Estimation and LQG control over unreliable network with acknowledgment randomly lost

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***Abstract*—In this paper, we study the state estimation and optimal control (i.e., linear quadratic Gaussian (LQG) control) problems for networked control systems in which control inputs, observations, and packet acknowledgments are randomly lost. The packet acknowledgment is a signal that is sent from the actuator to inform the estimator whether the actuator has successfully received the control packets or not. For such systems, we obtain the optimal estimator, which is consisted of exponentially increasing terms. For the solvability of the optimal LQG problem, we come to a conclusion that even when the optimal LQG control exists, it is impossible and unnecessary to be obtained as its calculation is not only technically difficult but also computationally prohibitive. This issue motivates us to design a sub-optimal LQG controller for the underlying systems. We first develop a sub-optimal estimator by using the estimator gain in each term of the optimal estimator. Then we derive the sub-optimal LQG controller and establish the conditions for stability of the closed-loop systems. Examples are given to illustrate the effectiveness and advantages of the proposed design scheme.**

***Index Terms*—networked control systems, optimal estimation and control, LQG, packet loss, Quasi-TCP-like system**

I. Introduction

Recently, increasing attention has been paid on the systems with their components (e.g., sensors, controllers, and actuators) connected via network, namely networked control systems (NCSs)[1]. The insertion of network brings numerous benifits, such as reduced system wiring, lower cost in maintenance, increased system agility, ease of information sharing, etc. However, it also causes some network-induced constraints, e.g., channel congestion, transmission delay, signal degradation, which may result in packet losses in data transmission [2, 3]. To deal with this challenging issue, a number of techniques have been developed, such as the H_∞ filter [4–8], the robust control [9–12], the predictive control [13, 14], and the fuzzy model-based approach [15, 16].

For NCSs with packet losses, two transport protocols are commonly deployed, i.e., the user datagram protocol (UDP) and transmission control protocol (TCP). NCSs with different protocol lead to distinct results in estimation and control [17]. The key difference between these two protocols lies in the existence of acknowledgment mechanism. In NCS with TCP protocol, there are acknowledgments (ACKs) sent from the actuator to inform the estimator whether the actuator has successfully received the control packets or not. Such system is usually called as the TCP-like system. The NCS with UDP protocol, that is, in which there is no ACK available for the estimator, is called as the UDP-like system. For the UDP-like system, the implementation of transport protocol is simplified and the additional energy consumption for the ACK signal transmission is avoided. However, the lack of ACKs not only makes difficult the analysis of the UDP-like system, but also degrades the performances of estimation and control [18, 19]. For the TCP-like system, the ACK mechanism facilitates theoretical analysis, but it is reported that for NCSs over unreliable network, subject to network jitter and transmission delay, it is impossible to send the ACK in time (without delay and loss) to implement the TCP scheme [20–24]. Therefore, the NCS with ACK randomly lost turns out to be a reasonable and practical model for many applications, and such system is called as the Quasi-TCP-like system [23, 24]. In this paper, we are concerned with the optimal and sub-optimal solutions to the estimation and linear quadratic Gaussian (LQG) control problems for the Quasi-TCP-like systems.

For the TCP-like system, its optimal estimator has been early known as the time-varying Kalman filter, and its stability depends on a critical value for the observation packet loss rate [25]. The critical value together with its upper and lower bounds has been further studied in [26, 27]. For the LQG problem, it has been pointed out in [17, 28] that "the separation principle holds, and the optimal LQG controller is a linear function of the estimated state." Thereafter, the LQG problem for the TCP-like systems has been extensively studied for various cases [29–31]. While, these results generally do not hold for the Quasi-TCP- or UDP-like systems.

For the UDP-like system, the optimal estimator and its stability were studied in our recent work [18] for a special case: the UDP-like system without observation packet loss. The structure of the optimal estimator is complex and its computation is time-consuming. Possibly due to the complexity of the optimal estimator, to our best knowledge, the optimal LQG problem is rarely studied. Instead of the optimal estimator, the linear minimum mean square error (LMMSE) estimator was used in studying the LQG problem (the LMMSE-estimator-based LQG problem). While it is still difficult to solve this problem except for some special cases [17]. Another sub-

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optimal linear LQG controllers were designed in [19, 21, 22]. However, in [21, 22] the finite horizon LQG problem is not studied, and the estimates used are the time-update prediction not the measurement-update estimation, which degrades the estimation and control performances. The controller propsed in [19] can stabilize the closed-loop system, only when the system is scalar and there is no observation packet loss.

For the Quasi-TCP-like system, an efficient sub-optimal estimator was developed in [32] but the optimal one is little studied. Thus, the LMMSE estimator was again used to study the LQG problem in [30], and the authors came to a conclusion that "in general, the separation principle does not hold, and the computation of the LQG controller requires solving a nonlinear optimization problem, and the resulting controller is a nonlinear function in the estimated state. If the observation equation is noise free and the matrix C is square and invertible, then the optimal control is a linear function of the estimates." The main reason is that when the system state is estimated by the measurement-update estimator, like the LMMSE estimator, the estimation error covariance (ECC) will be a nonlinear function of the control u_k, making it difficult to solve the LQG problem. Although the time-update predictor not the measurement-update estimator was used in [24] to avoid such difficulty, the resulting structure of the ECC is still complicated. Thus, the author had to introduce some approaximations in the derivation to obtain a sub-optimal LQG controller.

As far as we know, the optimal estimator for the Quasi-TCP-like system has not yet been obtained. Therefore, what is the structure of it? Whether does there exist a solution to the optimal LQG problem, and is the solution solvable? We also wonder that whether or not there is a sub-optimal measurement-update estimator, based on which the LQG problem can be solved and the resulting LQG controller can be presented in the traditional and familar way like the classic LQG controller. These questions, to our best knowledge, remain unsolved. Hence, motivated by these issues, we study the optimal as well as sub-optimal solutions to the estimation and LQG problems for the Quasi-TCP-like system. Our main results and contributions are summarized as follows:

- *Optimal estimator:* We derive the optimal estimator for the Quasi-TCP-like system, and show that its ECC can be decoupled into two parts: one is the EEC for the TCP-like system and the other is a summation consisting of exponentially increasing terms.
- *Solvability of the optimal LQG problem:* Up to now, it is little known about the solution to the optimal LQG problem. We make a conclusion on its solvability that in general, even when the optimal LQG control exists, it is impossible and unnecessary to obtain the optimal solutions for both the estimation and LQG control, as their calculations are not only technically difficult but also computationally prohibitive. We show that the difficulties in solving the optimal LQG problem differ from the well known ones to obtain the LMMSE-estimator-based LQG control.
- *Sub-optimal LQG controller:* Although the optimal estimator cannot be used in practice, in its complex structure we find a component and then use it to design the desired sub-optimal estimator. Based on it, the LQG controller is obtained, which can be presented in the form similar to the classic LQG controller. Finally, conditions for the stability of the closed-loop systems are given.

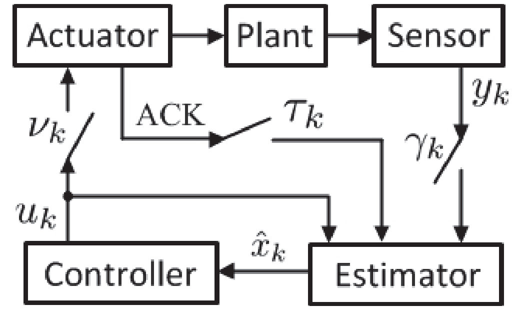

Fig. 1. The Quasi-TCP-like systems.

The rest of the paper is organized as follows: In Section II, the system setup and problems are formulated. In Section III, the optimal estimator and the LQG problems are studied. Sub-optimal estimator and LQG controller are developed in Section IV. In Section V, numerical examples are given to illustrate the effectiveness of the proposed design methods. The conclusions are presented in Section VI.

Notations:

- $\mathcal{N}_x(\mu, P)$ denotes the Gaussian pdf of the random variable x with mean μ and covariance P.
- $x \sim \mathcal{N}_x(\mu, P)$ means that the pdf of the random variable x is $\mathcal{N}_x(\mu, P)$.
- $\mathbb{P}(\cdot)$ denotes probability measure.
- $p(\cdot)$ and $p(\cdot|\cdot)$ denote the pdf and the conditional pdf, respectively.
- $\mathbb{E}[\cdot]$ denotes probability expectation.
- $(\cdot)'$ denotes the transpose of a matrix or a vector.
- $\mathrm{tr}(\cdot)$ denotes the trace of matrix.
- Let M be a matrix. $(\cdot)^2_M$ denotes the quadratic form of $(\cdot)M(\cdot)'$. $(\cdot)^2_I$ with the identity matrix I means $(\cdot)(\cdot)'$.
- $\lambda(M), \lambda_M$: $\lambda(M)$ denotes the spectral radius of M, and $\lambda_M \triangleq \lambda(M)$.

II. System setup and problem formulation

A. System setup

Consider the following discrete-time Quasi-TCP-like linear system:

$$x_{k+1} = Ax_k + \nu_k B u_k + \omega_k \tag{1a}$$

$$y_k = \begin{cases} Cx_k + \upsilon_k, & \text{for } \gamma_k = 1 \\ \phi, & \text{for } \gamma_k = 0 \end{cases} \tag{1b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^q$, and $y_k \in \mathbb{R}^p$ are system state, control input, and observation, respectively. ϕ denotes empty set. ω_k and υ_k are zero mean Gaussian noises with covariances $Q \geq 0$ and $R > 0$, respectively. ν_k, γ_k, and τ_k (see, Fig. 1) are i.i.d. Bernoulli random sequences with $\mathbb{P}(\nu_k = 1) = \nu$, $\mathbb{P}(\gamma_k = 1) = \gamma$, and $\mathbb{P}(\tau_k = 1) = \tau$. They describe the packet losses in the controller-to-actuator (C/A) channel, the sensor-to-estimator (S/E) channel, and the ACK channel, respectively. That is,

- $\gamma_k = 1$ stands for that the observation y_k has been successfully received by estimator, otherwise $\gamma_k = 0$.
- $\nu_k = 1$ indicates that the control input u_k has been successfully delivered to the actuator, otherwise $\nu_k = 0$.
- $\tau_k = 1$ means that the ACK signal, i.e., the value of ν_k, has been successfully delivered to the estimator; Otherwise, $\tau_k = 0$ and ν_k is unavailable for the estimator.

For the system described in (1), some standard assumptions are made as follows.

