an insurer with loss aversion. In this paper, we derive the optimal strategy with a general utility function, a special case of our model can be reduced to that in Guo (2014), and we incorporate the proportional reinsurance for an insurer in our model. The decision maker's preference is represented by a two-piece utility function with a reference point. When the utility function is convex, we set a lower bound for the wealth. If the utility takes a concave preference, the lower bound can be ignored. Comparing to results in the relative literatures, the result in this paper is more general. We do not assume a specified function form for the utility function in this paper, the specified utility functions are given as examples for illustration. For some utility functions, for example, a two-piece power utility or exponential utility, we are able to obtain corresponding analogue results as those in Browne (1995) and Guo (2014). We give a uniform expression for concave utility function where the result is always coincident. But in some special two-piece concave utility case, the result is different and relies on the corresponding parameters of the positive part and negative part of utility function. In concave-convex utility function case, the result is seriously affected by the lower bound. In this paper, we apply traditional martingale technique, which is widely used in mathematical finance, to work out the closed form of the optimal strategy and the optimal wealth process.

The rest of this paper is arranged as follows. Section 2 describes the economy for the insurance company. The maximization problem of investment and reinsurance is presented in Section 3. The explicit expression of the optimal strategy and the

optimal wealth process are obtained in Section 4. In Section 5, we present some examples for illustration purpose.

2. The market. In this section, We define a continuous-time financial market on a finite-time horizon $\mathcal{T} := [0, T]$, where $T < \infty$ is the terminal time of decision making process. The uncertainty of the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}(t)|t \in \mathcal{T}\}$ is the collection of information until time t and \mathbb{P} is a real-world probability measure. All the processes defined below are presumed to be adapted to \mathbb{F} . We denote by $\mathbb{E}[\cdot]$ the expectation under \mathbb{P} .

The financial market consists of a risk-free asset and a risky asset which can be traded continuously on the time horizon \mathcal{T} . The price process of the risky-free asset $B := \{B(t)\}_{t \geq 0}$ evolves according to

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1,$$

where $r(t)$ denotes the risk-free interest rate for borrowing and is assumed to be deterministic and uniformly bounded. The price process of risky asset $S := \{S(t)|t \in \mathcal{T}\}$ is governed by a geometric Brownian motion

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)], \quad S(0) = s > 0,$$

where $b(t)$ and $\sigma(t)$ denote the appreciation rate and the volatility of the asset at time t respectively and satisfy $b(t) > r(t), \sigma(t) > 0$; $b(t)$ and $\sigma(t)$ are assumed to be deterministic and uniformly bounded; $W := \{W(t)|t \in \mathcal{T}\}$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

The surplus process of an insurer, $U := \{U(t)|t \in \mathcal{T}\}$, is assumed to be the classical Cramér-Lunderberg model, namely

$$dU(t) = c(t)dt - dL(t),$$

where $c(t) > 0$ is the premium rate at time t ; $L := \{L(t) = \sum_{i=1}^{N(t)} Y_i|t \in \mathcal{T}\}$ is a compound poisson process defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\{Y_i\}_{i=1}^{\infty}$ is an *i.i.d.* sequence of non-negative random variables, $N := \{N(t)|t \in \mathcal{T}\}$ is a poisson process with intensity $\lambda(t) > 0$ and represents the number of claims up to time t . Here N is assumed to be independent of Y_i and $\mathbb{E}(Y_i) = \mu_1 < \infty$, $\mathbb{E}(Y_i^2) = \mu_2 < \infty$. According to the expected premium principle, we set

$$\int_0^t c(s)ds = \mu_1(1 + \eta) \int_0^t \lambda(s)ds,$$

where $t \in [0, T]$ and $\eta > 0$ represents the safety loading. Thus the surplus process $\{U(t)|t \in \mathcal{T}\}$ is governed by the following equation

$$dU(t) = \mu_1(1 + \eta)\lambda(t)dt - d\left[\sum_{i=1}^{N(t)} Y_i\right].$$

It is well-known that the surplus process above can be approximated by following Brownian motion with drifted (see Grandll (1991))

$$\begin{aligned} dU(t) &= \mu_1(1 + \eta)\lambda(t)dt - \mu_1\lambda(t)dt + \sqrt{\mu_2\lambda(t)}dW_0(t) \\ &= \mu_1\eta\lambda(t)dt + \sqrt{\mu_2\lambda(t)}dW_0(t), \end{aligned}$$

where $W_0 := \{W_0(t)|t \in \mathcal{T}\}$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Moreover, we assume that W_0 and W are stochastically independent. Thus the filtration $\{\mathcal{F}(t)|t \in \mathcal{T}\}$ can be regarded as augmentation of the filtration $\{\mathcal{F}_t^{W(t), W_0(t)}|t \in \mathcal{T}\}$ that generated by (W, W_0) .

The insurer is allowed to invest its surplus in the financial market and purchase reinsurance to control its risk. In this paper, we restrict our attention to the proportional reinsurance and let $p(t)$ denote the reinsurance proportion, that is $1 - p(t)$ portion of the insurance risk is divided to the reinsurer business. Here $p(t)$ is restricted to be non-negative and $p(t) > 1$ means taking new reinsurance business from insurance market.

The insurer's objective is to choose an \mathbb{F} -adapted process $\pi(t)$, representing the amount invested in the risky stock, and an \mathbb{F} -adapted process $p(t)$, the proportional reinsurance process, so as to maximize the expected utility of terminal wealth at time T . The reinsurance and investment strategy is a two-dimensional stochastic process $u := \{u(t) | t \in \mathcal{T}\} = \{(p(t), \pi(t)) | t \in \mathcal{T}\}$.

Definition 2.1. A proportional reinsurance and investment strategy u is said to be admissible if $(\pi(t), p(t))$ is \mathbb{F} -adapted process such that

$$\begin{aligned} \int_0^T \pi^2(t) dt &< \infty, \quad \mathbb{P} - a.s.; \\ \int_0^T p^2(t) dt &< \infty, \quad \mathbb{P} - a.s., \quad p(t) \geq 0. \end{aligned}$$

The set of all admissible strategies is denoted by Π .

The wealth process $X(t)$, associated with an admissible strategy u , takes the following form

$$\begin{aligned} dX(t) &= \mu_1 \eta \lambda(t) dt - (1 - p(t)) \mu_1 \theta \lambda(t) dt + \sqrt{\mu_2 \lambda(t)} p(t) dW_0(t) \\ &\quad + \pi(t) \frac{dS(t)}{S(t)} + (X(t) - \pi(t)) \frac{dB(t)}{B(t)}, \quad X(0) = x_0, \end{aligned}$$

where x_0 denotes the initial wealth; θ represents the safety loading of the reinsurer and, in general, $\theta \geq \eta$, otherwise there are arbitrage opportunities. For simplicity, we restrict our analysis to the cheap reinsurance case: $\theta = \eta$, that is, the reinsurance company uses the same safety loading as the cedent. Thereafter, $X(t)$ is of the following form

$$\begin{aligned} dX(t) &= \theta \mu_1 \lambda(t) p(t) dt + \sqrt{\mu_2 \lambda(t)} p(t) dW_0(t) + \pi(t) \frac{dS(t)}{S(t)} + (X(t) - \pi(t)) \frac{dB(t)}{B(t)} \\ &= [\theta \mu_1 \lambda(t) p(t) + \pi(t) b(t) + r(t) X(t) - r(t) \pi(t)] dt \\ &\quad + \sqrt{\mu_2 \lambda(t)} p(t) dW_0(t) + \pi(t) \sigma(t) dW(t), \end{aligned} \tag{1.1}$$

where $\lambda(t), b(t), r(t), \sigma(t)$ are assumed to be deterministic and uniformly bounded on $[0, T]$.

