The Cartesian analytical solutions for the N-dimensional damped compressible Euler equations

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Abstract

Employing matrix formulation and decomposition technique, we theoretically provide essential conditions for the existence of general analytical solutions for [redacted] N-dimensional damped compressible Euler equations arising in fluid mechanics. We also investigate the effect of damping on the solutions,

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density and pressure. There are two merits of this approach: First, this kind of solutions can be expressed by an explicit formula~~e~~ $\boldsymbol{u} = \boldsymbol{b}(t) + A(t)\boldsymbol{x}$ and no additional constraint on the dimension of the damped compressible Euler equations is needed. Second, we transform analytically the process of solving the Euler equations into algebraic construction of an appropriate matrix $A(t)$. Once the required matrix $A(t)$ is chosen, the solution $\boldsymbol{u}$ is obtained directly. Here we overcome the difficulty of solving matrix differential equations by utilizing decomposition and reduction techniques. In particular, we find two important solvable relations between the dimension of the Euler equations and the pressure parameter: $\gamma = 1 - 2/N$ in the damped case and $\gamma = 1 + 2/N$ for no damping. These two cases constitute a full range of solvable parameter $0 \leq \gamma < +\infty$. Special cases of our results also include several interesting conclusions: 1. If the velocity field $\boldsymbol{u}$ is a linear transformation on the Euclidean spatial vector $\boldsymbol{x} \in R^N$, then the pressure p is a quadratic form of $\boldsymbol{x}$. 2. The damped compressible Euler equations admit the Cartesian solutions if $A(t)$ is an anti-symmetric matrix. 3. The pressure p possesses ~~radial~~ symmetric forms if $A(t)$ is an anti-symmetrically orthogonal matrix.

radially

MSC: 35Q31, 35C05, 76B03, 76M60 remove 'ly'

Key Words: The damped compressible Euler equations, Cartesian solutions, symmetric and anti-symmetric matrix, quadratic form, curve integration.

1 Introduction

We consider a general system of N-dimensional damped compressible Euler equations (or compressible Euler equations with damping) [1]-[13]

$$\begin{aligned} &\rho_t + \operatorname{div}(\rho\boldsymbol{u}) = 0, \\ &\rho[\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] + \nabla p + 2\alpha\rho\mathbf{u} = 0, \end{aligned} \tag{1.1}$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_N)^T \in R^N$, $\boldsymbol{u} = (u_1, u_2, \cdots, u_N)^T$, $\rho(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ denote respectively the velocity, density, and pressure of the fluid at a position $\boldsymbol{x}$. The damping $\alpha\rho\mathbf{u}$ is proportional to the momentum and $\alpha \geq 0$ is a frictional constant. For $\alpha = 0$, $\alpha \neq 0$ the system (1.1) ~~are~~ called compressible and damped compressible

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Euler equations respectively. Here we take the coefficient as 2α in order to give convenient expression for calculation. Subsequent S

S Similar to the compressible and incompressible Euler equations, the damped compressible Euler equations (1.1) also model important physical phenomena. For example, the system describes compressible gas flow passing through a porous medium with friction force proportional to the linear momentum in the opposite direction [1, 4, 9]. For the one dimensional case, the system can be written in the Lagrangian coordinates as follows

$$v_t - u_x = 0, \quad u_t + p(v)_x = -au, \tag{1.2}$$

where $v = 1/\rho$ is the specific volume. It was shown that the system (1.2) was asymptotically equivalent to the porous media equation [1]. In many situation in S gas dynamics, we may consider the case where the density ρ and pressure p satisfy a relation

$$p(\rho) = K\rho^{\gamma}, \tag{1.3}$$

with ~~where~~ $K > 0$ and the constant $\gamma = c_p/c_\gamma$, where c_p, c_γ are the specific heat capacities ~~per unit mass~~ under constant pressure and constant volume respectively. In particular, ~~when $\gamma = 1$ the fluid is called isothermal.~~ ~~It~~ The flow can be used for constructing models with non-degenerate isothermal cores, which have a ~~[illegible]~~ connection with the so-called Schonberg-Chandrasekhar model [14]. For γ = 2, the system is mathematically equivalent to the classical shallow water equations of water waves theory [15], where the amplitude is allowed to be fully nonlinear and dispersion is neglected. For γ = 5/3 or 7/5, the system then describes the dynamics of monoatomic and diatomic gases respectively [16, 17].

The compressible Euler equations (1.1) ($\alpha = 0$) have been extensively investigated in the term of both weak solutions and exact solutions. A remark ~~Remark~~ on the existence results of global solutions of compressible Euler equations will be relevant for the present discussion. The existence and explicit structure of global solutions with antisymmetry continuous initial data was constructed in [18]. Chen proved the existence of global weak solution with symmetry outside of a circular core with the

for γ = 1 and the equation of state, the fluid flow is then isothermal.

center at the origin [19]. Li studied global solution of an initial-value problem for two-dimensional compressible Euler equations [20]. Zelik's work gave details of the existence of weak solutions for the unbounded domain [21, 22].

Exact solutions of the Euler equations for inviscid flow are very important for understanding how a real fluid will flow. In 1995, Zhang and Zheng obtained spiral solutions for the 2D compressible Euler equations with $p = K\rho^{\gamma}$ and $\gamma = 2$ [16] In 2003, Gibbon, Moore and Stuart obtained infinite energy, blow-up solutions for 3D Euler equations [23]. Recently, Li found Lax pairs for 2D and 3D Euler equations [24, 25]. Lou et al proposed Backlund transformation, Darboux transformation and exact solutions for the 2D Euler equations in vorticity form [26, 27]. Yuen constructed a class of exact solutions with elliptical symmetry by using the new characteristic method [28]. In 2011, Yuen further obtained a class of exact and rotational solutions for the incompressible and compressible 3D Euler equations [redacted] [29, 30]. Recently, Yuen found a kind of vortical and self-similar flows of the 2D compressible Euler equations [31].

