Stabilization and Entropy Reduction via SDP-Based Design of Fixed-Order Output Feedback Controllers and Tuning Parameters

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Abstract

This paper addresses the problem of designing fixed-order output feedback controllers and tuning parameters for reducing the instability of linear time-invariant (LTI) systems. Specifically, continuous-time (CT) and discrete-time (DT) LTI systems are considered, whose coefficients are rational functions of design parameters that are searched for in a given semi-algebraic set. Two instability measures are considered, the first defined as the spectral abscissa (CT case) or the spectral radius (DT case), and the second defined as the sum of the real parts of the unstable eigenvalues (CT case) or the product of the magnitudes of the unstable eigenvalues (DT case). Two sufficient conditions are given for establishing either the non-existence or the existence of design parameters that reduce the considered instability measure under a desired value. These conditions require to solve a semidefinite program (SDP), which is a convex optimization problem, and to find the roots of a multivariate polynomial, which is a difficult problem in general. To overcome this difficulty, a technique based on linear algebra operations is exploited, which easily provides the sought roots in common cases by taking into account the structure of the polynomial under consideration. Also, it is shown that these conditions are also necessary by increasing enough the size of the SDP under some mild assumptions. Lastly, it is explained how the proposed methodology can be used to search for design parameters that minimize a given cost function while reducing the instability.

I. Introduction

Instability measures play a key role in control systems. For LTI systems, a commonly used instability measure is the spectral abscissa (CT case) or the spectral radius (DT case). This instability measure, which will be referred to as spectral measure (SM), is important for several reasons, for instance because tells whether the system is asymptotically stable and reveals the speed of the least stable modes. Hence, the problem of reducing the SM is of fundamental importance. However, problems such as the design of fixed-order output feedback controllers or tuning parameters for achieving such a goal are notoriously difficult to solve. Indeed, by using classic stability conditions based on Lyapunov functions and linear matrix inequalities (LMIs) [3], one generally faces the problem of establishing feasibility of bilinear matrix inequalities (BMIs) due to the product of the coefficients of the Lyapunov function with the coefficients of the controller or the tuning parameters, which unfortunately involve non-convex optimization. For the case of static output feedback controllers, an approach based on the use of Hermite matrices is proposed in [15], where LMI relaxations based on the theory of moments are derived.

Another instability measure of interest for linear systems is the sum of the real parts of the unstable eigenvalues (CT case) or the product of the magnitudes of the unstable eigenvalues (DT case). This instability measure, that in the DT case is known as Mahler measure [19], will be referred to as entropy measure (EM) being strictly related to the entropy of LTI systems, see [2], [23] for details. The EM is important because allows one to establish whether a stabilizing controller can be designed in a number of scenarios characterized by the presence of communication constraints. Indeed, this is shown in the contexts of quantized feedback stabilization [12], data rate constrained mean square stabilizability [21], data rate constrained observability and stabilizability [27], stabilization with sector bound uncertainty [13], signal-to-noise ratio (SNR) constrained feedback stabilization [4], and stabilization with multirate sampling [5]. See Section II-B for details, and see also [18], [20] for other applications of the EM. However, similarly to the case of the SM, design problems for reducing the EM generally involve non-convex optimization.

This paper proposes a novel framework for addressing the above mentioned problems, namely the design of fixed-order output feedback controllers and tuning parameters for reducing the SM and EM. Specifically, CT and DT LTI systems are considered, whose coefficients are rational functions of design parameters that are searched for in a given semi-algebraic set. First, two sufficient conditions are given for establishing either the non-existence or the existence of design parameters that reduce the considered instability measure under a desired value. These conditions require to solve an SDP, which is a convex optimization problem, and to find the roots of a multivariate polynomial, which is a difficult problem in general. To overcome this difficulty, a technique based on linear algebra operations is exploited, which easily provides the sought roots in common cases by taking into account the structure of the polynomial under consideration. These conditions are obtained by introducing eigenvalue combinations and modified stability tables, and by exploiting polynomials that can be expressed as sums of squares of polynomials (SOS). Second, it is shown that these conditions are also necessary by increasing enough the size of the SDP under some mild assumptions on the semi-algebraic set. Such assumptions concern the polynomial inequalities used to define the semi-algebraic set, and are shown to be automatically satisfied in typical cases. Third, it is explained how the proposed methodology can be used to search for design parameters that minimize a given cost function while reducing the instability.

This paper extends the preliminary conference version [9] which does not consider the case of DT systems, the SM measure, and the minimization of a given cost function. It is worth mentioning that the SM and EM are studied in [7], [8], [10], which address the determination of worst-cases values of these measures in the contexts of uncertain systems and nonlinear systems.

The paper is organized as follows. Section II introduces some preliminaries. Section III derives the sufficient conditions. Section IV investigates the necessity of these conditions. Section V discusses the specializations and extensions. Section VI presents some illustrative examples. Lastly, Section VII concludes the paper with some final remarks.

II. Preliminaries

This section provides the problem formulation, the motivation, and some information about SOS polynomials.

A. Problem Formulation

The notation used in the paper is as follows. The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the spaces of real numbers and complex numbers. We denote with 0 and I the null matrix and the identity matrix of size specified by the context. The transpose is denoted by A'. The expressions $A > 0$ and $A \geq 0$, where A is a real symmetric matrix, denote a positive definite matrix and a positive semidefinite matrix. The quantities $\Re(a)$, $\Im(a)$, and $|a|$ are the real part, imaginary part, and magnitude of $a \in \mathbb{C}$. The adjoint, determinant, image, null space, spectrum, and trace of a matrix A are denoted by $\mathrm{adj}(A)$, $\det(A)$, $\mathrm{img}(A)$, $\ker(A)$, $\mathrm{spec}(A)$, and $\mathrm{trace}(A)$. The function $\mathrm{sgn}(a)$, with $a \in \mathbb{R}$, denotes the sign function, i.e., 1 if $a > 0$, 0 if $a = 0$, and -1 if $a < 0$.

The expressions $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the largest integer not greater than $a \in \mathbb{R}$ and the smallest integer not smaller than $a \in \mathbb{R}$. The notation $\deg(p(v))$ denotes the degree of the polynomial $p(v)$. We say that an eigenvalue is unstable if it has positive real part (CT case) or magnitude greater than 1 (DT case). We say that a matrix is asymptotically stable if all its eigenvalues have negative real part (CT case) or magnitude less than 1 (DT case). We say that an univariate polynomial is asymptotically stable if all its roots have negative real part (CT case) or magnitude less than 1 (DT case).

Let us consider the parametric LTI system

$$\delta(x(t)) = A(v)x(t) \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x \in \mathbb{R}^n$ is the state, $\delta(\cdot)$ is the operator

$$\delta(x(t)) = \begin{cases} \dot{x}(t) & \text{(CT case)} \\ x(t+1) & \text{(DT case)}, \end{cases} \tag{2}$$

$v \in \mathbb{R}^m$ is the vector of design parameters, and $A : \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is a rational matrix function that we express as

$$A(v) = \frac{A_{num}(v)}{a_{den}(v)} \tag{3}$$

where $A_{num} : \mathbb{R}^m \to \mathbb{R}^{n \times n}$ and $a_{den} : \mathbb{R}^m \to \mathbb{R}$ are matrix polynomials.

In the sequel, the vector of design parameters will be searched for into the semi-algebraic set

$$\mathcal{V} = \{v \in \mathbb{R}^m : \ w_i(v) \geq 0 \ \forall i = 1, \ldots, n_w\} \tag{4}$$

where $w_i(v)$, $i = 1, \ldots, n_w$, are polynomials. Semi-algebraic sets, in fact, can represent a large class of sets, in particular sets that are connected or disconnected, convex or non-convex, bounded or unbounded.

In order to ensure that $A(v)$ does exist for all the admissible values of the design parameters, we introduce the well-posedness condition

$$|a_{den}(v)| \geq \zeta \quad \forall v \in \mathcal{V} \tag{5}$$

where $\zeta \in \mathbb{R}$, $\zeta > 0$, is a chosen threshold. Let us observe that the well-posedness condition (5) introduces a minor restriction on the problem addressed, as the threshold ζ can be chosen arbitrary small. We define the set

$$\mathcal{Z} = \{v \in \mathbb{R}^m : \ \text{(5) holds}\}. \tag{6}$$

We consider the following two instability measures of a matrix $X \in \mathbb{R}^{n\times n}$. The first is the spectral abscissa (CT case) or the spectral radius (DT case). This measure is referred to as spectral measure (SM), and is denoted by

$$\mu_{SM}(X) = \begin{cases} \max\limits_{i=1,\ldots,n} \Re\left(\lambda_i(X)\right) & \text{(CT case)} \\ \max\limits_{i=1,\ldots,n} |\lambda_i(X)| & \text{(DT case).} \end{cases} \tag{7}$$

The second instability measure is the sum of the real parts of the unstable eigenvalues (CT case) or the product of the magnitude of the unstable eigenvalues (DT case). This measure, that in the DT case is known as Mahler measure [19], is referred to as entropy measure (EM) being strictly related to the entropy of LTI systems [2], [23], and is denoted by

$$\mu_{EM}(X) = \begin{cases} \sum\limits_{i=1}^{n} \max\left\{0, \Re\left(\lambda_i(X)\right)\right\} & \text{(CT case)} \\ \prod\limits_{i=1}^{n} \max\left\{1, |\lambda_i(X)|\right\} & \text{(DT case).} \end{cases} \tag{8}$$

For brevity of presentation, we denote all these measures with the common function

$$\mu(X) = \begin{cases} \mu_{SM}(X) & \text{(SM case)} \\ \mu_{EM}(X) & \text{(EM case).} \end{cases} \tag{9}$$

Let us introduce the constant

$$\mu_0 = \begin{cases} -\infty & \text{(CT \& SM case)} \\ 0 & \text{(DT \& SM case)} \\ 0 & \text{(CT \& EM case)} \\ 1 & \text{(DT \& EM case).} \end{cases} \tag{10}$$

For a given $\psi \in \mathbb{R}$, $\psi > \mu_0$, let us define the set

$$\mathcal{U} = \left\{v \in \mathbb{R}^m : \ \mu(A(v)) < \psi\right\}. \tag{11}$$

The problem addressed in this paper is formulated as follows.

Problem 1: Establish whether the set of sought design parameters

$$\mathcal{S} = \mathcal{V} \cap \mathcal{Z} \cap \mathcal{U} \tag{12}$$

is non-empty and, if yes, find a vector v in this set.

