Robust Stability Analysis and Synthesis for Uncertain Discrete-Time Networked Control Systems Over Fading Channels

Lanlan Su, *Student Member, IEEE,* and Graziano Chesi, *Senior Member, IEEE*

***Abstract*—This paper investigates uncertain discrete-time networked control systems over fading channels. It is assumed that the plant is affected by polytopic uncertainty and is connected to the controller in closed-loop via fading channels which are modeled by multiplicative noise processes. Three contributions are proposed as follows. First, it is shown that robust stability in the mean square sense of the uncertain closed-loop networked control system is equivalent to the existence of a Lyapunov function in a certain class. Second, it is shown that the existence of a Lyapunov function in such a class is equivalent to the feasibility of a set of linear matrix inequalities (LMIs). Third, it is shown that the proposed condition can be exploited for the synthesis of robust controllers ensuring robust stability in the mean square sense of the uncertain closed-loop networked control system.**

***Index Terms*—Uncertain systems, Robust stability, Networked control systems.**

I. Introduction

In recent years, there has been a wide interest in networked control systems. In this research area, a fundamental issue is stabilization with information constraints in the channels, see, e.g., the special issue [1]. Considering different information constraints, numerous methods have been proposed by researchers for analysis and synthesis. Among the methods, one should mention [2], [3] which deal with quantized feedback control, [4], [5] which investigate the time delay, [6], [7] where constraints on the signal-to-noise ratio are considered, and [8], [9] where the problem of resource allocation is addressed. On the other hand, as it often happens when dealing with real systems, the model of the plant is partially or completely unknown, which leads to a further research problem, networked control of uncertain systems. See, for instance, [10] which studies delayed feedback control of uncertain systems, [11] where quantized feedback control for a class of uncertain autoregressive plants is considered, [12] which exploits the uncertainty-dependent Lyapunov function to handle the time-varying uncertainty caused by quantization.

A key issue that researchers have been recently started to deal with is the presence of fading channels in the networked control systems. Fading channels are frequently encountered in networked control systems, particularly in the wireless communication channels, and they arise due to several randomly time-varying factors, including signal attenuation, signal distortion, packet drop, and disturbance, see, e.g., [13]–[16] for more details. In the context of stochastic process including fading channel, stabilization has been studied in the mean square sense, see e.g. [16]–[19].

This paper investigates uncertain discrete-time networked control systems over fading channels. Specifically, we consider the model identical in [15], with the difference that the coefficients of the model are function of polytopic uncertainty. Three contributions are proposed as follows. First, it is shown that robust stability in the mean square sense of the uncertain closed-loop networked control system is equivalent to the existence of a Lyapunov function in a certain class. Second, it is shown that the existence of a Lyapunov function in such a class is equivalent to the feasibility of a set of LMIs. Third, it is shown that the proposed condition can be exploited for the synthesis of robust controllers ensuring robust stability in the mean square sense of the uncertain closed-loop networked control system. A conference version of this paper (without the controller synthesis) appeared as reported in [20]. It is worth remarking that sufficient and necessary conditions based on convex optimization for the problem considered in this paper have never been proposed in the literature.

L. Su and G. Chesi are with the Department of Electrical and Electronic Engineering, University of Hong Kong, Hong Kong (e-mail: llsu@eee.hku.hk).

II. Preliminaries

The notation is as follows. The sets of real numbers and nonnegative integers are denoted by $\mathbb{R}$ and $\mathbb{N}$. The set $\mathbb{R}_0^r$ is $\mathbb{R}^r\backslash\{0_r\}$. The identity matrix is denoted by I. For symmetric matrices A and B of same size, the notation $A \geq B$ (respectively, $A > B$) denotes that $A - B$ is positive semidefinite (respectively, positive definite). The trace and the determinant of a square matrix A are denoted by $\mathrm{tr}(A)$ and $\det(A)$, respectively. The symbols $\otimes$ and $\odot$ denote the Kronecker product and the Hadamard product, respectively. For scalars $x_1, \ldots, x_n$, the notation $\mathrm{diag}(x_1, \ldots, x_n)$ denotes the diagonal matrix having its (i,i)-th entry equal to x_i, $i = 1, \ldots, n$. For a vector $x = (x_1, \ldots, x_n)'$, the notation x^2 denotes the vector of squares $x = (x_1^2, \ldots, x_n^2)'$ and x^y denotes $\prod_{i=1}^r x_i^{y_i}$ with $y \in \mathbb{N}^r$. For a matrix polynomial $A(x)$, the notation $\deg(A)$ denotes the maximum of the degrees of the entries of $A(x)$. The mathematical expectation operator is denoted by $\mathcal{E}(\cdot)$. The acronym SOS stands for sum of squares of matrix polynomials.

