Robustly Strategic Consumption-Portfolio Rules with Informational Frictions*

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Abstract

This paper provides a tractable continuous-time constant-absolute-risk averse (CARA)-Gaussian framework to explore how the interactions of fundamental uncertainty, model uncertainty due to a preference for robustness (RB), and state uncertainty due to information-processing constraints (rational inattention or RI) affect strategic consumption-portfolio rules and precautionary savings in the presence of uninsurable labor income. Specifically, after solving the model explicitly, I compute and compare the elasticities of strategic asset allocation and precautionary savings to risk aversion, robustness, and inattention. Furthermore, for plausibly estimated and calibrated model parameters, I quantitatively analyze how the interactions of model uncertainty and state uncertainty affect the optimal share invested in the risky asset, and show that they can provide a potential explanation for the observed stockholding behavior of households with different education and income levels.

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1 Introduction

Intertemporal consumption-saving and portfolio choice is a fundamental topic in modern economics. In the real world, ordinary investors face pervasive uncertainty, and have to make consumption-saving-investment decisions in environments in which they are not only uncertain about the present or future states of the world (e.g., equity returns and uninsurable labor income), but are also concerned about the structure of the model economy. Many empirical studies found that incomplete information about the relevant variables plays an important role in affecting agents' optimal decisions. For example, direct survey evidence in King and Leape (1987) suggested that incomplete information about investment opportunities is a significant determinant of household portfolio composition.[1] Hong, Torous, and Valkanov (2007) found that investors in the stock market react gradually to information contained in industry returns about their fundamentals and that information diffuses only gradually across markets. Coibion and Gorodnichenko (2015) found pervasive evidence consistent with the information rigidity theories using the U.S. surveys of professional forecasters and other agents. As for model uncertainty, many empirical and experimental studies have repeatedly confirmed Ellsberg's conjecture that aversion to ambiguity (or concerns about model misspecification) would lead to a violation of the Savage axioms. For example, Ahn, Choi, Gale, and Kariv (2014) used a rich experiment data set to estimate a portfolio-choice model and found that about forty percentage of subjects display either statistically significant pessimism or ambiguity aversion. It is therefore critical for us to understand how inattentive investors make optimal financial decisions when facing various types of risk and uncertainty. This paper provides a tractable continuous-time constant absolute risk aversion (CARA)-Gaussian framework to explore how investors make strategic consumption-saving-asset allocation decisions when they face both *fundamental uncertainty* (uncertainty about the equity return and labor income uncertainty) and *induced uncertainty*. In this paper, we define induced uncertainty as the interaction of model uncertainty due to a preference for robustness (RB) and state uncertainty due to rational inattention (RI).[2]

Model uncertainty and state uncertainty arise from two major types of incomplete information:

[1] In the survey (Survey of Consumer Financial Decisions) conducted by SRI international, for each of four different assets – stocks, stock mutual funds, bonds, and bond funds – households not owning the asset were asked "Why doesn't anyone in your household hold any [stocks, etc.]?" More than forty percent of those who did not own stock or stock mutual funds said that it was because they did not know enough about it.

[2] Here we label model uncertainty or state uncertainty "induced uncertainty" because it is induced by the interactions of the preference for robustness or information-processing constraints with fundamental uncertainty.

one is incomplete information about the distribution of the state transition equation, and the other is incomplete information about the true value of the state. Hansen and Sargent (1995) first introduced the preference for robustness into linear-quadratic-Gaussian (LQG) economic models.[3] In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they choose their decisions as if the subjective distribution over shocks was chosen by an evil agent to minimize their utility. As discussed in Hansen, Sargent, and Tallarini (HST, 1999) and Luo and Young (2010), RB models can produce precautionary savings even within the class of LQG models, which leads to analytical simplicity. Sims (2003) first introduced rational inattention due to information-processing constraint into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation, agents have finite Shannon channel capacity, limiting their ability to process signals about the true state. As a result, a shock to the state induces only gradual responses by individuals. Another important implication of rational inattention is that attention is a scarce resource that is important for productivity. Specifically, people would be more productive at work if they have higher income and can own more distraction-saving goods and services at home (e.g., a good baby sitter).[4] Because RI introduces the endogenous noise due to finite capacity into economic models, RI by itself creates an additional demand for robustness. Furthermore, in the standard RI problem, the agent combines a pre-specified prior over the state with the new noisy state observations to construct the perceived value of the state, and is assumed to have only a single prior (i.e., no concerns about model misspecification). However, given the difficulty in estimating expected total wealth, the sensitivity of optimal decisions to finite capacity, and the substantial empirical evidence that agents are not neutral to ambiguity, it is important to consider inattentive investors with multiple priors who are concerned about model misspecification and hence desire robust decision rules that work well for a set of possible models. The key distinction between these two types of informational frictions can be seen from the following continuous-time transition equation of the true state (s_t):

$$ds_t = (As_t + Bc_t)\, dt + \sigma dB_t,$$

[3] It is worth noting that both the preference for "wanting robustness" proposed by Hansen and Sargent and "ambiguity aversion" proposed by Epstein and his coauthors (e.g., Chen and Epstein 2002) can be used to capture the same idea of the multiple priors model. In this paper, we use Hansen and Sargent's "wanting robustness" specification to introduce model misspecification.

[4] See Banerjee and Mullainathan (2008) for a discussion on the relationship among limited attention, productivity, and income distribution.

where s_t and c_t are state and control variables, respectively; A, B, and σ are constant coefficients; and B_t is a standard Brownian motion. Under RB, agents do not know the true data generating process driven by the random innovation (B_t), whereas agents under RI cannot observe the true state (s_t) perfectly.

As the first contribution of this paper, I construct a continuous-time theoretical framework in which there are (i) two fundamental risks: uninsurable labor income and the equity return, (ii) two types of induced uncertainty: model uncertainty due to the preference for robustness and state uncertainty due to rational inattention, and (iii) CARA utility. The main reason that I adopt the CARA utility specification is for technical convenience.[5] It is shown that the models with these features can be solved explicitly. In particular, when introducing state uncertainty due to RI, I derive the continuous-time version of the information-processing constraint (IPC) proposed in Sims (2003), and obtain the explicit expressions for the stochastic properties of the RI-induced noise and the Kalman filtering equation. This paper is therefore closely related to the literature on imperfect information, learning, asset allocation and asset pricing (see Gennotte 1986, Lundtofte 2008, and Wang 2009).

Second, I show that the optimal consumption/saving-portfolio choice problem under RI and RB can be formulated by making two additions to the standard full-information rational expectations (FI-RE) model: (i) imposing an additional constraint on the information-processing ability of the agent that gives rise to endogenous noises; and (ii) introducing an additional minimization over the set of probability models subject to the additional constraint. The additional constraint recognizes that the probability model of the perceived state is not unique. Furthermore, the additional minimization procedure reflects the preference for robustness of the agent who understands that he only has finite information-processing capacity. After solving the models explicitly, we can exactly inspect the mechanism through which these two types of induced uncertainty interact and affect different types of demand for the risky asset and the precautionary saving demand.[6] In particular, I find that the optimal allocation in the risky asset and the precautionary saving demand are more sensitive to risk aversion than RB, and are more sensitive to RB than RI when

[5] A few papers find closed-form solutions in CARA models with portfolio choice and uninsurable labor income. For example, Svensson and Werner (1993) study an infinite horizon consumption-investment model with normally distributed income. It is well known that there is no closed-form solution if we move away from the CARA specification (e.g., if we adopt the constant relative risk aversion or CRRA utility) and explicitly model uninsurable labor income in the infinite-horizon consumption-portfolio choice model.

[6] Maenhout (2004), Liu, Pan, and Wang (2005), Liu (2010), and Chen, Ju, and Miao (2014) examined how model uncertainty and ambiguity affects portfolio choices and/or asset prices.

the agent is not highly information-constrained.

Finally, the model presented in this paper has some testable implications. As discussed in Haliassos and Bertaut (1995) and Campbell (2006), the empirical evidence on the correlation between labor income and equity returns for different population groups is difficult to reconcile with the observed stockholding behavior. Davis and Willen (2000) estimated that the correlation is between 0.1 and 0.3 for college-educated males and is about -0.25 for male high school dropouts. Since negative correlation between earnings and equity returns implies increased willingness to invest in the risky asset, less-educated investors should be more heavily invested in the stock market while college graduates should put less wealth in the stock market. In contrast, the empirical evidence on stock market participation shows a significantly positive correlation between education level and stockholding. Table 1 shows that the mean value of stockholding increases with the education level in the past eight surveys of consumer finances (SCF), and Table 2 shows a similar pattern in the latest 2013 SCF after controlling for income and net worth.[7] I find that incorporating induced uncertainty due to the interaction of RB and RI into the otherwise standard model can have the potential to help reconcile the model with the empirical evidence. Specifically, poorer and less well-educated investors probably face greater induced uncertainty; consequently, they rationally choose to invest less in the stock market even if the correlation between their labor income and equity returns is negative and they have stronger incentive to hedge against their earnings risk.