Assumption 1: The pair $(A, Q^{1/2})$ is controllable, and the pair (A, C) is observable. The initial state $x_0 \sim \mathcal{N}_{x_0}(\bar{x}_0, P_0)$, and x_0, ω_k, υ_k, ν_k, γ_k, and τ_k are mutually independent.

B. Problems formulation

Define the information set $\mathcal{I}_k \triangleq \{\mathrm{y}^k, \gamma^k, \tau^k\}$ with $\mathcal{G}_0 \triangleq \phi$ (empty set), where $\mathrm{y}^k \triangleq \{y_k, \cdots, y_1\}$, $\gamma^k \triangleq \{\gamma_k, \cdots, \gamma_1\}$, and $\tau^k \triangleq \{\tau_k, \cdots, \tau_0\}$. In this paper, we study the following four problems.

Problem 1: (Optimal estimation) Determine the optimal state estimation, denoted by $\hat{x}_k$, in the minimum mean square error (MMSE) sense. That is, to find $\hat{x}_k$, minimizing $\mathbb{E}\big[\|x_k - \hat{x}_k|\mathcal{I}_k\|^2\big]$.

The optimal LQG problem is formulated as follows. Given a integer N, let $\{W_k\}$ and $\{\Lambda_k\}$ for $1 \leq k \leq N$ be positive definite matrices. Define the cost functions as follows:

$$J_N(\pi_{N-1}, \bar{x}_0, P_0) = \mathbb{E}\big[x_N' W_N x_N + \textstyle\sum_{k=0}^{N-1} x_k' W_k x_k + \nu_k u_k' \Lambda_k u_k | \pi_{N-1}, \bar{x}_0, P_0\big]$$

where $\pi_{N-1} = \{u_0, \ldots, u_{N-1}\}$ is a sequence of the control inputs, and u_k is a function of $\mathcal{I}_k$, i.e., $u_k = f_k(\mathcal{I}_k)$.

Assumption 2: The pair (A, B) is stabilizable, and the pair $(A, W^{1/2})$ is detectable.

Problem 2: (Optimal LQG control) Determine the optimal control sequence, denoted by π_{N-1}^*, that minimizes the cost function J_N, i.e.,

$$J_N^* = J_N(\pi_{N-1}^*, \bar{x}_0, P_0) = \min_{\pi_{N-1}} J_N(\pi_{N-1}, \bar{x}_0, P_0).$$

As we will see later, based on the optimal estimator, it is difficult, sometimes impossible, to solve the optimal LQG problem. Thus, we consider the following sub-optimal LQG problem.

Problem 3: (Sub-optimal LQG control) Whether does there exist a sub-optimal linear estimator, based on which the LQG controller can be obtained?

An important property for a contrller is that whether or not it can stablize the closed-loop system. Since the Quasi-TCP-like system is a stochastic system, the stability is considered in the mean sense as follows:

Definition 1: The closed-loop system is said to be mean square stable, if both $\mathbb{E}[\|x_k\|^2]$ and $\mathbb{E}[\|\hat{x}_k\|^2]$ are bounded.

Problem 4: (Mean square stability) Determine the condition under which the closed-loop system is mean square stable.

The answers to Problems 1 – 4 are presented in Theorems 1 – 4, respectively.

III. The optimal estimator and solvability of the optimal LQG controller

A. Optimal estimator

It is well known in [33] that the desired optimal estimation $\hat{x}_k$ is given by $\mathbb{E}[x_k|\mathcal{I}_k]$. Thus, we first derive the pdf of x_k in Lemma 3, and then compute the optimal estimation in Theorem 1. Let $\bar{x}_k \triangleq \mathbb{E}[x_k|\mathcal{I}_{k-1}]$ denote the state prediction, and let $\bar{P}_k$ and P_k denote the prediction and estimation error covariances, respectively.

It is shown later that the pdf of x_k is a Gaussian mixture. Two uesful results on Gaussian and Gaussian mixture pdfs are formulated in the following two lemmas.

Lemma 1: [34, pp. 44] Given two independent random variables $X \sim \mathcal{N}_X(m, P)$ and $Z \sim \mathcal{N}_Z(0, W)$. Let $Y = CX + Z$ where C is a constant matrix. Then

$$P_Y = CPC' + W \quad (2a)$$
$$p(Y) = \mathcal{N}_Y(Cm, CPC' + W) \quad (2b)$$
$$p(X|Y) = \mathcal{N}_X\big(m + K(y - Cm), (I - KC)P\big) \quad (2c)$$

where $K = PC'(CPC' + W)^{-1}$, and P_Y is the covariance of Y.

Lemma 2: [33, pp. 213] Consider the following discrete-time linear system:

$$x_k = Ax_{k-1} + B_{k-1}u_{k-1} + \omega_{k-1}$$
$$y_k = Cx_k + \upsilon_k$$

where the B_k is a time-varying *deterministic* parameter. $\omega_k \sim \mathcal{N}(0, Q)$ and $\upsilon_k \sim \mathcal{N}(0, R)$ are mutually independent. Then the following facts hold.

(i) If $p(x_{k-1}) = \sum_{i=1}^N \alpha_{k-1}^{[i]} \mathcal{N}_{x_{k-1}}(m_{k-1}^{[i]}, M_{k-1})$, then the time-update pdf

$$p(x_k) = \textstyle\sum_{i=1}^N \bar{\alpha}_k^{[i]} \mathcal{N}_{x_k}(\bar{m}_k^{[i]}, \bar{M}_k), \quad (3)$$

where $\bar{\alpha}_k^{[i]} = \alpha_{k-1}^{[i]}$, $\bar{m}_k^{[i]} = Am_{k-1}^{[i]} + B_{k-1}u_{k-1}$, and $\bar{M}_k = AM_{k-1}A' + Q$.

(ii) If $p(x_k)$ takes the form as in (3), then the measurement-update pdf

$$p(x_k|y_k) = \textstyle\sum_{i=1}^N \alpha_k^{[i]} \mathcal{N}_{x_k}(m_k^{[i]}, M_k), \quad (4)$$

where $m_k^{[i]} = \bar{m}_k^{[i]} + K_k(y_k - C\bar{m}_k^{[i]})$, $K_k = \bar{M}_k C'(C\bar{M}_k C' + R)^{-1}$, $M_k = (I - K_k C)\bar{M}_k$, $\alpha_k^{[i]} = \bar{\alpha}_k^{[i]} \phi_k^{[i]}/c$, $\phi_k^{[i]} = \mathcal{N}_{y_k}(C\bar{m}_k^{[i]}, P_k^Y)$, $P_k^Y = C\bar{M}_k C' + R$, and $c = \sum_{j=1}^N \bar{\alpha}_k^{[j]} \phi_k^{[j]}$.

(iii) Denote $\hat{x}_k = \mathbb{E}[x_k|y_k]$ and $P_k = \mathbb{E}[(x_k - \hat{x}_k)_I^2|y_k]$. If $p(x_k|y_k)$ takes the form as in (4), then

$$\hat{x}_k = \textstyle\sum_{i=1}^N \alpha_k^{[i]} m_k^{[i]}, P_k = B + \sum_{i=1}^N \alpha_k^{[i]} (m_k^{[i]} - \hat{x}_k)_I^2.$$

1) Probability density function of x_k: Denote by n_k the number of the lost ACK signals during time 0 to $k-1$.

Lemma 3: Let $N_k = 2^{n_k}$. For $1 \leq i \leq N_k$,

$$p(x_k|\mathcal{I}_{k-1}) = \textstyle\sum_{i=1}^{N_k} \bar{\alpha}_k^{[i]} \mathcal{N}_{x_k}(\bar{m}_k^{[i]}, \bar{M}_k) \quad (5a)$$
$$p(x_k|\mathcal{I}_k) = \textstyle\sum_{i=1}^{N_k} \alpha_k^{[i]} \mathcal{N}_{x_k}(m_k^{[i]}, M_k), \quad (5b)$$

Algorithm 1 Prediction-update step:

$$\bar{M}_k = AM_{k-1}A' + Q. \tag{6}$$

- If $\tau_{k-1} = 1$, then $N_k = N_{k-1}$ and

$$\bar{m}_k^{[i]} = Am_{k-1}^{[i]} + \nu_{k-1}Bu_{k-1}, 1 \leq i \leq N_k \tag{7a}$$
$$\bar{\alpha}_k^{[i]} = \alpha_{k-1}^{[i]}, 1 \leq i \leq N_k. \tag{7b}$$

- If $\tau_{k-1} = 0$, then $N_k = 2 * N_{k-1}$ and

$$\bar{m}_k^{[i]} = \begin{cases} Am_{k-1}^{[i]}, 1 \leq i \leq N_{k-1} \\ Am_{k-1}^{[i-N_{k-1}]} + Bu_{k-1}, N_{k-1}+1 \leq i \leq N_k \end{cases} \tag{8a}$$

$$\bar{\alpha}_k^{[i]} = \begin{cases} \bar{\nu}\alpha_{k-1}^{[i]}, & 1 \leq i \leq N_{k-1} \\ \nu\alpha_{k-1}^{[i-2^{k-1}]}, & N_{k-1}+1 \leq i \leq N_k. \end{cases} \tag{8b}$$

where $\bar{\nu} \triangleq 1 - \nu$.

Algorithm 2 Measurement-update step:

For $1 \leq i \leq N_k$,

$$m_k^{[i]} = \bar{m}_k^{[i]} + \gamma_k K_k (y_k - C\bar{m}_k^{[i]}) \tag{9}$$

where

$$K_k = \bar{M}_k C' \left(C\bar{M}_k C' + R\right)^{-1} \tag{10a}$$
$$M_k = \bar{M}_k - \gamma_k K_k C \bar{M}_k \tag{10b}$$

and

$$\alpha_k^{[i]} = \left(\frac{\phi_k^{[i]}}{\sum_{j=1}^{N_k} \phi_k^{[j]}\bar{\alpha}_k^{[j]}}\right)^{\gamma_k} \bar{\alpha}_k^{[i]} \tag{11}$$

where $\phi_k^{[i]} \triangleq \mathcal{N}_{y_k}(C\bar{m}_k^{[i]}, P_k^Y)$ and $P_k^Y \triangleq C\bar{M}_k C' + R$.

where $\{\bar{\alpha}_k^{[i]}, \alpha_k^{[i]}\}$, $\{\bar{m}_k^{[i]}, m_k^{[i]}\}$, and $\{\bar{M}_k, M_k\}$ are computed by Algorithms 1 and 2 with $\alpha_0^{[1]} = 1$, $m_0^{[1]} = \bar{x}_0$, and $M_0 = P_0$.

