ON THE ANDREWS-ZAGIER ASYMPTOTICS FOR PARTITIONS WITHOUT SEQUENCES

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Abstract. In this paper, we establish asymptotics of radial limits for certain functions of Wright. These functions appear in bootstrap percolation and the generating function for partitions without sequences of k consecutive part sizes. We specifically establish asymptotics numerically obtained by Zagier in the case $k = 3$.

1. Introduction and statement of results

Holroyd, Liggett, and Romik [8] introduced the following probability models: Let $0 < s < 1$ and $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be independent events with probabilities

$$\mathbf{P}_s(\mathcal{C}_n) := 1 - e^{-ns}$$

under a certain probability measure $\mathbf{P}_s$. Let A_k be the event

$$A_k := \bigcap_{j=1}^{\infty} (\mathcal{C}_j \cup \mathcal{C}_{j+1} \cup \cdots \cup \mathcal{C}_{j+k-1})$$

that there is no sequence of k consecutive $\mathcal{C}_j$ that do not occur. With $q := e^{-s}$ throughout the remainder of the paper, set

$$g_k(q) := \mathbf{P}_s(A_k).$$

To solve a problem in bootstrap percolation, Holroyd, Liggett, and Romik established an asymptotic for $\log(g_k(e^{-s}))$.

Interestingly, the above described probability model also appears in the study of integer partitions [4, 8]. In particular,

$$G_k(q) = g_k(q) \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

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is the generating function for the number of integer partitions without k consecutive part sizes. Partitions without 2 consecutive parts have a celebrated history in relation to the famous Rogers-Ramanujan identities. See MacMahon's book [10] or the works of Andrews [1, 2, 3] for more about such partitions.

Andrews [3] found that the key to understanding the function if $k = 2$ lies in Ramanujan's mock theta function

$$\chi(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}\left(1 - q^j + q^{2j}\right)}.$$

Namely, he proved that

$$g_2(q) = \chi(q) \prod_{n=1}^{\infty} \frac{\left(1 + q^{3n}\right)}{\left(1 - q^n\right)\left(1 - q^{2n}\right)}.$$

From this, an asymptotic expansion for $g_2(e^{-s})$ may obtained (see [5]). Using additional q-series identities if $k > 2$, Andrews made the following conjecture.

Conjecture 1.1 (Andrews [3])**.** *For each $k \geq 2$, there exists a positive constant C_k such that, as $s \to 0$,*

$$g_k\left(e^{-s}\right) \sim C_k s^{-\frac{1}{2}} \exp\left(-\frac{\pi^2}{3k(k+1)s}\right).$$

This conjecture proved difficult to establish via standard q-series techniques. The asymptotic of [8] was improved by Mahlburg and the first author [6] . Finally, Daniel Kane and the fourth author [9], using a technique similar to the transfer matrix method of statistical mechanics, proved Conjecture 1.1 with $C_k = \sqrt{2\pi}/k$.

Zagier [18], using a formula for g_k found by Andrews [3], did extensive computations of these asymptotics. He numerically found that, as $s \to 0$,

$$g_3(e^{-s}) \sim \sqrt{\frac{2\pi}{s}} e^{-\frac{\pi^2}{36s} + \frac{s}{24}} \left(\frac{1}{3} + c_1 s^{\frac{1}{3}} t_1(s) + c_2 s^{\frac{2}{3}} t_2(s)\right),$$

where

$$\begin{aligned}
t_1(s) :=& 1 - \frac{7}{2^6 3}s - \frac{97}{2^8 3^3}s^2 - \frac{40061}{2^{15}3^4}s^3 - \frac{18915331}{2^{19}3^6 5}s^4 - \frac{13796617247}{2^{27}3^6 5}s^5 - \cdots,\\
t_2(s) :=& 5 - \frac{29}{2^4 3}s + \frac{19435}{2^{11}3^3}s^2 - \frac{14885}{2^{12}3^3}s^3 + \frac{51970999}{2^{18}3^6}s^4 - \frac{28436136277}{2^{24}3^7 5}s^5 + \cdots,
\end{aligned}$$

and

$$c_1 := \frac{3^{-\frac{1}{6}}\Gamma\left(\frac{1}{3}\right)}{8\pi} \quad \text{and} \quad c_2 := \frac{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}{32\pi}. \tag{1.1}$$

The computations of Zagier are tantalizing because of the rational values appearing in the expansion of $t_1(s)$ and $t_2(s)$ and curious because of the powers of $s^{1/3}$ which are atypical in similar partition problems. We establish Zagier's numerics and its generalizations for all k.