□

B. Motivation

Problem 1 includes important problems in control systems. Hereafter we present two of them, namely the design of stabilizing fixed-order output feedback controllers and the design of tuning parameters for reducing the entropy. It turns out that these problems are difficult to solve being non-convex optimization problems.

1) **Design of stabilizing fixed-order output feedback controllers:** One of the problems included in Problem 1 is the design of stabilizing fixed-order output feedback controllers. Indeed, let us denote the plant as

$$\left\{ \begin{array}{rcl} \delta(x_{pla}(t)) & = & A_{pla}x_{pla}(t) + B_{pla}u(t) \\ y(t) & = & C_{pla}x_{pla}(t) + D_{pla}u(t) \end{array} \right. \tag{13}$$

where $x_{pla}(t) \in \mathbb{R}^{n_{pla}}$ is the plant state, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the output, and $A_{pla} \in \mathbb{R}^{n_{pla} \times n_{pla}}$, $B_{pla} \in \mathbb{R}^{n_{pla} \times n_u}$, $C_{pla} \in \mathbb{R}^{n_y \times n_{pla}}$ and $D_{pla} \in \mathbb{R}^{n_y \times n_u}$ are given matrices. Then, let us denote the fixed-order output feedback controller as

$$\left\{ \begin{array}{rcl} \delta(x_{con}(t)) & = & A_{con}(v)x_{con}(t) + B_{con}(v)y(t) \\ u(t) & = & C_{con}(v)x_{con}(t) + D_{con}(v)y(t) \end{array} \right. \tag{14}$$

where $x_{con}(t) \in \mathbb{R}^{n_{con}}$ is the controller state, $v \in \mathbb{R}^m$ is the vector of design parameters, and $A_{con}(v) \in \mathbb{R}^{n_{con} \times n_{con}}$, $B_{con}(v) \in \mathbb{R}^{n_{con} \times n_y}$, $C_{con}(v) \in \mathbb{R}^{n_u \times n_{con}}$ and $D_{con}(v) \in \mathbb{R}^{n_u \times n_y}$ are matrix polynomials that define the desired structure of the controller matrices. Also, let us impose that the closed-loop system is well-posed by constraining v with

$$|\det(E(v))| \geq \zeta \tag{15}$$

where

$$E(v) = I - D_{con}(v)D_{pla} \tag{16}$$

and $\zeta \in \mathbb{R}$, $\zeta > 0$, is a chosen threshold. It follows that the closed-loop system obtained from (13)–(14) can be expressed as in (1) where

$$\begin{cases} x(t) = \begin{pmatrix} x_{pla}(t) \\ x_{con}(t) \end{pmatrix} \\ A_{num}(v) = \\ \left(\begin{array}{c} \det(E(v))A_{pla} + B_{pla}\mathrm{adj}(E(v))D_{con}(v)C_{pla} \\ B_{con}(v)\left(\det(E(v))I + D_{pla}\mathrm{adj}(E(v))D_{con}(v)\right)C_{pla} \end{array} \right. \\ \left. \begin{array}{c} B_{pla}\mathrm{adj}(E(v))C_{con}(v) \\ \det(E(v))A_{con}(v) + B_{con}(v)D_{pla}\mathrm{adj}(E(v))C_{con}(v) \end{array} \right) \\ a_{den}(v) = \det(E(v)). \end{cases} \tag{17}$$

Problem 1 boils down to the search for v (and, hence, the controller (14)) such that the closed-loop system is asymptotically stable by simply choosing

$$\mu(\cdot) = \mu_{SM}(\cdot) \tag{18}$$

and

$$\psi = \begin{cases} 0 & \text{(CT \& SM case)} \\ 1 & \text{(DT \& SM case)}. \end{cases} \tag{19}$$

This situation is considered in Examples 2 and 3 in Section VI.

2) **Design of tuning parameters for reducing the entropy:** Another problem included in Problem 1 is the design of tuning parameters for reducing the entropy. Indeed, let us consider the case where v in the system (1) represents a vector of tuning parameters that can be selected in order to achieve a desired performance. In particular, the target is to select v such that the EM is smaller than a certain value. Problem 1 boils down to such a problem by simply choosing

$$\mu(\cdot) = \mu_{EM}(\cdot). \tag{20}$$

This situation is considered in Example 4 in Section VI.

Reducing the entropy is important in a number of scenarios characterized by the presence of communication constraints. Hereafter we mention some of them.

a) **Quantized feedback stabilization**: as explained in Theorem 2.2 of [12], the coarsest quantizer that quadratically stabilizes a single input DT system is logarithmic, and the optimal logarithmic base is given by the ratio between the EM minus one and the EM plus one.

b) **Data rate constrained mean square stabilizability**: as explained in Theorem 2.1 of [21] for the case of DT stochastic systems with noise, a necessary condition for stabilizability in the mean square sense is that the logarithm of the EM is smaller than the data rate of the channel.

c) **Data rate constrained observability and stabilizability**: as explained in Propositions 3.1–3.2 of [27] for the case of DT systems, a necessary condition for asymptotical observability and asymptotical stabilizability is that the logarithm of the EM is smaller than the data rate of the channel.

d) **Stabilization with sector bound uncertainty**: as explained in Theorem 2.1 of [13] for the case of DT single input systems, there exists a quadratically stabilizing state feedback controller in the presence of sector bound uncertainty if and only if the sector bound is smaller than the inverse of the EM.

e) **SNR constrained feedback stabilization**: as explained in Theorem II.1 of [4] for the case of single input CT systems, there exists a stabilizing state feedback controller such that the power of the sent signal is not larger than a desired value if and only if the EM is not larger than a half of the ratio between such a value and the power spectral density of the noise.

f) **Stabilization with multirate sampling**: as explained in Theorems 4.1–4.2 in [5] for the case of CT systems with multirate sampling, there exists a stabilizing state feedback controller if and only if the EM is not greater than the total network capacity.

C. SOS Polynomials

Here we provide some information about SOS polynomials, see for instance [6] for details and references. Let us start by introducing the following definition.

Definition 1: A polynomial $p:\mathbb{R}^m\to\mathbb{R}$ is said to be SOS if there exist polynomials $p_i:\mathbb{R}^m\to\mathbb{R}$, $i=1,\ldots,k$, such that

$$p(v)=\sum_{i=1}^{k}p_i(v)^2. \tag{21}$$

$\square$

SOS polynomials have gained a lot of interest in last years for two main reasons. First, SOS polynomials can be used to express (and, hence, recognize) non-negative polynomials. Second, establishing whether a polynomial is SOS amounts to solving a convex optimization problem. Indeed, a necessary and sufficient condition for establishing whether $p(v)$ is SOS can be obtained via an LMI feasibility test.

In fact, let d be a non-negative integer such that $\deg(p(v)) \leq 2d$. Then, $p(v)$ can be expressed as

$$p(v) = b(v)' P b(v) \tag{22}$$

where $b(v) \in \mathbb{R}^{\sigma(m,d)}$ is a vector containing all monomials of degree less than or equal to d in v, $\sigma(m,d)$ is the number of such monomials given by

$$\sigma(m,d) = \frac{(m+d)!}{m!d!}, \tag{23}$$

and $P \in \mathbb{R}^{\sigma(m,d)\times\sigma(m,d)}$ is a symmetric matrix. The representation (22) is known as Gram matrix method and as square matrix representation (SMR). It follows that $p(v)$ is SOS if and only if there exists $P = P'$ such that

$$\begin{cases} P \geq 0 \\ \text{(22) holds.} \end{cases} \tag{24}$$

The condition (24) is an LMI subject to a linear equality. The number of free decision variables in this condition is given by the number of independent entries of P minus the number of linear constraints imposed by (22), and turns out to be

$$\tau(m,2d) = \frac{1}{2}\sigma(m,d)\left(\sigma(m,d)+1\right) - \sigma(m,2d). \tag{25}$$

III. Proposed Methodology: Sufficiency

The first step of the proposed methodology consists of partitioning the set $\mathcal{Z}$ into two subsets in order to tackle separately the cases where $a_{den}(v)$ is either positive or negative. To this end, let us define the set

$$\Theta = \{-1, 1\}. \tag{26}$$

For $\theta \in \Theta$ we define the polynomial

$$f(v) = \theta a_{den}(v) - \zeta \tag{27}$$

where ζ has been introduced in the well-posedness condition (5). Hence, we partition $\mathcal{Z}$ in (6) as

$$\mathcal{Z} = \bigcup_{\theta\in\Theta} \tilde{\mathcal{Z}} \tag{28}$$

where

$$\tilde{\mathcal{Z}} = \{v \in \mathbb{R}^m : \ f(v) \geq 0\}\,. \tag{29}$$

The second step of the proposed methodology consists of defining a family of matrices for recasting both SM and EM into a common instability measure, in particular the SM. To this end, the eigenvalues of such matrices have to be the sums (CT case) or the products (DT case) of the possible subsets of the eigenvalues of a given matrix $X \in \mathbb{R}^{n\times n}$. Hence, let us define the set

$$\mathcal{K} = \begin{cases} \{1\} & \text{(SM case)} \\ \{1,\dots,n\} & \text{(EM case).} \end{cases} \tag{30}$$

For $k \in \mathcal{K}$, let $\Omega_k(X)$ be a matrix that satisfies the property

$$\operatorname{spec}(\Omega_k(X)) = \begin{cases} \left\{ \sum_{i=1}^{k} \lambda_{a_i}(X),\ a \in \mathcal{T}_k \right\} & \text{(CT case)} \\ \left\{ \prod_{i=1}^{k} \lambda_{a_i}(X),\ a \in \mathcal{T}_k \right\} & \text{(DT case)} \end{cases} \tag{31}$$

where $\mathcal{T}_k$ is the set of k-tuples in $\{1,\dots,n\}$, i.e.,

$$\begin{aligned} \mathcal{T}_k \ = \ & \{(a_1,\dots,a_k) : \ a_i \in \{1,\dots,n\} \\ & \text{and } a_i < a_{i+1}\ \forall i = 1,\dots,k-1\}\,. \end{aligned} \tag{32}$$

One way to build $\Omega_k(X)$ is described in [1] and is as follows. Let $Y \in \mathbb{R}^{k\times k}$ be the submatrix of X built with the rows indexed by $z(i)$ and the columns indexed by $z(j)$. Moreover, let Z be the submatrix of X built similarly to Y by removing from $z(i)$ and $z(j)$ the common entries. Lastly, let z be the difference between the sums of the positions of the common entries in $z(j)$ and in $z(i)$. Then, the (i,j)-th entry of $\Omega_k(X)$ is given by, in the CT case,

$$(\Omega_k(X))_{i,j} = \begin{cases} \operatorname{trace}(Y) & \text{if } i = j \\ (-1)^z Z & \text{else if } Z \text{ has size } 1\times 1 \\ 0 & \text{else} \end{cases} \tag{33}$$

and, in the DT case,

$$(\Omega_k(X))_{i,j} = \det(Y). \tag{34}$$

Some comments about $\Omega_k(X)$ are as follows:

1) the size of $\Omega_k(X)$ is $c_k \times c_k$, where c_k is the binomial coefficient

$$c_k = \frac{n!}{(n-k)!k!}; \tag{35}$$

2) some special cases of $\Omega_k(X)$ are

$$\Omega_1(X) = X \tag{36}$$

and

$$\Omega_n(X) = \begin{cases} \mathrm{trace}(X) & \text{(CT case)} \\ \det(X) & \text{(DT case);} \end{cases} \tag{37}$$

3) $\Omega_k(X)$ is linear in the CT case, and polynomial of degree k in the DT case;

4) in the SM case, only $\Omega_1(X)$ is needed since $\mathcal{K} = \{1\}$.