Proof: We prove this lemma by mathematical induction. *Step 1:* Consider the case $k=1$. Then $x_1 = Ax_0 + \nu_0 Bu_0 + \omega_0$.

- If $\tau_0 = 1$, then the value of ν_0 is known and $n_1 = 0$. From (2b) in Lemma 1, it follows that $p(x_1) = \mathcal{N}_{x_1}(\bar{x}_1, \bar{P}_1)$, where $\bar{x}_1 = A\bar{x}_0$ and $\bar{P}_1 = AP_0A' + Q$. By computing (6) and (7) with $k = 1$, we can obtain $\bar{\alpha}_1^{[1]}$, $\bar{m}_1^{[1]}$, and $\bar{M}_1$. Substituting them into (5a) yields $p(x_1) = \mathcal{N}_{x_1}(\bar{x}_1, \bar{P}_1)$. Thus, (5a), (6), and (7) hold for $k = 1$ and $\tau_0 = 1$.
- If $\tau_0 = 0$, then the value of ν_0 is unknown. Then $n_1 = 1$ and $N_1 = 2$. By the total probability law, we have

$$p(x_1) = p(x_1|\{\nu_0 = 0\})p(\{\nu_0 = 0\}) + p(x_1|\{\nu_0 = 1\})p(\{\nu_0 = 1\}) \tag{12}$$

In $p(x_1|\{\nu_0 = 0\})$, ν_0 takes the value 0 and is a deterministic quantity. By (2b), $p(x_1|\{\nu_0 = 0\}) = \mathcal{N}_{x_1}(A\bar{x}_0, \bar{M}_1)$ where $\bar{M}_1 = AP_0A' + Q$. Similarly, by using (2b) again, $p(x_1|\{\nu_0 = 1\}) = \mathcal{N}_{x_1}(A\bar{x}_0 + Bu_0, \bar{M}_1)$. If we set $\bar{\alpha}_1^{[1]} = \bar{\nu}$, $\bar{\alpha}_1^{[2]} = \nu$, $\bar{m}_1^{[1]} = A\bar{x}_0$, and $\bar{m}_1^{[2]} = A\bar{x}_0 + Bu_0$, then (12) can be rewritten as

$$p(x_1) = \bar{\alpha}_1^{[1]}\mathcal{N}_{x_1}(\bar{m}_1^{[1]}, \bar{M}_1) + \bar{\alpha}_1^{[2]}\mathcal{N}_{x_1}(\bar{m}_1^{[2]}, \bar{M}_1). \tag{13}$$

It is easy to verify that $p(x_1)$ computed by (5a), (6), and (8) with $k = 1$ is equal to (13). Hence, (5a), (6), and (8) hold for $k = 1$ and $\tau_0 = 0$.

Consequently, (5a), (6), (7), and (8) hold for $k = 1$.

Step 2: In Step 1, we have proved that (5a) holds at $k = 1$, that is,

$$p(x_1) = \sum_{i=1}^{N_1} \bar{\alpha}_1^{[i]}\mathcal{N}_{x_1}(\bar{m}_k^{[i]}, \bar{M}_1). \tag{14}$$

- If $\gamma_1 = 0$, there is no observation y_1 and thus $p(x_1|\mathcal{I}_1) = p(x_1)$. Let $p(x_1|\mathcal{I}_1)$ take the form

$$p(x_1|\mathcal{I}_1) = \sum_{i=1}^{2^{N_1}} \alpha_1^{[i]}\mathcal{N}_{x_1}(m_1^{[i]}, M_1). \tag{15}$$

It is evident that $\alpha_1^{[i]} = \bar{\alpha}_1^{[i]}, m_1^{[i]} = \bar{m}_1^{[i]}$, and $M_1 = \bar{M}_1$, since $p(x_1|\mathcal{I}_1) = p(x_1)$. Hence, (5b), (9), (10), and (11) hold at $k = 1$ and $\gamma_1 = 0$.
- If $\gamma_1 = 1$, with the observation y_1, $p(x_1|\mathcal{I}_1)$ can be derectly obtained from $p(x_1)$ in (14) by using Lemma 2 (ii). We still let $p(x_1|y_1)$ take the form as in (15). It is easy to check that $p(x_1|\mathcal{I}_1)$ and the parameters $\{\alpha_1^{[i]}, m_1^{[i]}, M_1\}$, obtained from $p(x_1)$ in (14) by using Lemma 2 (ii), are completely indentical to those computed by (5b), (9)-(11) at $k = 1$ and $\gamma_1 = 1$.

From Steps 1 and 2, it follows that (5)-(11) hold at $k = 1$. Suppose that (5)-(11) hold for $1, \ldots, n$. We check the case $k = n+1$ as follows.

Step 3: For $k = n+1$, $x_{n+1} = Ax_n + \nu_n Bu_n + \omega_n$.

- If $\tau_n = 1$, then the value of ν_n is known and $n_{n+1} = n_n$. $p(x_{n+1}|\mathcal{I}_n)$ can be obtained from $p(x_n|\mathcal{I}_n)$ by using Lemma 2 (i). It is easy to verify that the $p(x_{n+1}|\mathcal{I}_n)$ obtained is equal to the $p(x_{n+1}|\mathcal{I}_n)$ computed by (5a), (6), and (7) with $k = n+1$. Thus, (5a), (6), and (7) hold at $k = n+1$ and $\tau_n = 1$.
- If $\tau_n = 0$, then the value of ν_n is unknown to the estimator, $n_{n+1} = n_n + 1$, and $N_{n+1} = 2N_n$. By using the total probability law,

$$p(x_{n+1}|\mathcal{I}_n) = p(x_{n+1}|\mathcal{I}_n, \{\nu_n = 0\})p(\{\nu_n = 0\}) + p(x_{n+1}|\mathcal{I}_n, \{\nu_n = 1\})p(\{\nu_n = 1\}). \tag{16}$$

By applying Lemma 2 (i) to $p(x_{n+1}|\mathcal{I}_n, \{\nu_n = 0\})$ and $p(x_{n+1}|\mathcal{I}_n, \{\nu_n = 1\})$, we have

$$p(x_{n+1}|\mathcal{I}_n, \{\nu_n = 0\}) = \sum_{i=1}^{2^{n_n}} \alpha_n^{[i]}\mathcal{N}_{x_{n+1}}(\bar{m}_{n+1}^{[i]}, \bar{M}_{n+1}) \tag{17}$$

where $\bar{m}_{n+1}^{[i]} = Am_n^{[i]}$ and $\bar{M}_{n+1} = AM_nA' + Q$, for $1 \leq i \leq 2^{n_n}$;

$$p(x_{n+1}|\mathcal{I}_n, \{\nu_n = 1\}) = \sum_{i=1}^{2^{n_n}} \alpha_n^{[i]}\mathcal{N}_{x_{n+1}}(\bar{m}_{n+1}^{[i]}, \bar{M}_{n+1}) \tag{18}$$

where $\bar{m}_{n+1}^{[i]} = Am_n^{[i]} + Bu_n$, for $1 \leq i \leq 2^{n_n}$. By substituting (17) and (18) into (16), $p(x_{n+1}|\mathcal{I}_n)$ can be rewritten as: $p(x_{n+1}|\mathcal{I}_n) = \sum_{i=1}^{2^{n_{n+1}}} \bar{\alpha}_{n+1}^{[i]}\mathcal{N}_{x_{n+1}}(\bar{m}_{n+1}^{[i]}, \bar{M}_{n+1})$ where $\{\bar{m}_{n+1}^{[i]},$

$\bar{\alpha}_{n+1}^{[i]}$, $\bar{M}_{n+1}\}$ are equal to (6) and (8) with $k = n+1$, which means that (5a), (6), and (8) hold for $k = n+1$ and $\tau_n = 0$.

Step 4: By using Lemma 2 (ii) and following the same line of argument in Step 2, it is easy to verify that (5b), (9), (10), and (11) hold at $k = n+1$. For the save of space, the proof is not presented here.

From Steps 3 and 4, it follows that (5)-(11) hold at $k = n+1$, which completes the proof. ■

2) Optimal estimator for the Quasi-TCP-like system:

Theorem 1 (Optimal estimator): The optimal estimator for the Quasi-TCP-like system is the following:

$$\hat{x}_k = \sum_{i=1}^{2^{n_k}} \alpha_k^{[i]} m_k^{[i]} \tag{19a}$$

$$P_k = M_k + \sum_{i=1}^{2^{n_k}} \alpha_k^{[i]} (m_k^{[i]} - \hat{x}_k)_I^2 \tag{19b}$$

where $\alpha_k^{[i]}$, $m_k^{[i]}$, and M_k can be computed by Algorithms 1 and 2.

Proof: Since $p(x_k|\mathcal{I}_k)$ in (5a) is a Gaussian mixture, (19) can be readily obtained by applying Lemma 2 (iii) to (5a). ■

Remark 1: The impacts of the random loss of ACK are formulated as follows:

- It is known in [17] that M_k in (10), in fact, is the estimation error covariances for the TCP-like system. Hence, the summation part in (19b) can be viewed as the degradation of estimation performance caused by the random losses of ACK signals.
- From Algorithms 1 and 2, it follows that the number of the terms in the Gaussian mixture pdfs (5) doubles at the time when the ACK signal is lost. As time passes, the number of the terms will exponentially increases. Consequently, the computation of $\hat{x}_k$ and P_k requires exponentially increasing time, and eventually exhausts computer's memory. Thus, in general, the optimal estimation cannot be used in practice.

B. Solvability of the Optimal LQG problem

For the UDP- or Quasi-TCP-like system, it is only known that it is difficult to obtain the LMMSE-estimator-based LQG control [17, 23], whereas it is little known about the optimal solution. In this section, by an example we reveal the difficulties in solving the optimal LQG problem, and then make a conclusion on its solvability in Theorem 2.

We define the optimal value function $V_k(x_k)$ as follows and use the cost-to-go dynamic programming approach to derive the optimal control.

$$V_N(x_N) = \mathbb{E}[x_N' W_N x_N|\mathcal{I}_N] \tag{20a}$$

$$\begin{aligned} V_k(x_k) = \min_{u_k} & \mathbb{E}[x_k' W_k x_k \\ & + \nu_k u_k' \Lambda_k u_k + V_{k+1}(x_{k+1})|\mathcal{I}_k]. \end{aligned} \tag{20b}$$

Example 1: Consider a simple scalar system [17] with $A = B = C = 1$, $W_N = W_k = 1$, $\Lambda_k = 0$, $R = 1$. We further assume that there is no system noise (i.e., $\omega_k \equiv 0$ and $Q = 0$) and no observation packet losses (i.e., $\gamma_k \equiv 1$). Without loss of generality, we suppose that $\tau_{N-1} = 0$ and $\tau_{N-2} = 1$.