This remainder of the paper is organized as follows. Section 2 provides a literature review. Section 3 presents the setup of a continuous-time consumption and portfolio choice model with uninsurable labor income when investors have rational inattention. Section 4 examines how the interactions of model uncertainty due to RB and state uncertainty due to RI affect robustly strategic consumption-portfolio rules. Section 5 discusses quantitative, empirical and policy implications of the interactions of these two types of induced uncertainty. Section 6 concludes.

[7] Here I report the empirical evidence on the amount of stockholdings because the CARA-Gaussian model predicts that the amount of wealth invested in the risky asset is independent of total wealth. See the note in Table 1 for the definitions of stock holdings. Haliassos and Bertaut (1995) also found that the share invested in the stock market is substantially larger among those with at least a college degree compared to those with less than high school education at all income levels.

2 Related Literature

This paper contributes to the literature on consumption-saving-portfolio decisions with incomplete information and uninsurable labor income. It is well-known in the literature that although human wealth, the expected present value of their current and future labor income, constitutes a major fraction of ordinary investors's total wealth, moral hazard and adverse selection problems prevent the emergence of markets that can insure these investors against their idiosyncratic labor income. Such market incompleteness has stimulated substantial research interest in the behavior of precautionary saving. The recent empirical evidence on household portfolios in the U.S. and major European countries has also stimulated research in generalizing the single asset precautionary saving model to allow for portfolio choice between risky and risk-free financial assets.[8] For example, Heaton and Lucas (2000) studied how the presence of background risks influences portfolio allocations. They found that labor income is the most important source of wealth and labor income risk is weakly positively correlated with equity returns. Viceira (2001) examined the effects of labor income risk on optimal consumption and portfolio choice for both employed and retired investors.

Furthermore, this paper is also closely related to Maenhout (2004) and Wang (2009). Maenhout (2004) explored how model uncertainty due to a preference for robustness affects optimal portfolio choice, and showed that robustness significantly reduces the demand for the risky asset and increases the equilibrium equity premium. Wang (2009) studied the effects of incomplete information about the income growth rate on the agent's consumption/saving and portfolio choice in an incomplete-market economy. He found that the estimation risk arising from the agent's learning process leads to additional precautionary savings demand and the agent can partially hedge against both the income risk and the estimation risk by investing in the risky asset. Unlike Maenhout (2004), the present paper explores how the interaction of model uncertainty and state uncertainty affects the consumption/saving-portfolio decisions in the presence of uninsurable labor income. The model presented in this paper can therefore be used to study the relationship between the correlation between the labor income risk and the equity return risk and the stockholding behavior. Unlike Wang (2009), this paper considers model uncertainty due to robustness. In addition, the state uncertainty considered in this paper is not only from the income process

[8] The empirical research on household portfolios documented that the stock market participation rate was increasing in the U.S. and Europe and the importance of the precautionary saving motive for portfolio choice. See Guiso, Haliassos, and Jappelli (2002).

but also from the equity return.

Finally, this paper is also related with the work on robust/risk-sensitive/rational inattention permanent income models such as HST (1999), Luo (2008), and Luo and Young (2010, 2016). The key difference between this paper and the papers mentioned above is that they adopted the LQG framework with constant asset returns to study consumption and saving dynamics and did not consider the portfolio choice problem.

3 Rational Inattention: The Reference Model for Inattentive Consumers

3.1 Benchmark: Full-Information Rational Expectations

In this paper, we follow Wang (2009) and consider a continuous-time version of the Caballero-type model (1990) with portfolio choice. The typical investor facing uninsurable labor income in the model economy makes optimal consumption-saving-asset allocation decisions. Specifically, we assume that the investor can access: one risk-free asset and one risky asset, and also receive uninsurable labor income. Labor income (y_t) is assumed to follow an Ornstein-Uhlenbeck process:

$$dy_t = \lambda\left(\overline{y} - y_t\right)dt + \sigma_y dB_{y,t}, \tag{1}$$

where the unconditional mean and variance of y_t are $\overline{y}$ and $\sigma_y^2/\left(2\lambda\right)$, respectively, when y_t is a stationary process; the persistence coefficient λ governs the speed of convergence or divergence from the steady state;[9] $B_{y,t}$ is a standard Brownian motion on the real line $\mathcal{R}$; and σ_y is the unconditional volatility of the income change over an incremental unit of time.

The agent can invest in both a risk-free asset with a constant interest rate r and a risky asset (i.e., the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by

$$dr_t^e = \left(r + \pi\right)dt + \sigma_e dB_{e,t}, \tag{2}$$

where π is the market risk premium; $B_{e,t}$ is a standard Brownian motion; and σ_e is the standard deviation of the market return. Let ρ_{ye} be the contemporaneous correlation between the labor

[9] If $\lambda > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\lambda \leq 0$, income is non-stationary. The $\lambda = 0$ case corresponds to a simple Brownian motion without drift. The larger λ is, the less y tends to drift away from $\overline{y}$. As λ goes to ∞, the variance of y goes to 0, which means that y can never deviate from $\overline{y}$.

income process and the return of the risky asset. When $\rho_{ye} = 0$, the labor income risk is idiosyncratic and is uncorrelated with the risky market return; when $\rho_{ye} = 1$, the labor income risk is perfectly correlated with the risky market return. The agent's financial wealth evolution is then given by

$$dw_t = (rw_t + y_t - c_t)\, dt + \alpha_t\, (\pi dt + \sigma_e dB_{e,t}), \tag{3}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

To simplify the model, we define a new state variable, s_t: $s_t \equiv w_t + h_t$, where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r: $h_t \equiv E_t \left[\int_t^\infty \exp\left(-r\,(s-t)\right) y_s ds\right]$. For the given income process, (1), it is straightforward to show that $h_t = y_t / \left(r + \lambda\right) + \lambda \overline{y} / \left(r\left(r + \lambda\right)\right)$.[10] Following the state-space-reduction approach proposed in Luo (2008) and using this new state variable, we can rewrite (3) as

$$ds_t = (rs_t - c_t + \pi\alpha_t)\, dt + \sigma dB_t, \tag{4}$$

where $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$, $\sigma_s = \sigma_y / \left(r + \lambda\right)$, and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t} \tag{5}$$

is the unconditional variance of the innovation to s_t.[11]

The utility function of the typical consumer takes the CARA form: $u(c_t) = -\exp\left(-\gamma c_t\right)/\gamma$, where $\gamma > 0$ is the coefficient of absolute risk aversion.[12] If the consumer trusts the model and observes the state perfectly, i.e., there is no model uncertainty and no state uncertainty, he is assumed to maximize the following expected lifetime utility:

$$J\left(s_0\right) = \sup_{c_t, \alpha_t} E_0 \left[\int_{t=0}^\infty \exp\left(-\delta t\right) u(c_t) dt\right], \tag{6}$$

[10] Here we need to impose the restriction that $r > -\lambda$ to guarantee the finiteness of human wealth.

[11] The main advantage of this state-space-reduction approach is to allow us to solve the model with both model uncertainty and state uncertainty explicitly and help better inspect the mechanism by which the informational frictions interact and affect optimal consumption-portfolio rules. It is worth noting that if we only consider model uncertainty, the reduced univariate model and the original multivariate model are observationally equivalent in the sense that they lead to the same consumption-portfolio rules. The detailed proof is available from Online Appendix A.

[12] It is well-known that the CARA utility specification is tractable for deriving the consumption function or optimal consumption-portfolio rules in different settings. See Merton (1971), Caballero (1990), Svensson and Werner (1993), Wang (2003, 2004, 2009), and Chen, Miao, and Wang (2010).

subject to (4), where $J(s_0)$ is the value function. The following proposition summarizes the solution to this benchmark full-information rational expectations (FI-RE) model:

Proposition 1 *Under FI-RE, the optimal consumption and portfolio rules are*

$$c_t = r\left(s_t - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{r\sigma_e^2}\right) + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} - \Gamma, \tag{7}$$

and

$$\alpha = \frac{\pi}{r\gamma\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{8}$$

where

$$\Gamma \equiv \frac{1}{2}r\gamma\left(1 - \rho_{ye}^2\right)\sigma_s^2, \tag{9}$$

is the investor's precautionary saving demand.

Proof. The proof is the same as that for Model I of Wang (2009). It is also available from the corresponding author by request. ■

Expression (7) shows that the consumption function can be decomposed into four components: (i) the annuity value of the risk-adjusted expected total wealth, $r\left(s_t - \pi\rho_{ye}\sigma_s\sigma_e/\left(r\sigma_e^2\right)\right)$, (ii) the effect of the relative impatience measured by $(\delta - r)/(r\gamma)$, (iii) the wealth effect of investing in the risky asset, $\pi^2/\left(2r\gamma\sigma_e^2\right)$, and (iv) the precautionary saving premium, $-\Gamma$.

The first term in (8) is the standard speculation demand for the risky asset, which is positively correlated with the risk premium of the risky asset over the risk-free asset and is negatively correlated with the degree of risk aversion and the variance of the return to the risky asset. The second term in (8) is the labor income hedging demand of the risky asset. When $\rho_{ye} \neq 0$, i.e., the income shock is not purely idiosyncratic, the desirability of the risky asset depends not only on its expected excess return relative to its variance, but also on its ability to hedge consumption against bad realizations of labor income. Following the literature of precautionary savings, we measure the demand for precautionary saving, Γ in (9), as the amount of saving due to the interaction of the degree of risk aversion and non-diversifiable labor income risk.[13] If labor income is perfectly correlated with the return to the risky asset (i.e., $\rho_{ye} = \pm 1$), the market is complete and the consumer can fully hedge his or her labor income risk; consequently, his or her demand for precautionary saving is 0.