The next lemma clarifies how $\Omega_k(X)$ can be used to study the instability measure $\mu(X)$.

Lemma 1: One has $\mu(X) < \psi$ if and only if

$$\mu_{SM}\left(\Omega_k(X)\right) < \psi \quad \forall k \in \mathcal{K}. \tag{38}$$

Proof. Let us start by supposing that $\mu(X)$ is the SM. Then, $\mu(X) < \psi$ is equivalent to (38) since $\mathcal{K} = \{1\}$ and $\Omega_1(X) = X$. Next, let us continue by supposing that $\mu(X)$ is the EM. One has $\mu(X) < \psi$ if and only if the sum of the real parts of the unstable eigenvalues of X (CT case) or the product of the magnitudes of the unstable eigenvalues of X (DT case) is smaller than ψ. This holds if and only if the sum of the real parts of any subset of eigenvalues of X (CT case) or the product of the magnitudes of any subsets of eigenvalues of X (DT case) is smaller than ψ. Since X is real, the spectrum is symmetric with respect to the real axis. This implies that the previous condition holds if and only if the sum of any subset of eigenvalues of X (CT case) or the product of any subsets of eigenvalues of X (DT case) is smaller than ψ. Therefore, (38) holds. □

The third step of the proposed methodology consists of introducing a family of polynomials that are asymptotically stable if and only if the vector of design parameters belongs to the set $\mathcal{U}$ in (11). To this end, let us define the matrix polynomial

$$G_k(v) = \begin{cases} \theta(\Omega_k(A_{num}(v)) - \psi a_{den}(v) I) & \text{(CT case)} \\ \dfrac{\theta \Omega_k(A_{num}(v))}{\psi} & \text{(DT case)} \end{cases} \tag{39}$$

where θ and ψ have been introduced in (27) and (11). Observe that $G_k(v)$ is well-defined also in the DT case since $\psi > \mu_0$ and $\mu_0 > 0$. Let $\lambda \in \mathbb{C}$, and define the polynomial

$$g_k(\lambda, v) = \begin{cases} \det(\lambda I - G_k(v)) & \text{(CT case)} \\ \det(\lambda\theta a_{den}^k(v) I - G_k(v)) & \text{(DT case).} \end{cases} \tag{40}$$

The following lemma explains how the polynomial $g_k(\lambda, v)$ can be used to investigate Problem 1.

Lemma 2: Let $v \in \mathcal{Z}$. One has $v \in \mathcal{U}$ if and only if the polynomial $g_k(\lambda, v)$ is asymptotically stable (in the variable λ) for all $k \in \mathcal{K}$ with

$$\theta = \operatorname{sgn}\left(a_{den}(v)\right). \tag{41}$$

Proof. From (3), (15) and Lemma 1 it follows that $v \in \mathcal{U}$ holds if and only if

$$\mu_{SM}\left(\Omega_k\left(\frac{A_{num}(v)}{a_{den}(v)}\right)\right) < \psi \quad \forall k \in \mathcal{K}.$$

Let us start by considering the CT case. Since $\Omega_k(X)$ is linear in this case, it follows that the previous condition holds if and only if

$$\mu_{SM}\left(\Omega_k\left(\theta A_{num}(v)\right)\right) < |a_{den}(v)|\psi \quad \forall k \in \mathcal{K}$$

with θ given by (41). This holds if and only if $G_k(v)$ is asymptotically stable for all $k \in \mathcal{K}$. Since $g_k(\lambda, v)$ is the characteristic polynomial of $G_k(v)$, this holds if and only if $g_k(\lambda, v)$ is asymptotically stable (in the variable λ) for all $k \in \mathcal{K}$.

Next, let us continue by considering the DT case. Since $\Omega_k(X)$ is polynomial of degree k in this case, it follows that $v \in \mathcal{U}$ holds if and only if

$$\mu_{SM}\left(\Omega_k\left(\theta A_{num}(v)\right)\right) < |a_{den}(v)|^k\psi \quad \forall k \in \mathcal{K}$$

with θ given by (41). This holds if and only if $G_k(v)|a_{den}(v)|^{-k}$ is asymptotically stable for all $k \in \mathcal{K}$. Since $g_k(\lambda, v)$ is the product of $|a_{den}(v)|^{c_k k}$ times the characteristic polynomial of $G_k(v)|a_{den}(v)|^{-k}$, this holds if and only if $g_k(\lambda, v)$ is asymptotically stable (in the variable λ) for all $k \in \mathcal{K}$. $\square$

The fourth step of the proposed methodology consists of imposing asymptotic stability on the family of polynomials $g_k(\lambda, v)$ by introducing suitable stability tables. To this end, let us express $g_k(\lambda, v)$ as

$$g_k(\lambda, v) = \sum_{j=0}^{c_k} h_{c_k-j,k}(v)\lambda^j \tag{42}$$

where $h_{0,k}(v), \ldots, h_{c_k,k}(v) \in \mathbb{R}$ are its coefficients. In the CT case, we introduce a modified Routh-Hurwitz table by defining the quantities

$$\begin{cases} m_{0,j,k}(v) &= h_{2j,k}(v) \quad \forall j = 0, \ldots, \left\lfloor \frac{c_k}{2} \right\rfloor \\ m_{1,j,k}(v) &= h_{2j+1,k}(v) \quad \forall j = 0, \ldots, \left\lfloor \frac{c_k - 1}{2} \right\rfloor \\ m_{i,j,k}(v) &= m_{i-1,0,k}(v) m_{i-2,j+1,k}(v) \\ & \quad - m_{i-2,0,k}(v) m_{i-1,j+1,k}(v) \\ & \quad \forall i = 2, \ldots, c_k - 1 \ \forall j = 0, \ldots, \left\lfloor \frac{c_k - i}{2} \right\rfloor \\ m_{c_k,0,k}(v) &= m_{c_k-2,1,k}(v). \end{cases} \tag{43}$$

In particular, $m_{i,j,k}(v)$ is the entry of the table in the i-th row and j-th column. This table is modified with respect to the standard Routh-Hurwitz table [16], [24] because no division is made when obtaining the entries of one row from those of the previous two rows, except for the last row. This ensures that the entries of the modified table are polynomials in v. In the DT case, we introduce a modified Jury table by defining the quantities

$$\begin{cases} m_{0,j,k}(v) &= h_{j,k}(v) \quad \forall j = 0, \ldots, c_k \\ m_{1,j,k}(v) &= h_{c_k-j,k}(v) \quad \forall j = 0, \ldots, c_k \\ m_{2i,j,k}(v) &= m_{2i-2,j,k}(v) m_{2i-1,c_k+1-i,k}(v) \\ & \quad - m_{2i-1,j,k}(v) m_{2i-2,c_k+1-i,k}(v) \\ & \quad \forall i = 1, \ldots, c_k \ \forall j = 0, \ldots, c_k - i \\ m_{2i+1,j,k}(v) &= m_{2i,c_k-i-j,k}(v) \quad \forall i = 1, \ldots, c_k \\ & \quad \forall j = 0, \ldots, c_k - i. \end{cases} \tag{44}$$

In particular, $m_{2i,j,k}(v)$ is the entry of the table in the $2i$-th row and j-th column. This table is modified with respect to the standard Jury table [17] because no division is made when obtaining the entries of one row from those of the previous two rows. As in the CT case, this ensures that the entries of the modified table are polynomials in v. Let us define the set

$$\mathcal{I}_k = \begin{cases} \{1, 2, \ldots, c_k\} & \text{(CT case)} \\ \{0, 2, \ldots, 2c_k\} & \text{(DT case).} \end{cases} \tag{45}$$

The following lemma explains how one can impose asymptotic stability on $g_k(\lambda, v)$ by using the constructed tables.

Lemma 3: The polynomial $g_k(\lambda, v)$ is asymptotically stable (in the variable λ) if and only if

$$m_{i,0,k}(v) > 0 \quad \forall i \in \mathcal{I}_k. \tag{46}$$

Proof. Let us start by considering the CT case. Let us observe that the entry $m_{i,j,k}(v)$ satisfies

$$m_{i,j,k}(v) = \tilde{m}_{i,j,k}(v) \prod_{\substack{l=i-1,i-3,\ldots \\ l \geq 1}} m_{l,0,k}(v) \quad \forall i = 0, \ldots, c_k - 1$$

and

$$m_{c_k,0,k}(v) = \tilde{m}_{c_k,0,k}(v) \prod_{\substack{l=i-3,i-5,\ldots \\ l \geq 1}} m_{l,0,k}(v)$$

where $\tilde{m}_{i,j,k}(v)$ is the entry in the i-th row and j-th column of the standard Routh-Hurwitz table. This implies that $g_k(\lambda, v)$ is asymptotically stable (in the variable λ) if and only if

$$\tilde{m}_{i,0,k}(v) > 0 \quad \forall i \in \mathcal{I}_k,$$

and this conditions holds if and only if (46) holds.

Next, let us continue by considering the DT case. Let us observe that the entry $m_{2i,j,k}(v)$ satisfies

$$m_{2i,j,k}(v) = \tilde{m}_{2i,j,k}(v) \prod_{l=0,\ldots,i-1} m_{2l,0,k}(v)$$

where $\tilde{m}_{2i,j,k}(v)$ is the entry in the $2i$-th row and j-th column of the standard Jury table. This implies that $g_k(\lambda, v)$ is asymptotically stable (in the variable λ) if and only if

$$\tilde{m}_{i,0,k}(v) > 0 \quad \forall i \in \mathcal{I}_k,$$

and this conditions holds if and only if (46) holds. $\square$

The fifth step of the proposed methodology consists of introducing a certificate, based on convex optimization, for establishing that a polynomial is non-negative whenever some polynomials are. To this end, we exploit the Positivstellensatz, see for instance [22], [25]. Let us define such a certificate as follows.