1) Calculations of $V_N(x_N), V_{N-1}(x_{N-1})$, and $V_{N-2}(x_{N-2})$: $V_N(x_N) = \mathbb{E}[x_N' W_N x_N|\mathcal{I}_N] = \mathbb{E}[x_N^2|\mathcal{I}_N]$.

$$\begin{aligned} & V_{N-1}(x_{N-1}) = \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + V_N(x_N)|\mathcal{I}_{N-1}] \\ & = \min_{u_{N-1}} \mathbb{E}[x_{N-1}^2 + (x_{N-1} + \nu_{N-1} u_{N-1})^2|\mathcal{I}_{N-1}] \\ & = \min_{u_{N-1}} \mathbb{E}[2x_{N-1}^2|\mathcal{I}_{N-1}] + \nu u_{N-1}^2 + 2\nu u_{N-1}\hat{x}_{N-1}. \end{aligned}$$

By solving $\partial V_{N-1}(x_{N-1})/\partial u_{N-1} = 0$, the optimal control is obtained as $u_{N-1}^* = -\hat{x}_{N-1}$. $V_{N-1}(x_{N-1}) = \mathbb{E}[2x_{N-1}^2|\mathcal{I}_{N-1}] - \nu \hat{x}_{N-1}^2 = \mathbb{E}[(2-\nu)x_{N-1}^2|\mathcal{I}_{N-1}] + \nu P_{N-1}$.

By virtue of $\tau_{N-2} = 1$, ν_{N-2} is known to the estimator and controller.

$$\begin{aligned} & V_{N-2}(x_{N-2}) = \min_{u_{N-2}} \mathbb{E}[x_{N-2}^2 + V_{N-1}(x_{N-1})|\mathcal{I}_{N-2}] \\ & = \min_{u_{N-2}} \mathbb{E}[x_{N-2}^2 + (2-\nu)(x_{N-2} + \nu_{N-2}u_{N-2})^2 \\ & \quad + \nu P_{N-1}|\mathcal{I}_{N-2}] \\ & = \min_{u_{N-2}} \mathbb{E}[(3-\nu)x_{N-2}^2|\mathcal{I}_{N-2}] + 2\nu_{N-2}(2-\nu)\hat{x}_{N-2}u_{N-2} \\ & \quad + \nu_{N-2}(2-\nu)u_{N-2}^2 + \nu \mathbb{E}[P_{N-1}|\mathcal{I}_{N-2}]. \end{aligned} \tag{21}$$

To obtain V_{N-2}, we proceed to compute P_{N-1} and $\mathbb{E}[P_{N-1}|\mathcal{I}_{N-2}]$.

2) Computation of P_{N-1}: Note that $\gamma_k \equiv 1$, $\tau_{N-1} = 0$. From (8b), we have $\bar{\alpha}_{N-1}^{[i]} = \alpha_{N-2}^{[i]}$. Let $c \triangleq \sum_{i=1}^{2^{n_{N-1}}} \alpha_{N-2}^{[i]} \phi_{N-1}^{[i]}$. From (9), we have for $1 \le i \le 2^k$,

$$\alpha_{N-1}^{[i]} = \frac{1}{c}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]} \tag{22a}$$

$$m_{N-1}^{[i]} = \mathbb{A}_{N-1} m_{N-2}^{[i]} + \Gamma_A, \tag{22b}$$

where $\Gamma_A \triangleq \nu_{N-2}\mathbb{K}_{N-2} B u_{N-2} + K_{N-1} y_{N-1}$.

From (19a) and (22a),

$$\hat{x}_{N-1} = \sum_{i=1}^{2^{n_{N-1}}} \alpha_{N-1}^{[i]} m_{N-1}^{[i]} = \mathbb{A}_{N-1} x_{N-2}^* + \Gamma_A, \tag{23}$$

where $x_{N-2}^* \triangleq \frac{1}{c}\sum_{i=1}^{2^{n_{N-1}}} \phi_{N-1}^{[i]} \alpha_{N-2}^{[i]} m_{N-2}^{[i]}$.

By (19b), (22a) and (23), we have

$$\begin{aligned} P_{N-1} &= M_{N-1} + \sum_{i=1}^{2^{n_{N-1}}} \alpha_{N-2}^{[i]} (m_{N-2}^{[i]} - \hat{x}_{N-2})^2 \\ &= M_{N-1} + \mathbb{A}_{N-1}\Gamma_B, \end{aligned}$$

where $\Gamma_B \triangleq \sum_{i=1}^{2^{n_{N-1}}} \frac{1}{c}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]}(m_{N-2}^{[i]} - x_{N-2}^*)^2$.

$$\begin{aligned} \Gamma_B = & \sum_{i=1}^{2^{n_{N-1}}} \tfrac{1}{c}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]}\big((m_{N-2}^{[i]})^2 - 2m_{N-2}^{[i]} x_{N-2}^* \\ & + (x_{N-2}^*)^2\big) \\ = & (x_{N-2}^*)^2 + \sum_{i=1}^{2^{n_{N-1}}} \tfrac{1}{c}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]}(m_{N-2}^{[i]})^2 \\ & - \sum_{i=1}^{2^{n_{N-1}}} 2\tfrac{1}{c}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]} m_{N-2}^{[i]} x_{N-2}^* \\ = & -(x_{N-2}^*)^2 + \sum_{i=1}^{2^{n_{N-1}}} \tfrac{1}{c}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]}(m_{N-2}^{[i]})^2. \end{aligned}$$

3) Computation of $\mathbb{E}[P_{N-1}|\mathcal{I}_{N-2}]$:
Since $y_k = Cx_k + \upsilon_k$, $p(y_k|\mathcal{I}_{k-1})$ can be obtained from $p(x_k|\mathcal{I}_{k-1})$ by applying Lemma 2 (i) to (5a). That is,

$$\begin{aligned} p(y_k|\mathcal{I}_{k-1}) = & \sum_{i=1}^{2^{n_k}} \mathcal{N}_{y_k}(C\bar{m}_k^{[i]}, P_k^Y)\bar{\alpha}_k^{[i]} \\ = & \sum_{i=1}^{2^{n_k}} \phi_k^{[i]}\bar{\alpha}_k^{[i]}. \end{aligned}$$

Then we have $p(y_{N-2}|\mathcal{I}_{N-2})=c$ due to $\bar{\alpha}_{N-1}^{[i]}=\alpha_{N-2}^{[i]}$. Note that $\mathcal{I}_{N-2}=\{y_{N-2},\ldots,y_1,\tau_{N-2},\ldots,\tau_0\}$. From (11) and (19b), it is clear that P_{N-1} contains y_{N-1}. Thus

$$\begin{aligned}\mathbb{E}[P_{N-1}|\mathcal{I}_{N-2}] &= \textstyle\int_{-\infty}^{\infty} P_{N-1}p(y_{N-1}|\mathcal{I}_{N-2})\mathrm{d}y_{N-1}\\ &= \textstyle\int_{-\infty}^{\infty}(\Gamma_C+\Gamma_D)\mathrm{d}y_{N-1},\end{aligned}$$

where

$$\Gamma_C = -c(x_{N-2}^*)^2 = \frac{(\sum_{i=1}^{2^{n_{N-1}}}\alpha_{N-2}^{[i]}\phi_{N-1}^{[i]}m_{N-2}^{[i]})^2}{\sum_{i=1}^{2^{n_{N-1}}}\alpha_{N-2}^{[i]}\phi_{N-1}^{[i]}}$$

$$\Gamma_D = cM_{N-1}+\mathbb{A}_{N-1}\textstyle\sum_{i=1}^{2^{n_{N-1}}}\phi_{N-1}^{[i]}\alpha_{N-2}^{[i]}(m_{N-2}^{[i]})^2.$$

Since c is a Gaussian mixture function and $\phi_{N-1}^{[i]}$ is a Gaussian function, there is a analytic expression for $\int_{-\infty}^{\infty}\Gamma_D\mathrm{d}y_{N-1}$.

4) Three difficulties are presented as follows:
   - The Gaussian function $\phi_{N-1}^{[i]}$ occurs in both the numerator and denominator of Γ_C. From the knowledge of calculus, it is clear that there is no analytic expression for $\int_{-\infty}^{\infty}\Gamma_C\mathrm{d}y_{N-1}$, even when $n_{N-1}=1$. Moreover, with the random losses of ACK signals, the number of $\phi_{N-1}^{[i]}$ will exponentially increase, which further confirms the fact that there is no analytic expression for $\int_{-\infty}^{\infty}\Gamma_C\mathrm{d}y_{N-1}$ and $\mathbb{E}[P_{N-1}|\mathcal{I}_{N-2}]$ either.
   - Note that the desired optimal u_{N-1}^* minimizing $V_{N-2}(N-2)$ in (21) is in fact a function not a deterministic quantity. Without analytic expression for $\mathbb{E}[P_{N-1}|\mathcal{I}_{N-2}]$, the nonlinear optimization cannot be further performed to obtain the optimal control u_{N-1}^* in $V(N-2)$;
   - The number of the Gaussian functions $\phi_{N-1}^{[i]}$ in Γ_C will exponentially increase, making its computation time-consuming. Meanwhile, in solving the LQG problem, the optimal estimation $\hat{x}_k$ is required, and its computation also is time-consuming.

For this simplified system, these three difficulties are enough to prevent the optimal LQG problem from being solved. ■

Theorem 2 (Solvability of the optimal LQG problem): For the general Quasi-TCP-like system, it is impossible to solve the optimal LQG problem.

Proof: It is clear that for the general Quasi-TCP-like system, these three difficulties mentioned above still exist. Both the calculation of $\int_{-\infty}^{\infty}\Gamma_C\mathrm{d}y_{N-1}$ and the aforementioned nonlinear optimization are technically difficult. More importantly, due to the random losses of ACK, the time for the computation will tend to infinity, and computer's memory will be eventually exhausted. Therefore, we claim that it is impossible to solve the optimal LQG problem. ■

Remark 2: The conclusion in Theorem 2 is also applicable to the UDP-like system, since the ACK in the UDP-like system is completely lost.

IV. SUB-OPTIMAL LQG CONTROL

The conclusion in Theorem 2 motivates us to develop a sub-optimal but efficient solution to the estimation and LQG problems for the Quasi-TCP-like system. In the following, we first develop a sub-optimal linear estimator, and then based on it we derive the LQG controller. Finally, we establish the conditions for the mean square stability of the closed-loop system.