Theorem 1.2. *For every $k \in \mathbb{N}$ with $k > 1$, and $N \in \mathbb{N}_0$, we have, as $s \to 0$,*

$$g_k(e^{-s}) = \frac{1}{k+1}\sqrt{\frac{2\pi}{s}}e^{-\frac{\pi^2}{3k(k+1)s}+\frac{s}{24}}\left(\frac{k+1}{k} + \sum_{j=1}^{kN}\beta_k(j)s^{\frac{j}{k}} + O\left(s^N\right)\right),$$

where

$$\beta_k(j) := b_k(j)(k+1)^{-j}k^{\frac{j(k+1)}{k}} + \sum_{kr+\ell=j} b_k(\ell)\sum_{n=1}^{\infty} a_{n,r}(-\ell)^n(k+1)^{n-\ell}k^{\frac{\ell(k+1)}{k}-n}$$

with

$$b_k(j) := \frac{k+1}{k\pi j!}(-1)^{j+1}\sin\left(\frac{\pi j(k-1)}{k}\right)\Gamma\left(\frac{j(k+1)}{k}\right) \tag{1.2}$$

and the $a_{n,r}$ are rational numbers defined in (4.2). *Moreover, for each $0 < j < k$ and $m \in \mathbb{N}$, the values $\beta_k(j + mk)/\beta_k(j) \in \mathbb{Q}$.*

Remark. Theorem 1.2 confirms Zagier's numerics in the case $k = 3$.

As is the case of g_2, modular forms (and mock modular forms) arise as generating functions in many partition problems. Knowing that certain generating functions are modular gives one access to deep theoretical tools to prove results in other areas. On the other hand our knowledge of q-hypergeometric series currently fall far short of a comprehensive theory to describe the interplay between them and modular forms. A recent conjecture of W. Nahm [12] relates the modularity of such series to K-theory. In the situation of interest, with the exception of the case $k = 2$, there are no modularity results for g_k.

Our proof technique demonstrates the connection between the series g_k and Wright's generalization of the Bessel function

$$\phi(\rho, \beta; z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\beta - \rho n)} \tag{1.3}$$

with $\rho < 1$ and $\beta \in \mathbb{C}$. In particular, as $s \to 0$, we establish that the leading term in the relative error R_k (equation (3.2)) is proportional to the real part of a Wright function

$$g_k(q) \sim \frac{2}{k+1}\sqrt{\frac{2\pi}{s}} e^{-\frac{\pi^2}{3k(k+1)s}+\frac{s}{24}} \operatorname{Re}\left(\phi\left(\frac{k}{k+1}, 1; \frac{(k+1)^{\frac{k}{k+1}}}{k} e^{\frac{\pi i k}{k+1}} s^{-\frac{1}{k+1}}\right)\right).$$

In particular, for $k = 3$, a result of Wright [17, equation (3.5) and Section 4] gives, as $s \to 0$

$$\frac{1}{2}\operatorname{Re}\left(\phi\left(\frac{3}{4}, 1; \frac{4^{\frac{3}{4}}}{3} e^{\frac{3\pi i}{4}} s^{-\frac{1}{4}}\right)\right) \sim \frac{1}{3} + c_1 s^{\frac{1}{3}} + 5c_2 s^{\frac{2}{3}} + O(s), \tag{1.4}$$

where c_1 and c_2 are as in (1.1). These are Zagier's asymptotics up to $O(s)$. We believe that such comparison and application of other q analogues of generalized hypergeometric functions may be useful in other asymptotic problems.

The paper is organized as follows. Section 2 contains notation and basic results about the q-functions used throughout the paper. Section 3 defines the relative error between the series g_k and the expected main term. Section 4 shows that the relative error can be approximated by the Wright function and its "moments".

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2. Notation and Preliminary Results

This section contains some preliminary results that we require for the proof of Theorem 1.2 as well as some q-series notation. Wright [14, 15, 17] established asymptotics for $\phi(\rho, \beta; z)$, defined in (1.3), in all domains. Unfortunately, a direct application of these asymptotics in our setting (see equation (1.4)) produces a degenerate answer. Namely, from [17, Theorem 1] with $Y = -\frac{1}{k(k+1)s}$

$$\phi\left(\frac{k}{k+1}, 1; \frac{(k+1)^{\frac{k}{k+1}}}{k} e^{\frac{\pi ik}{k+1}} s^{-\frac{1}{k+1}}\right) \sim i\sqrt{k(k+1)s}e^{\frac{1}{k(k+1)s}} \sum_{m=0}^{M-1} A_m(-1)^m (k(k+1)s)^m,$$

where the coefficients A_m are given in [17]. Taking real parts shows that

$$\mathrm{Re}\left(\phi\left(\frac{k}{k+1}, 1; \frac{(k+1)^{\frac{k}{k+1}}}{k} e^{\frac{\pi ik}{k+1}} s^{-\frac{1}{k+1}}\right)\right) = O(1).$$

A little more nuance needs to be applied to Wright's work to obtain a meaningful estimate.