Definition 2: Let $p, q_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n_q$, be polynomials, and d be a non-negative integer. Let us define

$$\mathcal{Q}(v) = \left\{ q_i(v) \ \forall i = 1, \ldots, n_q \right\}. \tag{47}$$

We denote with

$$\mathrm{incone}(p(v), \mathcal{Q}(v), d) \tag{48}$$

the condition

$$\begin{array}{c} \exists \text{ polynomials } r_i(v),\ i = 1, \ldots, n_q : \\ \left\{ \begin{array}{l} r_i(v) \text{ is SOS } \forall i = 1, \ldots, n_q \\ s(v) \text{ is SOS} \\ \deg(q_i(v) r_i(v)) \leq 2(d_0 + d) \end{array} \right. \end{array} \tag{49}$$

where $s(v)$ is the polynomial

$$s(v) = p(v) - \sum_{i=1}^{n_q} q_i(v) r_i(v) \tag{50}$$

and d_0 is the integer

$$d_0 = \left\lceil \frac{1}{2} \max \left\{ \deg(p(v)), \deg(q_1(v)), \ldots, \deg(q_{n_q}(v)) \right\} \right\rceil. \tag{51}$$

□

Definition 2 introduces the condition $\mathrm{incone}(p(v), \mathcal{Q}(v), d)$ which establishes the existence of SOS polynomials $r_i(v)$ with degree bounded by $\deg(q_i(v) r_i(v)) \leq 2(d_0 + d)$ such that the polynomial $s(v)$ in (50) is SOS. From (21)–(24) it follows that the condition $\mathrm{incone}(p(v), \mathcal{Q}(v), d)$ is equivalent to establish feasibility of a finite system of LMIs with finite dimensions.

The following lemma summarizes how the condition $\mathrm{incone}(p(v), \mathcal{Q}(v), d)$ can be used.

Lemma 4: The condition $\mathrm{incone}(p(v), \mathcal{Q}(v), d)$ implies that

$$p(v) \geq 0 \quad \forall v \in \mathbb{R}^m : \ q_i(v) \geq 0 \ \forall i = 1, \ldots, n_q. \tag{52}$$

Proof. Let us suppose that $\text{incone}(p(v), \mathcal{Q}(v), d)$ holds. This implies that $r_i(v)$ and $s(v)$ are SOS and, consequently, non-negative. Let $\tilde{v} \in \mathbb{R}^m$ be such that

$$q_i(\tilde{v}) \geq 0 \ \forall i = 1, \ldots, n_q.$$

From (50) it follows that

$$\begin{aligned} 0 &\leq s(\tilde{v}) \\ &= p(\tilde{v}) - \sum_{i=1}^{n_q} q_i(\tilde{v}) r_i(\tilde{v}) \\ &\leq p(\tilde{v}), \end{aligned}$$

i.e., $p(\tilde{v})$ is non-negative. Therefore, (52) holds. □

The sixth step of the proposed methodology consists of defining an SDP for investigating Problem 1. The goal is to maximize one of the entry of the first column of the modified Routh-Hurwitz table (CT case) or the modified Jury table (DT case) over the set of vector of design parameters that make positive the remaining entries in this column. In fact, if the result of this maximization is positive, one can say that the maximizer of this maximization makes all entries of the first column of these tables positive since the maximizer is a feasible point of the maximization. On the other hand, if the result of this maximization is non-positive, one can say that there does not exist any vector of design parameters that makes all entries of the first column of these tables positive since the chosen entry is negative over the feasible set or the feasible set is empty.

In order to achieve this goal, let us denote with $i_{cost} \in \mathcal{I}_{k_{cost}}$ and $k_{cost} \in \mathcal{K}$ the indices i and k that identify the chosen entry to be maximized. Also, let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be a chosen lower bound for the remaining entries. Let us introduce the set of polynomials

$$\begin{aligned} \mathcal{M}(v) &= \{ m_{i,0,k}(v) - \varepsilon \ \ \forall (i,k) \in \mathcal{I}_k \times \mathcal{K} : \\ & \quad (i,k) \neq (i_{cost}, k_{cost}) \} . \end{aligned} \tag{53}$$

Also, let us define the set of polynomials

$$\mathcal{Q}(v) = \{f(v)\} \cup \mathcal{M}(v) \cup \mathcal{W}(v) \tag{54}$$

where $f(v)$ is given by (27), $\mathcal{M}(v)$ is defined in (53), and

$$\mathcal{W}(v) = \{ w_i(v) \ \ \forall i = 1, \ldots, n_w \} . \tag{55}$$

Let $\gamma \in \mathbb{R}$, and let us define the polynomial

$$p(v) = \gamma - m_{i_{cost},0,k_{cost}}(v). \tag{56}$$

Let d be a non-negative integer, and let us define the optimization problem

$$\begin{aligned} \gamma^* \quad = \quad & \inf_{\gamma \in \mathbb{R}} \ \gamma \\ & \text{s.t.} \ \ \mathrm{incone}(p(v), \mathcal{Q}(v), d). \end{aligned} \tag{57}$$

The optimization problem (57) is an SDP since the cost function is linear and the constraint is equivalent to a finite system of LMIs with finite dimensions.

The following theorem provides a sufficient condition for establishing the non-existence of a vector of design parameters that solves Problem 1.

Theorem 1: Let $\varepsilon = 0$, where ε is used in (53). The set $\mathcal{S}$ in (12) is empty if, for all $\theta \in \Theta$, there exists a non-negative integer d such that

$$\gamma^* \leq 0. \tag{58}$$

Proof. Let us start by considering θ fixed in Θ. Let us suppose that there exists a non-negative integer d such that (58) holds with $\varepsilon = 0$. Let us define

$$\mathcal{F} = \left\{ v \in \mathbb{R}^m : \ q_i(v) \geq 0 \ \ \forall i = 1, \ldots, n_q \right\}.$$

First, let us suppose that $\mathcal{F}$ is non-empty. From Lemma 4 it follows that

$$m_{i_{cost},0,k_{cost}}(v) \leq 0 \ \ \forall v \in \mathcal{F}.$$

From Lemma 3 it follows that $g_{k_cost}(\lambda, v)$ cannot be asymptotically stable for any v in the set $\mathcal{F}$. This means that $g_{k_cost}(\lambda, v)$ cannot be asymptotically stable for any v that belongs $\mathcal{V}$, satisfies $f(v) \geq 0$, and makes $g_k(\lambda, v)$ asymptotically stable for all $k \in \mathcal{K} \setminus \{k_{cost}\}$. From Lemma 2 this implies that there does not exist any $v \in \mathcal{V} \cap \tilde{\mathcal{Z}} \cap \mathcal{U}$.

Second, let us suppose that $\mathcal{F}$ is empty. The fact that $\mathcal{F}$ is empty implies that there does not exist any v that belongs $\mathcal{V}$, satisfies $f(v) \geq 0$, and makes $g_k(\lambda, v)$ asymptotically stable for all $k \in \mathcal{K}$, if $c_{k_{cost}} > 1$, or for all $k \in \mathcal{K} \setminus \{k_{cost}\}$ if $c_{k_{cost}} = 1$. As in the previous case, this implies that there does not exist any $v \in \mathcal{V} \cap \tilde{\mathcal{Z}} \cap \mathcal{U}$.

Next, let us continue by considering that (58) holds for all $\theta \in \Theta$. From the previous part of the proof and (28) it follows that the set $\mathcal{S}$ is empty. □

The condition of Theorem 1 certifies that the set $\mathcal{S}$ is empty. In particular, for a fixed value of $\theta \in \Theta$, (58) guarantees that $\mathcal{V} \cap \tilde{\mathcal{Z}} \cap \mathcal{U}$ is empty.

For any chosen $\theta \in \Theta$ and non-negative integer d, the condition provided by Theorem 1 requires to solve the SDP (57) and to check whether the found γ^* is non-positive. Let us observe that this includes the case where $\gamma^* = -\infty$.

As it will be shown in Section IV, the condition provided by Theorem 1 is not only sufficient but also necessary for a sufficiently large d by introducing some assumptions on Problem 1.

Next, let us suppose that γ^* is finite for some $\theta \in \Theta$ (if not, then $\gamma^* = -\infty$ for all $\theta \in \Theta$, and Theorem 1 certifies the non-existence of a vector of design parameters in the region $\mathcal{Z}$). Let us denote with $r_i^*(v)$ and $s^*(v)$ the optimal values of the polynomials $r_i(v)$ and $s(v)$ in the SDP (57) introduced via Definition 2, and let us observe that $r_i^*(v)$ and $s^*(v)$ are SOS polynomials.

The following theorem provides a sufficient condition for establishing the existence of a vector of design parameters that solves Problem 1.

Theorem 2: The set $\mathcal{S}$ is non-empty if there exist $\theta \in \Theta$, a non-negative integer d, $\varepsilon > 0$ and $v^* \in \mathbb{R}^m$ such that

$$\left\{ \begin{array}{rcl} \gamma^* & > & 0 \\ q_i(v^*) & \geq & 0 \quad \forall i = 1, \ldots, n_q \\ q_i(v^*) r_i^*(v^*) & = & 0 \quad \forall i = 1, \ldots, n_q \\ s^*(v^*) & = & 0. \end{array} \right. \tag{59}$$

Moreover, $v^* \in \mathcal{S}$.

Proof. Let us consider such θ, d, ε and v^* in the sequel of this proof. From the third and fourth conditions in (59) it follows that

$$\begin{array}{rcl} 0 & = & s^*(v^*) \\ & = & \gamma^* - m_{i_{cost},0,k_{cost}}(v^*) - \sum_{i=1}^{n_q} q_i(v^*) r_i^*(v^*) \\ & = & \gamma^* - m_{i_{cost},0,k_{cost}}(v^*) \end{array}$$

which implies from the first constraint in (59) that

$$m_{i_{cost},0,k_{cost}}(v^*) > 0.$$

From the second constraint in (59) and $\varepsilon > 0$ it follows that

$$m_{i,0,k}(v^*) > 0 \quad \forall (i,k) \in \mathcal{I}_k \times \mathcal{K}.$$

The second constraint in (59) also implies that

$$\begin{cases} w_i(v^*) & \geq & 0 \ \forall i = 1, \ldots, n_w \\ f(v^*) & \geq & 0, \end{cases}$$

i.e., $v^* \in \mathcal{V} \cap \tilde{\mathcal{Z}}$. Hence, Lemma 3 implies that the polynomial $g_k(\lambda, v^*)$ is asymptotically stable (in the variable λ) for all $k \in \mathcal{K}$. Let us observe that (41) is satisfied with $v = v^*$ since $v^* \in \mathcal{V} \cap \tilde{\mathcal{Z}}$. Hence, from Lemma 2 it follows that $v^* \in \mathcal{U}$, i.e., $v^* \in \mathcal{S}$. $\square$

Theorem 2 certifies that the set $\mathcal{S}$ is non-empty, and provides a vector of design parameters v^* in such a set. In particular, this vector belongs to $\mathcal{V} \cap \tilde{\mathcal{Z}} \cap \mathcal{U}$.