3. The insurer's maximization problem. In this section, we introduce the insurer's utility function and propose the maximization problem. It is well known that one of the conventional assumptions in the theory of optimal reinsurance-investment strategy is that the utility function is a smooth, concave and increasing function over terminal wealth $X(T)$. In this paper, we consider a two-piece utility function, that is, the insurer is assumed to be an investor with following preference

$$U(x) = \begin{cases} U_1(x), & x > 0, \\ U_2(x), & x \leq 0, \end{cases} \tag{1.2}$$

where $U_1(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}$, is strictly increasing, concave and twice differentiable with $U_1'(+\infty) = 0$; $U_2(\cdot) : \mathbb{R}^- \mapsto \mathbb{R}$, is strictly increasing, twice differentiable and $U_1(0) = U_2(0)$. We consider two cases in this paper: the first case is that $U_2(\cdot)$ is a concave function, the second case is that $U_2(\cdot)$ is a convex function.

The utility in (1.2) can be viewed as a generalization of the common concave utility and the utility with loss aversion. When $U_1(\cdot) = U_2(\cdot)$ are both concave functions, the two-piece utility $U(\cdot)$ reduces to the common concave utility, that means, the insurer shows an invariable risk-averse attitude towards gain and loss.

In the case of convex preference for loss part and $U_1'(0+) = U_2'(0-) = \infty$, originating from the classical CPT theory in Tversky and Kahneman (1992), the utility $U(\cdot)$ is an S -shaped function, that is, the insurer is an investor with loss aversion. Moreover, if $U_2(\cdot)$ is always steeper than $U_1(\cdot)$, that reflects that the investor is more sensitive to losses than gains. Due to the convexity of $U_2(\cdot)$, the agent is risk-averse in the gain domain and risk-seeking in the loss domain. We will consider this special case in Section 5.

Given the terminal wealth $X(T) \in \mathcal{F}(T)$, the objective function for the insurer is given by

$$\begin{aligned} V(X(T)) &:= \mathbb{E}[U(X(T) - k)] \\ &= \mathbb{E}[U_1(X(T) - k)\mathbf{I}_{X(T) > k} + U_2(X(T) - k)\mathbf{I}_{X(T) \leq k}], \end{aligned}$$

where k represents the reference point which divides the utility into two parts; \mathbf{I} is an indicator function.

Following the expected utility maximization criterion, the problem of choosing an optimal reinsurance-investment strategy for an insurer is formulated as

$$\begin{cases} \max_{u \in \Pi} & \mathbb{E}[U(X^u(T) - k)] \\ \text{s.t.} & X(t) \text{ satisfies (1.1),} \\ & X(t) \geq L, \quad L \in (-\infty, +\infty), \quad \forall t \in [0, T], \end{cases} \quad (1.3)$$

where L denotes the lower bound of $X(t)$. Usually the value of L is equal to zero, indicating that the insurance company is not bankrupt throughout the investment period $[0, T]$.

4. The optimal strategy choice. In this section we derive the optimal terminal wealth and optimal proportional reinsurance and investment strategy. The steps to achieve the goal can be listed as follows: we reduce the original problem (1.3) to a static optimization problem which is subject to a linear constraint and then apply martingale technique to solve the static problem and derive the optimal terminal wealth. Finally, the optimal proportional reinsurance and investment strategy u^* can be obtained.

Define

$$\begin{aligned} H(t) &:= \exp \left\{ \int_0^t - \left(\frac{\theta^2 \mu_1^2 \lambda(s) \sigma^2(s) + \mu_2 (r(s) - b(s))^2 + 2r(s) \mu_2 \sigma^2(s)}{2\mu_2 \sigma^2(s)} \right) ds \right. \\ &\quad \left. - \int_0^t \theta \mu_1 \sqrt{\frac{\lambda(s)}{\mu_2}} dW_0(s) + \int_0^t \left(\frac{r(s) - b(s)}{\sigma(s)} \right) dW(s) \right\}. \end{aligned} \quad (1.4)$$

We have two similar propositions as those in Guo (2014).

Proposition 1. *If $H(t)$ is defined by (1.4) on $[0, T]$, $H(t)X(t)$ is a martingale under probability measure \mathbb{P} .*

Proof. Differentiate $H(t)$, we have

$$dH(t) = H(t) \left[-r(t)dt - \theta\mu_1 \sqrt{\frac{\lambda(t)}{\mu_2}} dW_0(t) + \left(\frac{r(t) - b(t)}{\sigma(t)} \right) dW(t) \right].$$

Applying *Itô* formula, we obtain

$$\begin{aligned} d(H(t)X(t)) &= H(t)dX(t) + X(t)dH(t) + d[H(t), X(t)] \\ &= \left[H(t)\sqrt{\mu_2\lambda(t)}p(t) - X(t)H(t)\theta\mu_1\sqrt{\frac{\lambda(t)}{\mu_2}} \right] dW_0(t) \\ &\quad + \left[H(t)\pi(t)\sigma(t) + X(t)H(t)\frac{r(t) - b(t)}{\sigma(t)} \right] dW(t), \end{aligned} \quad (1.5)$$

where $[H(t), X(t)]$ is the quadratic covariation process of $H(t)$ and $X(t)$. This shows that $H(t)X(t)$ can be represented as an *Itô* integral with respect to two-dimensional Brownian motion $(W_0(t), W(t))$. Therefore $H(t)X(t)$ is a martingale under \mathbb{P} . \square

Proposition 2. *Let the initial wealth x_0 be given, then for any $\mathcal{F}(T)$ random variable ξ with a lower bound L satisfying*

$$\mathbb{E}[H(T)\xi] = x_0,$$

there exists an admissible strategy u such that

$$X^u(T) = \xi.$$

Proof. Let us define a martingale

$$M(t) = \mathbb{E}[H(T)\xi | \mathcal{F}_t].$$

From martingale representative theorem (Karatzas and Shreve 1991), there exist two progressive measurable processes $\varphi : \Omega \times [0, T] \mapsto \mathfrak{R}$; $\psi : \Omega \times [0, T] \mapsto \mathfrak{R}$ satisfying

$$\begin{aligned} \int_0^T |\varphi(s)|^2 ds &< \infty, \\ \int_0^T |\psi(s)|^2 ds &< \infty, \end{aligned}$$

such that

$$\begin{aligned} M(t) &= \mathbb{E}[H(T)\xi] + \int_0^t \varphi(s) dW_0(s) + \int_0^t \psi(s) dW(s) \\ &= x_0 + \int_0^t \varphi(s) dW_0(s) + \int_0^t \psi(s) dW(s). \end{aligned}$$