Proposition 2.1. *If $\frac{1}{2} \leq \rho < 1$ with $|\arg(-e^{2\pi i\rho})| < \frac{\pi}{2}(1+\rho)$ and $z > 0$, then, for $L \in \mathbb{N}$,*

$$\mathrm{Re}\left(\phi\left(\rho, 1; ze^{\pi i\rho}\right)\right) = \frac{1}{2\rho} + \frac{1}{2\pi\rho}\sum_{\ell=1}^{L-1} \frac{(-1)^{\ell+1}}{\ell!}\Gamma\left(\frac{\ell}{\rho}\right) z^{-\frac{\ell}{\rho}} \sin\left(\frac{\pi\ell(2\rho-1)}{\rho}\right) + O\left(z^{-\frac{L}{\rho}}\right).$$

Proof. We apply the identity

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\pi}\sin(\pi z) \tag{2.1}$$

and the double angle formula to show that

$$\mathrm{Re}\left(\phi(\rho, 1; ze^{\pi i\rho})\right) = \frac{1}{2\rho} + \frac{1}{2\pi}\mathrm{Im}\left(D\left(ze^{2\pi i\rho}\right)\right),$$

where

$$D(w) := \gamma\left(\frac{1}{\rho} - 1\right) + \frac{1}{\rho}\log(-w) + \sum_{m=1}^{\infty} \frac{w^m\Gamma(\rho m)}{m!}.$$

Equation (3.5) of [17] states that if $|\arg(-w)| < \pi/2(1+\rho)$, then

$$D(w) = \frac{1}{\rho}\sum_{m=1}^{L-1} \frac{(-1)^m}{m!}\Gamma\left(\frac{m}{\rho}\right)(-w)^{-\frac{m}{\rho}} + O\left(w^{-\frac{L}{\rho}}\right).$$

Note that Wright [17] used the notation $\sigma = \rho$ and $\beta = 1$. Moreover, our $D(w)$ is $d(w)$ adjusted for the $t = 0$ singularity. This adjustment is discussed Section 4 of the same paper.

$\square$

Throughout, we use the following q-notation ($z \in \mathbb{C}$):

$$\begin{aligned}
(z;q)_\infty &:= \prod_{m=0}^{\infty} (1 - zq^m),\\
(q;q)_z &:= \frac{(q;q)_\infty}{(q^{z+1};q)_\infty},\\
\theta(z,q) &:= \sum_{n\in\mathbb{Z}} (-1)^n z^n q^{n^2},\\
\Gamma_q(z) &:= (q;q)_{z-1}(1-q)^{1-z}.
\end{aligned} \tag{2.2}$$

The Jacobi function has the product expansion (see (100.2) of [13])

$$\theta(z,q) = \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\left(1 - zq^{2n-1}\right)\left(1 - z^{-1}q^{2n-1}\right), \tag{2.3}$$

and satisfies the following inversion formula (with $z := e^{2\pi i u}$) (see (38.2) of [13])

$$\theta(z,q) = \sqrt{\frac{\pi}{s}} \sum_{n \text{ odd}} e^{-\frac{\pi^2}{4s}(n+2u)^2}. \tag{2.4}$$

Next, we recall two identities due to Euler, which state that [2, equations (2.25) and (2.2.6)]

$$\begin{aligned}
\frac{1}{(z;q)_\infty} &= \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n},\\
(z;q)_\infty &= \sum_{n=0}^{\infty} \frac{(-1)^n z^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}.
\end{aligned}$$

Moreover, we require the following asymptotic behavior

$$(q;q)_\infty = \sqrt{2\pi} s^{-\frac{1}{2}} \exp\left(-\frac{\pi^2}{6s} + \frac{s}{24}\right)\left(1 + O\left(e^{-\frac{4\pi^2}{s}}\right)\right), \tag{2.5}$$

which is easily derived from the transformation formula

$$(q;q)_\infty = \sqrt{\frac{2\pi}{s}} e^{-\frac{\pi^2}{6s} + \frac{s}{24}} \prod_{n=1}^{\infty} \left(1 - e^{-\frac{4\pi^2 n}{s}}\right) \tag{2.6}$$

(see (118.5) of [13]).

The following lemma is used in Section 3 to identify terms which can be asymptotically ignored in a q-hypergeometric expression for g_k.

Lemma 2.2. *As $s \to 0$ and $x \to \infty$, we have*

$$\frac{1}{(q;q)_{x-3}(q;q)_{-x}} = O\left(sq^{-\frac{x(x-3)}{2}}\right).$$

Proof. By (2.3)

$$\left(q^{x-2};q\right)_\infty \left(q^{1-x};q\right)_\infty (q;q)_\infty = \theta\left(q^{x-\frac{3}{2}}, q^{\frac{1}{2}}\right).$$

Dividing by $(q;q)_\infty^3$ and using (2.2) then results in

$$\frac{1}{(q)_{x-3}(q)_{-x}} = \frac{\theta\left(q^{x-\frac{3}{2}}, q^{\frac{1}{2}}\right)}{(q;q)_\infty^3}. \tag{2.7}$$

By (2.5)

$$\frac{1}{(q;q)_\infty^3} = \frac{s^{\frac{3}{2}}}{\sqrt{8\pi^3}} \exp\left(\frac{\pi^2}{2s} - \frac{s}{8}\right)\left(1 + O\left(e^{-\frac{4\pi^2}{s}}\right)\right).$$