For any chosen $\theta \in \Theta$ and non-negative integer d, the condition of Theorem 2 requires to solve the SDP (57) and to check whether there exists $v^* \in \mathbb{R}^m$ such that (59) holds.

The search for $v^* \in \mathbb{R}^m$ satisfying (59) can be addressed via linear algebra operations once that the SDP (57) has been solved. Specifically, one determines the candidates for v^* that satisfy the fourth constraint in (59), i.e., the zeros of $s^*(v)$. This operation can be done with the method proposed in [11] for solving systems of polynomial equations as explained hereafter:

1) once that the SDP (57) has been solved, one obtains from the SDP solver a positive semidefinite Gram matrix of $s^*(v)$, i.e., a symmetric matrix $S^* \geq 0$ such that $s^*(v) = b(v)' S^* b(v)$, where $b(v)$ is a vector of monomials in v;
2) since $S^* \geq 0$, one has that $s^*(v) = 0$ if and only if $b(v) \in \ker(S^*)$. Hence, the problem of finding the zeros of $s^*(v)$ is equivalent to the problem of finding vectors of monomials in $\ker(S^*)$;
3) the problem of finding vectors of monomials in $\ker(S^*)$ can be addressed by pivoting operations that reduce the problem to finding the roots of a polynomial in a single variable whenever the dimension of $\ker(S^*)$ is smaller than a certain value as shown in [11]. Alternatively, this step can be solved by computing Cholesky factorizations, column echelon forms, and Schur decompositions as explained in [14].

The following example clarifies the above procedure for determining the zeros of $s^*(v)$.

Example 1. Let us suppose that

$$s^*(v) = (3 - v_1^3 - 2v_2^2)^2 + (1 - 2v_1v_2 - v_1^3 + v_2^3)^2.$$

A positive semidefinite Gram matrix S^* of $s^*(v)$ can be obtained from an SDP solver as explained in Section II-C. By using the SDP solver specified at the beginning of Section VI, the found S^* provides[1]

$$\ker(S^*) = \mathrm{img}(S_1^*),\ \ S_1^* = \begin{pmatrix} -0.355 & 0.330 & -0.085 \\ -0.236 & 0.252 & 0.205 \\ 0.403 & 0.361 & 0.083 \\ -0.238 & 0.190 & 0.402 \\ 0.223 & 0.311 & -0.174 \\ -0.385 & 0.427 & -0.426 \\ -0.294 & 0.136 & 0.596 \\ 0.182 & 0.265 & -0.307 \\ -0.161 & 0.334 & -0.106 \\ 0.507 & 0.427 & 0.332 \end{pmatrix}$$

where $b(v)$ is chosen as $b(v) = (1, v_1, v_2, v_1^2, v_1v_2, v_2^2, v_1^3, v_1^2v_2, v_1v_2^2, v_2^3)'$. Following the method proposed in [11], one builds an equivalent representation of $\ker(S^*)$ by means of pivoting operations, in particular

$$\ker(S^*) = \mathrm{img}(S_2^*),\ \ S_2^* = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 1.656 & 0.117 & -0.790 \\ 0.000 & 1.000 & 0.000 \\ 2.269 & 0.086 & -1.382 \\ -0.612 & 0.642 & 0.658 \\ 0.000 & 0.000 & 1.000 \\ 3.000 & 0.000 & -2.000 \\ -1.072 & 0.494 & 1.032 \\ 0.510 & 0.233 & 0.192 \\ 0.776 & 1.284 & -0.685 \end{pmatrix}.$$

[1]The number of LMI scalar variables is 28, and the computational time is less than 1 second.

From the last row of S_2^* one has that any v such that $b(v) \in \ker(S^*)$ must satisfy

$$0.776 + 1.284v_2 - 0.685v_2^2 - v_2^3 = 0.$$

By simply computing the roots of this equation in v_2, and by reading the value of v_1 in the vector $S_2^*(1, v_2, v_2^2)'$ built for each one of these roots, one obtains that the zeros of $s^*(v)$ are included in the set

$$\left\{ \begin{pmatrix} 0.346 \\ -1.216 \end{pmatrix}, \begin{pmatrix} 0.818 \\ 1.108 \end{pmatrix}, \begin{pmatrix} 1.327 \\ -0.576 \end{pmatrix} \right\}.$$

Lastly, one simply substitutes the vectors of this set into $s^*(v)$, concluding that all of them are zeros of $s^*(v)$. □

See also Section VI where the search for $v^* \in \mathbb{R}^m$ satisfying (59) is illustrated in other numerical examples. Once the candidates for v^* have been determined, one just checks whether any of them satisfies the other constraints in (59).

As it will be shown in Section IV, the condition provided by Theorem 2 is not only sufficient but also necessary for a sufficiently large d by introducing some assumptions on Problem 1.

IV. Proposed Methodology: Necessity

In this section we analyze the conservatism of the sufficient conditions provided by Theorems 1 and 2. Let us start by introducing the following assumption.

Assumption 1: The set $\mathcal{V}$ is non-empty and compact. Moreover, the polynomials $w_i(v)$, $i = 1, \ldots, n_w$, in (4) have even degree and their highest degree forms have no common root except zero. □

It is important to observe that Assumption 1 introduces minor restrictions. Indeed, the set of admissible controllers parameters v has to be non-empty. Moreover, for numerical computation and practical implementation of the controller, it is reasonable to require that this set is compact.

Lastly, the requirement that the polynomials $w_i(v)$ have even degree and their highest degree forms have no common root except zero is automatically satisfied for typical sets such as spheres,

$$\mathcal{V} = \{v \in \mathbb{R}^m : \ v'v \leq 1\}$$
$$\Updownarrow$$
$$\left\{ \begin{array}{rcl} w_1(v) & = & 1 - v'v \\ n_w & = & 1 \end{array} \right.$$

and multi-interval sets,

$$\mathcal{V} = \left\{v \in \mathbb{R}^m : \ [v_i^-, v_i^+], \ i = 1, \ldots, m\right\}$$
$$\Updownarrow$$
$$\left\{ \begin{array}{rcl} w_i(v) & = & (v_i^- - v)(v - v_i^+) \\ n_w & = & m. \end{array} \right.$$

Next, we introduce a change on the construction of the SDP (57) as follows.

Change C1. Any polynomial $q_i(v)$ with odd degree in the set $\mathcal{Q}(v)$ in (54) is replaced in such a set by

$$q_i(v) \quad \rightarrow \quad q_i(v)z(v) \tag{60}$$

where $z(v)$ is any affine linear function such that

$$z(v) > 0 \quad \forall v \in \mathcal{V}. \tag{61}$$

$\square$

Let us observe that Change C1 has the effect of making all the polynomials $q_i(v)$ in the set $\mathcal{Q}(v)$ having even degree, without changing the set of v for which all these polynomials are non-negative (since $\mathcal{V}$ is included in such a set). Also, let us observe that the affine linear function $z(v)$ always exists since $\mathcal{V}$ is compact.

The following result analyzes the necessity of the condition provided by Theorem 1 under Assumption 1.

Theorem 3: Let us suppose that Assumption 1 holds, and let us modify the set $\mathcal{Q}(v)$ in (54) according to Change C1. Let $\varepsilon = 0$. The set $\mathcal{S}$ is empty only if, for all $\theta \in \Theta$, there exists a non-negative integer d such that (58) holds.

Proof. Let us suppose that the set $\mathcal{S}$ is empty. Let $\theta \in \Theta$. From Lemma 2 it follows that, for all $v \in \mathcal{V}$, the polynomial $g_k(\lambda, v)$ is not asymptotically stable for some $k \in \mathcal{K}$ or (41) does not hold. From Lemma 3 one has that, for all $v \in \mathcal{V}$, (46) or (41) does not hold. Let $\mathcal{F}$ be the set introduced in the proof of Theorem 1. First, let us consider the case where $\mathcal{F}$ is non-empty. It follows that

$$m_{i_{cost},0,k_{cost}}(v) \leq 0 \ \forall v \in \mathcal{V} \cap \tilde{\mathcal{Z}}.$$

From Assumption 1, Change C1, and Putinar's Positivstellensatz [22], it follows that

$$\forall \gamma > 0 \ \exists r_i(v) : \ \begin{cases} r_i(v) \text{ is SOS } \forall i = 1, \ldots, n_q \\ s(v) \text{ is SOS}. \end{cases}$$

Since the degrees of the polynomials $r_i(v)$ in (49) arbitrarily increase by increasing d, it follows that $\gamma^* \leq 0$ for a sufficiently large d.

Second, let us consider the case where $\mathcal{F}$ is empty. Without loss of generality, let us suppose that the polynomials $w_i(v)$, $i = 1, \ldots, n_w$, are the first n_w polynomials in the list $q_i(v)$, $i = 1, \ldots, n_q$, i.e.,

$$q_i(v) = w_i(v) \ \forall i = 1, \ldots, n_w.$$

It follows that

$$\mathcal{F} = \mathcal{V} \cap \tilde{\mathcal{F}}$$

where

$$\tilde{\mathcal{F}} = \left\{ v \in \mathbb{R}^m : \ q_i(v) \geq 0 \ \ \forall i = n_w + 1, \ldots, n_q \right\}.$$

Since $\mathcal{F}$ is empty and $\mathcal{V}$ is non-empty, it follows that

$$\forall v \in \mathcal{V} \ \exists i = n_w + 1, \ldots, n_q : \ q_i(v) < 0.$$

Hence, exploiting again Putinar's Positivstellensatz, it follows that

$$\forall \gamma < 0 \ \exists r_i(v) : \ \begin{cases} r_i(v) \text{ is SOS } \forall i = 1, \ldots, n_q \\ s(v) \text{ is SOS}. \end{cases}$$

$\square$

Theorem 3 states that the sufficient condition provided by Theorem 1, which certifies that the set $\mathcal{S}$ is empty, is also necessary for a sufficiently large d, at least when Assumption 1 holds and the set $\mathcal{Q}(v)$ in (54) is modified according to Change C1.