Compare $dW_0(t)$ -term and $dW(t)$ -term with those in (1.5), let $t = T$, we have

$$\begin{aligned} \varphi(t) &= H(t)\sqrt{\mu_2\lambda(t)}p(t) - X(t)H(t)\theta\mu_1\sqrt{\frac{\lambda(t)}{\mu_2}}, \\ \psi(t) &= H(t)\pi(t)\sigma(t) + X(t)H(t)\frac{r(t) - b(t)}{\sigma(t)}, \end{aligned}$$

which imply that

$$p(t) = \frac{\varphi(t) + X(t)H(t)\theta\mu_1\sqrt{\frac{\lambda(t)}{\mu_2}}}{H(t)\sqrt{\mu_2\lambda(t)}},$$

$$\pi(t) = \frac{\psi(t) - X(t)H(t)\frac{r(t)-b(t)}{\sigma(t)}}{H(t)\sigma(t)}.$$

The admissibility of u can be obtained from the corresponding results in Guo (2014). \square

According to the above propositions, any $\mathcal{F}(t)$ random variable ξ , satisfying $\mathbb{E}[H(T)\xi] = x_0$, can be financed via trading an admissible strategy u . Thus the dynamic maximization problem (1.3) can be transformed into following static optimization problem

$$\begin{cases} \max_{\xi \geq L} & \mathbb{E}[U(\xi - k)] \\ \text{s.t.} & \mathbb{E}[H(T)\xi] \leq x_0, \quad \xi \geq L. \end{cases} \quad (1.6)$$

After a simple change, the above problem (1.6) turns to be

$$\begin{cases} \max_{\xi' \geq L'} & \mathbb{E}[U(\xi')] \\ \text{s.t.} & \mathbb{E}[H(T)\xi'] \leq x'_0, \quad \xi' \geq L', \end{cases} \quad (1.7)$$

where $\xi' := \xi - k$, $x'_0 := x_0 - k\mathbb{E}[H(T)]$ and $L' := L - k$. We only consider the case $k = 0$ in the following. We apply a divide formulation to split problem (1.6) into two parts

Positive Part Problem. A problem with parameters (A, x_+) :

$$\begin{cases} \max_{\xi_+ \geq L} & \mathbb{E}[U_1(\xi_+)] \\ \text{s.t.} & \mathbb{E}[H(T)\xi_+] = x_+, \quad \xi_+ \geq 0, \quad \xi_+ = 0 \text{ on } A^c. \end{cases} \quad (1.8)$$

Negative Part Problem. A problem with parameters $(A^c, x_0 - x_+)$:

$$\begin{cases} \max_{\xi_- \geq L} & \mathbb{E}[U_2(\xi_-)] \\ \text{s.t.} & \mathbb{E}[H(T)\xi_-] = x_0 - x_+, \quad \xi_- \leq 0, \quad \xi_- = 0 \text{ on } A, \end{cases} \quad (1.9)$$

where $x_+ \geq x_0$, $x_+ \geq 0$ and $A \in \mathcal{F}(T)$ is given.

If the lower bound $L > 0$, problem (1.7) turns to be a single positive problem (1.8) with additional constraint on the value of L . If the lower bound $L \leq 0$, we need to consider both problems (1.8) and (1.9), and make comparisons to obtain the optimal wealth. We solve problem (1.8) and (1.9) respectively in the following.

Proposition 3. Suppose $\mathbb{P}(A) < 1$, $\mathbb{E}[H(T)L] \leq x_+$ and $U_1(\cdot)$ satisfy condition $U_1'(+\infty) = 0$, then the optimal wealth of the insurer for positive part problem (1.8) is given by

$$\xi_+^* = (U_1')^{-1}(y^*H(T))\mathbf{I}_{y^*H(T) \leq (U_1')_+(L \vee 0)}\mathbf{I}_A + (L \vee 0)\mathbf{I}_{y^*H(T) > (U_1')_+(L \vee 0)}\mathbf{I}_A,$$

where $L \vee 0 := \max\{L, 0\}$ and y^* is the unique solution of

$$\mathbb{E}[H(T)\xi_+^*] = x_+.$$

Proof. We first solve the point-wise maximization problem

$$\tilde{U}_1(H(T)) = \max_{\xi_+ \geq L \vee 0} [U_1(\xi_+) - yH(T)\xi_+], \quad y > 0. \quad (1.10)$$

where $L \vee 0 := \max\{L, 0\}$. Due to the concavity of utility function $U_1(\cdot)$ and the restriction that $\xi_+ \geq L \vee 0$, the maximizer ξ_+^* to problem (1.10) is given by

$$\xi_+^* = \begin{cases} (U_1')^{-1}(yH(T)), & \text{on } A \cap \{yH(T) \leq (U_1')_+(L \vee 0)\}, \\ L \vee 0, & \text{on } A \cap \{yH(T) > (U_1')_+(L \vee 0)\}, \\ 0, & \text{on } A^c, \end{cases}$$

where $U_1'(\cdot)$ denotes the derivative of $U_1(\cdot)$; $(U_1')^{-1}(\cdot)$ denotes the inverse of $U_1'(\cdot)$ and $(U_1')_+(L \vee 0)$ denotes the right-hand derivative of $U_1(\cdot)$ at point $L \vee 0$.

We then show that ξ_+^* is the candidate optimal wealth for problem (1.8). Suppose ξ_+' represents any possible optimal solution satisfying the static budget equation in (1.8), then we have

$$\begin{aligned} \mathbb{E}[U_1(\xi_+^*)] - \mathbb{E}[U_1(\xi_+')] &= \mathbb{E}[U_1(\xi_+^*)] - yx_+ - \{\mathbb{E}[U_1(\xi_+')] - yx_+\} \\ &= \mathbb{E}[U_1(\xi_+^*)] - y\mathbb{E}[H(T)\xi_+^*] - \{\mathbb{E}[U_1(\xi_+')] - y\mathbb{E}[H(T)\xi_+']\} \\ &= \mathbb{E}[(U_1(\xi_+^*) - yH(T)\xi_+^*) - (U_1(\xi_+') - yH(T)\xi_+')] \\ &\geq 0, \end{aligned}$$

which manifests the optimality of ξ_+^* .

Finally we verify that, for any $x_+ \geq x_0$ and $x_+ > 0$, there exists a unique $y > 0$ satisfying the budget constraint in (1.8). Since $U_1(\cdot)$ is strictly increasing, concave, twice differentiable, and defined on \mathbb{R}^+ , its derivative has a strictly decreasing, continuous inverse $(U_1')^{-1}(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$. Hence, for any $y_1 > y_2 > 0$, and due to the non-negative of $H(T)$, we have

$$\begin{aligned} 0 \leq H(T)(U_1')^{-1}(y_1H(T)) &< H(T)(U_1')^{-1}(y_2H(T)) \\ &\text{on } A \cap \{yH(T) \leq (U_1')_+(L \vee 0)\}. \end{aligned}$$

Therefore

$$\varphi(y) := H(T)(U_1')^{-1}(yH(T))\mathbf{I}_{yH(T) \leq (U_1')_+(L \vee 0)} + H(T)(L \vee 0)\mathbf{I}_{yH(T) > (U_1')_+(L \vee 0)}$$

is decreasing with respect to y .