Moreover (2.4) yields

$$\theta\left(q^{x-\frac{3}{2}}, q^{\frac{1}{2}}\right) = \sqrt{\frac{8\pi}{s}} \mathrm{Re}\left(\exp\left(\frac{\left(\pi i + s\left(x - \frac{3}{2}\right)\right)^2}{2s}\right)\right)\left(1 + O\left(e^{-\frac{4\pi^2}{s}}\right)\right).$$

Combining these approximations with (2.7) gives

$$\begin{aligned}\frac{1}{(q;q)_{x-3}(q;q)_{-x}} &= \frac{s}{\pi}\mathrm{Re}\left(\exp\left(\frac{\left(\pi i + s\left(x-\frac{3}{2}\right)\right)^2 + \pi^2}{2s}\right)\right)\left(1 + O\left(e^{-\frac{4\pi^2}{s}}\right)\right)\\ &= -\frac{sq^{-\frac{\left(x-\frac{3}{2}\right)^2}{2}}}{\pi}\sin(\pi x)\left(1 + O\left(e^{-\frac{4\pi^2}{s}}\right)\right) = O\left(sq^{-\frac{x(x-3)}{2}}\right).\end{aligned}$$

□

The following is derived from [11, Theorem 2] after applying (2.5) (see also [19]).

Theorem 2.3. *For $x \in \mathbb{R}\backslash\{-\mathbb{N}_0\}$ and $N \in \mathbb{N}_0$, as $|s| \to 0$,*

$$\frac{\Gamma(x)}{\Gamma_q(x)}\left(\frac{1-q}{s}\right)^{1-x} q^{\frac{x(x-1)}{2}} = q^{\frac{x(x-1)}{4}} \exp\left(-\sum_{j=1}^N \frac{B_{2j}B_{2j+1}(x)}{2j(2j+1)!}s^{2j} + O_N\left(|s|^{2N+1}\right)\right),$$

where $B_k(x)$ are the Bernoulli polynomials and B_k are the Bernoulli numbers. Moreover, this asymptotic can be taken to hold on compact subsets of the complex s-plane.

3. The Relative Error

In this section, we asymptotically approximate g_k and define a relative error term which is then compared to the Wright function.

We start by representing g_k as an infinite sum of theta functions (see equation (3.3) in [3])

$$g_k(q) = \frac{1}{(q^k;q^k)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{km(m+1)}{2}} \left(q^{k+1-km};q^{k+1}\right)_\infty}{(q^k;q^k)_m} \theta\left(q^{km}, q^{\frac{k(k+1)}{2}}\right). \tag{3.1}$$

Turning to the asymptotic expansion of g_k, it follows from Conjecture 1.1, with the constant as established in [9], and (2.5) that

$$g_k\left(e^{-s}\right) \sim \frac{k+1}{k} \frac{\left(q^{k+1};q^{k+1}\right)_\infty}{(q^k;q^k)_\infty} \sqrt{\frac{2\pi}{k(k+1)s}} e^{-\frac{\pi^2}{2k(k+1)s}}.$$

Thus it is natural to define the relative error

$$R_k(q) := g_k(q) \frac{\left(q^k;q^k\right)_\infty}{\left(q^{k+1};q^{k+1}\right)_\infty} \sqrt{\frac{k(k+1)s}{2\pi}} e^{\frac{\pi^2}{2k(k+1)s}} \tag{3.2}$$

and hence $\lim_{q\to 1} R_k(q) = (k+1)/k$.

The next lemma transforms the theta term in (3.1) to identify a leading term for the relative error R_k in terms of the q-series

$$\mathcal{I}_n(s) := \sum_{m=0}^{\infty} \frac{(-1)^m e^{\frac{\pi i m n}{k+1}} q^{\frac{km(m+1)}{2} - \frac{km^2}{2(k+1)}}}{(q^k;q^k)_m (q^{k+1};q^{k+1})_{-\frac{km}{k+1}}}.$$

Remark. The function $\mathcal{I}_1$ is closely related to the q-Wright function defined in [7]. The main difference is that $(k+1)/k$ is not an integer in our case.

Lemma 3.1. *For every $q \in (0,1)$, we have*

$$R_k(q) = \sum_{n \text{ odd}} e^{-\frac{\pi^2\left(n^2-1\right)}{2k(k+1)s}} \mathcal{I}_n(s).$$

Proof. Rewriting (3.1), we obtain that

$$g_k(q) = \frac{\left(q^{k+1};q^{k+1}\right)_\infty}{\left(q^k;q^k\right)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{km(m+1)}{2}} \theta\left(q^{km}, q^{\frac{k(k+1)}{2}}\right)}{\left(q^k;q^k\right)_m \left(q^{k+1};q^{k+1}\right)_{-\frac{km}{k+1}}}.$$

Lemma 3.1 now follows by applying the transformation law (2.4), to yield that

$$\theta\left(q^{km}, q^{\frac{k(k+1)}{2}}\right) = \sqrt{\frac{2\pi}{k(k+1)s}} q^{-\frac{km^2}{2(k+1)}} \sum_{n \text{ odd}} e^{\frac{\pi imn}{k+1}} e^{-\frac{\pi^2 n^2}{2k(k+1)s}}.$$

□

The next lemma bounds the terms in the summation for R_k in Lemma 3.1.