[13] Note that hedging with the risky asset ($\rho_{ye} \neq 0$) reduces the consumer's precautionary saving demand.

3.2 Rational Inattention due to Information-Processing Constraint

In the above FI-RE case, we considered the optimal behavior of investors who have unlimited information-processing capacity and thus can observe the state perfectly. However, there is plenty of evidence that ordinary people only have limited information-processing capacity and face many competing demands for their attention. For example, Hong, Torous, and Valkanov (2007) provided strong evidence to show that investors in the stock market react gradually to information contained in industry returns about their fundamentals and that information diffuses only gradually across markets. Coibion and Gorodnichenko (2015) found pervasive evidence consistent with Sims' rational inattention model using the U.S. surveys of professional forecasters, ordinary consumers and investors.

In this section, following Sims (2003), I consider a situation in which the typical investor cannot observe the state (s) perfectly due to finite information-processing capacity (rational inattention, or RI). The main idea of Sims' RI theory is that agents with finite capacity react to the innovations to the state gradually and incompletely because the channel along which information flows cannot carry an infinite amount of information. In other words, the typical investor can neither observe s_t nor can he or she observe the source of innovation dB_t, included in the state transition equation, (4):

$$ds_t = (rs_t - c_t + \pi\alpha_t)\, dt + \sigma dB_t. \tag{10}$$

Following Peng (2004), Kasa (2006), and Reis (2011), we adopt the noisy-information specification and assume that the investor observes only a noisy signal containing imperfect information about s_t:

$$ds_t^* = s_t dt + d\xi_t, \tag{11}$$

where ξ_t is the noise shock, and is a Brownian motion with mean 0 and variance Λ (in the RI setting, the variance, Λ, is a choice variable for the agent). Note that here we assume that the investor receives signals on $s_t dt$ rather than on ds_t. As emphasized in Sims (1998) and discussed in Kasa (2006) and Reis (2011), the latter specification is not suitable to model state uncertainty due to finite capacity because this specification means that in any finite interval, arbitrarily large amounts of information can be passed through the investor's channel. In addition, following the RI literature, we assume that ξ_t is independent of the Brownian motion governing the fundamental shock, B_t. It is worth noting that there exists another information rigidity specification, the sticky-information specification, in the literature. The key assumption of this specification is that agents adjust optimal plans infrequently because they face cost functions.[14] In this paper I adopt the noisy-information specification to better capture Sims' original RI idea that some freely available information is not used, or imperfectly used due to limited information-processing capacity. (Note that both w_t and y_t are freely available information for ordinary investors.)

To model RI due to finite capacity, we follow Sims (2003) and impose the following constraint on the investor's information-processing ability:

$$\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_t\right)-\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_{t+\Delta t}\right)\leq\kappa\Delta t, \tag{12}$$

where κ is the investor's information channel capacity; $\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_t\right)$ denotes the entropy of the state prior to observing the new signal at $t+\Delta t$; and $\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_{t+\Delta t}\right)$ is the entropy after observing the new signal. κ imposes an upper bound on the amount of information – that is, the change in the entropy – that can be transmitted in any given period. Formally, entropy is defined as the expectation of the negative of the (natural) log of the density function of a random variable s, $-E\left[\ln\left(f\left(s\right)\right)\right]$. For example, the entropy of a discrete distribution with equal weight on two points is simply $E\left[\ln\left(f\left(X\right)\right)\right]=-0.5\ln\left(0.5\right)-0.5\ln\left(0.5\right)=0.69$, and the unit of information contained in this distribution is 0.69 "nats" or 1 bit.[15] In this case, an agent can remove all uncertainty about s if the capacity devoted to monitoring s is $\kappa=1$ bit. Since imperfect observations on the state lead to welfare losses, the information-processing constraint must be binding. In other words, rational investors use all of their channel capacity, κ, to reduce the uncertainty upon new observations. To apply this information constraint to the state transition equation, we first rewrite (10) in the time interval of $[t,t+\Delta t]$:[16]

$$s_{t+\Delta t}=\rho_{0,t}+\rho_1 s_t+\rho_2\sqrt{\Delta t}\varepsilon_{t+\Delta t}, \tag{13}$$

where $\rho_{0,t}=\left(-c_t+\pi\alpha_t\right)\left(1-\exp\left(r\Delta t\right)\right)/\left(-r\Delta t\right)$, $\rho_1=\exp\left(r\Delta t\right)$, $\rho_2=\sigma\sqrt{\left(1-\exp\left(2r\Delta t\right)\right)/\left(-2r\Delta t\right)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to permanent income. Taking conditional variances on both sides of (13) and substituting it into (12), we have

$$\ln\left(\rho_1^2\Sigma_t+\rho_2^2\right)-\ln\left(\Sigma_{t+\Delta t}\right)=2\kappa\Delta t,$$

[14] Coibion and Gorodnichenko (2015) found pervasive evidence consistant with both Sims' rational inattention model and the infrequent adjustment model (i.e., the sticky-information model) using the U.S. surveys of professional forecasters, consumers and investors.

[15] For alternative bases for the logarithm, the unit of information differs; with log base 2 the unit of information is the "bit" and with base 10 it is a "dit" or a "hartley." In the subsequent analysis, for simplicity I set the unit of κ to be "nat".

[16] Note that here we use the fact that $\Delta B_t=\varepsilon_t\sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

which reduces to

$$\dot{\Sigma}_t = 2(r-\kappa)\Sigma_t + \sigma^2,$$

as $\Delta t \to 0$, where $\Sigma_t = E_t\left[(s_t - \widehat{s}_t)^2\right]$ the conditional variance at t (see Appendix 7.1 for a proof). In the steady state in which $\dot{\Sigma}_t = 0$, the steady state conditional variance can be written as:[17]

$$\Sigma = \frac{\sigma^2}{2(\kappa - r)}. \tag{14}$$

To make optimal decisions, the investor is required to filter in the optimal way the value of s_t using the observed s_t^*. Although the setting of our CARA-Gaussian model is not a typical tracking problem, the filtering problem in this model could be similar to the tracking problem proposed in Sims (2003). Specifically, we may think that the model with imperfect state observations can be decomposed into a two-stage optimization procedure:[18]

1. The optimal filtering problem determines the optimal evolution of the perceived (estimated) state;

2. The optimal control problem in which the decision makers treat the perceived state as the underlying state when making optimal decisions.

Here we assume ex post Gaussian distributions and Gaussian noise but adopt exponential or CARA preferences. See Peng (2004), Van Nieuwerburgh and Veldkamp (2009), and Mondria (2010) for applications of this specification in RI models. Because both the optimality of ex post Gaussianity and the standard Kalman filter are based on the LQG specification, the applications of these results in the RI models with CARA preferences are only approximately valid. Under the LQG setting, as has been shown in Sims (2003), ex post Gaussian distribution, $s_t|\mathcal{I}_t \sim N(E[s_t|\mathcal{I}_t], \Sigma_t)$, where $\Sigma_t = E_t\left[(s_t - \widehat{s}_t)^2\right]$, is optimal. In the next section, after taking RB into account, we will show that the ex post Gaussianity is still optimal under RI-RB. As will been shown in the next section, although introducing RB can significantly change the RI model's consumption-portfolio rules, it does not change the key properties of the ex post distribution of the state and the RI-induced noise. The logic for modeling RI and RB jointly this way is that we first conjecture that RB does not change the optimality of ex post Gaussianity, and then verify that the conjecture is correct after incorporating RB into the RI model.

[17] Note that here we need to impose the restriction $\kappa - r > 0$. If this condition fails, the state is not stabilizable and the conditional variance diverges.

[18] See Liptser and Shiryayev (2001) for a textbook treatment on this topic and an application in a precautionary saving model in Wang (2004).