A. Sub-optimal Linear Estimator

The structure of the optimal estimator is complex, but the K_k in (10a) can be recursively calculated. By this K_k, we design a sub-optimal linear estimator as in Algorithm 3.

In Algorithm 3, the symbols $\bar{x}_k$ and $\hat{x}_k$ are recycled to denote the predicted and estimated system states for this sub-optimal estimator, respectively. Then $\bar{P}_k$ and P_k are recycled to denote the corresponding prediction and estimation error covariances, respectively. Define a function

$$g(\gamma,M)=AMA'-\gamma AMC'(CMC'+R)^{-1}CMA'+Q. \quad (24)$$

Algorithm 3 Sub-optimal Linear Estimator

Initial condition: $\hat{x}_0=\bar{x}_0$ $\bar{M}_0=P_0$

Prediction step: (LMMSE predictor)

$$\bar{x}_{k+1}=A\hat{x}_k+(\tau_k\nu_k+\bar{\tau}_k\nu)Bu_k \quad (25)$$

where $\bar{\tau}_k=1-\tau_k$.

Estimation step:

$$\hat{x}_{k+1}=\bar{x}_{k+1}+\gamma_{k+1}K_{k+1}(y_{k+1}-C\bar{x}_{k+1}) \quad (26)$$

where

$$K_{k+1}=\bar{M}_{k+1}C'\left(C\bar{M}_{k+1}C'+R\right)^{-1} \quad (27)$$

$$\bar{M}_{k+1}=g(\gamma_{k+1},\bar{M}_k) \quad (28)$$

Remark 3: In Algorithm 3, we use the LMMSE preditor to obtain the predicted system state. For the estimation step, the estimator gain K_k in (27) is in fact the K_k in (10a) computed by the $\bar{M}_k$, which occurs in each term of the Gaussian mixture pdf (5a). The way to design this K_k is inspired by [35]. In [35], by constructing auxiliary system states, a sub-optimal estimator was developed for the UDP-like without observation packet loss. The resulting estimator gain is identical to the one computed by the $\bar{M}_k$ in each term of the Gaussian mixture pdf. The benefit of this design method is that the K_k is not a nonlinear function of u_k, and thus the nonlinear optimization problem is circumvented.

Remark 4: Technically speaking, the estimation performance of the sub-optimal estimator in Algorithm 3 is inferior to that of the LMMSE estimator, but in the backgound of the LQG problem they are quite close, which will be shown and explained later in Section V.

Lemma 4: The prediction and estimation error covariances can be calculated as follows:

$$\bar{P}_{k+1}=AP_kA'+Q+\bar{\tau}_k\bar{\nu}\nu Bu_ku_k'B' \quad (29)$$

$$\begin{aligned}P_{k+1}=&(I-\gamma_{k+1}K_{k+1}C)\bar{P}_{k+1}(I-\gamma_{k+1}K_{k+1}C)'\\ &+\gamma_{k+1}K_{k+1}RK_{k+1}'. \end{aligned}\quad (30)$$

Proof: From (1a) and (25), we have

$$x_{k+1}-\bar{x}_{k+1}=A(x_k-\hat{x}_k)+\bar{\tau}_k(\nu_k-\nu)Bu_k+\omega_k. \quad (31)$$

If $\tau_k = 1$, then $\bar{\tau}_k = 0$. By applying (2a) to (31), we obtain the covariance of $x_{k+1} - \bar{x}_{k+1}$, i.e., $\bar{P}_{k+1} = AP_kA' + Q$. If $\tau_k = 0$, then $\bar{\tau}_k = 1$ and ν_k is an unknown random quantity. By using (2a) again, we have $\bar{P}_{k+1} = AP_kA' + Q + \bar{\nu}\nu Bu_ku_k'B'$. Therefore, (29) holds for $\tau_k = 1$ and 0.

From (1a) and (26), we have

$$\begin{aligned} & x_{k+1} - \hat{x}_{k+1} \\ &= x_{k+1} - \bar{x}_{k+1} - \gamma_{k+1}K_{k+1}(Cx_{k+1} + \upsilon_{k+1} - C\bar{x}_{k+1}) \\ &= (I - \gamma_{k+1}K_{k+1}C)(x_{k+1} - \bar{x}_{k+1}) - \gamma_{k+1}K_{k+1}\upsilon_{k+1}. \end{aligned} \tag{32}$$

The value of γ_{k+1} is known for the estimator. By applying (2a) to (32) and noting that $\gamma_{k+1}^2 = \gamma_{k+1}$, it is easy to verify that (30) holds. The proof is completed. ■

B. LQG control

To derive the finite horizon LQG controller, we first calculate $\mathbb{E}[\mathrm{tr}(HP_{k+1})|\mathcal{I}_k]$ in Lemma 5. Such quantity is required in Lemma 6 to derive the optimal value function $V_k(x_k)$.

Define $\mathbb{K}_k \triangleq I - K_kC$.

Lemma 5: Given a matrix H, let $T \triangleq \gamma\mathbb{K}_{k+1}'H\mathbb{K}_{k+1} + \bar{\gamma}H$. Then

$$\begin{aligned} \mathbb{E}[\mathrm{tr}(HP_{k+1})|\mathcal{I}_k] &= \mathrm{tr}\big(A'TAP_k\big) + \mathrm{tr}\big(TQ\big) \\ &\quad + \bar{\tau}\bar{\nu}\nu\mathrm{tr}\big(u_k'B'TBu_k\big) + \gamma\mathrm{tr}(K_{k+1}'HK_{k+1}R). \end{aligned}$$

Proof: By substituting (29) into (30) and then taking mathematical expectation to P_{k+1}, we have

$$\begin{aligned} \mathbb{E}[P_{k+1}|\mathcal{I}_k] &= \gamma\mathbb{K}_{k+1}(AP_kA' + Q + \bar{\tau}\bar{\nu}\nu Bu_ku_k'B')\mathbb{K}_{k+1}' \\ &\quad + \gamma K_{k+1}RK_{k+1}' + \bar{\gamma}(AP_kA' + Q + \bar{\tau}\bar{\nu}\nu Bu_ku_k'B'). \end{aligned}$$

By using the property that $\mathbb{E}[\mathrm{tr}(A)] = \mathrm{tr}(\mathbb{E}[A])$ and $\mathrm{tr}(BAPA) = \mathrm{tr}(ABAP)$,

$$\begin{aligned} & \mathbb{E}[\mathrm{tr}(HP_{k+1})|\mathcal{I}_k] = \mathrm{tr}(H\mathbb{E}[P_{k+1}|\mathcal{I}_k]) \\ &= \mathrm{tr}\big(A'(\gamma\mathbb{K}_{k+1}'H\mathbb{K}_{k+1} + \bar{\gamma}H)AP_k\big) \\ &\quad + \mathrm{tr}\big((\gamma\mathbb{K}_{k+1}'H\mathbb{K}_{k+1} + \bar{\gamma}H)Q\big) \\ &\quad + \bar{\tau}\bar{\nu}\nu\mathrm{tr}\big(u_k'B'(\gamma\mathbb{K}_{k+1}'H\mathbb{K}_{k+1} + \bar{\gamma}H)Bu_k\big) \\ &\quad + \gamma\mathrm{tr}(K_{k+1}'HK_{k+1}R) \\ &= \mathrm{tr}\big(A'TAP_k\big) + \mathrm{tr}\big(TQ\big) + \bar{\tau}\bar{\nu}\nu\mathrm{tr}\big(u_k'B'TBu_k\big) \\ &\quad + \gamma\mathrm{tr}(K_{k+1}'HK_{k+1}R). \end{aligned}$$

The proof is completed. ■

In the following, we use the cost-to-go dynamic programming approach to obtain the $V_k(x_k)$.

Lemma 6: Based on the sub-optimal estimator in Algorithm 3, $V_k(x_k)$ defined in (20) can be calculated as follows:

$$V_k(x_k) = \mathbb{E}[x_k'Z_kx_k|\mathcal{I}_k] + \mathrm{tr}(H_kP_k) + \Delta_k \tag{33}$$

where K_k is computed by (27) and (28), and

$$L_k = -(\Lambda_k + B'(Z_{k+1} + \bar{\tau}\bar{\nu}T_{k+1})B)^{-1}B'Z_{k+1}A \tag{34a}$$

$$T_{k+1} = \gamma\mathbb{K}_{k+1}'H_{k+1}\mathbb{K}_{k+1} + \bar{\gamma}H_{k+1} \tag{34b}$$

$$\begin{aligned} Z_k &= A'Z_{k+1}A + W_k - \nu A'Z_{k+1}B(\Lambda_k + \\ &\quad B'(Z_{k+1} + \bar{\tau}\bar{\nu}T_{k+1})B)^{-1}B'Z_{k+1}A \end{aligned} \tag{34c}$$

$$H_k = A'T_{k+1}A + W_k + A'Z_{k+1}A - Z_k \tag{34d}$$

$$\begin{aligned} \Delta_k &= \Delta_{k+1} + \mathrm{tr}\big(T_{k+1}Q\big) + \mathrm{tr}(Z_{k+1}Q) \\ &\quad + \mathrm{tr}\big((K_{k+1}'H_{k+1}K_{k+1})R\big) \end{aligned} \tag{34e}$$

with $Z_N = W_N$, $H_N = 0$, and $\Delta_N = 0$.