Lemma 3.2. *For all $n \in \mathbb{N}$ and $s > 0$, we have, as $s \to 0$,*

$$\mathcal{I}_n(s) = O\left(\frac{1}{s^3}\exp\left(\frac{\pi^2}{6k(k+1)s}\right)\right)$$

uniformly in n.

Proof. Let us first note that for $x > 1$, $(1-q^x)^{-1} < (1-q)^{-1}$, so that

$$(q;q)_x = \frac{(q;q)_{x+3}}{\left(1-q^{x+3}\right)\left(1-q^{x+2}\right)\left(1-q^{x+1}\right)} = O\left(\frac{(q;q)_{x+3}}{s^3}\right). \tag{3.3}$$

Applying Lemma 2.2 with $x = km/(k+1)$, yields, using (3.3),

$$\begin{aligned}
\frac{1}{\left(q^k;q^k\right)_m \left(q^{k+1};q^{k+1}\right)_{-\frac{km}{k+1}}} &= O\left(s \frac{\left(q^{k+1};q^{k+1}\right)_{\frac{km}{k+1}-3}}{\left(q^k;q^k\right)_m} q^{-\frac{k^2m^2}{2(k+1)} + \frac{3km}{2}}\right) \\
&= O\left(\frac{\left(q^{k+1};q^{k+1}\right)_{\frac{km}{k+1}}}{s^2\left(q^k;q^k\right)_m} q^{-\frac{k^2m^2}{2(k+1)}}\right) \\
&= O\left(\frac{\left(q^{k+1};q^{k+1}\right)_\infty}{\left(q^k;q^k\right)_\infty} \frac{1}{s^2} \frac{\left(q^k q^{km};q^k\right)_\infty}{\left(q^{km+k+1};q^{k+1}\right)_\infty} q^{-\frac{k^2m^2}{2(k+1)}}\right) \\
&= O\left(\frac{\left(q^{k+1};q^{k+1}\right)_\infty}{\left(q^k;q^k\right)_\infty s^2} q^{-\frac{k^2m^2}{2(k+1)}}\right).
\end{aligned}$$

The last equality follows since

$$\left(1-q^{k(m+j)}\right)<\left(1-q^{(k+1)(m+j)}\right),$$

which implies that

$$\frac{\left(q^kq^{km};q^k\right)_\infty}{\left(q^{km+k+1};q^{k+1}\right)_\infty}<1.$$

Combining the above gives

$$\mathcal{I}_n(s)=O\left(\frac{\left(q^{k+1};q^{k+1}\right)_\infty}{\left(q^k;q^k\right)_\infty s^2}\frac{1}{1-q^{\frac{k}{2}}}\right).$$

The claim then follows by (2.6). $\square$

The next lemma determines the main terms in the summation for R_k in Lemma 3.1 explicitly.

Lemma 3.3. *For $s>0$ and $N\in\mathbb{N}$, we have*

$$R_k(q)=\mathcal{I}_1(s)+\mathcal{I}_{-1}(s)+O\left(s^N\right)=2\mathrm{Re}\left(\mathcal{I}_1(s)\right)+O\left(s^N\right).$$

Proof. We have, using Lemma 3.1, Lemma 3.2, and the integral comparison test,

$$\begin{aligned}\left|R_k(q)-\mathcal{I}_1(s)-\mathcal{I}_{-1}(s)\right|&\leq 2\frac{1}{s^3}e^{\frac{\pi^2}{6k(k+1)s}}\sum_{\substack{n\text{ odd}\\ n\geq 3}}e^{-\frac{\pi^2\left(n^2-1\right)}{2k(k+1)s}}\\ &=O\left(\frac{1}{s^3}e^{\frac{2\pi^2}{3k(k+1)s}}\int_2^\infty e^{-\frac{\pi^2x^2}{2k(k+1)s}}dx\right)=O\left(s^N\right).\end{aligned}$$

$\square$

4. Relative Error in terms of the Wright Function

In this section, we continue the study of $\mathcal{I}_1(s)$, relating it, and thus the relative error R_k, to the Wright function. By definition

$$\mathcal{I}_{-1}(s)=\sum_{m=0}^\infty\frac{e^{\frac{\pi imk}{k+1}}q^{\frac{k^2m^2}{2(k+1)}}}{\Gamma_{q^k}(m+1)\Gamma_{q^{k+1}}\left(1-\frac{km}{k+1}\right)}\left(\frac{\left(1-q^{k+1}\right)^{\frac{k}{k+1}}q^{\frac{k}{2}}}{1-q^k}\right)^m.$$