By the dominated convergence theorem and monotone convergence theorem, it is easy to see that $\mathbb{E}[\varphi(y)]$ is continuous with limit

$$\begin{aligned} \lim_{y \rightarrow +\infty} \mathbb{E}[\varphi(y)] &\leq x_+, \\ \lim_{y \rightarrow 0} \mathbb{E}[\varphi(y)] &= +\infty. \end{aligned}$$

Thus there exists a unique y such that

$$\mathbb{E}[\varphi(y)] = x_+,$$

and we denote it by y^* . In conclusion we have

$$\xi_+^* = (U_1')^{-1}(y^*H(T))\mathbf{I}_{y^*H(T) \leq (U_1')_+(L \vee 0)} + (L \vee 0)\mathbf{I}_{y^*H(T) > (U_1')_+(L \vee 0)}\mathbf{I}_A.$$

□

Proposition 4. *Suppose $x_0 - x_+ < 0$ and $\mathbb{P}(A^c) > 0$.*

(1) *If $U_2(\cdot)$ is a strictly convex utility function, $L \in (-\infty, 0)$ and*

$$\mathbb{E}[H(T)L\mathbf{I}_{A^c}] \leq x_0 - x_+,$$

then the optimal wealth of the insurer in negative part problem (1.9) is given by

$$\xi_-^* = L\mathbf{I}_{y^*H(T)L > U_2(L) - U_2(0)}\mathbf{I}_{A^c},$$

where y^ is the unique solution of*

$$\mathbb{E}[H(T)\xi_-^*] = x_0 - x_+.$$

(2) *If $U_2(\cdot)$ is a strictly concave utility function, $L \in (-\infty, 0)$ and*

$$\mathbb{E}[H(T)L\mathbf{I}_{A^c}] \leq x_0 - x_+,$$

then the optimal wealth of the insurer in negative part problem (1.9) is given by

$$\xi_-^* = (U_2')^{-1}(y^*H(T))\mathbf{I}_{(U_2')_-(0) \leq y^*H(T) \leq (U_2')_+(L)}\mathbf{I}_{A^c} + L\mathbf{I}_{y^*H(T) > (U_2')_+(L)}\mathbf{I}_{A^c},$$

where y^ is the unique solution of*

$$\mathbb{E}[H(T)\xi_-^*] = x_0 - x_+.$$

Proof. (1) Consider the point-wise maximization problem

$$\tilde{U}_2(H(T)) = \max_{L \leq \xi_- \leq 0} [U_2(\xi_-) - yH(T)\xi_-], \quad y > 0, \quad (1.11)$$

and define

$$\phi(x) := U_2(x) - yH(T)x.$$

It is easy to see that $\phi'(x)$ is increasing with respect to x , thus the optimal maximizer ξ_-^* for problem (1.11) is located at one of the boundaries

$$\xi_-^* = 0, \quad \xi_-^* = L.$$

Comparing the value of $\phi(x)$ at $\xi_-^* = 0$ and $\xi_-^* = L$, we obtain when

$$yH(T)L \leq U_2(L) - U_2(0),$$

the optimal value is $\xi_-^* = L$ and when

$$yH(T)L > U_2(L) - U_2(0),$$

the optimal value is $\xi_-^* = 0$. Therefore, the optimal terminal wealth is given by

$$\xi_-^* = L\mathbf{I}_{y^*H(T)L \leq U_2(L) - U_2(0)}\mathbf{I}_{A^c},$$

where $y^* > 0$ is the unique solution of

$$\mathbb{E}\left[H(T)L\mathbf{I}_{y^*H(T)L \leq U_2(L) - U_2(0)}\mathbf{I}_{A^c}\right] = x_0 - x_+.$$

The optimality of ξ_-^* and uniqueness of y^* can be obtained by a similar proof as that in Proposition 3.

(2) Similar to Proposition 3, the result can be obtained. \square

It is worthy noting that when $U_2(\cdot)$ is a strictly convex function, the existence of lower bound L is essential otherwise the problem will be ill-posed, because the prospect value will be unbounded. The condition

$$\mathbb{E}[H(T)L\mathbf{I}_{A^c}] \leq x_0 - x_+$$

is to ensure the existence of optimal strategy, otherwise there is no strategy satisfying the static budget equation in (1.9).

The following theorem presents the optimal solution to (1.6).

Theorem 4.1. (1) Suppose $U_2(\cdot)$ is strictly convex and $L \in (-\infty, 0)$ with

$$\mathbb{E}[H(T)L] \leq x_0.$$

The optimal terminal wealth is given by

$$X^*(T) = (U'_1)^{-1}(y^*H(T))\mathbf{I}_{y^*H(T) \leq (U'_1)_+(0)}\mathbf{I}_{H(T) \leq H_T(y^*)} + L\mathbf{I}_{H(T) > H_T(y^*)},$$

where y^* , $H_T(y^*)$ is the unique solution of $\mathbb{E}[H(T)X^*(T)] = x_0$ and

$$\left\{ U_1[(U'_1)^{-1}(y^*H_T(y^*))] - y^*H_T(y^*)(U'_1)^{-1}(y^*H_T(y^*)) + y^*H_T(y^*)L \right\} \quad (1.12)$$

$$\mathbf{I}_{y^*H_T(y^*) \leq (U'_1)_+(0)} - U_2(L) + [U_1(0) + y^*H_T(y^*)L]\mathbf{I}_{y^*H_T(y^*) > (U'_1)_+(0)} = 0.$$

(2) Suppose $U_2(\cdot)$ is strictly concave and $L \in (-\infty, 0)$ with

$$\mathbb{E}[H(T)L] \leq x_0.$$

①. If $(U'_2)_-(0) \geq (U'_1)_+(0)$, The optimal terminal wealth is given by

$$X^*(T) = (U'_1)^{-1}(y^*H(T))\mathbf{I}_{y^*H(T) \leq (U'_1)_+(0)} \\ + (U'_2)^{-1}(y^*H(T))\mathbf{I}_{(U'_2)_-(0) \leq y^*H(T) \leq (U'_2)_+(L)} + L\mathbf{I}_{y^*H(T) > (U'_2)_+(L)},$$

where y^* is the unique solution of $\mathbb{E}[H(T)X^*(T)] = x_0$.

②. If $(U'_2)_-(0) < (U'_1)_+(0)$, The optimal terminal wealth is given by

$$X^*(T) = (U'_1)^{-1}(y^*H(T))\mathbf{I}_{H(T) \leq H_T(y^*)} + L\mathbf{I}_{y^*H(T) > (U'_2)_+(L)}\mathbf{I}_{H(T) > H_T(y^*)} \\ + (U'_2)^{-1}(y^*H(T))\mathbf{I}_{(U'_2)_-(0) \leq y^*H(T) \leq (U'_2)_+(L)}\mathbf{I}_{H(T) > H_T(y^*)},$$

where y^* , $H_T(y^*) \in \left[\frac{(U'_1)_+(0)}{y^*}, \frac{(U'_2)_-(0)}{y^*} \right]$ is the unique solution of

$$\mathbb{E}[H(T)X^*(T)] = x_0$$

and

$$U_1 \left[(U'_1)^{-1}(y^*H_T(y^*))\mathbf{I}_{y^*H_T(y^*) \leq (U'_1)_+(0)} \right] - y^*H_T(y^*)(U'_1)^{-1}(y^*H_T(y^*)) \\ \mathbf{I}_{y^*H_T(y^*) \leq (U'_1)_+(0)} - U_2 \left[(U'_2)^{-1}(y^*H_T(y^*))\mathbf{I}_{(U'_2)_-(0) \leq y^*H_T(y^*) \leq (U'_2)_+(L)} \right. \\ \left. + L\mathbf{I}_{y^*H_T(y^*) > (U'_2)_+(L)} \right] + y^*H_T(y^*) \left[(U'_2)^{-1}(y^*H_T(y^*)) \right. \\ \left. \mathbf{I}_{(U'_2)_-(0) \leq y^*H_T(y^*) \leq (U'_2)_+(L)} + L\mathbf{I}_{y^*H_T(y^*) > (U'_2)_+(L)} \right] = 0.$$