Proof: We prove this lemma by mathematical induction. It is evident that (33) holds at time N. Suppose that (33) holds for $N, \cdots, k+1$. Now we check $V_k(x_k)$.

$$\begin{aligned} & V_k(x_k) \\ &= \min_{u_k}\mathbb{E}\big[x_k'W_kx_k + \nu_ku_k'\Lambda_ku_k + V_{k+1}(x_{k+1})|\mathcal{I}_k\big] \\ &= \min_{u_k}\mathbb{E}\big[x_k'(W_k + A'Z_{k+1}A)x_k + \omega_{k+1}'Z_{k+1}\omega_{k+1} \\ &\quad + \nu_ku_k'(\Lambda_k + B'Z_{k+1}B)u_k + 2\nu_ku_k'B'Z_{k+1}Ax_k \\ &\quad + \mathrm{tr}(H_{k+1}P_{k+1}) + \Delta_{k+1}|\mathcal{I}_k\big] \\ &\overset{(a)}{=} \min_{u_k}\mathbb{E}\big[x_k'(W_k + A'Z_{k+1}A)x_k|\mathcal{I}_k\big] + \mathrm{tr}(A'T_{k+1}AP_k) \\ &\quad + \big\{\Delta_{k+1} + \mathrm{tr}(Z_{k+1}Q) + \mathrm{tr}(T_{k+1}Q) \\ &\quad + \gamma\mathrm{tr}(K_{k+1}'H_{k+1}K_{k+1}R)\big\} \\ &\quad + \nu\bar{\nu}\bar{\tau}u_k'B'T_{k+1}Bu_k \\ &\quad + \nu u_k'(\Lambda_k + B'Z_{k+1}B)u_k + 2\nu u_k'B'Z_{k+1}A\hat{x}_k, \end{aligned} \tag{35}$$

where $\overset{(a)}{=}$ is obtained by using Lemma 5. Then we solve $\partial V_k(x_k)/\partial u_k = 0$ and get the u_k which minimizes $V_k(x_k)$ as follows

$$u_k = -(\Lambda_k + B'(Z_{k+1} + \bar{\tau}\bar{\nu}T_{k+1})B)^{-1}B'Z_{k+1}A\hat{x}_k.$$

The quantities in $\{\cdot\}$ of (35) are equal to the Δ_k in (34e). Substituting this u_k back into (35) yields

$$\begin{aligned} & V_k(x_k) \\ &= \mathbb{E}\big[x_k'(W_k + A'Z_{k+1}A)x_k|\mathcal{I}_k\big] + \Delta_k + \mathrm{tr}(A'T_{k+1}AP_k) \\ &\quad - \nu\hat{x}_k'A'Z_{k+1}B(\Lambda_k + B'(Z_{k+1} + \bar{\tau}\bar{\nu}T_{k+1})B)^{-1}B'Z_{k+1}A\hat{x}_k \\ &\overset{(a)}{=} \mathbb{E}\big[x_k'Z_kx_k|\mathcal{I}_k\big] + \Delta_k + \mathrm{tr}(A'T_{k+1}AP_k) \\ &\quad + \mathrm{tr}\big((W_k + A'Z_{k+1}A - Z_k)P_k\big) \\ &\overset{(b)}{=} \mathbb{E}\big[x_k'Z_kx_k|\mathcal{I}_k\big] + \Delta_k + \mathrm{tr}(H_kP_k), \end{aligned}$$

where $\overset{(a)}{=}$ is obtained by using (34c) and the existing result ([17], Lemma 4.1) that $\mathbb{E}[x_k'Sx_k] = \hat{x}_k'S\hat{x}_k + \mathrm{tr}(SP_k)$. The equality $\overset{(b)}{=}$ is obtained by using (34d). Hence, (33) holds for the time k. The proof is completed. ■

Based on Lemma 6, the results on the LQG control problem are formulated in the following theorem.

Theorem 3 (LQG control): For the Quasi-TCP-like system, based on the sub-optimal estimator in Algorithm 3,

- the finite horizon LQG controller is $u_k = L_k\hat{x}_k$, and the corresponding cost function

$$\begin{aligned} J_N &= x_0'Z_0x_0 + \mathrm{tr}\big((Z_0 + H_0)P_0\big) \\ &\quad + \textstyle\sum_{k=1}^{N}\mathrm{tr}\big((T_k + Z_k)Q + (K_k'H_kK_k)R\big), \end{aligned} \tag{36}$$

  where L_k, Z_k, T_k, H_k are computed by (34), and $\hat{x}_k$ and K_k are computed by Algorithm 3.
- There is no solution to the infinite horizon LQG problem.

Proof: From the dynamic programming approach, it follows that the control sequence $u_k = L_k\hat{x}_k$ obtained in

Lemma 6 is the desired optimal control that minimizes the cost function J_N, and $J_N = V_0(x_0)$. From (33) and (34), it is easy to obtain J_N as in (36).

Similar to the TCP-like system, the estimator gain K_k is a random quantity, making $\frac{1}{k}J_k$ and L_k unconvergent. Therefore, there is no solution to the infinite horizon LQG problem. The proof is completed. ■

Remark 5: (Separation principle) From Algorithm 3, it is known that the estimator gain K_k is independent of the design of the LQG controller. However, the LQG controller depends on K_k, and the estimation error covariance P_k depends on the control inputs. Consequently, the separation principle does not hold for the Quasi-TCP-like system.

C. Stability of the closed-loop systems

In the sequel, we show that under some conditions the controller $u_k = L_k\hat{x}_k$ can stabilize the closed-loop Quasi-TCP-like system, where L_k is computed by (34).

To study the stability of the closed-loop systems, we let $W_k = W$, $\Lambda_k = \Lambda$, and $G_k \triangleq H_k + Z_k$. Note that $S = g(1, S)$ is the standard algebraic Riccati equation, where $g(\cdot)$ defined in (24). It is well known that under Assumption 1, there is an unique positive definite solution S_∞ for $S_\infty = g(1, S_\infty)$. For the convenience of formulation, we define some symbols. Denote the maximum sigular value of $\mathbb{K}$ by $\lambda_{\mathbb{K}}$, where $\mathbb{K} = I - S_\infty C'(CS_\infty C' + R)^{-1}C$. Define $\eta \triangleq \gamma(\lambda_{\mathbb{K}})^2 + \bar{\gamma}$ and $\rho \triangleq \bar{\tau}\bar{\nu}\eta$.

Three condtions are given as follows: *Condtion 1:* B is square and invertible; *Condtion 2:* C is full column rank; and *Condtion 3:* $P_0 \geq S_\infty$.

Theorem 4 (Stability of the closed-loop systems): Consider the system in (1) with the LQG controller $u_k = L_k\hat{x}_k$ where L_k is computed by (34).

(i) If Z_k and G_k are bounded, then the system is mean square stable.

(ii) If Conditions 1, 2, and 3 are satified, then a sufficient condition for the boundedness of Z_k and G_k is

$$\lambda_A^2(\eta + \nu - 2\eta\nu) < (\eta + \nu - \eta\nu).$$

Proof of part (i): Let $\mathcal{K}_k = (I - \gamma_k K_k C)$. We start with calculating x_k and e_k. By substituting $u_k = L\hat{x}_k$ into (1) and using $e_k = x_k - \hat{x}_k$,

$$\begin{aligned} x_{k+1} &= Ax_k + \nu_k B u_k + \omega_k \\ &= (A + \nu_k BL_{k+1})x_k - \nu_k BL_{k+1}e_k + \omega_k. \end{aligned} \tag{37}$$

By combining (31) and (32), we have

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= \mathcal{K}_{k+1}(Ae_k + \bar{\tau}_k(\nu_k - \nu)BL_{k+1})\hat{x}_k + \omega_k) \\ &\quad - \gamma_{k+1}K_{k+1}\upsilon_{k+1} \\ &= \bar{\tau}_k(\nu_k - \nu)\mathcal{K}_{k+1}BL_{k+1}x_k \\ &\quad + \mathcal{K}_{k+1}(A - \bar{\tau}_k(\nu_k - \nu)BL_{k+1})e_k \\ &\quad + \mathcal{K}_{k+1}\omega_k - \gamma_{k+1}K_{k+1}\upsilon_{k+1}. \end{aligned} \tag{38}$$

The homogenous parts of (37) and (38) are the following:

$$x_{k+1} = (A + \nu_k BL_{k+1})x_k - \nu_k BL_{k+1}e_k \tag{39}$$

$$\begin{aligned} e_{k+1} &= \bar{\tau}_k(\nu_k + \nu)\mathcal{K}_{k+1}BL_{k+1}x_k \\ &\quad + \mathcal{K}_{k+1}(A - \bar{\tau}_k(\nu_k + \nu)BL_{k+1})e_k. \end{aligned} \tag{40}$$

Since $\mathbb{E}[\|\omega_k\|^2] = \mathrm{tr}(Q)$ and $\mathbb{E}[\|\upsilon_{k+1}\|^2] = \mathrm{tr}(R)$ in (37) and (38) are bounded, it was pointed out in [28] that if the homogenous parts of (37) and (38) are asymptotically stable, then the system equations (37) and (38) are mean square stable.

To study the asympotic stability of (39) and (40), we follow the similar line of augument developed in [28], which requires the calculation of $x_k' Z_k x_k + e_k' H_k e_k$. However, it would be tedious to compute this quantity directly via (39) and (40), which can be seen in [28]. Actually, majorities of the derivations for computing this quantity have been performed in calculating $V_k(x_k)$ in Lemma 6. Therefore, in the following we employ the results on $V_k(x_k)$ to compute this quantity.

Denote the optimal control by u_k^*. From (20), we have

$$V_k(x_k) = \mathbb{E}[x_k' W x_k + \nu_k (u_k^*)' \Lambda u_k^* + V_{k+1}(x_{k+1}) | \mathcal{I}_k].$$

According to the definition of the mean square stability, it is the $\mathbb{E}[\|x_k\|^2]$ not the $\mathbb{E}[\|x_k\|^2 | \mathcal{I}_k]$ that is considered. Thus, taking mathematical expectation with respect to $\mathcal{I}_k$ yields

$$\mathbb{E}[V_{k+1}(x_{k+1}) - V_k(x_k)] = -\mathbb{E}[x_k' W x_k + \nu(u_k^*)' \Lambda u_k^*]. \tag{41}$$

From (33) and by noting that $\mathbb{E}[e_k' H_k e_k] = \mathrm{tr}(H_k P_k)$, we obtain

$$\mathbb{E}[V_k(x_k)] = \mathbb{E}[x_k' Z_k x_k + e_k' H_k e_k] + \mathbb{E}[\Delta_k]. \tag{42}$$

Then

$$\begin{aligned} &\mathbb{E}[x_{k+1}' Z_{k+1} x_{k+1} + e_{k+1}' H_{k+1} e_{k+1} - (x_k' Z_k x_k + e_k' H_k e_k)] \\ &= \mathbb{E}[V_{k+1}(x_{k+1}) - \Delta_{k+1} - (V_k(x_k) - \Delta_k)] \\ &= -\mathbb{E}[x_k' W x_k + \nu(u_k^*)' \Lambda u_k^*] + (\mathrm{tr}(T_{k+1}Q) \\ &\quad + \mathrm{tr}(Z_{k+1}Q) + \mathrm{tr}((K_{k+1}' H_{k+1} K_{k+1})R)) \end{aligned} \tag{43}$$

where the last equality is obtained by (41) and (34e).