Define w by

$$\frac{(1-q^{k+1})^{\frac{k}{k+1}}q^{\frac{k}{2}}}{1-q^k}\sim\frac{(k+1)^{\frac{k}{k+1}}}{ks^{\frac{1}{k+1}}}=:w\quad\text{as }q\to 1$$

and write

$$\mathcal{I}_{-1}(s)=\sum_{m=0}^{\infty}\frac{w^m e^{\frac{\pi i m k}{k+1}}h_q(m)}{\Gamma(m+1)\Gamma\left(1-\frac{km}{k+1}\right)},$$

where, for $z\in\mathbb{C}$,

$$h_q(z):=q^{\frac{k^2z^2}{2(k+1)}+\frac{kz}{2}}\frac{\Gamma(z+1)\Gamma\left(1-\frac{kz}{k+1}\right)}{\Gamma_{q^k}(z+1)\Gamma_{q^{k+1}}\left(1-\frac{kz}{k+1}\right)}\left(\frac{1-q^{k+1}}{(k+1)s}\right)^{\frac{kz}{k+1}}\left(\frac{ks}{1-q^k}\right)^{z}.$$

For every $s>0$, $\Gamma_q(z)$ is, as a function of z, a nonzero meromorphic function with simple poles only if $q^{z+m}=1$ for some $m\in\mathbb{N}_0$. Therefore, $\Gamma(z)/\Gamma_q(z)$ can be continued to an entire function in z and thus the same is true for $h_q(z)$. Hence, it is possible to define z-Taylor coefficients for $h_q(z)$. Namely,

$$h_q(z)=a_0(s)+a_1(s)z+a_2(s)z^2+a_3(s)z^3+\cdots. \tag{4.1}$$

We must then expand each $a_n(s)$ in terms of powers of s and show that while $a_0(s)=1$, $a_n(s)=O(s)$ as $s\to 0$.

Lemma 4.1. *With $a_n(s)$ defined in (4.1), there exists numbers $a_{n,j}$, such that*

$$a_n(s)=\begin{cases}1 & \text{if } n=0,\\ a_{n,\lceil\frac{n}{2}\rceil}s^{\lceil\frac{n}{2}\rceil}+a_{n,\lceil\frac{n}{2}\rceil+1}s^{\lceil\frac{n}{2}\rceil+1}+\cdots+a_{n,N}s^N+O_N\left(s^{N+1}\right) & \text{if } n>0,\end{cases}$$

for $N\in\mathbb{N}_0$. Furthermore $a_n(s)\ll s^{\frac{n}{2}}$ uniformly in n as $s\to 0$.

Proof. Since the z-Taylor series of $h_q(z)$ converges uniformly on compact subsets, we obtain $h_q(z)=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}a_{n,j}s^jz^n$. Applying Theorem 2.3 twice gives

$$h_q(z)=q^{-\frac{kz^2}{4(k+1)}+\frac{kz}{2}}\exp\left(-\sum_{j=1}^{N}f_{2j}(z)s^{2j}+O\left(|s|^{2N+2}\right)\right), \tag{4.2}$$

where

$$f_{2j}(z):=\frac{B_{2j}\left(B_{2j+1}(1+z)k^{2j}+B_{2j+1}\left(1-\frac{kz}{k+1}\right)(k+1)^{2j}\right)}{2j(2j+1)!}.$$

Since

$$\lim_{s\to 0}h_q\left(\frac{z}{\sqrt{s}}\right)=e^{\frac{k}{4(k+1)}z^2}$$

by Cauchy's Theorem one obtains

$$a_{2j}(s) \sim s^j \left(\frac{k}{4(k+1)} \right)^j$$

and $a_{2j+1}(s) = o(s^j)$. It then follows that $a_j(s) \ll s^{\frac{j}{2}}$ uniformly in j. This is the claimed expansion for $a_n(s)$ with $n \geq 1$. Finally, observe that $h_q(0) = a_0(s) = 1$. In particular,

$$h_q(z) = 1 - \frac{k}{2} sz - s\frac{k^2}{4(k+1)} z^2 + O\left(s^2\right).$$

□

Using Lemma 3.3 the relative error becomes

$$R_k(q) = \sum_{j=0}^{\infty} a_j(s) 2\mathrm{Re}\left(\phi_j\left(\frac{k}{k+1}, 1; e^{-\frac{\pi i k}{k+1}} w\right)\right) + O\left(s^N\right),$$

where

$$\phi_j(\rho, \beta; z) := \sum_{m=0}^{\infty} \frac{m^j z^m}{\Gamma(m+1)\Gamma(\beta - \rho m)}.$$

Note that $\phi_0(\rho, \beta; z) = \phi(\rho, \beta; z)$ is the usual Wright function given in the introduction. Define

$$W_j(w) := 2\mathrm{Re}\left(\phi_j\left(\frac{k}{k+1}, 1; e^{-\frac{\pi i k}{k+1}} w\right)\right).$$

In this notation, (4.1) and Lemma 4.1 yield

$$R_k(q) = W_0(w) + \sum_{j=1}^{\infty} a_j(s) W_j(w) + O\left(s^N\right). \tag{4.3}$$

Since $\frac{1}{2} \leq k/(k+1) \leq 1$ and $w \to \infty$ as $s \to 0$ we are interested in the behavior of the Wright function for $\frac{1}{2} \leq \rho < 1$ as $w \to \infty$. Proposition 2.1 applies directly with $\rho = \frac{k}{k+1}$ and yields the following.