In stage 1, consumers need to estimate the unobserved state (s_t) using its prior distribution and all processed and available information (i.e., their noisy observations, $\mathcal{F}_t = \left\{ s_j^* \right\}_{j=0}^{t}$). Specifically, consumers rationally compute the conditional distribution of the unobserved state and represent the original optimization problem as a Markovian one. Given the Gaussian prior $s_0 \sim N(\widehat{s}_0, \Sigma_0)$, finding the posterior distribution of s_t becomes a standard filtering problem that can be solved using the Kalman-Bucy filtering method. Specifically, the optimal estimate for s_t given $\mathcal{F}_t = \left\{ s_j^* \right\}_{j=0}^{t}$ in the mean square sense coincides with the conditional expectation: $\widehat{s}_t = E_t[s_t]$, where $E_t[\cdot]$ is based on $\mathcal{F}_t$. Applying Theorem 12.1 in Liptser and Shiryaev (2001), we can obtain the filtering differential equations for $\widehat{s}_t$ and Σ_t as follows:

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t)\, dt + K_t d\eta_t, \tag{15}$$

$$\dot{\Sigma}_t = -\Lambda K_t^2 + 2r\Sigma_t + \sigma^2, \tag{16}$$

given $s_0 \sim N(\widehat{s}_0, \Sigma_0)$, where

$$K_t = \frac{\Sigma_t}{\Lambda} \tag{17}$$

is the Kalman gain and

$$d\eta_t = \sqrt{\Lambda} dB_t^*, \tag{18}$$

with mean $E[d\eta_t] = 0$ and $\mathrm{var}(d\eta_t) = \Lambda dt$, where B_t^* is a standard Brownian motion and Λ is to be determined. Note that η_t is a Brownian motion with mean 0. Although the Brownian variable, ξ_t, is not observable, the innovation process, η_t, is observable because it is derived from observable processes (i.e., ds_t^* and $(r\widehat{s}_t - c_t + \pi\alpha_t)\, dt$). In this case, the path of the conditional expectation, $\widehat{s}_t$, is generated by the path of the innovation process, η_t. In the steady state, we have the following proposition:

Proposition 2 *Given finite capacity κ, in the steady state, the evolution of the perceived state can be written as:*

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t)\, dt + \widehat{\sigma} dB_t^*, \tag{19}$$

where

$$\widehat{\sigma} \equiv \Sigma/\sqrt{\Lambda} = f(\kappa)\,\sigma, \tag{20}$$

$f(\kappa) = \sqrt{\kappa/(\kappa - r)} > 1$ *(i.e., the standard deviation of the estimated state is greater than that of the true state),*

$$\Lambda = \frac{\sigma^2}{4\kappa(\kappa - r)} \tag{21}$$

is the steady state conditional variance, and

$$K = 2\kappa \tag{22}$$

is the corresponding Kalman gain.

Proof. In the steady state in which $\dot{\Sigma}_t = 0$, substituting the definition of the Kalman gain, (17), into $-\Lambda K_t^2 + 2r\Sigma_t + \sigma^2 = 0$ and using $\Sigma = \frac{\sigma^2}{2(\kappa - r)}$, we can easily obtain that:

$$\Lambda = \frac{\sigma^2}{4\kappa(\kappa - r)} \text{ and } K = 2\kappa.$$

■

It is worth noting that the above RI case can be observationally equivalent to the traditional signal extraction (SE) model with exogenously specified noises (i.e., the steady state variance of the noise Λ or the signal-to-noise ratio (SNR) σ^2/Λ are specified exogenously) in the sense that they lead to the same model dynamics when the signal-to-noise ratio and finite capacity satisfy some restriction. (See Online Appendix B for the detailed proof.) In other words, RI can provide a microfoundation for the exogenously specified SNR in the traditional SE models.

There is no direct survey evidence on the value of channel capacity of ordinary households in the economics literature. However, there exist some quantitative studies inferring and calibrating the value of capacity. For example, in the RI literature, to explain the observed aggregate fluctuations and the effects of monetary policy on the macroeconomy, the calibrated values of K are well below 1 (the FI-RE case). For example, Luo (2008) found that if $\kappa = 0.5$ bits, the otherwise standard permanent income model generates realistic relative volatility of consumption to income. Luo (2010) showed that to generate the observed share invested in the risky asset and realistic joint dynamics of aggregate consumption and asset returns, the degree of attention must be as low as 10% (i.e., the corresponding channel capacity is 0.08 bits). Coibion and Gorodnichenko (2015) proposed an empirical model to test information rigidities including the RI specification for a variety of economic agents such as consumers, firms, and financial market participants for whom forecast data are available. They found that the RI specification is likely to play a pervasive role in determining macroeconomic dynamics, and the model can fit the data quite well when $\kappa = 0.5$ bits in their discrete-time forecasting model.

In the next section, we will discuss how the interaction between RI and RB affects optimal consumption-portfolio rules and precautionary savings by robustifying this RI model. In addition, we will also show that the RB-RI model can better explain the empirical evidence than the RI model presented in this section.

4 Robust Consumption-Portfolio Rules under Rational Inattention

4.1 Incorporating Model Uncertainty due to Robustness

In this section, we assume that the typical inattentive investor has concerns about the innovation to perceived total wealth. As argued in HST (1999) and Hansen, Sargent, and Wang (henceforth, HSW, 2002), the simplest version of robustness considers the question of how to make optimal decisions when the decision-maker does not know the true probability model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the approximating model governing the evolution of the state variables is the true model, but also perform reasonably well when there is some type of model misspecification. To introduce robustness into our model proposed above, we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004) to assume that investors are concerned about the model misspecifications and take Equation (19) as the approximating model. The corresponding distorting model can thus be obtained by adding an endogenous distortion $\upsilon(\widehat{s}_t)$ to (19):

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha)\,dt + \widehat{\sigma}\left(\widehat{\sigma}\upsilon(\widehat{s}_t)\,dt + d\widetilde{B}_t\right), \tag{23}$$

It is worth noting that here we assume that the typical investor understands that he only has limited capacity when processing relevant information. As a result, he accepts the approximating model, (19), as the best approximating model, but is still concerned that it is misspecified. He therefore wants to consider a range of models surrounding the approximating model when computing the continuation payoff. A preference for robustness is then achieved by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $\upsilon(\widehat{s}_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (23) and (ii) an entropy penalty:

$$\inf_{\upsilon_t}\left[\mathcal{D}J(\widehat{s}_t) + \upsilon(\widehat{s}_t)\widehat{\sigma}^2 J_s + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{24}$$

where

$$\mathcal{D}J(\widehat{s}_t) = J_{\widehat{s}}(r\widehat{s}_t - c_t + \pi\alpha_t) + \frac{1}{2}J_{\widehat{s}\widehat{s}}\widehat{\sigma}^2. \tag{25}$$

the first two terms are the expected continuation payoff when the state variable follows (23), $\mathcal{H} = (\upsilon(\widehat{s}_t)\widehat{\sigma})^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term.[19] As shown in AHS (2003), $\mathcal{D}J$ can be thought of as $E\left[dJ\right]/dt$ and is easily obtained using Itô's lemma. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable. ϑ_t is fixed and state-independent in AHS (2003), whereas it is state-dependent in Maenhout (2004). In this paper, we also specify that ϑ_t is state-dependent ($\vartheta\left(\widehat{s}_t\right)$) in the CARA-Gaussian setting. Note that the evil agent's minimization problem, (24), becomes invariant to the scale of total resource, $\widehat{s}_t$, when using the state-dependent specification of ϑ_t. The main reason for this specification is to guarantee homotheticity and scale invariance of the decision problem, which makes robustness not diminish as the value of the total wealth increases.[20] Another reason for adopting this state-dependent specification of ϑ is that the multiplier specification of robustness is closely related to (i) the constraint specification of robustness proposed in HSTW (2006) and (ii) the ambiguity aversion theory proposed in Chen and Epstein (2002), and these three specifications all lead to constant portfolio rules and capture the same idea of the multiple priors model that the decision maker is concerned about their model is misspecified. (See Online Appendices C and D for the solutions to these two problems.)

It is worth noting that in the RB-RI model, the prior variance of s, $\widehat{\sigma}^2 = f\left(\kappa\right)\sigma$, is affected by the optimal portfolio choice, α^*, which is to be determined after solving the whole model with both RB (ϑ) and RI (κ). Given the value of κ, the value of the variance of the noise (Λ) should also be endogenously determined by α^*. The following is the two-stage procedure to solve the robust optimization problem of the inattentive investor:

1. First, guess that the ex post Gaussian distribution of the true state is still optimal when the agent has a preference for RB. Given the optimality of ex post Gaussianity and Gaussian noise, we can apply the standard Kalman filter and use the robust dynamic programming

[19] The last term in (24) is due to the investor's preference for robustness. Note that the $\vartheta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem. See AHS (2003) and Hansen, Sargent, Turmuhambetova, and Williams (2006) (henceforth, HSTW) for detailed discussions on the differences between this specification and another closely related robustness specification, the *constraint* robust control specification. See Online Appendix C for detailed discussions on the relatioship between these two specifications in this model. In this paper, to keep the model tractable, we focus on the multiplier formulation of RB in the subsequent analysis.

[20] See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness". Note that the impact of robustness wears off if we assume that ϑ_t is constant. It is clear from the procedure of solving the robust HJB proposed in Appendix 7.2.

to solve the RI-RB model explicitly. In addition, given finite SNR, we guess that the optimal portfolio choice under RB and RI is time-invariant, i.e., $\alpha_t = \alpha$. Consequently, $\sigma = \sqrt{\sigma_e^2 \alpha^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha}$ is also time-invariant. The investor with imperfect information about the state (SNR > 0) understands that he cannot observe s_t perfectly and needs to use the Kalman filter, (19), to update the perceived state when making decisions. In other words, (19) is regarded as the approximating model in this RB-RI model. The consumer solves the following HJB:

$$0 = \sup_{c_t, \alpha} \inf_{\upsilon_t} \left[-\frac{1}{\gamma} \exp\left(-\gamma c_t\right) - \delta J\left(\widehat{s}_t\right) + \mathcal{D} J\left(\widehat{s}_t\right) + \upsilon\left(\widehat{s}_t\right) \widehat{\sigma}^2 J_s + \frac{1}{2\vartheta\left(\widehat{s}_t\right)} \upsilon\left(\widehat{s}_t\right)^2 \widehat{\sigma}^2 \right], \quad (26)$$

subject to the distorted model: (23), where $\widehat{\sigma} \equiv f(\kappa)\,\sigma$ and $f(\kappa) = \sqrt{\frac{\kappa}{\kappa - r}} > 1$, and the transversality condition (TVC), $\lim_{t\to\infty} E\left|\exp\left(-\delta t\right) J\left(\widehat{s}_t\right)\right| = 0$, holds at optimum. (See Online Appendix E for the proof.)