In Lemma 6, x_k and e_k are determined by (37) and (38). While what we consider is their homogenous parts, i.e., (41) and (42), in which there is no noise, which is equivalent to letting $Q = R = 0$ in $V_k(x_k)$. Therefore, for the homogenous parts (41) and (42), by letting $Q = R = 0$ in (43),

$$\begin{aligned} &\mathbb{E}[x_{k+1}' Z_{k+1} x_{k+1} + e_{k+1}' H_{k+1} e_{k+1} - (x_k' Z_k x_k + e_k' H_k e_k)] \\ &= -\mathbb{E}[x_k' W x_k + \nu_k (u_k^*)' \Lambda u_k^*]. \end{aligned}$$

Summing up this equality for $k = 0$ to $n - 1$ yields

$$\begin{aligned} &\mathbb{E}[x_n' Z_n x_n + e_n' H_n e_n - (x_0' Z_0 x_0 + e_0' H_0 e_0)] \\ &= -\textstyle\sum_{k=0}^{n-1} \mathbb{E}[x_k' W x_k + \nu(u_k^*)' \Lambda u_k^*]. \end{aligned}$$

Due to $\mathbb{E}[x_n' Z_n x_n + e_n' H_n e_n] \geq 0$, we have $\mathbb{E}[x_0' Z_0 x_0 + e_0' H_0 e_0] \geq \sum_{k=0}^{n-1} \mathbb{E}[x_k' W x_k + \nu(u_k^*)' \Lambda u_k^*]$. By the hypothesis that $\{Z_k$ and $G_k\}$ are bounded, we have $\bar{Z} \geq Z_0$ and $\bar{G} \geq G_0 = Z_0 + H_0 \geq H_0$. Then $\mathbb{E}[x_0' \bar{Z} x_0 + e_0' \bar{G} e_0] \geq \sum_{k=0}^{n-1} \mathbb{E}[x_k' W x_k + \nu(u_k^*)' \Lambda u_k^*]$. The boundedness of the series $\sum_{k=0}^{n-1} \mathbb{E}[x_k' W x_k]$ implies $\lim_{k\to\infty} \mathbb{E}[x_k' W x_k] = 0$. Due to $W > 0$, $\mathbb{E}[x_k' x_k] = \mathbb{E}[\|x_k\|^2] \to 0$. Since $\mathbb{E}[x_k' W x_k] = \hat{x}_k' W \hat{x}_k + \mathbb{E}[e_k' W e_k]$, we have $\lim_{k\to\infty} \mathbb{E}[\hat{x}_k' W \hat{x}_k] = 0$, i.e., $\mathbb{E}[\|\hat{x}_k\|^2] \to 0$, which implies the asympotic stability of (39) and (40). Hence, (37) and (38) are mean square stable. The proof of part (i) is completed.

After introducing some preliminaries and lemmas as follows, we continue the proof of part (ii). ■

To study the boundedness of Z_k and G_k, we reverse the time index in (34) and then rewrite (34) as follows:

$$L_{k+1} = -(\Lambda + B'(Z_k + \bar{\tau}\bar{\nu}T_k)B)^{-1}B'Z_kA \quad (44a)$$
$$Z_{k+1} = \Phi_X(Z_k, Z_k + \bar{\tau}\bar{\nu}T_k) \quad (44b)$$
$$G_{k+1} = A'(\gamma\mathbb{K}_k'H_k\mathbb{K}_k + \bar{\gamma}H_k + Z_k)A + W \quad (44c)$$
$$\Delta_{k+1} = \Delta_k + \mathrm{tr}\big(T_kQ + (K_k'H_kK_k)R + Z_kQ\big) \quad (44d)$$

with $Z_0 = W$ and $H_0 = 0$, where

$$\Phi_X(Z, Y) \triangleq A'ZA + W - \nu A'ZB(\Lambda + B'YB)^{-1}B'ZA.$$

Define two operators as follows:

$$\begin{aligned}\Phi_Z(Z, G, \rho) &\triangleq \Phi_X(Z, (1-\rho)Z + \rho G)\\ &= A'ZA + W - \nu A'ZB(\Lambda\\ &\quad + B'((1-\rho)Z + \rho G)B)^{-1}B'ZA\\ \Phi_G(Z, G, \eta) &\triangleq (1-\eta)A'GA + \eta A'ZA + W.\end{aligned}$$

Lemma 7: Some results on $g(1, X)$, Φ_X, Φ_Z, and Φ_G are formulated as follows ([17, pp. 182] and [36, Theorems 10.6 and 10.7]):

(i) $g(1, X)$, Φ_X, Φ_Z, and Φ_G are monotonically increasing functions. Namely, if $Z_1 \geq Z_2$ and $Y_1 \geq Y_2$, then $g(1, Z_1) \geq g(1, Z_2)$, $\Phi_X(Z_1, Y_1) \geq \Phi_X(Z_2, Y_2)$, $\Phi_Z(Z_1, Y_1, \rho) \geq \Phi_Z(Z_2, Y_2, \rho)$, and $\Phi_G(Z_1, Y_1, \eta) \geq \Phi_G(Z_2, Y_2, \eta)$.

(ii) If Condition 1 is satisfied, then a necessary and sufficient condition for the convergences of $Z_{k+1} = \Phi_Z(Z_k, G_k, \rho)$ and $G_{k+1} = \Phi_G(Z_k, G_k, \eta)$ is $\lambda_A^2(\eta + \nu - 2\eta\nu) < (\eta + \nu - \eta\nu)$.

(iii) If $S_0 \geq S_\infty$, then $S_0 \geq S_k \geq S_\infty$.

Lemma 8: Let $X > 0$ and $Y \geq 0$, and C is a matrix with compatible dimension. Then

(i) ([37], Theorem 7.7.3 and Corollary 7.7.4)
The following three inequalities are equivalent:
$\lambda(YX^{-1}) < 1 \Leftrightarrow X > Y \Leftrightarrow Y^{-1} > X^{-1}$.

(ii) ([38], pp. 213) The matrix inverse lemma:
$XC'(CXC' + Y)^{-1} = (X^{-1} + C'Y^{-1}C)^{-1}C'Y^{-1}$.

In the sequel, we assume that Conditions 1, 2, and 3 are satified.

Lemma 9: Let $\bar{M}_0 = P_0 \geq S_\infty$. The following facts hold.

(i) Let $S_{k+1} = g(1, S_k)$ with $S_0 = \bar{M}_0 = P_0$. Then $S_\infty \leq S_k \leq \bar{M}_k$.

(ii) $F(\bar{M}_k) = \mathbb{K}_k'H_k\mathbb{K}_k$ is monotonically decreasing, and thus $\mathbb{K}_k'H_k\mathbb{K}_k \leq (\lambda_{\mathbb{K}})^2H_k$.

Proof:

(i) We prove this lemma by mathematical induction. For $k = 0$, this lemma holds. Suppose that it holds for $0, \ldots, n$. We check the case $k = n+1$ as follws. By the hypothesis that $S_n \leq \bar{M}_n$ and Lemma 7 (i), $S_{n+1} = g(1, S_n) \leq g(1, \bar{M}_n) \leq g(\gamma_{n+1}, \bar{M}_n) = \bar{M}_{n+1}$. Consequently, we have $S_k \leq \bar{M}_k$. From Lemma 7 (iii), it follows that $S_\infty \leq S_k \leq \bar{M}_k$. The proof is completed.

(ii) Define $h(S) \triangleq (S^{-1} + C'R^{-1}C)^{-1}C'R^{-1}C$, $f(S) \triangleq I - h(S)$, and $F(S) \triangleq f(S)'H_kf(S)$. By Lemma 8 (ii), we have $h(\bar{M}_k) = K_kC$. Thus, $f(\bar{M}_k) = \mathbb{K}_k$ and $F(\bar{M}_k) = \mathbb{K}_k'H_k\mathbb{K}_k$.
Suppose that $S_1 > S_2$. By Lemma 8 (ii), we have $S_1^{-1} < S_2^{-1}$ and thus $S_1^{-1} + C'R^{-1}C < S_2^{-1} + C'R^{-1}C$. Let $Y = C'R^{-1}C$, and Y^{-1} exists by virtue of the assumption that C is full column rank. By using Lemma 8 (ii) again, we have

$$\begin{aligned}&(S_1^{-1} + Y)^{-1} > (S_2^{-1} + Y)^{-1}\\ &\overset{(a)}{\Rightarrow} \lambda\big((S_2^{-1} + Y)^{-1}YY^{-1}(S_1^{-1} + Y)\big) < 1\\ &\Rightarrow \lambda(h(S_2)h(S_1)^{-1}) < 1 \overset{(b)}{\Rightarrow} h(S_1) > h(S_2)\\ &\overset{(c)}{\Rightarrow} f(S_1) < f(S_2) \overset{(d)}{\Rightarrow} \lambda(f(S_1)f(S_2)^{-1}) < 1,\end{aligned}$$

where the inequalities on the right-hand side of $\overset{(a)}{\Rightarrow}$, $\overset{(b)}{\Rightarrow}$, and $\overset{(d)}{\Rightarrow}$ are obtained by using Lemma 8 (i), and $\overset{(c)}{\Rightarrow}$ is obtained by noting that $f(S_1) < f(S_2)$ due to $f(S) = I - h(S)$.
To compare $f(S_1)'H_kf(S_1)$ with $f(S_2)'H_kf(S_2)$, we consider the following inequalities.

$$\begin{aligned}&\lambda\Big((f(S_1)f(S_2)^{-1})'H_k(f(S_1)f(S_2)^{-1})H_k^{-1}\Big)\\ &\leq \lambda\Big((f(S_1)f(S_2)^{-1})'\Big)\lambda\Big(H_kf(S_1)(f(S_2)^{-1}H_k^{-1}\Big)\\ &= \Big(\lambda\big(f(S_1)f(S_2)^{-1}\big)\Big)^2 < 1.\end{aligned}$$

From Lemma 8 (i), it follows that $(f(S_1)f(S_2)^{-1})'H_kf(S_1)(f(S_2))^{-1} < H_k$, which means that $f(S_1)'H_kf(S_1) < f(S_2)'H_kf(S_2)$, i.e., $F(S_1) < F(S_2)$. From the result in part (i), we have $\mathbb{K}_k'H_k\mathbb{K}_k = F(\bar{M}_k) \leq F(S_\infty) = \mathbb{K}'H_k\mathbb{K} \leq (\lambda_{\mathbb{K}})^2H_k$.