Proposition 4.2. *For $z > 0$ and $L \in \mathbb{N}$,*

$$W_0(z) = \frac{k+1}{k} + \sum_{\ell=1}^{L} b_k(\ell) z^{-\frac{\ell(k+1)}{k}} + O\left(z^{-\frac{L(k+1)}{k}}\right),$$

where $b_k(\ell)$ is defined in (1.2).

The following theorem is a generalization of Proposition 4.2.

Proposition 4.3. *For every $z > 0$ and $j, L \in \mathbb{N}$*

$$W_j(z) = \sum_{\ell=1}^{L-\left\lfloor \frac{jk}{k+1}\right\rfloor -1} \left(-\ell\frac{k+1}{k}\right)^j b_k(\ell) z^{-\frac{\ell(k+1)}{k}} + 2\sum_{m=1}^{\left\lfloor j-\frac{L(k+1)}{k}\right\rfloor} \frac{m^j z^m}{m!}\cos\left(-\frac{\pi m k}{k+1}\right) + O\left(z^{j-\frac{L(k+1)}{k}}\right),$$

where the implied constant is uniform in j.

Remark. If $j \leq \frac{L(k+1)}{k}$, then the second sum is empty, while if $j \geq \frac{L(k+1)}{k}$, then the first sum is empty.

Proof. The proof is similar to the proof of Proposition 2.1. We first use (2.1) to rewrite

$$W_j(z) = \frac{1}{2\pi}\operatorname{Im}\left(D_j\left(ze^{2\pi i\rho}\right)\right),$$

where $\rho = \frac{k}{k+1}$ and, setting $\delta_{j,1} = 1$ if $j = 1$ and $\delta_{j,1} = 0$ otherwise,

$$D_j(w) := \delta_{j,1} + \sum_{m=1}^{\infty} \frac{m^j w^m \Gamma(\rho m)}{m!}.$$

Note that since t^j vanishes at $t = 0$, we do not need the modification for the $t = 0$ singularity which occurred in the proof of Proposition 2.1. The goal now is to use a result similar to (3.5) of [17] for the modified function $D_j(w)$ instead of $D(w)$. As pointed out by Wright, equation (3.5) of [17] is a direct application of Theorem 6 of [16].

Theorem 6 of [16] may also be directly applied to $D_j(w)$, but unfortunately the error term is not necessarily uniform in j. In order to obtain a uniform bound in j, we investigate Theorem 6 of [16] further. Wright proved Theorem 6 as a direct corollary of Theorem 5 of [16], which is more general and a careful analysis of the proof of Theorem 5 yields the desired uniform bound. Setting $\kappa := 1 - \rho$, define for $j \in \mathbb{N}_0$

$$\varphi_j(t) := \frac{\Gamma(1-\kappa t)\Gamma(\rho t)t^j}{\Gamma(t+1)}$$

and

$$k_j(t) := \frac{\pi\varphi_j(t)(-w)^t}{\sin(\pi t)\Gamma(1-\kappa t)}.$$

Then for $j \geq 1$

$$D_j(w) = \sum_{m=0}^{\infty} \mathrm{Res}_{t=m}\left(k_j(t)\right).$$

Setting $h_j := j - \frac{L}{\rho}$, we claim that

$$\begin{aligned} D_j(w) &= \sum_{\ell=1}^{\left\lfloor \frac{h_j}{\rho} \right\rfloor} \mathrm{Res}_{t=-\ell\rho}\left(k_j(t)\right) + \delta_{j,1}\delta_{h_j\geq 0} + \sum_{m=1}^{\lfloor h_j \rfloor} \frac{m^j w^j \Gamma(\rho m)}{m!} \\ &= \frac{1}{\rho}\sum_{\ell=1}^{L-\lfloor j\rho\rfloor -1} \frac{(-1)^\ell}{\ell!}\left(-\frac{\ell}{\rho}\right)^j \Gamma\left(\frac{\ell}{\rho}\right)(-w)^{\frac{\ell}{\rho}} + \delta_{j,1}\delta_{h_j\geq 0} + \sum_{m=1}^{\left\lfloor j-\frac{L}{\rho}\right\rfloor} \frac{m^j w^j \Gamma(\rho m)}{m!} + O\left(w^{j-\frac{L}{\rho}}\right), \end{aligned} \tag{4.4}$$

where the implied constant is independent of j and $\delta_{h_j\geq 0} = 1$ if and only if $h_j \geq 0$. Wright's asymptotic in Theorem 5 (see page 444 of [16]) is obtained by choosing some $h \in \mathbb{R}$ and taking a contour integral of $k_j(t)$ along $\mathrm{Re}(t) = h$ with $|t| < m + \frac{1}{2}$ and then along the arc $\mathcal{C}_m$ of the circle $|t| = m + \frac{1}{2}$ with $\mathrm{Re}(t) \geq h$. Letting $m \to \infty$, Lemma 11 (ii) of [16] states that

$$\lim_{m\to\infty} \int_{\mathcal{C}_m} k_j(t)dt,$$

and hence (note that k_j has a removable singularity at $t = 0$), as given in the last equation on page 444 of [16],

$$\frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} k_j(t)dt = D_j(w) - \sum_{m=1}^{\lfloor h\rfloor} \frac{m^j w^m \Gamma(\rho m)}{m!} + \sum_{\ell=1}^{\left\lfloor \frac{h}{\rho}\right\rfloor} \mathrm{Res}_{t=-\ell\rho}\left(k_j(t)\right).$$