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It is noticed that the existing exact solutions of the velocity function $\boldsymbol{u}$ mentioned above are in a linear form of spatial coordinates $\boldsymbol{x}$. Such [redacted] solutions have a long history in fluid flows, especially for the Euler and Navier-Stokes equations [32]-[40]. One of the important results in this direction is the work of Craik and Criminale [40] wherein a comprehensive analysis of solutions to the incompressible Navier-Stokes equations was given. Hence it is natural to enquire whether the compressible Euler/ Navier-Stokes equations admit similar nonlinear form of solution in spatial variables $\boldsymbol{x}$ and whether they admit the general analytical solutions in Cartesian coordinates. To answer these two questions, we constructed several novel nonlinear exact solutions for the 2D incompressible Euler equations by using the Clarkson-Kruskal reduction method [41]. Based on matrix theory and decomposition technique, we theoretically show the existence of the Cartesian rotational solutions

the

$$\mathbf{u} = \mathbf{b}(t) + A\mathbf{x} \tag{1.4}$$

for the general N-dimensional compressible Euler equations (1.1) [42].

Regarding the damped compressible Euler equations (1.1) ($\alpha \neq 0$), there are several well-known mathematical results on weak solutions in Sobolev space. As an illustrative example, Huang and Pan et al studied the large-time asymptotic behavior for the solutions in vacuum. In fact, they showed that the L^∞ weak-entropy solutions in vacuum for the Cauchy problem converged to the Barenblatts profile of the porous medium equation strongly in L^p [4, 6, 7]. Sideris et al studied the effect of damping on the large-time behavior of solutions to the Cauchy problem for the three-dimensional compressible Euler equations. It is proved that damping prevents the development of singularities in small amplitude classical solutions, using an equivalent reformulation of the Cauchy problem to obtain effective energy estimates [3]. Wang and Yang studied the [redacted] asymptotic behavior of solutions for the isentropic Euler equations with damping in multi-dimension [2]. Exact solutions usually more useful and serve as better model for practical applications and physics than weak solutions in Sobolev space. However, exact solutions of damped compressible Euler equations (1.1) have not been investigated fully to our knowledge.

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Based on these considerations [redacted], the goals of our present paper can be explained:

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First, we extend our previous results [42] to a general damped compressible Euler equations. Based on algebraic and decomposition techniques on vectors, matrices and curve integration, we theoretically show the existence of the analytical solutions (1.4) for the general N-dimensional damped compressible Euler equations.

Second, we investigate the effects of damping on the solution and pressure parameter γ. In the absence of [42], we found an important relation

damping

$$\gamma = 1 + \frac{2}{N} \tag{1.5}$$

between the dimension of equations and the pressure parameter which implies that $\gamma \geq 1$ for $N \geq 2$. In fact, the importance of the relation (1.5) provides a general rule for different physical phenomena and solvable cases where $\gamma = 2, \gamma = 5/3,\ \gamma = 7/5$ for example, see a series of papers [15, 16, 17]. In our present paper, we obtain another important relation

$$\gamma = 1 - \frac{2}{N} \tag{1.6}$$

between the dimension of equations and the pressure parameter, which implies that $0 \leq \gamma < 1$ for $N \geq 2$. For the damped case, we can thus construct exact solutions for $\gamma < 1$. Relations (1.6), (1.5) cover all real positive values of the ratio of specific heat capacities parameter. if N is taken as a real number

Third, we show explicitly that damping will have a dramatic influence on the properties of the exact solutions. This feature is also demonstrated in the work of Sideris et al, where damping will alter the dynamics of the Cauchy problem for the three dimensional damped compressible Euler equation [3].

This paper is arranged as follows. In section 2, we show that the damped compressible Euler equations admit the Cartesian solutions if A satisfies appropriate matrix equations. Two solvable cases of the matrix equations are considered in section 3 and section 4. In section 3, if A is a special anti-symmetric constant matrix, then the damped compressible Euler equations admit Cartesian solutions. We establish the relation $\gamma = 1 - 2/N$ between the dimension of equations and pressure parameter. In section 4, to find more general solutions, decomposition technique is used to reduce the matrix equations into solvable ones, where we find another important relation $\gamma = 1 + 2/N$ between the dimension of equations and pressure parameter. We also provide some illustrative examples for two solvable cases in section 3 and section 4. In section 5, we give some conclusions and remarks.

2 Existence of the Cartesian solutions

Before deriving the exact solutions, we change the damped compressible Euler equations (1.1) into a more convenient form. From the gas law of (1.3), it is easy to see that we may take $K = 1$ without loss of generality by using a simple transformation $\rho \to K^{-1/\gamma}\rho$. Let

$$\bar{p} = \begin{cases} \ln \rho, & for \quad \gamma = 1, \\ \frac{\gamma}{\gamma-1}\rho^{\gamma-1}, & for \quad \gamma \neq 1, \end{cases} \quad \Longleftrightarrow \quad \rho = \begin{cases} \exp(\bar{p}), & for \quad \gamma = 1, \\ \mu \bar{p}^{\frac{1}{\gamma-1}}, & for \quad \gamma \neq 1, \end{cases} \tag{2.1}$$

with $\mu = (\frac{\gamma-1}{\gamma})^{\frac{1}{\gamma-1}}$, the compressible Euler equations (1.1) can then be written in the form

$$\rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \tag{2.2}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \bar{p} + 2\alpha \boldsymbol{u} = 0. \tag{2.3}$$

The challenge is to deduce an appropriate function $\bar{p}(\boldsymbol{x})$, where we can establish exact solutions for the velocity linear in the Cartesian coordinates for the damped compressible Euler equations (2.2)-(2.3)

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x},$$

where the N-dimensional vector function $\boldsymbol{b}(t)$ and $N \times N$ matrix function A are defined by

$$\boldsymbol{b}(t) = (b_1(t), b_2(t), \cdots, b_N(t))^T. \quad A = (a_{ij}(t))_{N \times N},$$

Elements $b_i(t)$ and $a_{ij}(t)$ $(i, j = 1, 2, \cdots, N)$ are functions ~~about~~ of t. Due to the equivalence between $\bar{p}$ and ρ defined by (2.1), we mainly deal with $\bar{p}$ in solving the compressible equations (2.2)-(2.3).