We consider uncertain discrete-time LTI plants of the form

$$\left\{ \begin{array}{rcl} x(t+1) & = & A(s)x(t) + B(s)u(t) \\ y(t) & = & C(s)x(t) \\ s & \in & \mathcal{S} \end{array} \right. \quad (1)$$

where the integer t denotes the time, $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ is the plant input, $y(t) \in \mathbb{R}^p$ is the plant output, and $s \in \mathbb{R}^r$ is the time-invariant uncertainty. Let us assume that the uncertainty s is constrained into the simplex $\mathcal{S}$, i.e.,

$$\mathcal{S} = \left\{ s \in \mathbb{R}^r : \sum_{i=1}^{r} s_i = 1, \ s_i \geq 0 \ \forall i = 1, \ldots, r \right\}. \tag{2}$$

The matrix functions $A : \mathbb{R}^r \to \mathbb{R}^{n \times n}$, $B : \mathbb{R}^r \to \mathbb{R}^{n \times m}$, $C : \mathbb{R}^r \to \mathbb{R}^{p \times n}$ are matrix polynomials, i.e., matrices where each entry is a polynomial. Without loss of generality, we assume that $(A(s), B(s), C(s))$ is stabilizable and detectable for all $s \in \mathcal{S}$.

We consider that the uncertain plant (1) is controlled over fading channels in closed-loop via output feedback. The controller is assumed to be LTI[1], described by

$$\left\{ \begin{array}{rcl} x_c(t+1) & = & A_c x_c(t) + B_c y(t) \\ v(t) & = & C_c x_c(t) + D_c y(t) \end{array} \right. \tag{3}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, $v(t) \in \mathbb{R}^m$ is the controller output and $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$ are the controller matrices.

The fading channels between the output of the controller and the input of the plant are modeled via

$$u(t) = \Xi(t) v(t). \tag{4}$$

The matrix function $\Xi(t) \in \mathbb{R}^{m \times m}$ represents the channel fading and has the diagonal structure

$$\Xi(t) = \mathrm{diag}\{\xi_1(t), \xi_2(t), \ldots, \xi_m(t)\}, \tag{5}$$

where $\xi_i(t)$, $i = 1, 2, \ldots, m$ are scalar noise processes with

$$\mu_i = \mathcal{E}\left(\xi_i(t)\right) > 0 \quad \forall i = 1, \ldots, m. \tag{6}$$

Each noise process $\xi_i(t)$, $i = 1, \ldots, m$ presents the fading effect of each channel in the form of multiplicative memoryless noise, i.e., the $i - th$ output of the controller is sent to the $i - th$ input of the plant over the $i - th$ channel represented by $\xi_i(t)$. However, the fading experience of different channels is allowed to be correlated in case a non-orthogonal access scheme is adopted. Let us further define

$$\left\{ \begin{array}{rcl} \Pi & = & \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_m\} \\ \Sigma & = & [\sigma_{ij}]_{i,j=1,2,\ldots,m} \\ \sigma_{ij} & = & \mathcal{E}((\xi_i(t) - \mu_i)(\xi_j(t) - \mu_j)) \ \forall i, j = 1, \ldots, m. \end{array} \right. \tag{7}$$

Let us stack the states of the plant and controller into $x_{cl} \in \mathbb{R}^{n_{cl}}$ defined as

$$x_{cl}(t) = \begin{pmatrix} x(t)' & x_c(t)' \end{pmatrix}' \tag{8}$$

where $n_{cl} = n + n_c$. According to the above description of the plant (1) and the controller (3), the overall closed-loop system obtained by connecting them over the fading channels (4)-(7) can be described by

$$\left\{ \begin{array}{rcl} x_{cl}(t+1) & = & A_{cl}(s) x_{cl}(t) \\ s & \in & \mathcal{S} \end{array} \right. \tag{9}$$

[1] We restrict our attention to LTI controllers as done in [15] because the problem under consideration is challenging and unsolved also in this case. Nevertheless, we think that it is not restrictive to consider LTI controllers given the nature of the plant and fading channel.

where

$$A_{cl}(s) = E(s) + F(s)\Xi(t)G(s) \tag{10}$$

and

$$\left\{ \begin{array}{rcl} E(s) & = & \begin{pmatrix} A(s) & 0_{n \times n_c} \\ B_c C(s) & A_c \end{pmatrix} \\ F(s) & = & \begin{pmatrix} B(s) \\ 0_{n_c \times m} \end{pmatrix} \\ G(s) & = & \begin{pmatrix} D_c C(s) & C_c \end{pmatrix}. \end{array} \right. \tag{11}$$

Let us further define

$$X_{cl}(t) = \mathcal{E}(x_{cl}(t) x_{cl}(t)'). \tag{12}$$

Extending the definition of mean square stability in [16] to the presence of uncertainties as considered in (1), we introduce the following definition.