The following result analyzes the necessity of the condition provided by Theorem 2 under Assumption 1.

Theorem 4: Let us suppose that Assumption 1 holds, and let us modify the set $\mathcal{Q}(v)$ in (54) according to Change C1. The set $\mathcal{S}$ is non-empty only if there exist $\theta \in \Theta$, a non-negative integer d, $\varepsilon > 0$ and $v^* \in \mathbb{R}^m$ such that (59) holds. Moreover, $v^* \in \mathcal{S}$.

Proof. Let us suppose that the set $\mathcal{S}$ is non-empty, and let $v \in \mathcal{S}$. From Lemma 2 it follows that the polynomial $g_k(\lambda, v)$ is asymptotically stable for all $k \in \mathcal{K}$ with θ given by (41). From Lemma 3 one has that (46) holds. Hence, there exists $\varepsilon > 0$ such that

$$\mathcal{F} \neq \emptyset$$

where $\mathcal{F}$ is the set introduced in the proof of Theorem 1. Let us consider such an ε in the sequel of this proof. Since $\mathcal{F}$ is compact due to Assumption 1, we can define

$$\tilde{\gamma} = \max_{v \in \mathcal{F}} \; m_{i_{cost},0,k_{cost}}(v)$$

and let $\tilde{v}$ be the maximizer in this optimization problem, i.e.,

$$\left\{ \begin{array}{lcl} \tilde{v} & \in & \mathcal{F} \\ \tilde{\gamma} & = & m_{i_{cost},0,k_{cost}}(\tilde{v}). \end{array} \right.$$

It follows that

$$\tilde{\gamma} > 0.$$

Let us observe that

$$\mathrm{incone}(p(v), \mathcal{Q}(v), d) \quad \Rightarrow \quad \gamma \geq \tilde{\gamma}$$

which implies

$$\gamma^* \geq \tilde{\gamma} \quad \forall d \geq 0.$$

From Assumption 1, Change C1, and Putinar's Positivstellensatz [22], it follows that there exists a non-negative integer d such that

$$\gamma^* = \tilde{\gamma}.$$

Let us consider such a d in the sequel of this proof. Since $s^*(v)$ and $r_i^*(v)$ are SOS polynomials, it follows that

$$\begin{array}{rcl} 0 & \leq & s^*(\tilde{v}) \\ & = & \gamma^* - m_{i_{cost},0,k_{cost}}(\tilde{v}) - \sum_{i=1}^{n_q} q_i(\tilde{v}) r_i^*(\tilde{v}) \\ & = & -\sum_{i=1}^{n_q} q_i(\tilde{v}) r_i^*(\tilde{v}) \\ & \leq & 0 \end{array}$$

since $q_i(\tilde{v}) \geq 0$. This implies that

$$\left\{ \begin{array}{rcl} q_i(\tilde{v}) r_i^*(\tilde{v}) & = & 0 \quad \forall i = 1, \ldots, n_q \\ s^*(\tilde{v}) & = & 0. \end{array} \right.$$

Therefore, (59) holds with $v^* = \tilde{v}$, and $\tilde{v} \in \mathcal{S}$. □

Theorem 4 states that the sufficient condition provided by Theorem 2, which certifies that the set $\mathcal{S}$ is non-empty and provides a vector of design parameters v^* in such a set, is also necessary for a sufficiently large d at least when Assumption 1 holds and the set $\mathcal{Q}(v)$ in (54) is modified according to Change C1.

V. Remarks, Simplifications, and Extensions

This section provides some remarks about the methodology proposed in Sections III and IV, and investigates some simplifications and extensions of interest.

A. Remarks

The first remark concerns the integers i_{cost} and k_{cost} introduced in the definition of the set $\mathcal{M}(v)$ in (53) and in the polynomial $p(v)$ in (56). These integers can be freely selected in the sets $\mathcal{I}_{k_{cost}}$ and $\mathcal{K}$, and have the role of identifying the entry of the first column of the modified Routh-Hurwitz table (CT case) or Jury table (DT case) to be maximized over the set of vector of design parameters that make positive the remaining entries in this column. A criterion for selecting i_{cost} and k_{cost} can be the maximization of the degree of the polynomial $p(v)$ in (56). This criterion is based on the fact that, by maximizing the degree of $p(v)$, one can reduce the degree of $s(v)$ in (50) for multipliers $r_i(v)$ of the same degree.

The second remark is about the scalar ε introduced in the set $\mathcal{M}(v)$ in (53). The role of this scalar is to define a chosen lower bound for the entries of the first column of the modified Routh-Hurwitz table (CT case) or Jury table (DT case), in order to impose that they are positive. In Theorems 1 and 3, this scalar does not need to be selected since $\varepsilon = 0$ is the only possibility. In Theorems 2 and 4, ε has to be positive, and can be chosen as small as the computer precision in order to minimize the conservatism.

The third remark is about the difference between the CT and DT cases. In the DT case, the computational burden of the SDP (57) is larger than in the CT case because the polynomials in the set $\mathcal{Q}(v)$ have higher degree. This is due to the stability tables used for handling the CT and DT cases. Another reason is that the matrix polynomial $\Omega_k(X)$ in (31) is linear in X in the CT case, and polynomial of degree k in the DT case.

B. Simplifications

The first simplification concerns $A(v)$ in the system (1) and its expression in (3). Indeed, the methodology proposed in Sections III and IV can be simplified in any of the following situations:

1) $A(v)$ is a matrix polynomial, i.e., $a_{den}(v) = 1$;
2) $A(v)$ is a rational matrix function and $a_{den}(v)$ is positive for all $v \in \mathcal{V}$.

In the context of fixed-order output feedback controllers design, the situations just mentioned occur whenever one of the following situations occurs:

1) the plant (13) or the controller (14) are strictly proper, i.e., $D_{pla} = 0$ or $D_{con}(v) = 0$;
2) the determinant of $E(v)$ in (16) is positive for all $v \in \mathcal{V}$.

In such situations, one does not need to investigate the case where $a_{den}(v)$ is negative. This means that the SDP (57) needs to be solved only for $\theta = 1$ in Theorems 1–4. Hence, provided that

$$a_{den}(v) \geq \zeta \quad \forall v \in \mathcal{V}, \tag{62}$$

the methodology proposed in Sections III and IV can be simplified by introducing the following change.

Change C2. The sets Θ in (26) and $\mathcal{Q}(v)$ in (54) are replaced by

$$\left\{ \begin{array}{rcl} \Theta & = & \{1\} \\ \mathcal{Q}(v) & = & \mathcal{M}(v) \cup \mathcal{W}(v). \end{array} \right. \tag{63}$$

□

The second simplification is about the polynomials $m_{i,0,k}(v)$ included in the set $\mathcal{M}(v)$ in (53) and in the polynomial $p(v)$ in (56). Let us observe that:

1) one does not need to include in the set $\mathcal{M}(v)$ or in the polynomial $p(v)$ the polynomials $m_{i,0,k}(v)$ that are positive for all $v \in \mathcal{V}$, such as positive constants. Indeed, v is searched for such that $m_{i,0,k}(v)$ is positive over $\mathcal{V}$. This leads to a reduction of the number of multipliers $r_i(v)$ in the condition $\mathrm{incone}(p(v), \mathcal{Q}(v), d)$ and, hence, to a reduction of the number of LMI scalar variables in the SDP (57);
2) if at least one of the polynomials $m_{i,0,k}(v)$, $i \in \mathcal{I}_k$ and $k \in \mathcal{K}$, is known to be non-positive for all $v \in \mathcal{V}$, then the condition (58) is automatically satisfied, and the set $\mathcal{V} \cap \tilde{\mathcal{Z}} \cap \mathcal{U}$ is empty.

C. Extensions

In the previous sections we have addressed Problem 1, which aims at finding a vector of design parameters in the set $\mathcal{S}$. It turns out that the proposed methodology can be extended to find a vector of design parameters in the set $\mathcal{S}$ that minimizes a given cost function. Indeed, let us formulate the problem as follows.

Problem 2: For given $\psi \in \mathbb{R}$, $\psi > \mu_0$, and polynomial $c : \mathbb{R}^m \to \mathbb{R}$, solve

$$c^* = \inf_{v \in \mathcal{S}} c(v). \tag{64}$$

□

In Problem 2, $c(v)$ is a given cost function that one aims at minimizing over the set $\mathcal{S}$. For instance, if $c(v)$ is chosen as $v'v$, one aims at determining the vector of design parameters with the smallest Euclidean norm that satisfies the required constraints. Problem 2 can be addressed by introducing the following change.

Change C3. The set $\mathcal{M}(v)$ in (53) is replaced by

$$\mathcal{M}(v) = \{m_{i,0,k}(v) - \varepsilon \ \ \forall (i,k) \in \mathcal{I}_k \times \mathcal{K}\}\,, \tag{65}$$

the polynomial $p(v)$ in (56) by

$$p(v) = c(v) - \gamma, \tag{66}$$

and the SDP (57) by

$$\begin{aligned} \gamma^* \ \ = \ \ & \sup_{\gamma \in \mathbb{R}} \ \gamma \\ & \text{s.t.} \ \ \mathrm{incone}(p(v), \mathcal{Q}(v), d). \end{aligned} \tag{67}$$

□

In short, Change C3 redefines the set of polynomials $\mathcal{Q}(v)$ by including the polynomial $m_{i_{cost},0,k_{cost}}(v)$ in the set $\mathcal{M}(v)$ that was previously absent, the polynomial $c(v)$ since now we aim at minimizing $c(v)$ rather than maximizing $m_{i_{cost},0,k_{cost}}(v)$, and the SDP (57) since γ is now a lower bound and must be maximized. It follows that, for any non-negative integer d,

$$\gamma^* \leq c^*. \tag{68}$$

Moreover, under Assumption 1, γ^* converges to c^*. The proof is analogous to those of Theorems 1–4 and is omitted for brevity.

VI. Examples

In this section we present some illustrative examples of the proposed methodology. The SDP (57) is solved with the toolbox SeDuMi [26] for Matlab on a standard computer with Windows 10, Intel Core i7, 3.4 GHz, 8 GB RAM. The degree of the polynomial multipliers $r_i(v)$ are bounded according to the last constraint in (49) with $d = 0$ unless specified otherwise. The numbers i_{cost} and k_{cost} are chosen as explained in the first remark in Section V-A. The scalar ε is chosen as $\varepsilon = 0.1$ unless specified otherwise.