Theorem 1 *We define B as part of a matrix Riccati equation*

$$B = (A_t + A^2 + 2\alpha A)/2. \tag{2.4}$$

If A and B satisfy the following matrix differential equations

$$B^T = B, \tag{2.5}$$

$$B_t + (\gamma - 1)\mathrm{tr}(A)B + BA + A^T B = 0, \tag{2.6}$$

then the compressible Euler equations (2.2)-(2.3) have explicit solutions in the form

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x}, \tag{2.7}$$

$$\bar{p} = -\boldsymbol{x}^T(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) - \boldsymbol{x}^T B\boldsymbol{x} + c(t), \tag{2.8}$$

where the vector function $\boldsymbol{b}(t)$ and scalar function $c(t)$ satisfy ordinary differential equations

$$(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b})_t + [(\gamma - 1)\mathrm{tr}(A)I + A^T](\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) + 2B\boldsymbol{b} = 0, \tag{2.9}$$

$$c_t + (\gamma - 1)\mathrm{tr}(A)c - \boldsymbol{b}^T(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) = 0. \tag{2.10}$$

Proof. We first prove that the proposed solution (2.7) will lead to (2.8) through the equation (2.3). Substituting (2.7) into (2.3) produces

$$\begin{aligned} & \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\bar{p} + 2\alpha\boldsymbol{u} \\ & = \boldsymbol{b}_t + A_t\boldsymbol{x} + [(\boldsymbol{b}+A\boldsymbol{x})\cdot\nabla](\boldsymbol{b}+A\boldsymbol{x}) + \nabla\bar{p} + 2\alpha(\boldsymbol{b}+A\boldsymbol{x}), \\ & = \boldsymbol{b}_t + A_t\boldsymbol{x} + (\ \boldsymbol{b}\cdot\nabla)A\boldsymbol{x} + (A\boldsymbol{x}\cdot\nabla)A\boldsymbol{x} + \nabla\bar{p} + 2\alpha\boldsymbol{b} + 2\alpha A\boldsymbol{x} \\ & = \boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b} + (A_t + A^2 + 2\alpha A)\boldsymbol{x} + \nabla\bar{p} = 0. \end{aligned} \tag{2.11}$$

For convenience, we introduce an auxiliary matrix

$$B = (b_{ij})_{N\times N} = \frac{1}{2}(A_t + A^2 + 2\alpha A), \quad b_{ij} = \frac{1}{2}\left(a_{ij,t} + 2\alpha a_{ij} + \sum_{k=1}^{N} a_{ik}a_{kj}\right),$$

and re-write the equation (2.11) into component form

$$Q_i(x_1,\cdots,x_N) \equiv -b_{it} - 2\alpha b_i - \sum_{k=1}^{N}(a_{ik}b_k + 2b_{ik}x_k) = \frac{\partial p}{\partial x_i}, \quad i = 1,2,\cdots,N. \tag{2.12}$$

In order to solve for $\bar{p}(\boldsymbol{x})$ from (2.12), these N equations should be compatible with each other, that is, the vector functions $(Q_1, Q_2, \cdots, Q_N)$ should constitute a potential field of $\bar{p}(\boldsymbol{x})$. The sufficient and necessary conditions are

$$\frac{\partial Q_j(x_1,\cdots,x_N)}{\partial x_i} = \frac{\partial Q_i(x_1,\cdots,x_N)}{\partial x_j}, \quad i,j = 1,2,\cdots,N, \tag{2.13}$$

which hold if and only if

$$b_{ji} = b_{ij}, \quad i,j = 1,2,\cdots,N,$$

which implies that B is a symmetric matrix, i.e. the condition (2.5).

It follows from the condition (2.13) that the function $\bar{p}(\boldsymbol{x})$ is a complete differential

$$d\bar{p}(\boldsymbol{x}) = \sum_{i=1}^{N}\frac{\partial\bar{p}(\boldsymbol{x})}{\partial x_i}dx_i = \sum_{i=1}^{N} Q_i(x_1,\cdots,x_N)dx_i.$$

Hence the second kind of curvilinear integral of $p(\boldsymbol{x})$ is independent of path. In particular, we may take a special integration route and directly obtain

$$
\begin{aligned}
p(\boldsymbol{x}) &= \sum_{i=1}^{N}\int_{(0,0,\cdots,0)}^{(x_1,x_2,\cdots,x_N)} Q_i(x_1,x_2,\cdots,x_N)dx_i \\
&= \int_0^{x_1} Q_1(x_1,0,\cdots,0)dx_1, + \int_0^{x_2} Q_2(x_1,x_2,0,\cdots,0)dx_2 \\
&+\cdots+\int_0^{x_N} Q_N(x_1,x_2,\cdots,x_N)dx_N \\
&= -\sum_{i=1}^{N}[b_{i,t}+\sum_{k=1}^{N}a_{ik}b_k+2\alpha b_i]x_i-\sum_{i=1}^{N}b_{ii}x_i^2-2\sum_{i,k=1,\ i<k}^{N}b_{ik}x_ix_k+c(t) \\
&= -\boldsymbol{x}^T(\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})-\boldsymbol{x}^TB\boldsymbol{x}+c(t).
\end{aligned}
$$

Next, we prove that the functions (2.7)-(2.8) will satisfy the equation (2.2). For $\gamma > 1$, by using (2.6), (2.9) and (2.10), we have

$$
\begin{aligned}
&\rho_t+\mathrm{div}(\rho\boldsymbol{u}) = \rho_t+\rho\mathrm{tr}(A)+\boldsymbol{u}\cdot\nabla\rho \\
&= -\frac{\mu}{\gamma-1}\bar{p}^{\frac{1}{\gamma-1}-1}\{x^T[B_t+(\gamma-1)\mathrm{tr}(A)B+2A^TB]x \\
&+x^T[(\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})_t+[(\gamma-1)\mathrm{tr}(A)I+A^T](\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})+2B\boldsymbol{b}] \\
&-[c_t+(\gamma-1)\mathrm{tr}(A)c-\boldsymbol{b}^T(\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})]\}=0,
\end{aligned}
\tag{2.14}
$$

where we have used the condition

$$x^T[B_t+(\gamma-1)\mathrm{tr}(A)B+2A^TB]x=0,$$

which is equivalent to

$$[B_t+(\gamma-1)\mathrm{tr}(A)B+2A^TB]^T=-[B_t+(\gamma-1)\mathrm{tr}(A)B+2A^TB],$$

or

$$B_t+(\gamma-1)\mathrm{tr}(A)B+BA+A^TB=0.$$

The case for $\gamma = 1$ can be proved in a similar way, just by replacing $\mu\bar{p}^{\frac{1}{\gamma-1}}$ by $\exp(\bar{p})$ in the proof of equation (2.14). We have established explicitly the solutions thus (2.7)-(2.8) for the N-dimensional Euler equations (2.2)-(2.3). □

The condition (2.6) is a matrix differential equation involving N^2 components. In the next section, special solutions are identified for simple cases of the matrix A.