Definition 1. The uncertain closed-loop system (9) is said to be robustly stable in the mean square sense for all $s \in \mathcal{S}$ if $X_{cl}(t)$ is well-defined and

$$\lim_{t \to \infty} X_{cl}(t) = 0 \quad \forall x_{cl}(0) \in \mathbb{R}^{n_{cl}} \ \forall s \in \mathcal{S}. \tag{13}$$

The problems addressed in this paper are as follows.

Problem 1. Establish whether the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$.

Problem 2. Design a robust controller (3) such that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$.

Remark 1. The formulation of the uncertain plant (1) is equivalent to the case where the uncertainty belongs to a generic bounded convex polytope instead of the simplex which is a special class of bounded convex polytope. Indeed, consider

$$\left\{ \begin{array}{rcl} x(t+1) & = & \hat{A}(\theta) x(t) + \hat{B}(\theta) u(t) \\ y(t) & = & \hat{C}(\theta) x(t) \\ \theta & \in & \Theta \end{array} \right. \tag{14}$$

where Θ is a bounded convex polytope and $\hat{A}(\theta)$, $\hat{B}(\theta)$ and $\hat{C}(\theta)$ are matrix polynomials in θ. The vectors θ in Θ can be parametrized through a linear function $l(s)$ over $\mathcal{S}$. Therefore, (14) can be rewritten as in (1) by expressing θ as $l(s)$.

III. Proposed Approach

A. Equivalence Result via Polynomial Lyapunov Functions

Let us start by addressing Problem 1. Let $V \in \mathbb{R}^{n_{cl} \times n_{cl}}$ be a symmetric matrix, and define the symmetric matrix polynomial

$$U(V, s) = V - H(V, s) - G(s)' J(V, s) G(s) \tag{15}$$

where H, $J : \mathbb{R}^{n_{cl} \times n_{cl}} \times \mathbb{R}^r \to \mathbb{R}^{n_{cl} \times n_{cl}}$ are the symmetric matrix polynomials

$$H(V, s) = \left(E(s) + F(s)\Pi G(s)\right)' V \left(E(s) + F(s)\Pi G(s)\right) \tag{16}$$

and

$$J(V, s) = \Sigma \odot \left(F(s)' V F(s)\right) \tag{17}$$

where E, F, G are defined in (11). Next, let $M : \mathbb{R}^r \to \mathbb{R}^{n_1 \times n_2}$ be a matrix polynomial, and let $N : \mathbb{R}^r \to \mathbb{R}^{n_1 \times n_2}$ be the matrix homogeneous polynomial that satisfies

$$\left\{ \begin{array}{rcl} N(s) & = & M(s) \ \forall s \in \mathcal{S} \\ \deg(N) & = & \deg(M). \end{array} \right. \tag{18}$$

Such a matrix homogeneous polynomial $N(s)$ can be simply built by multiplying each monomial of $M(s)$ by a suitable power of $\sum_{i=1}^r s_i$, since $\sum_{i=1}^r s_i = 1$ over $\mathcal{S}$. We denote the operator that returns $N(s)$ from $M(s)$ as

$$N(s) = \hom(M(s)). \tag{19}$$

Theorem 1: The uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$ if and only if there exists a symmetric matrix homogeneous polynomial $V : \mathbb{R}^r \to \mathbb{R}^{n_{cl} \times n_{cl}}$ such that

$$\forall s \in \mathcal{S} \left\{ \begin{array}{rcl} V(s) & > & 0 \\ U(V(s), s) & > & 0. \end{array} \right. \tag{20}$$

Proof. "$\Rightarrow$" Suppose that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. Then, from the item d) of Lemma 1 in [15] it directly follows that, for all $s \in \mathcal{S}$, there exists a symmetric matrix function $V(s)$ such that (20) holds. Following the proof of Lemma 2 in the same reference, let us define

$$Y_1(V(s), s) = H(V(s), s) + G(s)' J(V(s), s) G(s)$$

and the sequence

$$Z(t+1, s) = Y_1(Z(t, s), s) + Y_0(s)$$

for any initial condition $Z(0, s)$ and any symmetric matrix function $Y_0(s)$ that satisfy

$$\forall s \in \mathcal{S} \quad \left\{ \begin{array}{rcl} Z(0, s) & \geq & 0 \\ Y_0(s) & > & 0. \end{array} \right.$$

Since from (20) one has $Z(t, s) > Y_1(Z(t, s), s)$ for all $s \in \mathcal{S}$, and since this implies that $Z(t, s)$ is bounded for all $t \geq 0$ and for all $s \in \mathcal{S}$, it follows that the limit $Z(s)$ of $Z(t, s)$ for t that goes to infinity exists and satisfies

$$Z(s) = Y_1(Z(s), s) + Y_0(s).$$

This implies that the system (by choosing $Y_0(s) = I$)

$$\left\{ \begin{array}{rcl} U(W(s), s) & = & I \\ s & \in & \mathcal{S} \end{array} \right.$$