2. Second, after solving for optimal consumption-portfolio rules and the value function under RB and RI, we can verify that the loss function due to RI is approximately quadratic and consequently the optimality of the ex post Gaussianity of the state could still hold in the RI-RB model. (See Online Appendix F for the proof.) Furthermore, we can also verify that whether or not the resulting portfolio rule is time-variant. If yes, our guess in the first step is correct and can thus rationalize the above procedure that we used to derive the stochastic property of the endogenous noise, Λ. The key reason is that time-invariant α yields time-invariant variance of the fundamental shock (σ).

Solving first for the infimization part of (26) yields: $\upsilon^*\left(\widehat{s}_t\right) = -\vartheta\left(\widehat{s}_t\right) J_{\widehat{s}}$, where $\vartheta\left(\widehat{s}_t\right) = -\vartheta / J\left(\widehat{s}_t\right) > 0$ and ϑ is a constant (see Appendix 7.2 for the derivation). Following Maenhout (2004) and Liu, Pan, and Wang (2005), here we can also define "$1/J\left(s_t\right)$" in the $\vartheta\left(s_t\right)$ specification as a *normalization* factor that is introduced to convert the relative entropy (i.e., the distance between the approximating model and the distorted model) to units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model.

4.2 Theoretical Implications

The following proposition summarizes the solution to the optimizing problem, (26):

Proposition 3 *Given ϑ and K, the robust consumption and portfolio rules for a typical inattentive investor are*

$$c_t^* = r\widehat{s}_t - \frac{\pi \rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} - \Gamma \tag{27}$$

and

$$\alpha^* = \frac{\pi}{r\widetilde{\gamma} f(\kappa)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{28}$$

respectively, where $\widehat{s}_t$ is governed by (23) and $f(\kappa) = \sqrt{\kappa/(\kappa - r)}$, we use the fact that $\widehat{\sigma}^2 = f(\kappa)^2\sigma^2$, and $\widetilde{\gamma} \equiv (1+\vartheta)\gamma$, and the precautionary savings premium, Γ, is

$$\Gamma = \frac{1}{2} r\widetilde{\gamma} f(\kappa)^2 \left(1 - \rho_{ye}^2\right)\sigma_s^2. \tag{29}$$

Finally, the worst possible distortion can be written as $v^* = -r\gamma\vartheta$ and the value function is

$$J(\widehat{s}_t) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 \widehat{s}_t\right),$$

where $\alpha_1 = r\gamma$ and $\alpha_0 = \frac{\delta - r}{r} + \frac{\pi[\pi/f(\kappa) - \rho_{ye}\sigma_s\sigma_e r\gamma(1+\vartheta)]}{(1+\vartheta)r\sigma_e^2} - \frac{1}{2}rf(\kappa)^2(1+\vartheta)\gamma^2\left(\sigma_e^2\alpha^* + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha^*\right)$.

Proof. See Appendix 7.2. ■

From (27), it is clear that robustness does not change the marginal propensity to consume (MPC) out of perceived permanent income ($\widehat{s}_t$), but affects the amount of precautionary savings. In other words, in the continuous-time setting, consumption is not sensitive to unanticipated income shocks. This conclusion is different from that obtained in the discrete-time robust-LQG permanent income model in which the MPC is increased via the interaction between RB and income uncertainty, and consumption is more sensitive to unanticipated shocks.[21]

Expression (28) shows that finite capacity (κ) affects the speculation demand invested in the risky asset (i.e., the first term in (28)) by a factor, $f(\kappa)$. Since $f(\kappa) = \sqrt{\kappa/(\kappa - r)} > 1$, we can see that RI reduces the share invested in the risky asset. The intuition behind this result is that consumption reacts to the income and asset return shocks gradually and with delay due to extracting useful information about the true state from noisy observations. Expression (28) also shows that RB reduces the optimal speculation demand by a factor, $1 + \vartheta$, but does not affect the hedging demand of the risky asset. In other words, RB increases the relative importance of the income hedging demand to the speculation demand by increasing the effective coefficient of absolute risk aversion ($\widetilde{\gamma}$). Furthermore, we can see from (28) that RI and RB affect strategic portfolio choice in the same direction.

[21] See HST (1999) and Luo and Young (2010) for detailed discussions on how RB affects consumption and precautionary savings in the discrete-time robust-LQG models.

Expression (29) shows that the precautionary savings demand increases with the degree of RI governed by $f(\kappa)$. It clearly shows that Γ decreases with the value of κ for different values of γ and ϑ. Expression (29) also shows that the precautionary savings premium increases with the degree of robustness (ϑ) by increasing the value of $\widetilde{\gamma}$ and interacting with two types of fundamental uncertainty: labor income uncertainty (σ_s^2) and the correlation between labor income and the equity return (ρ_{ye}).

It is straightforward to show that for some plausible values of κ, RI only has minor impact on both optimal portfolio rule and precautionary savings. For example, using the estimated value of κ, 0.5, from Coibion and Gorodnichenko (2015), we can calculate that $f(\kappa) = 1.02$. In other words, the impact of RI on the consumption-saving-portfolio rules could be marginal when the investor has moderate information-processing capacity. Furthermore, when κ is reduced from 0.5 to 0.1, $f(\kappa)$ increases to 1.12. It is worth noting that although $\kappa = 0.1$ is a very low number and is well below the total information-processing ability of human beings, it is not unreasonable in practice for ordinary individuals because they also face many other competing demands on capacity. For an extreme case, a young worker who accumulates balances in his retirement savings account (e.g., 401(k)) might pay no attention to the behavior of the stock market until he retires. In addition, here for simplicity we only consider one shock to expected total wealth s, while in reality ordinary investors face substantial idiosyncratic shocks that we do not explicitly model in this paper. Therefore, the exogenous capacity assumed in this paper can be regarded as a shortcut to small fractions of investors' total capacity used to monitor the evolution of the state variable. We will further explore the quantitative and empirical implications of RB and RI for robust consumption-saving-portfolio rules after calibrating the RB parameter in the next section.

An interesting question here is the relative sensitivity of the precautionary savings premium to CARA (γ), RB (ϑ) and RI (κ), holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their relative sensitivity. Specifically, using (29), we have the following proposition:

Proposition 4 *The relative sensitivity of precautionary saving to CARA (γ), robustness (RB, ϑ) and rational attention (RI, κ) can be measured by:*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{1+\vartheta}{\vartheta} > 1, \tag{30}$$

$$\mu_{\vartheta\kappa} \equiv -\frac{e_\vartheta}{e_\kappa} = \frac{\vartheta}{1+\vartheta}\frac{\kappa - r}{r}, \tag{31}$$

where $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$, $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$, *and* $e_\kappa \equiv \frac{\partial\Gamma/\Gamma}{\partial\kappa/\kappa}$ *are the elasticities of the precautionary saving* *demand to CARA, RB, and RI, respectively. (30) means that the precautionary savings demand is more sensitive to absolute risk aversion measured by γ than RB measured by ϑ. Furthermore, (31) means that when*

$$\kappa < (\geq)\, r\left(2 + \frac{1}{\vartheta}\right),$$

the precautionary saving demand is more sensitive to RI than RB. In other words, when finite capacity is sufficiently low, the precautionary saving demand becomes more sensitive to RI. Note that Expressions (30) and (31) can also measure the relative sensitivity of portfolio choice to CARA, RB, and RI.

Proof. The proof is straightforward using (30) and (31), and the facts that $\mu_{\vartheta\kappa} \equiv \left(\frac{\partial\Gamma/\Gamma}{\partial\widetilde{\vartheta}/\widetilde{\vartheta}} \frac{\partial\widetilde{\vartheta}/\widetilde{\vartheta}}{\partial\vartheta/\vartheta}\right) / \left(\frac{\partial\Gamma/\Gamma}{\partial f/f} \frac{\partial f/f}{\partial\kappa/\kappa}\right)$, where $\widetilde{\vartheta} = 1 + \vartheta$. ■

Using (30), it is simple to show that $\partial\mu_{\gamma\vartheta}/\partial\vartheta > 0$, which means that $\mu_{\gamma\vartheta}$ is increasing with the degree of RB, ϑ. Using (31), it is straightforward to show that:

$$\frac{\partial\mu_{\vartheta\kappa}}{\partial\vartheta} > 0 \text{ and } \frac{\partial\mu_{\vartheta\kappa}}{\partial\kappa} > 0,$$

which mean that $\mu_{\vartheta\kappa}$ is increasing with the degree of RB, ϑ, while is decreasing with the degree of inattention (i.e., less values of κ).