3 First reduction: constant matrix A

We re-write (2.6) in the form

$$B_t + (\gamma - 1)\mathrm{tr}(A)B + [B, A] + (A + A^T)B = 0, \tag{3.1}$$

where $[B, A] = BA - AB$ denotes the Lie bracket between A and B.

Theorem 2 *If $A + \alpha I$ is an anti-symmetric constant matrix, and the parameters satisfy*

$$[N(\gamma - 1) + 2]\alpha = 0, \tag{3.2}$$

then the compressible Euler equations (2.2)-(2.3) admit a general solution

$$\boldsymbol{u} = \boldsymbol{b}(t) + A\boldsymbol{x}, \tag{3.3}$$

$$\bar{p} = -\boldsymbol{x}^T(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) - \boldsymbol{x}^T B\boldsymbol{x} + c(t), \tag{3.4}$$

where the vector function $\boldsymbol{b}(t)$ and scalar function $c(t)$ are given by the following two cases for different α,

(i) For $\alpha = 0$, which corresponds to non-damped compressible Euler equations, $\boldsymbol{b}(t)$ and $c(t)$ possess solutions in the form of polynomials [redacted]

$$\boldsymbol{b}(t) = \boldsymbol{c}_1 t + \boldsymbol{c}_2, \tag{3.5}$$

$$c(t) = \frac{1}{3}\boldsymbol{c}_1^T A\boldsymbol{c}_1 t^3 + \frac{1}{2}(\boldsymbol{c}_1^T\boldsymbol{c}_1 + \boldsymbol{c}_1^T A\boldsymbol{c}_2 + \boldsymbol{c}_2^T A\boldsymbol{c}_1)t^2 + \boldsymbol{c}_2^T A\boldsymbol{c}_2 t + c_3, \tag{3.6}$$

(ii) For $\alpha \neq 0$ and $\gamma = 1 - 2/N$, which corresponds to damped compressible Euler equations, $\boldsymbol{b}(t)$ and $c(t)$ possess solutions in the form of exponential function [redacted]

$$\boldsymbol{b}(t) = \boldsymbol{c}_1 e^{-2\alpha t} + \boldsymbol{c}_2, \tag{3.7}$$

$$\begin{aligned} c(t) = &-\frac{1}{2\alpha}\boldsymbol{c}_1^T A\boldsymbol{c}_1 e^{-4\alpha t} + c_4 e^{-2\alpha t} + \boldsymbol{c}_2^T(A + \alpha I)\boldsymbol{c}_2 \\ &+ \boldsymbol{c}_2^T A\boldsymbol{c}_1 + \boldsymbol{c}_1^T(A + \alpha I)\boldsymbol{c}_2 = 0, \end{aligned} \tag{3.8}$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are arbitrary constant vectors, c_3 and c_4 are arbitrary constants.

Remark 1 *In the case $\alpha \neq 0$, we have*

$$\gamma = 1 - \frac{2}{N},$$

which implies that

$$0 \leq \gamma < 1, \quad \text{for} \quad N \geq 2.$$

If $\gamma = 0$, then ρ is constant, which corresponds to the incompressible Euler equations. We conclude that the damping coefficient α not only affects γ, but also the form of solutions as (3.7)-(3.8), which are different from solutions (3.5)-(3.6). This new phenomenon, which was not discussed in our previous work [41], where we only considered non-damped case of $\gamma \geq 1$.

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Proof. We just need to verify that the conditions (2.5), (2.6), (2.9) and (2.10) are satisfied under the Theorem 2.

If $A + \alpha I$ is an anti-symmetric constant matrix, then we can rewrite B in the form

$$B = \frac{1}{2}(A^2 + 2\alpha A) = \frac{1}{2}\left[(A + \alpha I)^2 - \alpha^2 I\right], \tag{3.9}$$

which implies that $B^T = B$ is a symmetric matrix. Moreover, direct calculation shows that the Lie bracket between A and B is commutative, that is,

$$[A, B] = AB - BA = \frac{1}{2}A(A^2 + 2\alpha A) - \frac{1}{2}(A^2 + 2\alpha A)A = 0. \tag{3.10}$$

On the other hand, from the anti-symmetric condition

$$(A + \alpha I)^T = -(A + \alpha I),$$

we deduce that

$$A^T = -A - 2\alpha I. \tag{3.11}$$

Noticing that $\operatorname{tr}(A^T) = \operatorname{tr}(A)$, we have

$$\operatorname{tr}(A) = -\operatorname{tr}(\alpha I) = -N\alpha. \tag{3.12}$$

Finally by using (3.2), (3.10)-(3.12), we have

$$\begin{aligned}
&B_t + (\gamma - 1)\operatorname{tr}(A)B + BA + A^T B,\\
&= -\alpha N(\gamma - 1)B + BA - (A + 2\alpha I)B\\
&= -\alpha[N(\gamma - 1) + 2]B + [B, A] = 0,
\end{aligned}$$

which shows that matrix equation (2.6) is satisfied.

By using (3.2) and (3.11), we ~~know that~~ obtain

$$(\gamma-1)\mathrm{tr}(A)I+A^T=2\alpha I+A^T=-A. \tag{3.13}$$

Making use of (3.2), (3.9) and (3.13), equations (2.9) and (2.10) reduce to

$$\begin{aligned}&(\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})_t-A(\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})+(A^2+2\alpha A)\boldsymbol{b}\\&=\boldsymbol{b}_{tt}+2\alpha\boldsymbol{b}_t=0, &(3.14)\\&c_t+2\alpha c-\boldsymbol{b}^T(\boldsymbol{b}_t+A\boldsymbol{b}+2\alpha\boldsymbol{b})=0. &(3.15)\end{aligned}$$

We should have two cases to discuss: S

If $\alpha=0$, solving (3.14) leads to the solution (3.5). ~~s~~ubstituting (3.5) into (3.15) and solving for $c(t)$ give~~s~~ (3.6). (remove 's')

If $\alpha\neq 0$, equations (3.14) and (3.15) admit solutions (3.7) and (3.8). $\square$

The solution procedure is now complete. Illustrative examples are given.