admits a unique solution $W(s)$, i.e., the limit of $z(t, s)$, that satisfies

$$W(s) > 0 \ \ \forall s \in \mathcal{S}.$$

Since $U(W(s), s)$ is linear in $W(s)$ according to (15), and since $U(W_0, s)$ is polynomial in s for all symmetric matrices W_0, it follows that $W(s)$ is a rational function of s that we write as

$$W(s) = \frac{W_{num}(s)}{W_{den}(s)}$$

where $W_{num}(s)$ is a symmetric matrix polynomial and $W_{den}(s)$ is a polynomial. One has $W_{den}(s) \neq 0 \ \forall s \in \mathcal{S}$ given that the solution is unique. This implies that $W_{den}(s) > 0 \ \forall s \in \mathcal{S}$ or $W_{den}(s) < 0 \ \forall s \in \mathcal{S}$ since $W_{den}(s)$ is a polynomial on a compact and connected set $\mathcal{S}$. In the latter case, we change the sign of $W_{den}(s)$ and $W_{num}(s)$ since this does not change $W(s)$. It follows that W_{den} satisfies

$$W_{den}(s) > 0 \ \ \forall s \in \mathcal{S}.$$

Let us define

$$V_{pol}(s) = W_{num}(s).$$

It follows that $V_{pol}(s)$ is a symmetric matrix polynomial. Let us redefine $V(s)$ as follows:

$$V(s) = \hom(V_{pol}(s)).$$

It follows that $V(s)$ is a symmetric matrix homogeneous polynomial. Moreover, $V(s) > 0$ for all $s \in \mathcal{S}$ since both $W_{num}(s)$ and $W_{den}(s)$ satisfy such a condition and since $V(s) = V_{pol}(s)$ for all $s \in \mathcal{S}$. Also, since $U(V(s), s)$ is linear in $V(s)$, one has that the constructed $V(s)$ yields

$$U(V(s), s) = W_{den}(s) I \ \ \forall s \in \mathcal{S}$$

which means that $U(V(s), s) > 0$ for all $s \in \mathcal{S}$ given the positivity of $W_{den}(s)$ over $\mathcal{S}$. Hence, (20) holds with a symmetric matrix homogeneous polynomial $V(s)$.

"$\Leftarrow$" Suppose that there exists a symmetric matrix homogeneous polynomial $V(s)$ such that (20) holds. Then, from Lemma 1 in [15] it directly follows that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. □

Remark 2. Theorem 1 states that robust stability in the mean square sense of the uncertain closed-loop networked control system is equivalent to the existence of a Lyapunov function in a certain class, specifically the class of Lyapunov functions of the form

$$v(X_{cl}(t), s) = \mathrm{tr}\,(X_{cl}(t) V(s)) \tag{21}$$

where $V(s)$ is a symmetric matrix homogeneous polynomial and $v(X_{cl}(t), s)$ is the Lyapunov function associated with the $V(s)$ searched for in Theorem 1. Let us observe that this equivalence result cannot be obtained by simply using the existing works for the uncertainty-free case, which only imply the existence of a Lyapunov function (of some form) for the uncertain case. On the other hand, the fact that $V(s)$ can be chosen polynomial according to Theorem 1 is essential in order to derive a necessary and sufficient condition for robust stability in terms of LMIs as it will be explained in the sequel.

B. Equivalence Result via LMIs

Here we show how Theorem 1 can be exploited to derive a sufficient and necessary condition checkable through convex optimization for establishing whether the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. Let us define

$$T(V(s), s) = \hom(U(V(s), s)). \tag{22}$$

Theorem 2: The uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$ if and only if there exist a symmetric matrix homogeneous polynomial $V(s)$ and $\epsilon > 0$ such that

$$\left\{ \begin{array}{rcl} V(s^2) - \epsilon I ||s||^{2deg(V)} & \text{is} & \text{SOS} \\ T(V(s^2), s^2) - \epsilon I ||s||^{2deg(T)} & \text{is} & \text{SOS}. \end{array} \right. \tag{23}$$

Proof. "$\Leftarrow$" Assume that there exists $\epsilon > 0$ such that (23) holds. Then, one has that

$$\forall s \in \mathbb{R}_0^r \left\{ \begin{array}{rcl} V(s^2) & > & 0 \\ T(V(s^2), s^2) & > & 0. \end{array} \right.$$

Since $T(V(s), s)$ is linear in $V(s)$, from Theorem 1.17 in [21] one has

$$\forall s \in \mathcal{S} \left\{ \begin{array}{rcl} V(s) & > & 0 \\ T(V(s), s) & > & 0. \end{array} \right.$$

Hence, (20) holds with such $V(s)$ since

$$T(V(s), s) = U(V(s), s) \; \forall s \in \mathcal{S}.$$

Therefore, from Theorem 1 we can conclude that the closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$.