Proposition 5 *The observational equivalence between the discount factor and robustness established in the discrete-time Hansen-Sargent-Tallarini (1999) does not hold in our continuous-time CARA-Gaussian RB-RI model.*

Hansen, Sargent, and Tallarini (1999) (henceforth, HST) show that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG permanent income model. The reason is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or RB on consumption and investment.[22] In contrast, in the continuous-time CARA-Gaussian model with portfolio choice discussed in this section, the observational equivalence between the discount rate and the RB parameter no longer holds. This result can be readily obtained by inspecting the explicit expressions of consumption, precautionary savings, and

[22] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would change as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

portfolio choice, (27)-(29). The main reason for this result is that the preference for robustness governed by ϑ affects the portfolio rule via increasing $\widetilde{\gamma}$, while the discount rate (δ) has no impact on the portfolio rule in this Merton-type solution. It is well known that this type of intertemporal consumption-portfolio choice models leads to myopic portfolio rules, and the discount rate does not play a role in affecting asset allocation. It is straightforward to show that once we rule out the risky asset from our model, we can re-establish the observational equivalence between robustness and patience. Specifically, for given r and γ, when $\delta = r + 0.5\,(r\gamma)^2\,\vartheta\sigma_s$, this RB-RI model is observationally equivalent to the RI model with a lower discount rate ($\delta = r$) in the sense that they lead to the same consumption-saving decisions. In summary, although the observational equivalence between the discount rate (δ) and RB (ϑ) can be established in the sense that they lead to the same consumption function, they imply different portfolio choices and thus break down the observational equivalence between δ and ϑ in the consumption-portfolio choice model.

5 Quantitative, Empirical, and Policy Implications

5.1 Quantitative Implications

To fully explore how RB affects the joint behavior of portfolio choice, consumption, and labor income for given levels of inattention, we adopt the calibration procedure outlined in HSW (2002) and AHS (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. The model detection error probability (DEP) denoted by p is a measure of how far the distorted model, (23), can deviate from the approximating model, (19), without being discarded. Low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the agent to distinguish the two models. The general idea of the calibration procedure is thus to find a value of ϑ such that the DEP, $p\,(\vartheta; N)$, equals a given value (e.g., 20%) after simulating the approximating model and the distorted model.[23] (See Online Appendix G for the detailed calibration procedure.)

Following the literature, we set the parameter values for the processes of returns, volatility, and consumption as follows: $\mu = 0.08$, $r = 0.02$, $\delta = 0.02$, and $\sigma_e = 0.156$. For the labor income process, we follow Wang (2009) and set that $\sigma_y = 0.1$ in our benchmark model. When $\lambda = 0$, i.e.,

[23] The number of periods used in the calculation, N, is set to be the actual length of the data we study. For example, when we consider the post-war U.S. annual time series data provided by Robert Shiller from $1946 - 2010$, N is set to be 65.

when labor income follows a Brownian motion, we can compute that $\sigma_s = 5$. Figure 1 illustrates how p varies with the value of ϑ for different values of κ when $\gamma = 2$. The figure shows that for given values of κ, the stronger the preference for robustness (higher ϑ), the less the p is. For example, when $\gamma = 2$ and $\kappa = 0.1$, $p = 11\%$ when $\vartheta = 1.4$, while $p = 17\%$ when $\vartheta = 1$. Both values of p are reasonable as argued in AHS (2002), HSW (2002), and Maenhout (2004). Furthermore, for given values of ϑ, the lower the degree of attention (i.e., the less the value of κ), the larger the value of p. That is, RI by itself creates an additional demand for robustness that leads to greater induced uncertainty facing investors. It is worth noting that in the RB model p governs the amount of model uncertainty facing agents, and ϑ reflects the effect of RB on the model's dynamics.

Figure 2 illustrate how p varies with the value of ϑ for different values of γ, σ_s and ρ_{ye} when $\kappa = \infty$ (i.e., when the investor has infinite information-processing capacity) for the same parameter values set above.[24] Note that reducing κ from ∞ to a finite value does not change the calibration results significantly. This figure also shows that the higher the value of ϑ, the less the p is. In addition, to calibrate the same value of p, less values of γ and σ_s (i.e., more volatile or higher persistent labor income processes) or higher values of ρ_{ye} lead to higher values of ϑ. The intuition behind this result is that γ, σ_s and ϑ have opposite effects on p. To keep the same value of p, if one parameter increases, the other one must reduce to offset its effect on p. Combining these facts with (28), we can see that an increase in ρ_{ye} not only reduces the hedging demand directly, but also reduces the standard speculation demand of the risky asset by affecting the calibrated values of ϑ using the same values of p. In contrast, an increase in σ_s reduces the hedging demand, but increases the speculation demand. It is worth noting that in the RB model p can be used to measure the amount of model uncertainty, whereas ϑ is used to measure the degree of the agent's preference for RB. If we keep p constant when recalibrating ϑ for different values of other parameters, the amount of model uncertainty is held constant, i.e., the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly when the values of other parameters change. In other words, the amount of model uncertainty is "elastic" and will change accordingly when the fundamental factors change.

From the expression for robust portfolio rule, (28), we can see that plausible values of RB can significantly affect the share invested in the risky asset. The upper panel of Figure 3 shows how

[24] Note that since $\sigma_s = \sigma_y / (r + \lambda)$, the value of σ_s can measure the persistence (λ) and volatility (σ_y) of the labor income process.

the robust portfolio rule varies with the degree of RI (κ) for different values of RB (ϑ). It clearly shows that α^* increases with the value of κ for different values of ϑ. In addition, it is also clear from the same figure that α^* decreases with ϑ for a given value of κ. The lower panel of Figure 3 illustrates how the precautionary saving demand (Γ) varies with the degree of RI for different values of RB. It shows that Γ decreases with the value of κ for different values of ϑ. From this figure, it is clear that RI does not have significant impacts on portfolio choice and precautionary savings when κ is not very low. For example, when $\rho_{ye} = 0.18$, $\gamma = 1.5$, and $\vartheta = 1.5$, $\alpha^* = 26.44$ and $\Gamma = 0.94$ when $\kappa = 0.5$, while $\alpha^* = 25.42$ and $\Gamma = 1.00$ when $\kappa = 0.2$. In contrast, RB has more significant impacts on portfolio choice and precautionary saving. For example, when $\rho_{ye} = 0.18$, $\gamma = 1.5$, and $\kappa = 0.5$, $\alpha^* = 26.44$ and $\Gamma = 0.94$ when $\vartheta = 1.5$, while $\alpha^* = 14.36$ and $\Gamma = 1.51$ when $\vartheta = 3$.[25]

To further inspect the effects of the interaction of RI and RB, Table 3 reports how different values of κ affect calibrated values of ϑ, optimal allocation in the risky asset (α^*), the relative importance of the income hedging demand to the speculation demand ($|\alpha_h^*| / \alpha_s^*$), and precautionary saving demand (Γ) for different values of ρ_{ye}.[26] Specifically, for given values of σ_s and ρ_{ye}, when κ decreases (i.e., more information-constrained), the calibrated value of ϑ increases; consequently, the optimal share invested in the risky asset decreases and the relative importance of the income hedging demand to the speculation demand increases. In addition, the precautionary saving demand decreases with the value of κ. It is also clear from the tables that in the presence of RB, the correlation between the equity return and undiversified labor income not only affects the hedging demand for the risky asset, but also affects its standard speculation demand. The key reason is that given the same value of p, the correlation between labor income and the equity return increases the calibrated value of RB and thus reduces the optimal share invested in the risky asset. From Table 3, we can also see that for given values of κ, the precautionary saving demand decreases with the correlation between the equity return and labor income risk (ρ_{ye}), holding other factors constant. The intuition is that the higher the correlation coefficient, the less demand for the risky asset and thus precautionary saving. In Table 4, we can see that as labor income becomes more volatile, the optimal allocation in the risky asset increases and

[25] Wang (2004) examined how incomplete information about individual income affects consumption and precautionary savings in a CARA-Gaussian model. See Online Appendix H for a detailed comparison between this model with RB and the benchmark model in this paper.

[26] Here the values of ρ_{ye} are set to be 0, 0.18, and 0.35 according to Campbell and Viceira (Chapter 6, 2002), and the values of σ_y are chosen to be the same as those used in Wang (2009).

the precautionary saving demand decreases. The reason is that the higher the value of σ_s, the less the calibrated value of ϑ, holding other factors fixed; consequently, the effective coefficient of absolute risk aversion decreases, and thus the optimal share increases and the precautionary saving demand decreases.