Example 1 *For the 2D damped incompressible Euler equations ($\gamma=0$) with $N=2$, $c_2=0$, $c_4=\alpha/2$, we take anti-symmetric matrix with $A+\alpha I$ and* $\mathrm{tr}(A)=-2\alpha$, *which is in the form*

$$A=\begin{pmatrix}-\alpha & a_{12}\\ -a_{12} & -\alpha\end{pmatrix},$$

remove 'it'

where a_{12} is an arbitrary constant. Substituting ~~it~~ into (3.9) gives ~~that~~ remove 'that'

$$B=\begin{pmatrix}-\frac{1}{2}(\alpha^2-a_{12}^2) & 0\\ 0 & -\frac{1}{2}(\alpha^2-a_{12}^2)\end{pmatrix}.$$

According to (3.7) and (3.8), we have

$$b(t)=\begin{pmatrix}b_1\\ b_2\end{pmatrix}e^{-2\alpha t},\quad c(t)=\frac{\alpha}{2}[(b_1^2+b_2^2)e^{-4\alpha t}+e^{-2\alpha t}],$$

where b_1 and b_2 are arbitrary constants.

Finally, (3.3) and (3.4) give us the solution of the 2D Euler equation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\alpha & a_{12} \\ -a_{12} & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-2\alpha t},$$

$$\bar{p} = \frac{1}{2}(\alpha^2 - a_{12}^2)(x_1^2 + x_2^2) - [(\alpha b_1 - a_{12} b_2)x_1 + (a_{12} b_1 + \alpha b_2)x_2]e^{-2\alpha t}$$
$$+ \frac{\alpha}{2}[(b_1^2 + b_2^2)e^{-4\alpha t} + e^{-2\alpha t}].$$

Example 2 *For the 3D damped compressible Euler equations with $\gamma = 1/3,\ c_2 = 0,\ c_4 = 1$, we get*

$$A = \begin{pmatrix} -\alpha & 1 & -1 \\ -1 & -\alpha & 1 \\ 1 & -1 & -\alpha \end{pmatrix}, \quad b(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-2\alpha t}, \quad c(t) = \frac{3}{2} e^{-4\alpha t} + e^{-2\alpha t},$$

$$B = \frac{1}{2} \begin{pmatrix} -\alpha^2 - 2 & 1 & 1 \\ 1 & -\alpha^2 - 2 & 1 \\ 1 & 1 & -\alpha^2 - 2 \end{pmatrix}.$$

An exact solution with vorticity is obtained

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\alpha & 1 & -1 \\ -1 & -\alpha & 1 \\ 1 & -1 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-2\alpha t},$$

$$\bar{p} = (\alpha^2 + 2)(x_1^2 + x_2^2 + x_3^2) - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$$
$$+ \alpha(x_1 + x_2 + x_3)e^{-2\alpha t} + \frac{3}{2} e^{-4\alpha t} + e^{-2\alpha t}.$$

Example 3 *For the N-dimensional damped compressible Euler equations, if we let*

$\boldsymbol{b}(t)=\boldsymbol{0}$, $c(t)=0$, $\gamma=1-2/N$, and take

$$A=\begin{pmatrix} -\alpha & 1 & \cdots & 1 \\ -1 & -\alpha & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -\alpha \end{pmatrix},$$

$$B=\frac{1}{2}\begin{pmatrix} -\alpha^2-(N-1) & -(N-2) & \cdots & -(N-2) \\ -(N-2) & -\alpha^2-(N-1) & \cdots & -(N-2) \\ \cdots & \cdots & \cdots & \cdots \\ -(N-2) & -(N-2) & \cdots & -\alpha^2-(N-1) \end{pmatrix}$$

another exact solution is

$$\begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_N \end{pmatrix}=\begin{pmatrix} -\alpha & 1 & \cdots & 1 \\ -1 & -\alpha & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix},$$

$$\bar{p}=-\boldsymbol{x}^T B\boldsymbol{x}=\frac{\alpha^2+N-1}{2}\sum_{j=1}^{N}x_j^2+(N-2)\sum_{i,j=1,i<j}^{N}x_ix_j.$$

Remark 2 *The off-diagonal elements of the matrix A in Examples 2 and 3 can in fact be chosen in an arbitrary manner, as long as conditions $a_{ij}=-a_{ji}, i\neq j$ ($A+I$ being anti-symmetric) and $a_{ii}=-\alpha$ $(\mathrm{tr}(A)=-N\alpha)$ are satisfied.*

4 Second reduction: time-dependent matrix A

Theorem 3 *Suppose that the matrix A can be decomposed into*

$$A+\alpha I=D+E,\quad D^T=-D, \tag{4.1}$$

where $D = (A + \alpha I)^{off}$ *denotes non-diagonal part of* A, *and* $E = (A + \alpha I)^{diag}$ *its diagonal part. If* D *and* E *satisfy the following matrix differential equations*

$$D_t + DE + ED = 0, \tag{4.2}$$

$$\begin{aligned} &(E_t + D^2 + E^2)_t + (\gamma - 1)(\mathrm{tr}(E) - \alpha N)(E_t + D^2 + E^2 - \alpha^2 I) \\ &+ (E_t + D^2 + E^2 - \alpha I^2)(E + D - \alpha I) + (E - D - \alpha I)(E_t + D^2 + E^2 - \alpha^2 I) = 0. \end{aligned} \tag{4.3}$$

then

$$B = \frac{1}{2}(E_t + D^2 + E^2 - \alpha^2 I) \tag{4.4}$$

is symmetric matrix, and A, B *satisfy the conditions (2.5)-(2.6) of Theorem 1.*

a

Proof. By using (4.2), we have

$$\begin{aligned} B &= \frac{1}{2}\left[A_t + (A + \alpha I)^2 - \alpha^2 I\right] \\ &= \frac{1}{2}[(E_t + E^2 + D^2 - \alpha^2 I) + (D_t + ED + DE)] \\ &= \frac{1}{2}(E_t + D^2 + E^2 - \alpha^2 I), \end{aligned}$$

which together with (4.1) implies that B is a symmetric matrix since

$$B^T = \frac{1}{2}(E_t + D^2 + E^2 - \alpha^2 I)^T = B.$$

The condition (4.1) implies that

$$\mathrm{tr}(A) = \mathrm{tr}(D) + \mathrm{tr}(E) - \alpha \mathrm{tr}(I) = \mathrm{tr}(E) - \alpha N, \tag{4.5}$$

and

$$A^T = -D + E - \alpha I. \tag{4.6}$$

Substituting (4.1), (4.4), ~~and~~ (4.5) and (4.6) into the equation (2.6), we then obtain the equation (4.3). □

Further simplifications into scalar differential equations are now proposed.