Proof. (1) Compare

$$X_1(T) := (U'_1)^{-1}(yH(T))\mathbf{I}_{yH(T) \leq (U'_1)_+(0)}$$

with

$$X_2(T) := L$$

and observe that

$$\begin{aligned} g(H(T), y) &:= U_1(X_1(T)) - yH(T)X_1(T) - [U_2(X_2(T)) - yH(T)X_2(T)] \\ &= \left\{ U_1 \left[(U'_1)^{-1}(yH(T))\mathbf{I}_{yH(T) \leq (U'_1)_+(0)} \right] - yH(T) \right. \\ &\quad \left. \left[(U'_1)^{-1}(yH(T))\mathbf{I}_{yH(T) \leq (U'_1)_+(0)} \right] \right\} - [U_2(L) - yH(T)L] \\ &= \left\{ U_1 \left[(U'_1)^{-1}(yH(T)) \right] - yH(T)(U'_1)^{-1}(yH(T)) + yH(T)L \right\} \\ &\quad \mathbf{I}_{yH(T) \leq (U'_1)_+(0)} - U_2(L) + [U_1(0) + yH(T)L]\mathbf{I}_{yH(T) > (U'_1)_+(0)}. \end{aligned}$$

Define

$$g_1(x) := U_1 \left[(U'_1)^{-1}(yx) \right] - yx(U'_1)^{-1}(yx) + yxL$$

and on $\{yx \leq (U'_1)_+(0)\}$ we obtain that

$$\begin{aligned} g'_1(x) &= yx \frac{y}{(U'_1)''(yx)} - y(U'_1)^{-1}(yx) - yx \frac{y}{(U'_1)''(yx)} + yL \\ &= -y(U'_1)^{-1}(yx) + yL \\ &< 0, \end{aligned}$$

meanwhile, $\psi(x) := U_1(0) + yxL$ is decreasing with respect to x . Therefore, for a fixed y , $g(H(T), y)$ is decreasing with respect to $H(T)$.

Similarly, for a fixed $H(T)$, $g(H(T), y)$ is also decreasing with respect to y . Thus, for a fixed y , there exists a $H(T)$, denoted by $H_T(y)$, such that when $H(T) \leq H_T(y)$,

$$U_1(X_1(T)) - yH(T)X_1(T) \geq U_2(X_2(T)) - yH(T)X_2(T),$$

and when $H(T) > H_T(y)$,

$$U_1(X_1(T)) - yH(T)X_1(T) < U_2(X_2(T)) - yH(T)X_2(T).$$

Thus, the candidate optimal terminal wealth is given by

$$X^*(T) = (U'_1)^{-1}(y^*H(T))\mathbf{I}_{y^*H(T) \leq (U'_1)_+(0)} + L\mathbf{I}_{H(T) > H_T(y^*)},$$

where y^* is the unique solution of

$$\mathbb{E}[H(T)X^*(T)] = x_0.$$

Similar to Proposition 3, the optimality of $X^*(T)$ and uniqueness of y^* can be obtained.

(2) Compare

$$X_1(T) := (U'_1)^{-1}(yH(T))\mathbf{I}_{yH(T) \leq (U'_1)_+(0)}$$

with

$$X_2(T) := (U'_2)^{-1}(yH(T))\mathbf{I}_{(U'_2)_-(0) \leq yH(T) \leq (U'_2)_+(L)} + L\mathbf{I}_{yH(T) > (U'_2)_+(L)}$$

and observe that

$$\begin{aligned}
h(H(T), y) &:= U_1(X_1(T)) - yH(T)X_1(T) - [U_2(X_2(T)) - yH(T)X_2(T)] \\
&= U_1 \left[(U'_1)^{-1}(yH(T)) \mathbf{I}_{yH(T) \leq (U'_1)_+(0)} \right] - yH(T)(U'_1)^{-1}(yH(T)) \\
&\quad \mathbf{I}_{yH(T) \leq (U'_1)_+(0)} - U_2 \left[(U'_2)^{-1}(yH(T)) \mathbf{I}_{(U'_2)_-(0) \leq yH(T) \leq (U'_2)_+(L)} \right. \\
&\quad \left. + L \mathbf{I}_{yH(T) > (U'_2)_+(L)} \right] + yH(T) \left[(U'_2)^{-1}(yH(T)) \right. \\
&\quad \left. \mathbf{I}_{(U'_2)_-(0) \leq yH(T) \leq (U'_2)_+(L)} + L \mathbf{I}_{yH(T) > (U'_2)_+(L)} \right] \\
&= \left\{ U_1 \left[(U'_1)^{-1}(yH(T)) \right] - yH(T)(U'_1)^{-1}(y^*H(T)) \right\} \mathbf{I}_{yH(T) \leq (U'_1)_+(0)} \\
&\quad - \left\{ U_2 \left[(U'_2)^{-1}(yH(T)) \right] - yH(T)(U'_2)^{-1}(yH(T)) \right\} \\
&\quad \mathbf{I}_{(U'_2)_-(0) \leq yH(T) \leq (U'_2)_+(L)} - [U_2(L) - yH(T)L] \mathbf{I}_{yH(T) > (U'_2)_+(L)} \\
&\quad - U_2(0) \mathbf{I}_{yH(T) < (U'_2)_-(0)} + U_1(0) \mathbf{I}_{yH(T) > (U'_1)_+(0)}.
\end{aligned}$$

Define

$$\nu_i(x) := U_i \left[(U'_i)^{-1}(yx) \right] - yx(U'_i)^{-1}(yx),$$

and observe that

$$\nu'_i(x) = -y(U'_i)^{-1}(yx).$$

Thus

$$\begin{aligned}
\nu'_1(H(T)) &< 0, \quad \text{on } \{yH(T) \leq (U'_1)_+(0)\}, \\
\nu'_2(H(T)) &> 0, \quad \text{on } \{(U'_2)_-(0) \leq yH(T) \leq (U'_2)_+(L)\},
\end{aligned}$$

and $h(H(T), y)$ is decreasing with respect to $H(T)$.

Similarly, for a fixed $H(T)$, $h(H(T), y)$ is also decreasing with respect to y .

①. Assume $(U'_2)_-(0) \geq (U'_1)_+(0)$ and let $H_T(y)$ be any value on $\left[\frac{(U'_1)_+(0)}{y}, \frac{(U'_2)_-(0)}{y} \right]$.