The proof is completed. ■

Lemma 10: Define $\bar{Z}_{k+1} = \Phi_Z(\bar{Z}_k, \bar{G}_k, \rho)$ and $\bar{G}_{k+1} = \Phi_G(\bar{Z}_k, \bar{G}_k, \eta)$ with $\bar{Z}_0 = Z_0, \bar{G}_k = G_0$. Then

$$\bar{Z}_k \geq Z_k, \bar{G}_k \geq G_k. \quad (45)$$

Proof: From (44c) and by using Lemma 9, we have

$$\begin{aligned}G_{k+1} &= A'(\gamma\mathbb{K}_k'H_k\mathbb{K}_k + \bar{\gamma}H_k + Z_k)A + W\\ &\leq (\gamma(\lambda_{\mathbb{K}})^2 + \bar{\gamma})A'H_kA + A'Z_kA + W\\ &= \eta A'G_kA + (1-\eta)A'Z_kA + W\\ &= \Phi_G(Z_k, G_k, \rho). \quad (46)\end{aligned}$$

From (34b), $Z_k + \bar{\tau}\bar{\nu}T_k \leq \bar{\tau}\bar{\nu}(\gamma\lambda_{\mathbb{K}}^2 + \bar{\gamma})(H_k + Z_k - Z_k) + Z_k = \rho G_k + (1-\rho)Z_k$. By Lemma 7 (i),

$$\begin{aligned}Z_{k+1} &= \Phi_X(Z_k, \bar{\tau}\bar{\nu}T_k)\\ &\leq \Phi_X(Z_k, (1-\rho)Z_k + \rho G_k) = \Phi_Z(Z_k, G_k, \rho). \quad (47)\end{aligned}$$

We prove this lemma by mathematical induction. It is clear that (45) holds for $k = 0$. Suppose that it holds for $0, \ldots, n$. We check the case $k = n+1$ as follows. From (46) (47) and Lemma 7 (i), we have $G_{n+1} \leq \Phi_G(Z_n, G_n, \rho) \leq \Phi_G(\bar{Z}_n, \bar{G}_n, \rho) = \bar{G}_{n+1}$, and $Z_{n+1} \leq \Phi_Z(Z_n, G_n, \rho) \leq \Phi_Z(\bar{Z}_n, \bar{G}_n, \rho) = \bar{Z}_{n+1}$. The proof is completed. ■

Proof of part (ii) of Theorem 4: From Lemma 7 (ii), it follows that if Condition 1 is satisfied and the inequality $\lambda_A^2(\eta+\nu-2\eta\nu) < (\eta+\nu-\eta\nu)$ holds, then $\bar{Z}_k$ and $\bar{G}_k$ are convergent and thus are bounded. By lemma 10, Z_k and G_k are bounded as well. ■

V. Numerical Examples

In this section, by examples, we evaluate the performance of the proposed LQG controller and verify the main results we obtained.

A. Stability of the closed-loop system:

In the following, we present two examples to verify the mean square stability of the closed-loop system.

Example 2: Consider the unstable MIMO system in [17] with following parameters:

$$A=\begin{bmatrix}1.001 & 0.005 & 0 & 0\\ 0.350 & 1.001 & -0.135 & 0\\ -0.001 & 0 & 1.001 & 0.005\\ -0.375 & -0.001 & 0.590 & 1.001\end{bmatrix} B=\begin{bmatrix}0.001\\ 0.540\\ -0.002\\ -1.066\end{bmatrix}$$

$$C = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \; R = \mathrm{diag}(0.001, 0.001),\; Q = qq'$$

where $q = [0.003, 1, -0.005, -2.150]'$, $W = \mathrm{diag}(1, 0, 0, 0)$, $\Lambda = 2$.

Since Condistions 1 and 2 are not satisfied for this MIMO system, the boundedness of Z_k and G_k cannot be theoratically determined via Theorem 4 (ii). Thus, we check their boundedness by simulation. By runnig the simulation 1000 times with randomly generated $\{\gamma_k, \nu_k, \tau_k\}$, we found that Z_k and G_k are always bounded. One of these running results is shown in Fig. 2. From Theorem 4 (i), it follows that the closed-loop system is mean square stable, as shown in Fig. 2.

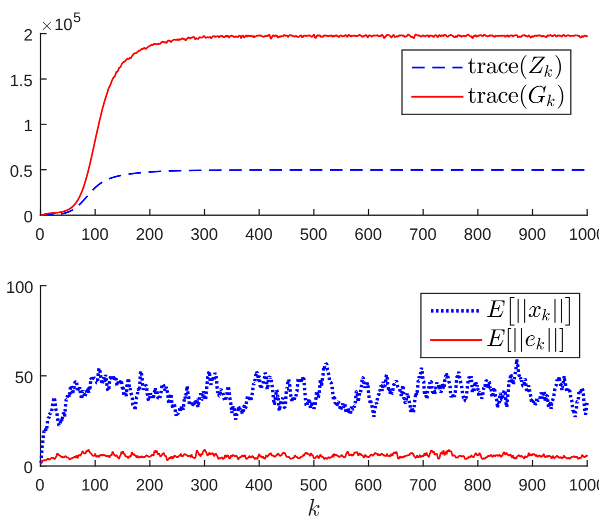

Fig. 2. The boundedness of Z_k and G_k, and the mean square stability of the MIMO system.

Example 3: Consider the scalar unstable system used in [19, 21] with the parameters: $A = 1.1$, $B = C = 1$, $Q = R = W = \Lambda = 1$. Let $\gamma = \nu = \tau = 0.8$. It is easy to check that for this system, Condistions 1 and 2 are satisfied, and the inequality $\lambda_A^2(\eta+\nu-2\eta\nu) < (\eta+\nu-\eta\nu)$ holds. It follows from Theorem 4 (ii) that Z_k and G_k are bounded, which guarantees the mean square stability of the closed-loop system. These results are illustrated in Fig. 3.

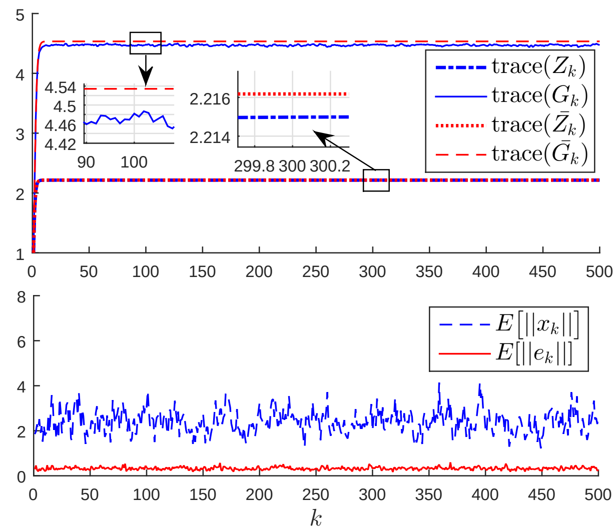

Fig. 3. The boundedness of Z_k and G_k, and the mean square stability of the scalar system.

B. Performance of the sub-optimal estimator and controller

Example 4: To compare the estimation performance of the sub-optimal estimator we proposed with that of the LMMSE estimator, we consider the quantity $\Delta_P = P_k - P_k^L$, where P_k^L denotes the estimation error covariance of the LMMSE estimator. For the scalar system above, Δ_P is shown in Fig.4. If the control inputs are deterministic and are independent of the estimates, like the cases $u_k \equiv 10$ or 20, then the larger the magnitude of control inputs is, the more apparant the difference between their estimation performances becomes. However, it is in the background of the LQG control problem that we propose this sub-optimal estimator. Once the control inputs are determined by the LQG controller we design, the performances between these two estimators are closer, as shown in Fig. 4. The main reason is that when the system is stabilized by the LQG controller, the system states are usually near zero and then the magnitude of the feedback control is small. From (29), it is known that $\bar{P}_k$ is closer to the prediction error covariance of the LMMSE estimator [23]. Thus, P_k is closer to P_k^L.

Example 5: For the LQG problem, J_k/k is usually adopted to evaluate the long term performance. For the scalar (i.e., SISO) and MIMO systems presented in Examples 1 and 2, the J_k/k is not convergent, which is shown in Fig. 5. When the ACK packet arrival rate τ is close to 1, the performance of the proposed LQG controller approches that of the optimal LQG controller for the TCP-like system (that is, the "Real TCP-like" case in Fig. 5).

VI. Conclusions

In this paper, for the general Quasi-TCP-like systems we have proposed the optimal estimator and given a conclusion

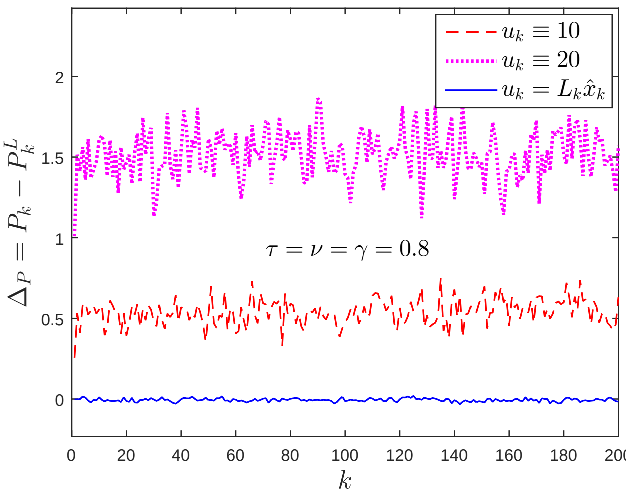

Fig. 4. The difference of the performances between the sub-optimal estimator we proposed and that of the LMMSE estimator.

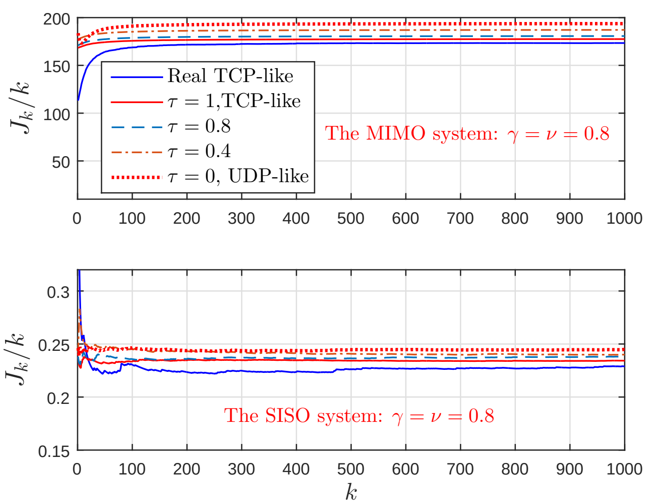

Fig. 5. The average cost function J_k/k.

on the solvability of the optimal LQG controller. Also a sub-optimal LQG controller is designed. Examples are given to demonstrate the potential and effectiveness of the proposed LQG controller.

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