"$\Rightarrow$" Suppose that the closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. Then, from Theorem 1 it follows that there exists a symmetric matrix homogeneous polynomial $V(s)$ such that (20) holds. Since $\mathcal{S}$ is compact, i.e., $\Sigma_{i=1}^r s_i = 1$ over $\mathcal{S}$, and

$$T(V(s), s) = U(V(s), s) \quad \forall s \in \mathcal{S},$$

it follows that there exists $\epsilon_1 > 0$ such that

$$\forall s \in \mathcal{S} \left\{ \begin{array}{rcl} V(s) - \epsilon_1 I \left(\sum_{i=1}^r s_i \right)^{\deg(V)} & > & 0 \\ T(V(s), s) - \epsilon_1 I \left(\sum_{i=1}^r s_i \right)^{\deg(T)} & > & 0. \end{array} \right.$$

From $Pólya's$ theorem [22], one has that a homogeneous matrix polynomial $M(s) : \mathbb{R}^r \to \mathbb{R}^{n \times n}$ is positive definite on the simplex $\mathcal{S}$ in (2) if and only if there exists $k \in \mathbb{N}$ such that all the matrix coefficients with respect to s of $M(s) \left(\sum_{i=1}^r s_i \right)^k$ are positive definite. We observe that any integer $k' \geq k$ will satisfy that all the matrix coefficients of $M(s)(\sum_{i=1}^r s_i)^{k'}$ are positive definite. Next, let us apply this to (20). For the positive definite matrix homogeneous polynomial $V(s) - \epsilon_1 I(\sum_{i=1}^r s_i)^{\deg(V)}$, one has that there exists an integer k_1 such that all the matrix coefficients of

$$\tilde{V}(s) = \left(V(s) - \epsilon_1 I \left(\sum_{i=1}^r s_i \right)^{\deg(V)} \right) \left(\sum_{i=1}^r s_i \right)^{k_1}$$

are positive definite, and hence $\tilde{V}(s)$ can be written as $\tilde{V}(s) = \sum_{y \in \zeta} C_y s^y$, where $\zeta = \{y \in \mathbb{N}^r : \sum_{i=1}^r y_i = \deg(V) + k_1\}$ and all $C_y \in \mathbb{R}^{n_{cl} \times n_{cl}}$ are the positive definite matrix coefficients. In this way, one has

$$\left\{ \begin{array}{rcl} \tilde{V}(s^2) & = & \sum_{y \in \zeta} C_y s^{2y} \\ C_y s^{2y} & = & (G_y s^y)'(G_y s^y) \end{array} \right.$$

where $G_y \in \mathbb{R}^{n_{cl} \times n_{cl}}$ is a Cholesky factor of C_y. This implies $\tilde{V}(s^2)$ is SOS. Therefore,

$$\left(V(s^2) - \epsilon_1 I \left(\sum_{i=1}^r s_i^2 \right)^{\deg(V)} \right) \left(\sum_{i=1}^r s_i^2 \right)^{k_1} \quad \text{is} \quad \text{SOS} .$$

For $T(V(s), s) - \epsilon_1 I(\sum_{i=1}^r s_i)^{\deg(T)}$, similarly, there exists an integer $k_2 \geq 0$ such that

$$\left(T(V(s^2), s^2) - \epsilon_1 I \left(\sum_{i=1}^r s_i^2 \right)^{\deg(T)} \right) (\sum_{i=1}^r s_i^2)^{k_2} \quad \text{is} \quad \text{SOS.}$$

Let us further define $k = \max\{k_1, k_2\}$

$$\left\{ \begin{array}{l} \left(V(s^2) - \epsilon_1 I \left(\sum_{i=1}^r s_i^2 \right)^{\deg(V)} \right) \left(\sum_{i=1}^r s_i^2 \right)^k \text{is SOS} \\ \left(T(V(s^2), s^2) - \epsilon_1 I \left(\sum_{i=1}^r s_i^2 \right)^{\deg(T)} \right) \left(\sum_{i=1}^r s_i^2 \right)^k \text{is SOS.} \end{array} \right.$$

Hence, (23) holds with $V(s)$ replaced by $V(s)(\sum_{i=1}^r s_i)^k$ and ϵ replaced by ϵ_1. $\square$

Remark 3. Theorem 2 provides a sufficient and necessary condition checkable through convex optimization for establishing whether the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. Indeed, the condition for a matrix polynomial depending linearly on some decision variables to be SOS can be equivalently expressed in terms of an LMI, see for instance [22] and references therein for details. Let us observe that the equivalence result provided by Theorem 2 is not obvious since it is known that there is gap between nonnegative polynomials and SOS polynomials, see for instance [23].

In practice, we start by testing the LMI condition for small degrees and repeat for larger degrees if the condition is not satisfied. Observe that the LMI condition is guaranteed to be nonconservative for some degree sufficiently large as proved in Theorem 2.