Theorem 4 *By taking*

$$E = fI, \tag{4.7}$$

then equation (4.2) admits solution

$$D = e^{-2\int f(t)dt}C, \tag{4.8}$$

an

where C is [redacted] anti-symmetric constant matrix, that is,

$$C = (c_{ij})_{N\times N},\ c_{ii} = 0,\quad c_{ij} = -c_{ji},\ i \neq j. \tag{4.9}$$

Equation (4.3) reduces to

$$\begin{aligned}&\{f_{tt} + 2ff_t + [N(\gamma-1)+2](f-\alpha)f_t + [N(\gamma-1)+2](f-\alpha)^2(f+\alpha)\}I\\ &+\{N(\gamma-1)+2](f-\alpha) - 4f\}e^{-4\int f(t)dt}C^2 = 0,\end{aligned} \tag{4.10}$$

or equivalently decomposing the damping parameter α part as the following form

$$\begin{aligned}&\{f_{tt} + [N(\gamma-1)+4]ff_t + [N(\gamma-1)+2]f^3\}I + [N(\gamma-1)-2]fe^{-4\int f(t)dt}C^2\\ &-\alpha[N(\gamma-1)+2][(f_t + \alpha f + f^2 - \alpha^2)I + e^{-4\int f(t)dt}C^2] = 0,\end{aligned} \tag{4.11}$$

Proof. Substituting (4.7) into (4.2) yields the reduction

$$D_t + 2fD = 0, \tag{4.12}$$

which can be solved and leads to the solution (4.8). Again substituting (4.7) and (4.8) into (4.3) directly leads to (4.10). □

E [redacted]quation (4.10) is key issue. We discuss four illustrative cases.

Case 1. In the case $\alpha = 0$ and $\gamma = 1 + 2/N$, the system (4.10) reduces to

$$f_{tt} + 6ff_t + 4f^3 = 0, \tag{4.13}$$

which has a solution

$$f = -\frac{1}{2t + c_3}. \tag{4.14}$$

Substituting (4.14) into (4.8) gives

$$D=\frac{1}{2t+c_3}C=\frac{1}{2t+c_3}(c_{ij})_{N\times N}, \tag{4.15}$$

where constants c_{ij} satisfy the relation (4.9). These results for non-damped compressible Euler equation were given in our previous paper [42].

Case 2. For $f=-\alpha\neq 0$ being a constant and $\gamma=1$, equation (4.10) is automatically satisfied. And (4.8) has an explicit expression

a

$$D=e^{2\alpha t}(c_{ij})_{N\times N},\quad c_{ii}=0,\quad c_{ji}=-c_{ij}, \tag{4.16}$$

where c_{ij} are arbitrary constants.

Example 4 *For the 3D compressible Euler equations, we take $\gamma=1,\ c_2=c_4=0$ and*

$$A=\begin{pmatrix}-\alpha & e^{2\alpha t} & e^{2\alpha t}\\ -e^{2\alpha t} & -\alpha & e^{2\alpha t}\\ -e^{2\alpha t} & -e^{2\alpha t} & -\alpha\end{pmatrix},$$

then according to (3.7), (3.8) and (3.9), we get

$$b(t)=\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-2\alpha t},\quad c(t)=\frac{3}{2}e^{-4\alpha t},$$

$$B=\frac{1}{2}\begin{pmatrix}-\alpha^2-2e^{4\alpha t} & -e^{4\alpha t} & -e^{4\alpha t}\\ -e^{4\alpha t} & -\alpha^2-2e^{4\alpha t} & -e^{4\alpha t}\\ -e^{4\alpha t} & -e^{4\alpha t} & -\alpha^2-2e^{4\alpha t}\end{pmatrix}.$$

Finally, according to (3.3) and (3.4), we directly obtain a rotational solution

$$\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=\begin{pmatrix}-\alpha & e^{2\alpha t} & e^{2\alpha t}\\ -e^{2\alpha t} & -\alpha & e^{2\alpha t}\\ -e^{2\alpha t} & -e^{2\alpha t} & -\alpha\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-2\alpha t},$$

$$\begin{aligned}\bar{p}=&(\alpha^2+2e^{4\alpha t})(x_1^2+x_2^2+x_3^2)+2(x_1x_2+x_1x_3+x_2x_3)e^{4\alpha t}\\ &+\alpha(x_1+x_2+x_3)e^{-2\alpha t}+\frac{3}{2}e^{-4\alpha t}.\end{aligned}$$

Case 3. If $C = 0$, the equation (4.10) reduce to

$$
\begin{aligned}
&f_{tt} + [N(\gamma-1)+4]ff_t + [N(\gamma-1)+2]f^3 \\
&+ \alpha[N(\gamma-1)+2](-f_t - \alpha f - f^2 + \alpha^2) = 0.
\end{aligned} \tag{4.17}
$$

We get a general solution

$$
A = (f-\alpha)I, \quad B = \frac{1}{2}(f_t + f^2 - \alpha^2)I,
$$

$$
\mathbf{u} = \boldsymbol{b}(t) + (f-\alpha)\boldsymbol{x}, \tag{4.18}
$$

$$
\bar{p} = -\boldsymbol{x}^T(\boldsymbol{b}_t + (f+\alpha)\boldsymbol{b}) - \frac{1}{2}\boldsymbol{x}^T(f_t + f^2 - \alpha^2)\boldsymbol{x} + c(t), \tag{4.19}
$$

where f is given by the equation (4.17), $\boldsymbol{b}(t)$ and $c(t)$ satisfy

$$
\begin{aligned}
&(\boldsymbol{b}_t + (f+\alpha)\boldsymbol{b})_t + [N(\gamma-1)+1](f-\alpha)(\boldsymbol{b}_t \\
&+ (f+\alpha)\boldsymbol{b}) + (f_t + f^2 - \alpha^2)\boldsymbol{b} = 0,
\end{aligned} \tag{4.20}
$$

$$
c_t + N(\gamma-1)(f-\alpha)c - \boldsymbol{b}^T(\boldsymbol{b}_t + (f+\alpha)\boldsymbol{b}) = 0. \tag{4.21}
$$

The damping α will affect the form of solution $f(t)$ for equation (4.17), we discuss two cases.