5.2 Empirical Implications

As discussed in Haliassos and Michaelides (2000) and Campbell (2006), the empirical evidence on the correlation between labor income and equity returns for different population groups is difficult to reconcile with the observed stockholding behavior. It is worth noting that estimating the correlation between individual labor income and the equity return is complicated by the lack of data on household portfolio choice that has both time-series and panel dimension. Nevertheless, several studies have tried to estimate these effects. For example, Davis and Willen (2000) estimated that the correlation is between 0.1 and 0.3 for college-educated males, is 0.25 or more for college-educated women, and is only about -0.25 for male high school dropouts. For both men and women, they found that the correlation between income shocks and equity returns tend to rise with education attainment. Heaton and Lucas (1999) found that the correlation between the entrepreneurial risk and the equity return was about 0.2.[27] Since negative correlation between earnings and equity returns implies increased willingness to invest in the risky asset, less educated investors should be more heavily invested in the stock market while college graduates and entrepreneurs should put less wealth in the stock market. In contrast, the empirical evidence on stock market participation shows a significant correlation between the education level and stockholding. Table 2 shows that the mean value of stockholding is substantially larger among those with at least a college degree compared to those with less than a high school education *at all income and net worth levels.* Furthermore, for any given education group, the mean value of financial wealth invested in the stock market is increasing with the income and net worth percentile. In other words, people with the same educational attainment level and higher income (or net worth) invest more in the stock market. Table 3 of Haliassos and Bertaut (1995) reported similar results on the share invested in the stock market. In addition, they also argued that more educated groups have higher information-processing capacities. This empirical evidence is consistent with our limited capacity theory. People with low income own less distraction-saving goods and ser-

[27] It is worth noting that estimating the correlation between individual labor income and the equity return is complicated by the lack of data on household portfolio choice that has both time-series and panel dimension, and by the difficulty of identifying households' unanticipated income shocks.

vices at home (e.g., they cannot afford a good baby sitter); consequently, they invested less in the stock market because they face greater state uncertainty.[28] In addition, people with higher education may have more efficient information-processing ability and thus face lower transition errors, which leads to higher effective channel capacity. Finally, people with higher education probably have more and better knowledge about the model economy, and are thus less concerned about the model specification.

The RB-RI model can have the potential to reconcile the model with the empirical evidence. Specifically, poorer and less well-educated investors probably face greater state uncertainty and model uncertainty, respectively; consequently, they rationally choose to invest less in the stock market even if the correlation between their labor income and equity returns is negative and they have stronger incentive to hedge against their earnings risk.[29] For example, the optimal amount invested in the risky asset (α^*) of a typical well-educated investor is 27.74 when $\vartheta = 1.5$, $\kappa = 1$, and $\rho_{ye} = 0.15$. (Here we also set $\mu = 0.08$, $r = 0.02$, $\sigma_s = 5$, $\sigma_e = 0.156$, and $\gamma = 1.5$.) In contrast, the optimal amount invested in the risky asset of a typical less well-educated investor is 26.39 when $\vartheta = 3$, $\kappa = 0.1$, and $\rho_{ye} = -0.25$.[30] In other words, although the negative correlation between earnings and equity returns increases the willingness of low-educated investors to invest in the risky asset, well-educated investors invest more wealth in the risky asset because they have more knowledge about the structure of the model and higher information-processing capacities, and face less induced uncertainty. It is worth noting that when we assume that inattentive consumers fully trust their model, we have $\alpha^* = 37.42$ for a typical less well-educated investor when $\vartheta = 1.5$, $\kappa = 0.1$, and $\rho_{ye} = -0.25$. It is clear that this conclusion still holds if the value of κ is higher than 0.1. That is, the model with only RI cannot explain the two seemingly contradictory observations even if the inattentive investor has very limited information-processing capacity. In summary, the introduction of induced uncertainty has the potential to explain the two seemingly contradictory observations: the correlation between labor income and equity returns and the stockholding behavior of less educated and well-educated investors.

[28] As argued in Banerjee and Mullainathan (2008), attention is a scarce resource that is important for labor productivity and income distribution.

[29] As documented in Campbell (2006), there is some evidence that households understand their own limitations and constraints, and avoid investment opportunities for which they feel unqualified.

[30] As we have discussed in Section 4.2, although $\kappa = 0.1$ is a very low number and is well below the total information-processing ability of human beings, it is not unreasonable in practice for ordinary investors because they also face many other competing demands on capacity.

5.3 Policy Implications

In this section, we discuss the effect of changes in the labor income tax rate on inattentive investors' precautionary saving and robustly strategic portfolio choice. Elmendorf and Kimball (2000) found that given decreasing absolute prudence (e.g., CRRA utility), even when labor income risk increases overall saving, it tends to lower investment in the risky asset. They also argued that realistic increases in the marginal tax rate on labor income can cause large enough reductions in the after-tax labor income risk, which leads to significant increases in investment in the risky asset. Using the same policy experiment conducted in their paper, we also consider the situation in which the marginal tax rate on labor income (τ) is increased from 0 to 10%. In this case, the labor income risk measured by $(1-\tau)\,\sigma_y$ is reduced from σ_y to $0.9\sigma_y$. Using the expressions for optimal portfolio choice and precautionary saving, (28) and (29), it is clear that the change in the tax rate leads to an increase in the risky investment and a reduction in precautionary savings, holding other parameter values fixed.

Specifically, when the labor income risk is reduced from σ_y to $0.9\sigma_y$, the value of human wealth is also reduced from σ_s to $0.9\sigma_s$. Consequently, Γ is reduced from $0.5r\widetilde{\gamma}f\left(\kappa\right)^2\left(1-\rho_{ye}^2\right)\sigma_s^2$ to about $0.4r\widetilde{\gamma}f\left(\kappa\right)^2\left(1-\rho_{ye}^2\right)\sigma_s^2$. The presence of induced uncertainty measured by $\left(1+\vartheta\right)f\left(\kappa\right)^2>1$ can amplify the impact of this taxation policy on the precautionary saving demand. For example, when $r=0.02$, $\vartheta=1.5$ and $\kappa=0.2$, $\left(1+\vartheta\right)f\left(\kappa\right)^2=2.78$. In other words, the policy impact on precautionary saving is almost tripled under RB and RI.

From (28), it is clear that the labor income risk does not directly interact with induced uncertainty due to RB and RI because σ_s does not enter the standard speculation demand function and the induced uncertainty term $\left(\left(1+\vartheta\right)f\left(\kappa\right)\right)$ does not enter the income-hedging demand function. However, the presence of induced uncertainty can offset the impact of the taxation policy on the optimal risky investment when ρ_{ye} is positive. The reason is that the taxation policy reduces the hedging demand while the presence of induced uncertainty reduces the standard speculation demand. Therefore, the policy impact on optimal portfolio choice can be mitigated under RB and RI.

6 Conclusion

This paper has developed a tractable continuous-time CARA-Gaussian framework to explore how induced uncertainty due to the interaction of RB and RI affects strategic consumption-portfolio rules and precautionary savings in the presence of uninsurable labor income. Specifically, I explored the relative sensitivity of strategic consumption-portfolio rules and precautionary savings with respect to the two types of induced uncertainty: (i) model uncertainty due to robustness and (ii) state uncertainty due to limited information-processing capacity, as well as risk aversion. In addition, I argued that both model uncertainty and state uncertainty are important for us to understand and design optimal household portfolios. In particular, I found that these two types of induced uncertainty reduce the optimal share invested in the risky asset, and thus can offer a potential explanation for two seemingly contradictory observations in the data: the negative correlation between labor income and equity returns and the low stock market participation rate of the less educated and lower income households.

7 Appendix

7.1 Deriving Continuous-time IPC

The IPC,

$$\ln\left(\rho_1^2\Sigma_t+\rho_2^2\right)-\ln\Sigma_{t+\Delta t}=2\kappa\Delta t,$$

can be rewritten as

$$\ln\left(\exp\left(2r\Delta t\right)\Sigma_t+\frac{1-\exp\left(2r\Delta t\right)}{-2r\Delta t}\Delta t\sigma^2\right)-\ln\Sigma_{t+\Delta t}=2\kappa\Delta t,$$

which can be reduced to

$$\Sigma_{t+\Delta t}-\Sigma_t=\left(\exp\left(2\left(r-\kappa\right)\Delta t\right)-1\right)\Sigma_t+\frac{\exp\left(2\left(r-\kappa\right)\Delta t\right)-\exp\left(-2\kappa\Delta t\right)}{2r}\sigma^2.$$

Dividing Δt on both sides of this equation and letting $\Delta t\to 0$, we have the following continuous-time updating equation for Σ_t:

$$\dot{\Sigma}_t=\lim_{\Delta t\to 0}\frac{\Sigma_{t+\Delta t}-\Sigma_t}{\Delta t}=2\left(r-\kappa\right)\Sigma_t+\sigma^2.$$

7.2 Solving the RB–RI Model

The Bellman equation associated with the optimization problem of the inattentive investor is

$$J\left(\widehat{s}_t\right)=\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)+\exp\left(-\delta dt\right)J\left(\widehat{s}_{t+dt}\right)\right],$$

subject to (23), where $J(\widehat{s}_t)$ is the value function. The HJB equation for this problem can thus be written as

$$0=\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(\widehat{s}_t\right)+\mathcal{D}J\left(s_t\right)\right],$$

where $\mathcal{D}J(\widehat{s}_t)$ is defined in (25). Under RB and RI, the HJB can be written as

$$0=\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(\widehat{s}_t\right)+\mathcal{D}J\left(\widehat{s}_t\right)+\upsilon\left(\widehat{s}_t\right)\widehat{\sigma}^2J_{\widehat{s}}+\frac{1}{2\vartheta\left(s_t\right)}\upsilon\left(\widehat{s}_t\right)^2\widehat{\sigma}^2\right]$$

subject to (23).