Thus

$$h(H_T(y), y) = 0,$$

and the candidate optimal terminal wealth is given by

$$\begin{aligned}
X^*(T) &= (U'_1)^{-1}(y^*H(T)) \mathbf{I}_{y^*H(T) \leq (U'_1)_+(0)} \\
&\quad + (U'_2)^{-1}(y^*H(T)) \mathbf{I}_{(U'_2)_-(0) \leq y^*H(T) \leq (U'_2)_+(L)} + L \mathbf{I}_{y^*H(T) > (U'_2)_+(L)}.
\end{aligned}$$

Consider a special case $U_1(\cdot) = U_2(\cdot) = U(\cdot)$, we obtain the candidate optimal terminal wealth

$$X^*(T) = (U')^{-1}(y^*H(T)) \mathbf{I}_{y^*H(T) \leq (U'_2)_+(L)} + L \mathbf{I}_{y^*H(T) > (U'_2)_+(L)},$$

where y^* is the unique solution of

$$\mathbb{E}[H(T)X^*(T)] = x_0.$$

②. Assume $(U'_2)_-(0) < (U'_1)_+(0)$. There exists a unique $H_T(y)$ on $\left[\frac{(U'_2)_-(0)}{y}, \frac{(U'_1)_+(0)}{y} \right]$ such that

$$h[H_T(y), y] = 0.$$

The candidate optimal terminal wealth is given by

$$\begin{aligned} X^*(T) = & (U'_1)^{-1}(y^* H(T)) \mathbf{I}_{H(T) \leq H_T(y^*)} + L \mathbf{I}_{y^* H(T) > (U'_2)_+(L)} \mathbf{I}_{H(T) > H_T(y^*)} \\ & + (U'_2)^{-1}(y^* H(T)) \mathbf{I}_{(U'_2)_-(0) \leq y^* H(T) \leq (U'_2)_+(L)} \mathbf{I}_{H(T) > H_T(y^*)}, \end{aligned}$$

where y^* is the unique solution of

$$\mathbb{E}[H(T)X^*(T)] = x_0.$$

Similar to Proposition 3, the optimality of $X^*(T)$ and uniqueness of y^* can be obtained. \square

We derive the optimal proportional reinsurance and investment strategy for an insurer under two-piece utility (1.2).

Theorem 4.2. *For an insurer with two-piece utility (1.2), the optimal proportional reinsurance and investment strategy u is given by*

$$\begin{cases} p(t) = -\frac{\partial C(t, H(t))}{\partial H(t)} \cdot \frac{H(t)\theta\mu_1}{\mu_2}, \\ \pi(t) = \frac{\partial C(t, H(t))}{\partial H(t)} \cdot \frac{H(t)(r(t) - b(t))}{\sigma^2(t)}. \end{cases}$$

(1) Suppose $U_2(\cdot)$ is a convex function and $L \in (-\infty, 0)$. $C(t, H(t))$ is given by

$$\begin{aligned} C(t, H(t)) = & \mathbb{E}_t \left[\rho_t (U'_1)^{-1}(y^* H(t) \rho_t) \mathbf{I}_{y^* H(t) \rho_t \leq (U'_1)_+(0)} \mathbf{I}_{H(t) \rho_t \leq H_T(y^*)} \right. \\ & \left. + \rho_t L \mathbf{I}_{H(t) \rho_t > H_T(y^*)} \right]. \end{aligned}$$

(2) Suppose $U_2(\cdot)$ is a concave function, $L \in (-\infty, 0)$ and $(U'_2)(0-) \geq (U'_1)(0+)$. $C(t, H(t))$ is given by

$$\begin{aligned} C(t, H(t)) = & \mathbb{E}_t \left[\rho_t (U'_1)^{-1}(y^* H(t) \rho_t) \mathbf{I}_{y^* H(t) \rho_t \leq (U'_1)_+(0)} + \rho_t (U'_2)^{-1}(y^* H(t) \rho_t) \right. \\ & \left. \mathbf{I}_{(U'_2)_-(0) \leq y^* H(t) \rho_t \leq (U'_2)_+(L)} + \rho_t L \mathbf{I}_{y^* H(t) \rho_t > (U'_2)_+(L)} \right]. \end{aligned}$$

(3) Suppose $U_2(\cdot)$ is a concave function, $L \in (-\infty, 0)$ and $(U'_2)(0-) < (U'_1)(0+)$. $C(t, H(t))$ is given by

$$\begin{aligned} & C(t, H(t)) \\ = & \mathbb{E}_t \left[\rho_t (U'_1)^{-1}(y^* H(T)) \mathbf{I}_{H(T) \leq H_T(y^*)} + \rho_t L \mathbf{I}_{y^* H(T) > (U'_2)_+(L)} \mathbf{I}_{H(T) > H_T(y^*)} \right. \\ & \left. + \rho_t (U'_2)^{-1}(y^* H(T)) \mathbf{I}_{(U'_2)_-(0) \leq y^* H(T) \leq (U'_2)_+(L)} \mathbf{I}_{H(T) > H_T(y^*)} \right]. \end{aligned}$$

Here $\rho_t := \frac{H(T)}{H(t)}$; $y^*, H_T(y^*)$ is given by theorem 4.1 and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ represents the conditional expectation with respect to $\mathcal{F}(t)$.

Proof. (1). From Proposition 1, $H(t)X(t)$ is a martingale under \mathbf{P} . Therefore

$$X(t) = \frac{1}{H(t)} \mathbb{E}_t \left\{ H(T) \left[(U'_1)^{-1}(y^* H(T)) \mathbf{I}_{y^* H(T) \leq (U'_1)_+(0)} \mathbf{I}_{H(T) \leq H_T(y^*)} + L \mathbf{I}_{H(T) > H_T(y^*)} \right] \right\}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ represents the conditional expectation with respect to $\mathcal{F}(t)$ and $y^*, H_T(y^*)$ is determined by (1.12).

Due to the form of $H(T)$ and Markov structure of the processes, $H(t)$ is independent of $\frac{H(T)}{H(t)}$. We denote $\frac{H(T)}{H(t)}$ by ρ_t , which takes the following form

$$\rho_t = \exp \left\{ \int_t^T - \left(\frac{\theta^2 \mu_1^2 \lambda(s) \sigma^2(s) + \mu_2 (r(s) - b(s))^2 + 2r(s) \mu_2 \sigma^2(s)}{2\mu_2 \sigma^2(s)} \right) ds - \int_t^T \theta \mu_1 \sqrt{\frac{\lambda(s)}{\mu_2}} dW_0(s) + \int_t^T \left(\frac{r(s) - b(s)}{\sigma(s)} \right) dW(s) \right\}$$

and follows a log-normal distribution. Thus

$$X(t) = \mathbb{E}_t \left[\rho_t (U'_1)^{-1}(y^* H(t) \rho_t) \mathbf{I}_{y^* H(t) \rho_t \leq (U'_1)_+(0)} \mathbf{I}_{H(t) \rho_t \leq H_T(y^*)} + \rho_t L \mathbf{I}_{H(t) \rho_t > H_T(y^*)} \right].$$