(i) If $\alpha = 0$, the equation (4.17) reduce to

$$
f_{tt} + [N(\gamma-1)+4]ff_t + [N(\gamma-1)+2]f^3 = 0, \tag{4.22}
$$

which has an explicit solution

$$
f = \frac{k}{t+c_4}, \tag{4.23}
$$

where $k = 1$ or $k = \frac{2}{N(\gamma-1)+2}$. It is obvious that there is no limit on N and γ, which is an extension of results given by our previous paper [42].

(ii) If $\alpha \neq 0$, the equation (4.17) admits a special solution

$$
f = \alpha \tanh(\alpha t), \tag{4.24}
$$

which is bounded and its form is different from (4.23) due to damping $\alpha \neq 0$.

Noting that $f_t + f^2 - \alpha^2 = 0$, from (4.20), we obtain

$$\boldsymbol{b}_t + \alpha(\tanh(\alpha t) + 1)\boldsymbol{b} = \tilde{\boldsymbol{b}}_1(|\cos(\alpha t)|e^{-\alpha t})^{N(\gamma-1)+1}, \tag{4.25}$$

where $\tilde{\boldsymbol{b}}_1$ is an arbitrary vector. Further integration yields

$$\boldsymbol{b}(t) = |\cos\alpha t|e^{-\alpha t}(\tilde{\boldsymbol{b}}_2 + \tilde{\boldsymbol{b}}_1 \int (|\cos(\alpha t)|e^{-\alpha t})^{N(\gamma-1)}dt), \tag{4.26}$$

where $\tilde{\boldsymbol{b}}_2$ is an arbitrary vector.

Substituting (4.25) and (4.26) into (4.21) and solving for c gives

$$c(t) = \tilde{c}(|\cos(\alpha t)|e^{-\alpha t})^{N(\gamma-1)} \int \boldsymbol{b}^T\tilde{\boldsymbol{b}}_1(|\cos(\alpha t)|e^{-\alpha t})^{N(\gamma-1)+1}dt. \tag{4.27}$$

$\tilde{c}$ is an arbitrary constant. Finally, we give a general solutions for N-dimensional damped compressible Euler equations

$$\begin{aligned} \mathbf{u} &= \boldsymbol{b}(t) + \alpha(\tanh(\alpha t) - 1)\boldsymbol{x}, \\ \bar{p} &= -(|\cos(\alpha t)|e^{-\alpha t})^{N(\gamma-1)+1}\tilde{\boldsymbol{b}}_1^T\boldsymbol{x} + c(t), \end{aligned}$$

where $\boldsymbol{b}(t)$ and $c(t)$ are given by (4.26) and (4.27) respectively. We see that $\bar{p}$ is also which linear with respect with $\boldsymbol{x}$ in this case. This is a new ~~phenomenon.~~ ~~that~~ did not arise in the no damping case [42]. phenomenon

Case 4. If C is an anti-symmetric and orthogonal matrix , then

$$C^T = -C, \quad C^TC = I, \quad C^2 = -C^TC = -I, \tag{4.28}$$

~~therefore the~~ hence the equation (4.10) reduce s to

$$\begin{aligned} &f_{tt} + [N(\gamma-1)+4]ff_t + [N(\gamma-1)+2]f^3 - [N(\gamma-1)-2]fe^{-4\int f(t)dt} \\ &+ \alpha[N(\gamma-1)+2](-f_t - \alpha f - f^2 + \alpha^2 + e^{-4\int f(t)dt}) = 0. \end{aligned} \tag{4.29}$$

Making a transformation

$$2\int f dt = \ln g, \tag{4.30}$$

then (4.29) will give a differential equation

$$\begin{aligned} &4g^2g_{ttt} + [N(\gamma-1)-2](2gg_tg_{tt} - g_t^3 - 4g_t) \\ &+ \alpha[N(\gamma-1)+2](2gg_t^2 - 4g^2g_{tt} - 4\alpha g^2g_t + 8\alpha^2g^3 + 8g) = 0. \end{aligned} \tag{4.31}$$

In this way, we obtain a more general solution

$$\mathbf{u} = \boldsymbol{b}(t) + A\boldsymbol{x}, \tag{4.32}$$

$$\bar{p} = -\boldsymbol{x}^T(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) - \boldsymbol{x}^T B\boldsymbol{x} + c(t), \tag{4.33}$$

$$(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b})_t + [(\gamma-1)\mathrm{tr}(A)I + A^T](\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) + 2B\boldsymbol{b} = 0, \tag{4.34}$$

$$c_t + (\gamma-1)\mathrm{tr}(A)c - \boldsymbol{b}^T(\boldsymbol{b}_t + A\boldsymbol{b} + 2\alpha\boldsymbol{b}) = 0, \tag{4.35}$$

$$A = (f-\alpha)I + D = -\frac{1}{2}g^{-1}(g_t + 2\alpha g)I + g^{-1}C, \tag{4.36}$$

$$\begin{aligned} B &= \frac{1}{2}[(f_t + f^2 - \alpha^2)I + D^2] \\ &= -\frac{1}{2}g^{-4}(2gg_{tt} - 2g_t^2 + 4\alpha^2 g^2 - g_t^2 + g^2C^2), \end{aligned} \tag{4.37}$$

where $C = (c_{ij})_{N\times N}$ constitutes an anti-symmetrically orthogonal matrix and g satisfies the equation (4.31).

We should explain the solvability of the equation (4.31) and existence of the matrix (4.28).