C. Controller Synthesis

Lastly, let us address Problem 2, i.e., to design a robust controller (3) such that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. Let us introduce an index to quantify the feasibility of the LMI condition provided in Theorem 2. Such an index can be defined via the semidefinite program (SDP)

$$\begin{array}{rcl} \epsilon^* & = & \sup_{V(s),\epsilon} \epsilon \\ & & \text{s.t.} \left\{ \begin{array}{ll} (23) & \text{holds} \\ \mathrm{tr}(V(s_0)) & = 1. \end{array} \right. \end{array} \tag{24}$$

where s_0 is a vector arbitrarily chosen in $\mathcal{S}$. Although the optimization problem (24) cannot be solved explicitly even for SISO plants with one uncertain variable, the solution can be easily found because SDPs are convex optimization problems.

From Theorem 2 we obtain the following result.

Corollary 1: There exists a robust controller (3) such that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$ if and only if there exist matrices A_c, B_c, C_c, D_c such that $\epsilon^* > 0$.

Corollary 1 provides a sufficient and necessary condition for the existence of a robust controller (3). This condition provides a strategy for the synthesis of such a robust controller, which consists of maximizing the index ϵ with respect to the matrices A_c, B_c, C_c, D_c. This step can be addressed in several ways, for instance by using gradient methods or randomized algorithms, where the SDP (24) is solved at each step. Let us observe that the constraint $\mathrm{tr}(V(s_0)) = 1$ can be introduced in the SDP (24) without loss of generality because the variables $V(s)$ and ϵ in the condition (23) are defined up to a positive scalar factor.

Hereafter we propose a method of solving Problem 2. To this end, let us observe that the matrix inequalities (20) can be rewritten in the form which is bilinear in the Lyapunov function and the controller. Indeed, exploiting the Schur complement lemma, let us define

$$\Gamma(s) = \begin{pmatrix} V(s) & \left((E(s)+F(s)\Pi G(s))'\,V(s)\right. & G(s)'J(V(s),s) \\ * & V(s) & 0 \\ * & * & J(V(s),s) \end{pmatrix} \tag{25}$$

and

$$\Omega(s) = \mathrm{hom}(\Gamma(s)). \tag{26}$$

Theorem 3: There exists a robust controller (3) such that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$ if and only if there exist a symmetric matrix homogeneous polynomial $V : \mathbb{R}^r \to \mathbb{R}^{n_{cl}\times n_{cl}}$ and matrices A_c, B_c, C_c, D_c such that $\hat{\gamma} > 0$ where

$$\begin{array}{rcl} \hat{\gamma} & = & \sup\limits_{\gamma} \gamma \\ & \text{s.t.} & \left\{ \begin{array}{rcl} \Omega(s^2) - \gamma I ||s||^{2\deg(\Omega)} & \text{is} & \text{SOS} \\ \mathrm{tr}(\Gamma(s_0)) & = & 1. \end{array} \right. \end{array} \tag{27}$$

Proof. " $\Leftarrow$ " Suppose that $\hat{\gamma} > 0$. This implies that

$$\Omega(s^2) > 0\,\forall s \in \mathbb{R}_0^r.$$

From Theorem 1.17 in [21] one can obtain

$$\Omega(s^2) > 0\,\forall s \in \mathcal{S}$$

which suggests

$$\Gamma(s) > 0\,\forall s \in \mathcal{S}.$$

According to the Schur complement lemma, the following two inequalities are equivalent:

$$\Gamma(s) > 0$$

and

$$\left\{ \begin{array}{rcl} V(s) & > & 0 \\ V(s) - H(V,s) - G(s)'J(V(s),s)G(s) & > & 0. \end{array} \right.$$

Therefore, the condition (20) in Theorem 1 holds.

"$\Rightarrow$" Assume that there exists a robust controller (3) such that the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $s \in \mathcal{S}$. Based on the necessity of the condition on Theorem 1 and Schur complement lemma, it is obvious that there exist a symmetric matrix homogeneous polynomial $V : \mathbb{R}^r \to \mathbb{R}^{n_{cl}\times n_{cl}}$ and matrices A_c, B_c, C_c, D_c such that $\Gamma(s) > 0\; s \in \mathcal{S}$. Since $\mathcal{S}$ is compact, then following the proof in Theorem 2, one can easily obtain $\hat{\gamma}$ defined in (27) should be positive. □

Theorem 3 provides a sufficient and necessary condition for the existence of a robust controller (3). This condition provides a strategy for the synthesis of such a robust controller, which consists of maximizing the index γ with respect to the matrices A_c, B_c, C_c, D_c and the symmetric matrix homogeneous polynomial $V(s)$. This step can be done by simply iterating between the variables (A_c, B_c, C_c, D_c) and $V(s)$ since the symmetric matrix homogeneous polynomial in (27) are bilinear with respect to the decision variables $V(s)$ and (A_c, B_c, C_c, D_c).