$X(t)$ can be viewed as a function of t and $H(t)$: $C(t, H(t))$. Hence

$$\begin{aligned} dX(t) &= \left(\frac{\partial C(t, H(t))}{\partial t} \right) dt + \left(\frac{\partial C(t, H(t))}{\partial H(t)} \right) dH(t) + \frac{1}{2} \left(\frac{\partial^2 C(t, H(t))}{\partial H^2(t)} \right) (dH(t))^2 \\ &= \left\{ -r(t)H(t) \frac{\partial C(t, H(t))}{\partial H(t)} + \frac{\partial^2 C(t, H(t))}{2\partial H^2(t)} \left[\frac{H^2(t) \theta^2 \mu_1^2 \lambda(t)}{\mu_2} + \frac{H^2(t)(r(t) - b(t))^2}{\sigma^2(t)} \right] + \frac{\partial C(t, H(t))}{\partial t} \right\} dt - \frac{\partial C(t, H(t))}{\partial H(t)} H(t) \\ &\quad \theta \mu_1 \sqrt{\frac{\lambda(t)}{\mu_2}} dW_0(t) + \frac{\partial C(t, H(t))}{\partial H(t)} H(t) \frac{r(t) - b(t)}{\sigma(t)} dW(t). \end{aligned} \tag{1.13}$$

Compare (1.1) with (1.13), we obtain that

$$\begin{cases} -\frac{\partial C(t, H(t))}{\partial H(t)} H(t) \theta \mu_1 \sqrt{\frac{\lambda(t)}{\mu_2}} = \sqrt{\mu_2 \lambda(t)} p(t), \\ \frac{\partial C(t, H(t))}{\partial H(t)} H(t) \frac{r(t) - b(t)}{\sigma(t)} = \pi(t) \sigma(t). \end{cases}$$

Therefore the optimal reinsurance-investment strategy is given by

$$\begin{cases} p(t) = -\frac{\partial C(t, H(t))}{\partial H(t)} \cdot \frac{H(t) \theta \mu_1}{\mu_2}, \\ \pi(t) = \frac{\partial C(t, H(t))}{\partial H(t)} \cdot \frac{H(t)(r(t) - b(t))}{\sigma^2(t)}. \end{cases}$$

The proof of (2), (3) is similar to that of (1). \square

It is easy to see that, for all cases in Theorem 4.1, $C(t, H(t))$ is always non-increasing with respect to $H(t)$. Thus we have that for an insurer whose wealth process evolves according to (1.1), with any two-piece utility function defined by (1.2) and under the expected value premium principle, the optimal proportional reinsurance strategy $p(t)$ is non-negative.

5. Some examples. In this section, we consider some common utility functions.

Example 5.1. A particular case of utility (1.2) is the two-piece power utility function, see Tversky and Kehneman (1992), which is represented by:

$$U(x) = \begin{cases} \alpha x^{\gamma_1}, & x > k, \\ -\beta(-x)^{\gamma_2}, & x \leq k, \end{cases}$$

where $0 < \gamma_1, \gamma_2 \leq 1$ are curvature parameters, $\alpha, \beta > 0$ and β stands for the loss aversion coefficients of the investor. This utility is widely used models that consider the loss aversion, for example, in Berkelaar (2004) and Guo (2014).

Let L be the lower bound of $X(T)$, satisfying $L < k$ and $\mathbb{E}[H(T)L] \leq x_0$, where x_0 represents the initial wealth. The utility function of the insurer is given by

$$\tilde{U}(X(T)) := U_1(X(T) - k)\mathbf{I}_{X(T) > k} + U_2(X(T) - k)\mathbf{I}_{X(T) \leq k},$$

where $U_1(x) := \alpha x^{\gamma_1}$ and $U_2(x) := -\beta(-x)^{\gamma_2}$. The maximization problem in this case can be reduced to

$$\begin{cases} \max & \mathbb{E}[\tilde{U}(\tilde{X}(T))] \\ \text{s.t.} & \mathbb{E}[H(T)\tilde{X}(T)] = \tilde{x}_0, \quad \tilde{X}(T) \geq \tilde{L}, \end{cases}$$

where

$$\begin{aligned} \tilde{X}(T) &:= X(T) - k, \\ \tilde{x}_0 &:= x_0 - k\mathbb{E}[H(T)], \\ \tilde{L} &:= L - k. \end{aligned}$$

The optimal terminal wealth is given by

$$\begin{aligned} \tilde{X}^*(T) &= \left(\frac{\alpha\gamma_1}{y^*H(T)} \right)^{\frac{1}{1-\gamma_1}} \mathbf{I}_{H(T) \leq H_T(y^*)} + \tilde{L} \mathbf{I}_{H(T) > H_T(y^*)}, \\ X^*(T) &= \left[\left(\frac{\alpha\gamma_1}{y^*H(T)} \right)^{\frac{1}{1-\gamma_1}} + k \right] \mathbf{I}_{H(T) \leq H_T(y^*)} + L \mathbf{I}_{H(T) > H_T(y^*)}, \end{aligned}$$

where y^* , $H_T(y^*)$ is the unique solution of

$$\begin{cases} \mathbb{E} \left\{ H(T) \left[\left(\frac{\alpha\gamma_1}{y^*H(T)} \right)^{\frac{1}{1-\gamma_1}} + k \right] \mathbf{I}_{H(T) \leq H_T(y^*)} + H(T)L \mathbf{I}_{H(T) > H_T(y^*)} \right\} = x_0, \\ \alpha \left(\frac{\alpha\gamma_1}{y^*H_T(y^*)} \right)^{\frac{\gamma_1}{1-\gamma_1}} - y^*H_T(y^*) \left(\frac{\alpha\gamma_1}{y^*H_T(y^*)} \right)^{\frac{1}{1-\gamma_1}} + y^*H_T(y^*)\tilde{L} + \beta(-\tilde{L})^{\gamma_2} = 0. \end{cases}$$

The optimal wealth process is given by

$$\begin{aligned} X(t) &= C(t, H(t)) \\ &= \mathbb{E}_t \left\{ \rho_t \left[\left(\frac{\alpha\gamma_1}{y^*H(t)\rho_t} \right)^{\frac{1}{1-\gamma_1}} + k \right] \mathbf{I}_{H(t)\rho_t \leq H_T(y^*)} + \rho_t L \mathbf{I}_{H(t)\rho_t > H_T(y^*)} \right\}, \end{aligned}$$

where $\rho(t) := \frac{H(T)}{H(t)}$. Thus

$$\frac{\partial C(t, H(t))}{\partial H(t)} = \frac{-1}{1 - \gamma_1} (H(t))^{\frac{\gamma_1 - 2}{1 - \gamma_1}} \cdot \mathbb{E}_t \left[\rho_t \left(\frac{\alpha \gamma_1}{y^* \rho_t} \right)^{\frac{1}{1 - \gamma_1}} \mathbf{I}_{H(t)\rho_t \leq H_T(y^*)} \right].$$

The optimal reinsurance-investment strategy is given by

$$\begin{cases} p(t) = \frac{1}{1 - \gamma_1} (H(t))^{\frac{-1}{1 - \gamma_1}} \cdot \mathbb{E}_t \left[\rho_t \left(\frac{\alpha \gamma_1}{y^* \rho_t} \right)^{\frac{1}{1 - \gamma_1}} \mathbf{I}_{H(t)\rho_t \leq H_T(y^*)} \right] \cdot \frac{\theta \mu_1}{\mu_2}, \\ \pi(t) = \frac{-1}{1 - \gamma_1} (H(t))^{\frac{-1}{1 - \gamma_1}} \cdot \mathbb{E}_t \left[\rho_t \left(\frac{\alpha \gamma_1}{y^* \rho_t} \right)^{\frac{1}{1 - \gamma_1}} \mathbf{I}_{H(t)\rho_t \leq H_T(y^*)} \right] \cdot \frac{r(t) - b(t)}{\sigma^2(t)}. \end{cases}$$