Remark 3 *C in the case 4 must be an even order anti-symmetric and orthogonal matrix, since*

$$C^T = -C, \quad \det(C) = \det(C^T) = \det(-C) = (-1)^N \det(C), \tag{4.38}$$

which implies that a degenerate situation will occur for odd N. Illustrative examples are

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \tag{4.39}$$

Remark 4 *To obtain elementary solutions of the equation (4.31), we employ the ansatz*

$$g = e^{kt}, \tag{4.40}$$

substituting (4.40) into (4.31), and noting e^{3kt} and e^{kt} ~~being~~ are *linear independent, we obtain a algebraic system*

$$k^3 - 2\alpha k - 4\alpha^2 k + 8\alpha^3 = 0, \tag{4.41}$$

$$2\alpha[N(\gamma-1)+2] - k[N(\gamma-1)-2] = 0. \tag{4.42}$$

The second equation (4.42) has a solution

$$k = 2\alpha\beta, \quad \beta = \frac{N(\gamma-1)+2}{N(\gamma-1)-2} \tag{4.43}$$

Again substituting (4.43) into the first equation (4.41) leads to

$$(\beta-1)^2(\beta+1) = 0,$$

which has a solution

$$\beta = -1$$

since $\beta \neq 1$ due to the relation (4.42). In this case, we have a constraint $\gamma = 1$. ~~Therefore~~ we get a special solution of ~~the~~ equation (4.31)

Consequently

$$g = e^{-2\alpha t}, \quad f = \frac{g_t}{2g} = -\alpha. \tag{4.44}$$

Example 5 ~~*Let's*~~ Let us *consider the 4D damped compressible Euler equations. For simplification, we take $\boldsymbol{b} = 0$, $c = 0$, two equations (4.32) and (~~4.32~~) are automatically satisfied. From (4.36) and (4.37), we construct* 4.33 ???

$$A = \frac{e^{2\alpha t}}{\sqrt{3}} \begin{pmatrix} -2\alpha\sqrt{3}e^{-2\alpha t} & 1 & 1 & -1 \\ -1 & -2\alpha\sqrt{3}e^{-2\alpha t} & 1 & 1 \\ -1 & -1 & -2\alpha\sqrt{3}e^{-2\alpha t} & -1 \\ 1 & -1 & 1 & -2\alpha\sqrt{3}e^{-2\alpha t} \end{pmatrix} \tag{4.45}$$

?

$$C = \frac{1}{2}g^2C^2 = -\frac{e^{4\alpha t}}{2}I = -\frac{e^{4\alpha t}}{2} \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}. \tag{4.46}$$

We then get a solution

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{e^{2\alpha t}}{\sqrt{3}} \begin{pmatrix} -2\alpha\sqrt{3}e^{-2\alpha t} & 1 & 1 & -1 \\ -1 & -2\alpha\sqrt{3}e^{-2\alpha t} & 1 & 1 \\ -1 & -1 & -2\alpha\sqrt{3}e^{-2\alpha t} & -1 \\ 1 & -1 & 1 & -2\alpha\sqrt{3}e^{-2\alpha t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

$$\bar{p} = -\frac{1}{2}e^{4\alpha t}(x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

Remark 5 *An explanation on the anti-symmetric nature of matrix D is necessary. Suppose that the matrix A can be decomposed into*

$$A + \alpha I = D + E, \quad D^T = D, \tag{4.47}$$

where $D = (A + \alpha I)^{off}$ denotes the non-diagonal part of A, with $E = (A + \alpha I)^{diag}$ being the diagonal part. If we take

$$E = fI, \quad D = e^{-2\int f(t)dt}C,$$

then the equation (4.3) reduces to

$$\begin{aligned} &\{f_{tt} + 2ff_t + [N(\gamma-1)+2](f-\alpha)f_t + [N(\gamma-1)+2](f-\alpha)^2(f+\alpha)\}I \\ &+ \{N(\gamma-1)+2](f-\alpha) - 4f\}e^{-4\int f(t)dt}C^2 \\ &+ 2(f_t + f^2 - \alpha^2)e^{-2\int f(t)dt}C + 2e^{-6\int f(t)dt}C^3 = 0, \end{aligned} \tag{4.48}$$

where C is a symmetric constant matrix. The last two terms arise due to (4.47) comparing with (4.10). If we further [redacted] $C^TC = I$ and [redacted] transformation

require

$$2\int f dt = \ln g,$$

implement the

then [redacted] (4.48) will result in two equations

$$\begin{aligned} &4g^2g_{ttt} + [N(\gamma-1)-2](2gg_tg_{tt} - g_t^3 - 4g_t) \\ &+ \alpha[N(\gamma-1)+2](2gg_t^2 - 4g^2g_{tt} - 4\alpha g^2g_t + 8\alpha^2g^3 + 8g) = 0, \end{aligned} \tag{4.49}$$

$$gg_{tt} - g_t^2 - 4\alpha^2g^2 + 4 = 0. \tag{4.50}$$

This system is over-determined and further research is necessary.

5 Conclusions and Remarks

By utilizing decomposition techniques and properties of special matrices, we have found necessary and sufficient conditions for the existence of exact solutions of the N-dimensional damped Euler equations in Cartesian coordinates. In many instances, these exact solutions are calculated explicitly. Generally the velocity field is linear in the spatial coordinates, while the pressure is expressed as a quadratic form.

Besides the intrinsic mathematical interests, these solutions will be applicable to many physical disciplines, e.g. fluid mechanics and astrophysics. High speed flows from the release of a localized amount of energy typically lead to a velocity field in similarity variables for large time and large distance from the origin. These similarity variable solutions usually match the form investigated in this paper. Despite the progress made here and other works in the literature, many challenges still remain ahead:

'italics' A

(1) Handling a matrix differential equation for the unknown A defined by equation (2.6) is key to the process of solving the compressible Euler equations. However, this step is difficult as N^2 (N = order of A) scalar equations are involved. Obviously this obstacle is especially nontrivial if N goes beyond 2.

'italics' N A

(2) Using a decomposition technique to separate a matrix into the diagonal portion and the anti-symmetric component may reduce the solution process to an investigation of a simpler system, but still highly nonlinear ordinary differential equations might result, and only isolated special exact solutions have been identified so far.

(3) A central theme of the present work is that the velocity field can be expressed as a linear function of the spatial coordinates. We do not know if analogous solutions in terms of quadratic, cubic or more general nonlinear function of the Cartesian coordinates can exist in the compressible case. If such solutions do exist, finding the corresponding pressure function will be an intriguing research project.

All these issues await efforts of researchers in the near future.

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