IV. Illustrative Examples

A. Example 1

Let us begin with the situation where the uncertain plant is described as in (14) with

$$\left\{ \begin{array}{c} \hat{A}(\theta) = \begin{pmatrix} -0.3+0.6\theta & 0.6-0.4\theta & -1+0.8\theta & 1 \\ 0 & 0.7 & 0.3\theta & 0 \\ 0.6-0.5\theta & -0.8 & 0 & 0 \\ -0.8\theta & 0.7\theta & -\theta & 0.5\theta \end{pmatrix} \\ \hat{B}(\theta) = \begin{pmatrix} \theta & 0 & 1 & 0 \\ 1 & 0 & 0 & \theta \end{pmatrix}',\ \hat{C}(\theta) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ \Theta = [-1, 1] \end{array} \right.$$

the controller (3) is static and given by

$$A_c = \emptyset,\ B_c = \emptyset,\ C_c = \emptyset,\ D_c = \begin{pmatrix} -0.6 & 0 \\ 0.3 & 0.5 \end{pmatrix},$$

and the fading channel (4)-(7) is described by

$$\Pi = \mathrm{diag}(1,2),\ \Sigma = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}.$$

The problem is to establish whether the uncertain closed-loop system (9) is robustly stable in the mean square sense for all $\theta \in \Theta$. First of all, let us observe that the uncertain plant is unstable for some admissible values of the uncertainty:

$$\theta = -1 \;\Rightarrow\; \mathrm{spec}(\hat{A}(\theta)) = \{0.078, 0.752, -0.765 \pm j1.018\}.$$

Then let us rewrite the uncertain plant as in (1). As explained in Remark 1, this can be done by replacing θ with $l(s) = s_2 - s_1$ where s belongs to the simplex with $r = 2$.

We solve the SDP (24) by using a symmetric matrix homogeneous polynomial $V(s)$ of degree 0, and we find that the solution of the SDP (24) is $\epsilon^* = 0.031$ (the number of LMI scalar variables in (24) is 26). Therefore, from Theorem 1 we can conclude that the uncertain plant (1) is robustly stabilized in the mean square sense for all $\theta \in \Theta$.

B. Example 2

Let us consider another example, in particular where the uncertain plant (1) is

$$A(s) = \begin{pmatrix} 1 & s_1 \\ s_2 & 1.2 \end{pmatrix},\ B(s) = \begin{pmatrix} 1 & s_1 \\ 1 & 1 \end{pmatrix},\ C(s) = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and the fading channel (4)-(7) is described by

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Sigma = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}.$$

The uncertain plant is unstable for some admissible values of the uncertainty since we have

$$s = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}' \Rightarrow \text{spec}(A(s)) = \{0.590, 1.610\}.$$

The problem is to synthesize the robustly stabilizing controller (A_c, B_c, C_c, D_c) where $A_c, B_c \in \mathbb{R}$ and $C_c, D_c \in \mathbb{R}^{2\times 1}$. With the initial value of the controller (A_0, B_0, C_0, D_0) set as $(0,\, 0,\, [0\ \ 0]', [0\ \ 0]')$, we solve the optimization problem (27) by using a symmetric matrix homogeneous polynomial $V(s)$ of degree 2. After 4 iterations between the variables (A_c, B_c, C_c, D_c) and $V(s)$, we obtain positive index γ as $\gamma = 0.63 > 0$. The found corresponding controller obtained is

$$A_c = 0,\ B_c = 0,\ C_c = \begin{pmatrix} 0 & 0 \end{pmatrix}',\ D_c = \begin{pmatrix} -0.401 & -0.097 \end{pmatrix}'$$

and such a controller is static. We can also choose other initial values of the controller. Indeed, with the initial value of the controller (A_0, B_0, C_0, D_0) set as $(1,\, 1,\, [-1\ -1]', [-1\ -1]')$, we solve the optimization problem (27) by using a symmetric matrix homogeneous polynomial $V(s)$ of degree 0. After 6 iterations between the variables (A_c, B_c, C_c, D_c) and $V(s)$, we obtain positive index γ as $\gamma = 0.043 > 0$. The found corresponding controller obtained is

$$A_c = 0.417,\ B_c = 0.876,\ C_c = \begin{pmatrix} -0.025 \\ 0.033 \end{pmatrix},\ D_c = \begin{pmatrix} -0.275 \\ -0.667 \end{pmatrix}$$

and such a controller is dynamic. With the dynamic controller derived, Figure 1 shows the trajectory of $||x_{cl}(t)||$ under different randomly generated initial conditions and different $s \in \mathcal{S}$ which make the plant unstable. As shown in the figure, the closed-loop state converges to origin within finite steps for all $s \in \mathcal{S}$ and all initial conditions.