Example 5.2. Consider a logarithmic utility

$$U(x) = \begin{cases} c + \gamma \ln(x + \delta), & x + \delta > 0, \\ -\infty, & x + \delta \leq 0, \end{cases}$$

where $\gamma > 0$. Let L be the lower bound of $X(T)$, satisfying $L \geq -\delta$ and $\mathbb{E}[H(T)L] \leq x_0$, where x_0 represents the initial wealth. The maximization problem in this case can be reduced to

$$\begin{cases} \max & \mathbb{E}[\tilde{U}(\tilde{X}(T))] \\ \text{s.t.} & \mathbb{E}[H(T)\tilde{X}(T)] = \tilde{x}_0, \quad \tilde{X}(T) \geq \tilde{L}, \end{cases}$$

where

$$\begin{aligned} \tilde{U}(x) &:= U_1(x) = c + \gamma \ln x, \\ \tilde{X}(T) &:= X(T) + \delta, \\ \tilde{x}_0 &:= x_0 + \delta \mathbb{E}[H(T)], \\ \tilde{L} &:= L + \delta. \end{aligned}$$

The optimal terminal wealth should be

$$\begin{aligned} \tilde{X}^*(T) &= \frac{\gamma}{y^* H(T)} \mathbf{I}_{y^* H(T) \leq (U'_1)_+(\tilde{L})} + \tilde{L} \mathbf{I}_{y^* H(T) > (U'_1)_+(\tilde{L})}, \\ X^*(T) &= \left[\frac{\gamma}{y^* H(T)} - \delta \right] \mathbf{I}_{y^* H(T) \leq (U'_1)_+(\tilde{L})} + L \mathbf{I}_{y^* H(T) > (U'_1)_+(\tilde{L})}, \end{aligned}$$

where y^* is the unique solution of

$$\mathbb{E} \left[\left(\frac{\gamma}{y^*} - H(T)\delta \right) \mathbf{I}_{y^* H(T) \leq (U'_1)_+(\tilde{L})} + H(T)L \mathbf{I}_{y^* H(T) > (U'_1)_+(\tilde{L})} \right] = x_0.$$

The optimal wealth process is given by

$$\begin{aligned} X(t) &= C(t, H(t)) \\ &= \frac{1}{H(t)} \mathbb{E}_t \left[\left(\frac{\gamma}{y^*} - H(T)\delta \right) \mathbf{I}_{y^* H(T) \leq (U'_1)_+(\tilde{L})} + H(T)L \mathbf{I}_{y^* H(T) > (U'_1)_+(\tilde{L})} \right], \end{aligned}$$

where $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ represents the conditional expectation with respect to $\mathcal{F}(t)$.

Thus

$$\frac{\partial C(t, H(t))}{\partial H(t)} = -\frac{1}{H^2(t)} \mathbb{E}_t \left[\frac{\gamma}{y^*} \mathbf{I}_{y^* H(t)\rho(t) \leq (U'_1)_+(\tilde{L})} \right],$$

where $\rho(t) := \frac{H(T)}{H(t)}$. The optimal reinsurance-investment strategy is given by

$$\begin{cases} p(t) = \frac{1}{H(t)} \mathbb{E}_t \left[\frac{\gamma}{y^*} \mathbf{I}_{y^* H(t) \rho(t) \leq (U'_1)_+(\bar{L})} \right] \frac{\theta \mu_1}{\mu_2}, \\ \pi(t) = -\frac{1}{H(t)} \mathbb{E}_t \left[\frac{\gamma}{y^*} \mathbf{I}_{y^* H(t) \rho(t) \leq (U'_1)_+(\bar{L})} \right] \frac{r(t) - b(t)}{\sigma^2(t)}. \end{cases}$$

Example 5.3. Consider an exponential utility

$$U(x) = c - \frac{\delta}{\gamma} e^{-\gamma x},$$

where $\delta > 0, \gamma > 0$. Let L be the lower bound of $X(T)$, satisfying $\mathbb{E}[H(T)L] \leq x_0$, where x_0 represents the initial wealth. The case can be viewed as

$$U(x) = U_1(x) = U_2(x).$$

The maximization problem in this case can be reduced to

$$\begin{cases} \max & \mathbb{E}[U(X(T))] \\ \text{s.t.} & \mathbb{E}[H(T)X(T)] = x_0, \quad X(T) \geq L. \end{cases}$$

The optimal terminal wealth should be

$$X(T) = -\frac{1}{\gamma} \ln \left(\frac{y^* H(T)}{\delta} \right) \mathbf{I}_{y^* H(T) \leq (U'_1)_+(L)} + L \mathbf{I}_{y^* H(T) > (U'_1)_+(L)},$$

where y^* is the unique solution of

$$\mathbb{E} \left[-\frac{H(T)}{\gamma} \ln \left(\frac{y^* H(T)}{\delta} \right) \mathbf{I}_{y^* H(T) \leq (U'_1)_+(L)} + H(T)L \mathbf{I}_{y^* H(T) > (U'_1)_+(L)} \right] = x_0.$$

The optimal wealth process is given by

$$\begin{aligned} X(t) &= C(t, H(t)) \\ &= \frac{1}{H(t)} \mathbb{E}_t \left[-\frac{H(T)}{\gamma} \ln \left(\frac{y^* H(T)}{\delta} \right) \mathbf{I}_{y^* H(T) \leq (U'_1)_+(L)} + H(T)L \mathbf{I}_{y^* H(T) > (U'_1)_+(L)} \right] \\ &= \mathbb{E}_t \left\{ -\frac{\rho(t)}{\gamma} \left[\ln(H(t)) + \ln \left(\frac{y^* \rho(t)}{\delta} \right) \right] \mathbf{I}_{y^* H(T) \leq (U'_1)_+(L)} \right. \\ &\quad \left. + \rho(t)L \mathbf{I}_{y^* H(T) > (U'_1)_+(L)} \right\}, \end{aligned}$$

where $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ represents the conditional expectation with respect to $\mathcal{F}(t)$. Thus

$$\frac{\partial C(t, H(t))}{\partial H(t)} = -\frac{1}{\gamma H(t)} \mathbb{E}_t \left[\rho(t) \mathbf{I}_{y^* H(t) \rho(t) \leq (U'_1)_+(L)} \right],$$

where $\rho(t) := \frac{H(T)}{H(t)}$. The optimal reinsurance-investment strategy is given by

$$\begin{cases} p(t) = \frac{\theta \mu_1}{\gamma \mu_2} \mathbb{E} \left[\rho(t) \mathbf{I}_{y^* H(t) \rho(t) \leq (U'_1)_+(L)} \right], \\ \pi(t) = -\frac{r(t) - b(t)}{\gamma \sigma^2(t)} \mathbb{E} \left[\rho(t) \mathbf{I}_{y^* H(t) \rho(t) \leq (U'_1)_+(L)} \right]. \end{cases}$$

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