A. Example 2

In this example we consider, in the CT case, the design of a static output feedback controller with structural constraints for stabilizing the plant (13) with

$$\begin{cases} A_{pla} = \begin{pmatrix} 1 & -3 & 3 \\ 2.5 & 0 & 6 \\ -0.5 & 2.5 & 0 \end{pmatrix}, \quad B_{pla} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \\ C_{pla} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad D_{pla} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

We consider the following two scenarios.

1) Scenario 1: Here the sought static output feedback controller is chosen as in (14) with

$$\begin{cases} A_{con}(v) = \emptyset, \quad B_{con}(v) = \emptyset \\ C_{con}(v) = \emptyset, \quad D_{con}(v) = \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix} \end{cases}$$

where $v = (v_1, v_2)'$ is the vector of design parameters. The closed-loop system can be expressed as in (1)–(3) with $A_{num}(v)$ and $A_{den}(v)$ given by (17).

The problem consists of finding a vector v in the set $\mathcal{S}$ in (12), where $\mathcal{V} = [-3, 3]^2$, $\mathcal{Z}$ is as in (6) with $\zeta = 0.1$, $\mathcal{U}$ is as in (11) with $\psi = -0.5$, and $\mu(\cdot)$ is the SM.

First of all, let us observe that this plant is unstable, in particular

$$\begin{cases} \text{spec}(A_{pla}) & = & \{-1.313 \pm j1.339, 3.625\} \\ \mu(A_{pla})) & = & 3.625. \end{cases}$$

From (16)–(17) one has

$$\begin{cases} A_{num}(v) = \begin{pmatrix} 1 & -3 & 3 \\ 2.5 + v_1 + v_2 & 0 & 6 + v_1 + v_2 \\ -0.5 + v_2 & 2.5 & v_2 \end{pmatrix} \\ a_{den}(v) = 1. \end{cases}$$

The set $\mathcal{Z}$ in (5) is $\mathcal{Z} = \mathbb{R}^2$.

Let us observe that, since (62) holds, the methodology proposed in Sections III and IV can be simplified with Change C2 in (63). Therefore, in the sequel of this scenario we consider $\Theta = \{1\}$.

We express $\mathcal{V}$ as in (4) by choosing $n_w = 2$ and $w_i(v) = 9 - v_i^2$ for all $i = 1, 2$. The set $\mathcal{Q}(v)$ in (63) is

$$\mathcal{Q}(v) = \{-2.5 - v_2, -10.125 - 6.75v_1 + 4.5v_2, 9 - v_1^2, 9 - v_2^2\}$$

and the polynomial $p(v)$ in (56) is

$$p(v) = \gamma - 20.75 - 5.5v_1 - v_2 + 0.5v_1v_2 - 0.5v_2^2.$$

Solving the SDP (57) we find $\gamma^* = -\infty$ with $\varepsilon = 0$. The polynomials $r_i(v)$ have degree 0, the number of LMI scalar variables is 5, and the computational time is less than 1 second. From Theorem 1 this implies that there does not exist any sought static output feedback controller, i.e., the set $\mathcal{S}$ is empty.

This result is verified by Figure 1 which shows the SM of the closed-loop system over the set $\mathcal{V}$. As it can be seen from Figure 1, the SM is always greater than ψ, indeed it is always positive.

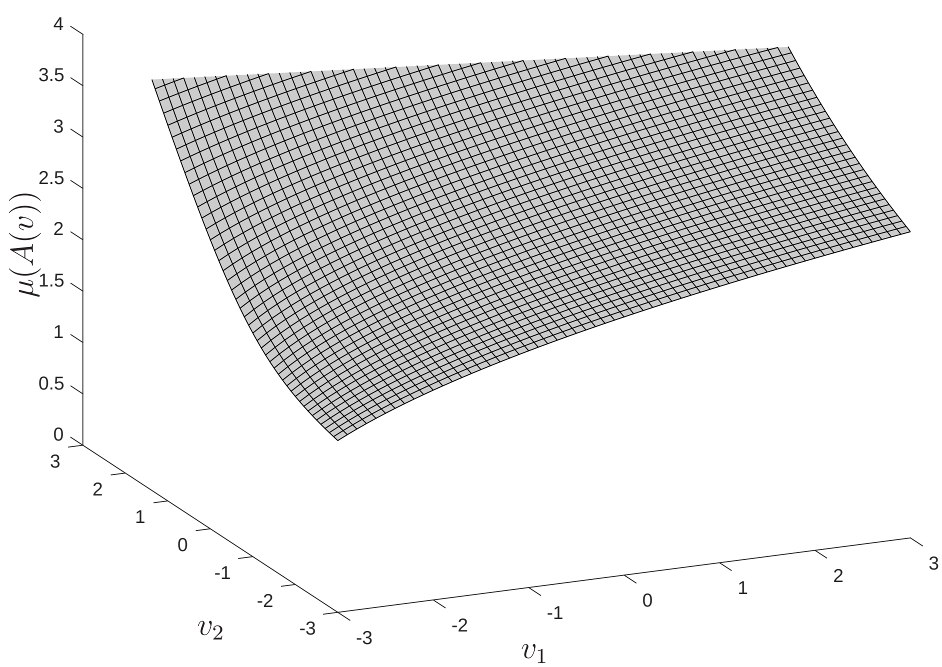

Fig. 1. Example 2, Scenario 1: SM of the closed-loop system over the set $\mathcal{V}$. As it can be seen, the SM is always positive in this case.

2) Scenario 2: Here we repeat the previous search by considering the presence of an additional design parameter in the sought static output feedback controller, specifically we consider that the matrix $D_{con}(v)$ has the form

$$D_{con}(v) = \begin{pmatrix} v_1 & 0 \\ v_2 & v_3 \end{pmatrix}$$

where $v = (v_1, v_2, v_3)'$ is the vector of design parameters constrained in $\mathcal{V} = [-3, 3]^3$.

It follows that

$$a_{den}(v) = 1 - v_3$$

and, hence, the set $\mathcal{Z}$ in (5) is

$$\mathcal{Z} = \left\{ v \in \mathbb{R}^3 : \ v_3 \in (-\infty, 0.9] \cup [1.1, \infty) \right\}.$$

Solving the SDP (57) we find $\gamma^* = 26.358$ with $\theta = -1$. The polynomials $r_i(v)$ have degree in the range $[0, 2]$, the number of LMI scalar variables is 72, and the computational time is less than 1 second.

At this point, we look for $v^* \in \mathbb{R}^3$ satisfying (59). As explained after Theorem 2, this can be done by looking for the zeros of $s^*(v)$, which can be addressed by looking for vector of monomials $b(v)$ in $\ker(S^*)$. It turns out that

$$S^* = \begin{pmatrix} 12.406 & 2.637 & -0.892 & -4.194 & \cdots \\ \star & 2.863 & -1.096 & -3.264 & \cdots \\ \star & \star & 2.503 & 0.004 & \cdots \\ \star & \star & \star & 5.643 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$b(v) = \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ v_1^2 \\ \vdots \end{pmatrix}, \quad \ker(S^*) = \mathrm{img} \begin{pmatrix} 0.054 \\ 0.161 \\ 0.161 \\ 0.104 \\ 0.482 \\ \vdots \end{pmatrix}.$$

By simply scaling, one obtains that the only possible candidate for v^* given by

$$v^* = (3.000, 3.000, 1.949)'.$$

We verify that this candidate satisfies (59). Hence, we conclude that $v^* \in \mathcal{S}$. Indeed,

$$\begin{cases} \det(E(v^*)) &= -0.949 \\ \mathrm{spec}(A(v^*)) &= \{-4.151, -1.060 \pm j3.136\} \\ \mu(A(v^*)) &= -1.060. \end{cases}$$

B. Example 3

In this example we consider, in the DT case, the design of a first-order output feedback controller with structural constraints for stabilizing the plant (13) with

$$\begin{cases} A_{pla} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_{pla} = \begin{pmatrix} 1 & 2 \end{pmatrix} \\ C_{pla} = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad D_{pla} = 0. \end{cases}$$

We look for a first-order output feedback controller with a pole in the origin, which can be expressed as in (14) with

$$\begin{cases} A_{con}(v) = 0, \quad B_{con}(v) = 1 \\ C_{con}(v) = v_1, \quad D_{con}(v) = v_2 \end{cases}$$

where $v = (v_1, v_2)'$ is the vector of design parameters. The closed-loop system can be expressed as in (1)–(3) with $A_{num}(v)$ and $A_{den}(v)$ given by (17).

The problem consists of finding a vector v in the set $\mathcal{S}$ in (12), where $\mathcal{V} = [-3, 3]^2$, $\mathcal{Z}$ is as in (6) with $\zeta = 0.1$, $\mathcal{U}$ is as in (11) with $\psi = 0.9$, and $\mu(\cdot)$ is the SM.

First of all, let us observe that this plant is unstable, in particular

$$\begin{cases} \mathrm{spec}(A_{pla}) &= \{0.382, 2.618\} \\ \mu(A_{pla})) &= 2.618. \end{cases}$$

Let us observe that, since the plant is strictly proper, the closed-loop system is well-posed for all $v \in \mathcal{V}$, in particular $a_{den}(v) = 1$. Hence, the methodology proposed in Sections III and IV can be simplified with Change C2 in (63). Therefore, in the sequel of this example we consider $\Theta = \{1\}$.

Solving the SDP (57) we find $\gamma^* = 0.002$. The polynomials $r_i(v)$ have degree in the range $[4, 6]$, the number of LMI scalar variables is 262, and the computational time is less than 1 second.

At this point, we look for $v^* \in \mathbb{R}^2$ satisfying (59). As explained after Theorem 2, this can be done by looking for the zeros of $s^*(v)$, which can be addressed by looking for vector of monomials $b(v)$ in $\ker(S^*)$. It turns out that

$$S^* = \begin{pmatrix} 0.291 & 6.319 & 1.12 & \cdots \\ \star & 142.247 & 25.371 & \cdots \\ \star & \star & 4.644 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$b(v) = \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_1^2 \\ \vdots \end{pmatrix}, \quad \ker(S^*) = \mathrm{img}\begin{pmatrix} 0.371 \\ -0.182 \\ 0.384 \\ 0.089 \\ \vdots \end{pmatrix}.$$

By simply scaling, one obtains that the only possible candidate for v^* given by

$$v^* = (-0.490, 1.035)'.$$

We verify that this candidate satisfies (59). Hence, we conclude that $v^* \in \mathcal{S}$. Indeed,

$$\begin{cases} \mathrm{spec}(A(v^*)) & = & \{0.584 \pm j0.523, 0.797\} \\ \mu(A(v^*)) & = & 0.797. \end{cases}$$

Figure 2 shows the set $\mathcal{S}$ found by brute force. As it can be seen, this set is quite small in this case.