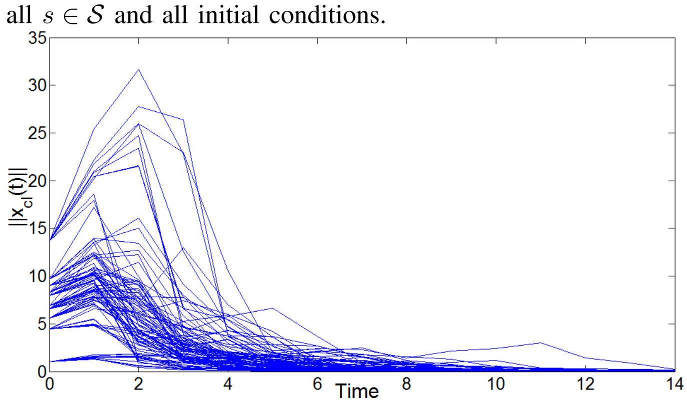

Fig. 1. $||x_{cl}(t)||$ versus t under 10 sets of initial conditions and 10 different uncertainty $s \in \mathcal{S}$ for each initial condition

V. Conclusions

This paper has studied uncertain discrete-time networked control systems over fading channels. A sufficient and necessary condition in terms of LMI is proposed to establish the robust stability in the mean square sense and it has been shown that the proposed condition can be exploited for the synthesis of robust controllers ensuring robust stability in the mean square sense of the uncertain closed-loop networked control system. Future work could investigate the case with time-varying uncertainty in the system using other classes of Lyapunov function [12]. Another direction could be incorporating the time delay in the model.

Acknowledgement

The authors thank the Associate Editor and the Reviewers for their useful comments.

References

[1] P. Antsaklis and J. Baillieul, "Guest editorial special issue on networked control systems," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1421–1423, 2004.

[2] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, 2001.

[3] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1698–1711, 2005.

[4] R. H. Middleton and D. E. Miller, "On the achievable delay margin using lti control for unstable plants," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1194–1207, 2007.

[5] D. Yue, E. Tian, and Q.-L. Han, "A delay system method for designing event-triggered controllers of networked control systems," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 475–481, 2013.

[6] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1391–1403, 2007.

[7] M. Barforooshan and R. M. Esfanjani, "Output feedback stabilization over finite-bandwidth communication channels with signal-to-noise ratio constraint," *Journal of Mathematical Control and Information*, p. dnv009, 2015.

[8] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 554–568, 2013.

[9] M. Tabbara, A. Rantzer, and D. Nešić, "On controller and capacity allocation co-design for networked control systems," *Syst. Control Lett.*, vol. 58, no. 9, pp. 672–676, 2009.

[10] D. Yue and Q.-L. Han, "Delayed feedback control of uncertain systems with time-varying input delay," *Automatica*, vol. 41, no. 2, pp. 233–240, 2005.

[11] X. Kang and H. Ishii, "Coarsest quantization for networked control of uncertain linear systems," *Automatica*, vol. 51, pp. 1–8, 2015.

[12] H. Gao and T. Chen, "A new approach to quantized feedback control systems," *Automatica*, vol. 44, no. 2, pp. 534–542, 2008.

[13] S. Dey, A. S. Leong, and J. S. Evans, "Kalman filtering with faded measurements," *Automatica*, vol. 45, no. 10, pp. 2223–2233, 2009.

[14] G. Gu and L. Qiu, "Networked feedback control over fading channels and the relation to H2 control," in *2012 International Conf. on Information and Automation (ICIA)*. IEEE, 2012, pp. 247–252.

[15] N. Xiao, L. Xie, and L. Qiu, "Feedback stabilization of discrete-time networked systems over fading channels," *IEEE Trans. Autom. Control*, vol. 57, no. 9, pp. 2176–2189, 2012.

[16] N. Elia, "Remote stabilization over fading channels," *Syst. Control Lett.*, vol. 54, no. 3, pp. 237–249, 2005.

[17] K. You, M. Fu, and L. Xie, "Mean square stability for kalman filtering with markovian packet losses," *Automatica*, vol. 47, no. 12, pp. 2647–2657, 2011.

[18] O. L. Costa and M. D. Fragoso, "Stability results for discrete-time linear systems with markovian jumping parameters," *Journal of mathematical analysis and applications*, vol. 179, no. 1, pp. 154–178, 1993.

[19] M. D. Fragoso and O. L. Costa, "A unified approach for stochastic and mean square stability of continuous-time linear systems with markovian jumping parameters and additive disturbances," *SIAM journal on control and optimization*, vol. 44, no. 4, pp. 1165–1191, 2005.

[20] L. Su and G. Chesi, "On the robust stability of uncertain discrete-time networked control systems over fading channels," in *American Control Conf.*, 2015, pp. 6010–6015.

[21] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems*. Springer, 2009.

[22] G. Chesi, "LMI techniques for optimization over polynomials in control: a survey," *IEEE Trans. Autom. Control*, vol. 55, no. 11, pp. 2500–2510, 2010.

[23] ——, "On the gap between positive polynomials and SOS of polynomials," *IEEE Trans. Autom. Control*, vol. 52, no. 6, pp. 1066–1072, 2007.