Choosing $h = h_j$ and computing the residues, one obtains

$$\sum_{\ell=1}^{\left\lfloor \frac{h}{\rho}\right\rfloor} \mathrm{Res}_{t=-\ell\rho}\left(k_j(t)\right) = -\frac{1}{\rho}\sum_{\ell=1}^{L-\lfloor j\rho\rfloor-1} \frac{(-1)^\ell}{\ell!}\left(-\frac{\ell}{\rho}\right)^j \Gamma\left(\frac{\ell}{\rho}\right)(-w)^{\frac{\ell}{\rho}},$$

which is precisely the negative of the main terms in the first sum on the right-hand side of (4.4).

It hence remains to show that for this choice of h

$$\frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} k_j(t)dt = O\left(w^{j-\frac{L}{\rho}}\right)$$

uniformly in j. To show this, we use Lemma 11 (i) of [16]. Suppose no poles of $k_j(t)$ occur in the region $\mathrm{Re}(t) \in (h_j - \varepsilon, h_j + \varepsilon)$. Then, under the assumption that there exist constants $C_0 > 0$ and $\sigma > 0$ (independent of j) for which (for $|t| \gg 1$)

$$|\varphi_j(t)| < C_0 |t|^{h_j - \varepsilon - \frac{1}{2}} e^{\kappa\sigma|t|}, \tag{4.5}$$

Lemma 11 (i) states that for t with $\mathrm{Re}(t) = h_j$, there exists a constant C, depending only on σ and C_0, for which

$$|k_j(t)| < C|t|^{-1-\varepsilon}\kappa^{h_j}\left(\rho^\rho|w|\right)^{\frac{h_j}{\rho}}. \tag{4.6}$$

Since $D(w)$ satisfies the conditions of Theorem 6 of [16], $\varphi_0(t)$ satisfies condition (4.5). We may hence choose C_0 and σ appropriately, depending on φ_0 and L (choosing $h = h_0 = -\frac{L}{\rho}$ for the $j = 0$ case). Since $h_j = h_0 + j$,

$$|\varphi_j(t)| = |t|^j\,|\varphi_0(t)|$$

implies that φ_j also satisfies (4.5) with the same constants C_0 and σ. Therefore (4.6) implies that (by slightly altering the integration path, we may assume without loss of generality that $h_j \neq 0$ for $\mathrm{Im}(t) < 1$)

$$\frac{1}{2\pi i}\int_{h_j-i\infty}^{h_j+i\infty} |k_j(t)|\,dt < C\kappa^{h_j}\left(\rho^\rho|w|\right)^{\frac{h_j}{\rho}}\int_{\mathbb{R}} \frac{1}{\left(y^2+h_j^2\right)^{\frac{1}{2}+\varepsilon}}dy.$$

Bounding $|h_j| > \delta > 0$, the remaining integral may be bound independent of j. Furthermore, since $0 < \kappa \leq \rho < 1$, we have $\kappa^j < 1$ and $\rho^{\rho j} < 1$, so κ^{h_j} and ρ^{h_j} may be bound independently of j. Hence we obtain

$$\frac{1}{2\pi i}\int_{h_j-i\infty}^{h_j+i\infty} |k_j(t)|\,dt = O\left(|w|^{\frac{h_j}{\rho}}\right)$$

with the implied constant independent of j. This completes the proof. □

We have now proven what we set out to prove.

Proof of Theorem 1.2. Recall that $h_q(z)$, as given in (4.2), represents an entire function which converges absolutely and uniformly in all compact subsets of the complex plane. A precise application of this is that

$$w^{\frac{(kN)(k+1)}{k}} \sum_{n=0}^{\infty} |a_n(s)| \left(2\frac{(kN)(k+1)}{k}\right)^n = O_N\left(w^{N(k+1)}\right) = O\left(s^N\right).$$

Furthermore, since for $k \geq 1$ we obtain from Lemma 4.1 that

$$a_j(s)w^j = O\left(s^{\frac{j}{2}} s^{-\frac{j}{k+1}}\right) = O(1),$$

the terms in the second sum of Proposition 4.3 do not contribute to the main asymptotic and for $L = kN$ the error term becomes $O(s^N)$. It then follows from Proposition 4.3, (4.2), and (4.3) that

$$R_k(q) = \frac{k+1}{k} + \sum_{\ell=1}^{kN-1} b_k(\ell) w^{-\frac{k+1}{k}} h_q\left(-\frac{\ell(k+1)}{k}\right) + O_N\left(s^N\right).$$

Theorem 1.2 now follows directly from (2.5), (3.2), (4.3), and Proposition 4.3. □

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