C. *Example 4*

In this example we consider, in the CT case, the design of tuning parameters for reducing the EM. Specifically, we consider the system (1) with

$$A(v) = \begin{pmatrix} 2 & 3 & v_1 \\ 1 - v_2 & -2 & -1 \\ -3 & 1 + v_3 & 1 \end{pmatrix}$$

where $v = (v_1, v_2, v_3)'$ is the vector of design parameters. We consider the following two scenarios.

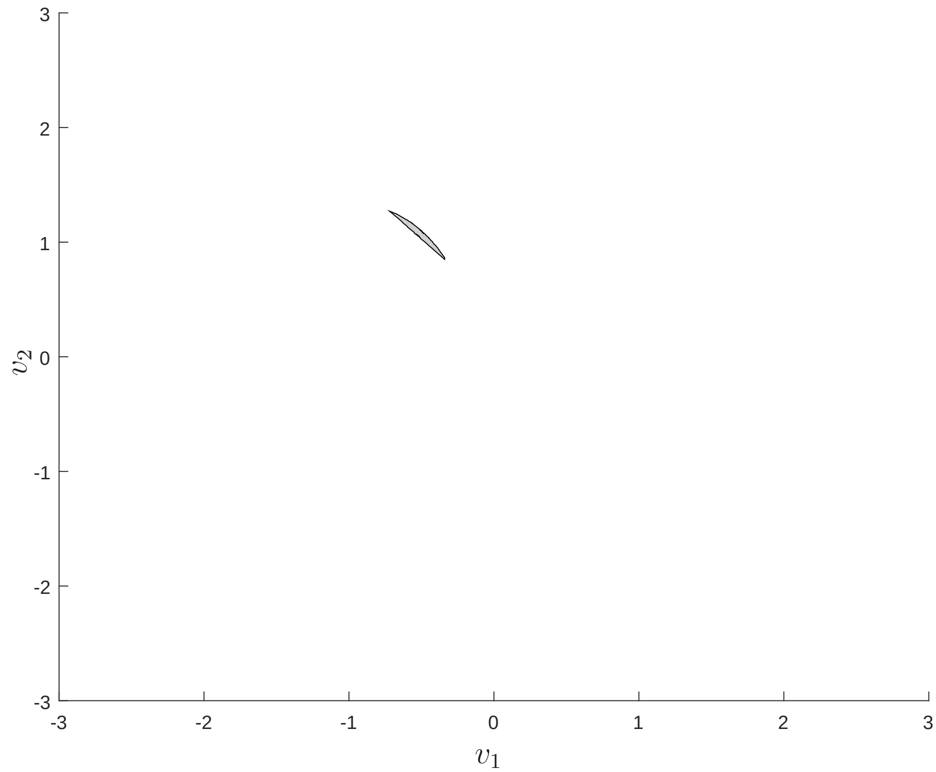

Fig. 2. Example 3: set $\mathcal{S}$. As it can be seen, the set $\mathcal{S}$ is quite small in this case.

1) Scenario 1: Here the problem consists of finding v in the set $\mathcal{S}$ in (12), where $\mathcal{V} = [-3, 3]^3$, $\mathcal{Z}$ is as in (6) with $\zeta = 0.1$, $\mathcal{U}$ is as in (11) with $\psi = 2$, and $\mu(\cdot)$ is the EM.

First of all, let us observe that the EM of the plant can be larger than the required value, indeed for $v = (0, 0, 0)'$ one has

$$\begin{cases} \mathrm{spec}(A(0)) & = & \{-3.220, 2.110 \pm j1.066\} \\ \mu(A(0)) & = & 4.220. \end{cases}$$

Let us observe that, since $A(v)$ is a matrix polynomial, one has $a_{den}(v) = 1$ in (17). Hence, the methodology proposed in Sections III and IV can be simplified with Change C2 in (63). Therefore, in the sequel of this example we consider $\Theta = \{1\}$.

Solving the SDP (57) we find $\gamma^* = 66.000$. The polynomials $r_i(v)$ have degree in the range $[0, 2]$, the number of LMI scalar variables is 54, and the computational time is less than 1 second.

At this point, we look for $v^* \in \mathbb{R}^3$ satisfying (59). As explained after Theorem 2, this can be done by looking for the zeros of $s^*(v)$, which can be addressed by looking for vector of monomials $b(v)$ in $\ker(S^*)$. It turns out that

$$S^* = \begin{pmatrix} 43.449 & -0.817 & -0.935 & 0.965 & \cdots \\ \star & 1.046 & 0.347 & -0.487 & \cdots \\ \star & \star & 0.874 & -0.237 & \cdots \\ \star & \star & \star & 0.939 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$b(v) = \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ v_1^2 \\ \vdots \end{pmatrix}, \quad \ker(S^*) = \operatorname{img} \begin{pmatrix} 0.044 \\ 0.132 \\ 0.132 \\ -0.132 \\ 0.397 \\ \vdots \end{pmatrix}.$$

By simply scaling, one obtains that the only possible candidate for v^* given by

$$v^* = (3.000, 3.000, -3.000)'.$$

We verify that this candidate satisfies (59). Hence, we conclude that $v^* \in \mathcal{S}$. Indeed,

$$\begin{cases} \operatorname{spec}(A(v^*)) & = & \{-0.641, 0.820 \pm j3.658\} \\ \mu(A(v^*)) & = & 1.641. \end{cases}$$

Figure 3 shows the set $\mathcal{S}$ found by brute force. As it can be seen, this set is non-convex in this case. The fact that the set $\mathcal{S}$ is non-convex can also be proved by observing that

$$\begin{cases} v^{(1)} = (-1.1, 3, -0.7)' \quad \Rightarrow \quad \mu(A(v^{(1)})) = 1.982 < \psi \\ v^{(2)} = (3, 3, -2.4)' \quad \Rightarrow \quad \mu(A(v^{(2)})) = 1.965 < \psi \\ v^{(3)} = \dfrac{v^{(1)} + v^{(2)}}{2} \quad \Rightarrow \quad \mu(A(v^{(3)})) = 2.273 \not< \psi. \end{cases}$$

2) Scenario 2: Here we consider the problem of minimizing the Euclidean norm of v under the constraints considered in the previous scenario. This problem can be addressed with Problem 2 by choosing $c(v) = v'v$ in (64). Hence, we make Change C3 in (65)–(66) and we solve the SDP (67) with $d = 2$. The polynomials $r_i(v)$ have degree in the range $[2, 4]$, the number of LMI scalar variables is 332, and the computational time is less than 2 seconds. At this point, we look for $v^* \in \mathbb{R}^3$ satisfying the fourth constraint in (59). We find that the only possible candidate for v^* is

$$v^* = (-1.141, 3.000, -0.427)'.$$

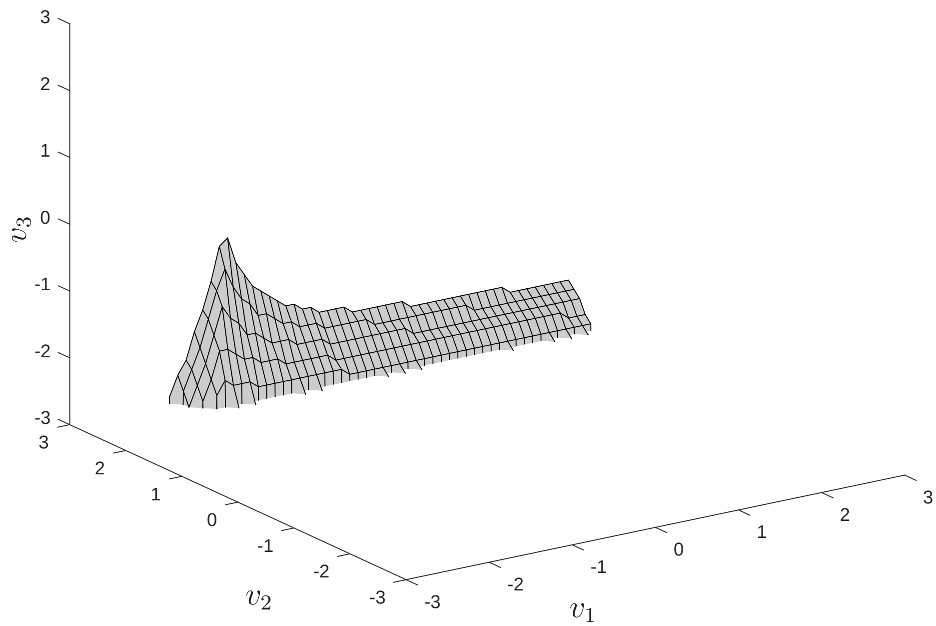

Fig. 3. Example 4, Scenario 1: set $\mathcal{S}$. As it can be seen, the set $\mathcal{S}$ is non-convex in this case.

The found vector of design parameters belongs to $\mathcal{S}$. Indeed,

$$\begin{cases} \operatorname{spec}(A(v^*)) &= \{-0.997, -0.003, 2.000\} \\ \mu(A(v^*)) &= 2.000. \end{cases}$$

Moreover, one has $c(v^*) = 10.484$, while the vector of design parameters found in the previous scenario achieves $c(v^*) = 27$.

VII. Conclusions

Two sufficient conditions have been given for establishing either the non-existence or the existence of designing parameters that reduce the SM and EM of LTI systems. These conditions require to solve an SDP, which is a convex optimization problem, and to find the roots of a multivariate polynomial, which is a difficult problem in general. To overcome this difficulty, a technique based on linear algebra operations has been exploited, which easily provides the sought roots in common cases by taking into account the structure of the polynomial under consideration. Also, it has been shown that these conditions are not only sufficient but also necessary by increasing enough the size of the SDP under some mild assumptions. Lastly, it has been explained how the proposed methodology can be used to search for design parameters that minimize a given cost function while reducing the instability.

Unfortunately, the computational burden of the proposed methodology quickly grows with the dimensions of the problem. This seems unavoidable in order to achieve conditions that are not only sufficient but also necessary through convex optimization.

Several directions can be explored in future work. One of these concerns the possibility of imposing that the eigenvalues of the system lie into a desired region. Another direction could attempt to achieve robust control design whenever the plant is affected by uncertainties. Lastly, another direction could explore the extension of the proposed methodology to the use of stability criteria based on Lyapunov functions.

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