Solving first for the infimization part of the problem yields: $\upsilon^*(\widehat{s}_t)=-\vartheta(\widehat{s}_t)J_{\widehat{s}}$. Given that $\vartheta(\widehat{s}_t)>0$, the perturbation adds a negative drift term to the state transition equation because $J_{\widehat{s}}>0$. Substituting for υ^* in the robust HJB equation gives:

$$0=\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(\widehat{s}_t\right)+\left(r\widehat{s}_t-c_t+\pi\alpha_t\right)J_{\widehat{s}}+\frac{1}{2}\widehat{\sigma}^2J_{\widehat{s}\widehat{s}}-\frac{1}{2}\vartheta\left(\widehat{s}_t\right)\widehat{\sigma}^2J_{\widehat{s}}^2\right]. \tag{32}$$

Performing the indicated optimization yields the first-order conditions for c_t and α_t:

$$c_t=-\frac{1}{\gamma}\ln\left(J_{\widehat{s}}\right), \tag{33}$$

$$\alpha_t=\frac{\pi J_{\widehat{s}}/f\left(\kappa\right)+\rho_{ye}\sigma_s\sigma_e\left(J_{\widehat{s}\widehat{s}}-\vartheta J_{\widehat{s}}^2\right)}{\left(\vartheta J_{\widehat{s}}^2-J_{\widehat{s}\widehat{s}}\right)\sigma_e^2}. \tag{34}$$

Substitute (33) and (34) back into (32) to arrive at the following PDE:

$$0=-\frac{J_{\widehat{s}}}{\gamma}-\delta J+\left(r\widehat{s}_t+\frac{1}{\gamma}\ln\left(J_{\widehat{s}}\right)+\pi\alpha_t\right)J_{\widehat{s}}+\frac{1}{2}\left(J_{\widehat{s}\widehat{s}}-\vartheta_tJ_{\widehat{s}}^2\right)f\left(\kappa\right)^2\sigma^2. \tag{35}$$

Conjecture that the value function is of the form $J(\widehat{s}_t)=-\frac{1}{\alpha_1}\exp(-\alpha_0-\alpha_1\widehat{s}_t)$, where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain $J_{\widehat{s}}=\exp(-\alpha_0-\alpha_1\widehat{s}_t)>0$ and $J_{\widehat{s}\widehat{s}}=-\alpha_1\exp(-\alpha_0-\alpha_1\widehat{s}_t)<0$. Furthermore, we guess that $\vartheta(\widehat{s}_t)=-\frac{\vartheta}{J(\widehat{s}_t)}=\frac{\alpha_1\vartheta}{\exp(-\alpha_0-\alpha_1\widehat{s}_t)}>0$.

Substituting these expressions into (35) yields:

$$\begin{aligned}-\delta\frac{1}{\alpha_1}=&-\frac{1}{\gamma}+\left\{r\widehat{s}_t-\left(\frac{\alpha_0}{\gamma}+\frac{\alpha_1}{\gamma}\widehat{s}_t\right)+\frac{\pi\left[\pi/f\left(\kappa\right)-\rho_{ye}\sigma_s\sigma_e\alpha_1\left(1+\vartheta\right)\right]}{\left(1+\vartheta\right)\alpha_1\sigma_e^2}\right\}\\&-\frac{1}{2}f\left(\kappa\right)^2\alpha_1\left(1+\vartheta\right)\left(\sigma_e^2\alpha_t^2+\sigma_s^2+2\rho_{ye}\sigma_s\sigma_e\alpha_t\right)\end{aligned}$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1=r\gamma, \tag{36}$$

$$\alpha_0=\frac{\delta-r}{r}+\frac{\pi\left[\pi/f\left(\kappa\right)-\rho_{ye}\sigma_s\sigma_er\alpha\left(1+\vartheta\right)\right]}{\left(1+\vartheta\right)r\sigma_e^2}-\frac{1}{2}rf\left(\kappa\right)^2\left(1+\vartheta\right)\gamma^2\left(\sigma_e^2\alpha_t+\sigma_s^2+2\rho_{ye}\sigma_s\sigma_e\alpha_t\right). \tag{37}$$

Substituting (36) and (37) into (33) and (34) yields the optimal portfolio and consumption rule, (28) and (27), respectively, in the main text. Using (37) and that fact that $\sigma_e^2\alpha_t^2+\sigma_s^2+2\rho_{ye}\sigma_s\sigma_e\alpha_t = \left(1-\rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{r^2\widetilde{\gamma}^2 f(\kappa)^2\sigma_e^2}$, we can obtain Expression (29) in the main text. Finally, it is straightforward to check that the investor's transversality condition, $\lim_{t\to\infty} E\left[\exp\left(-\delta t\right)\left|J\left(\widehat{s}_t\right)\right|\right] = 0$, is satisfied at optimum. (See Online Appendix E for the detailed proof.)

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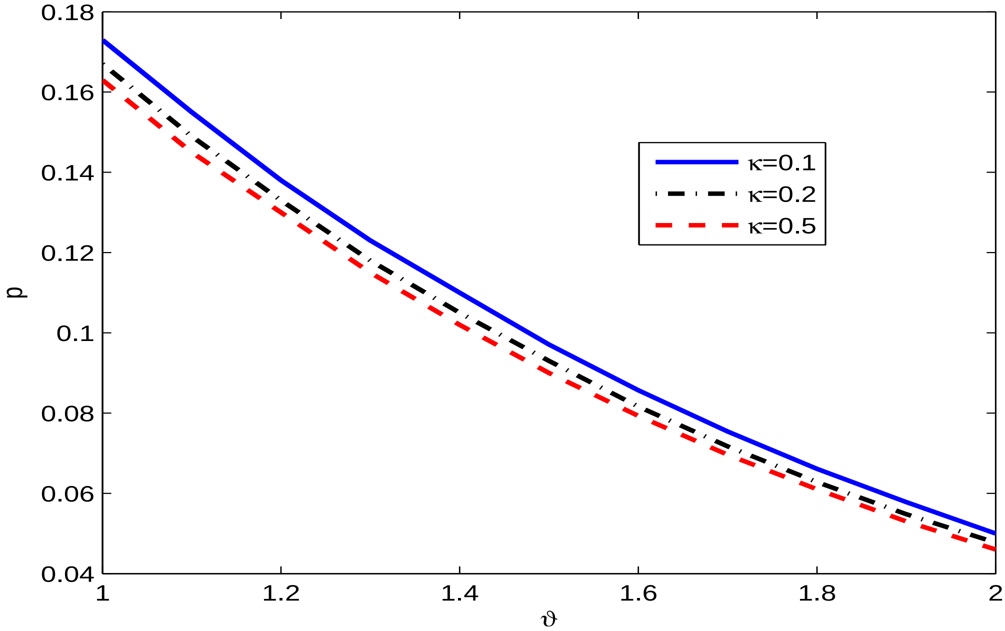

Figure 1: Relationship between ϑ and p under RI ($\gamma = 2$)

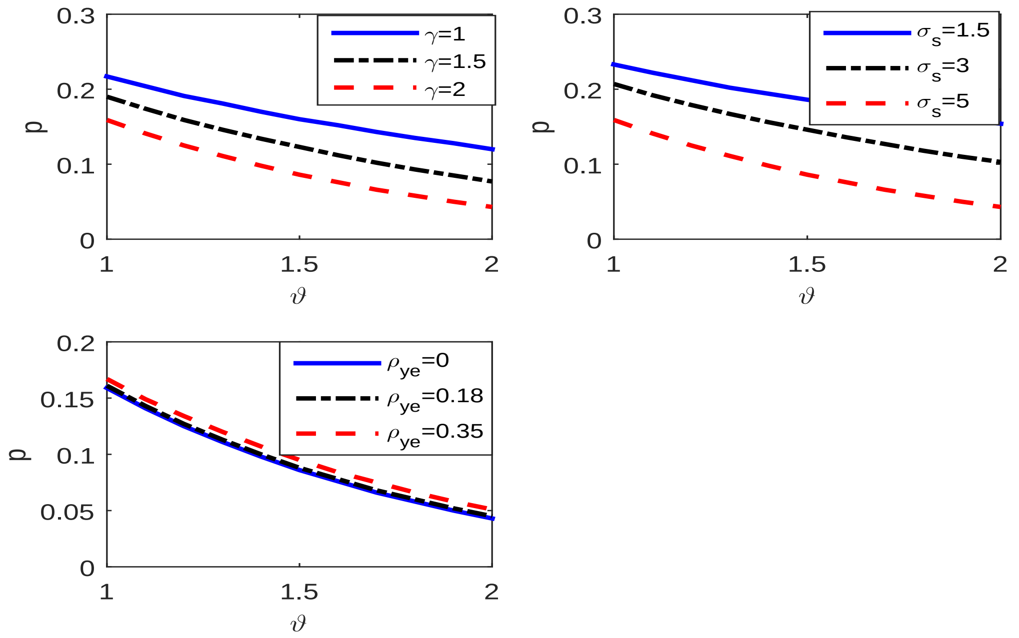

Figure 2: Relationship between ϑ and p $(\kappa = \infty)$

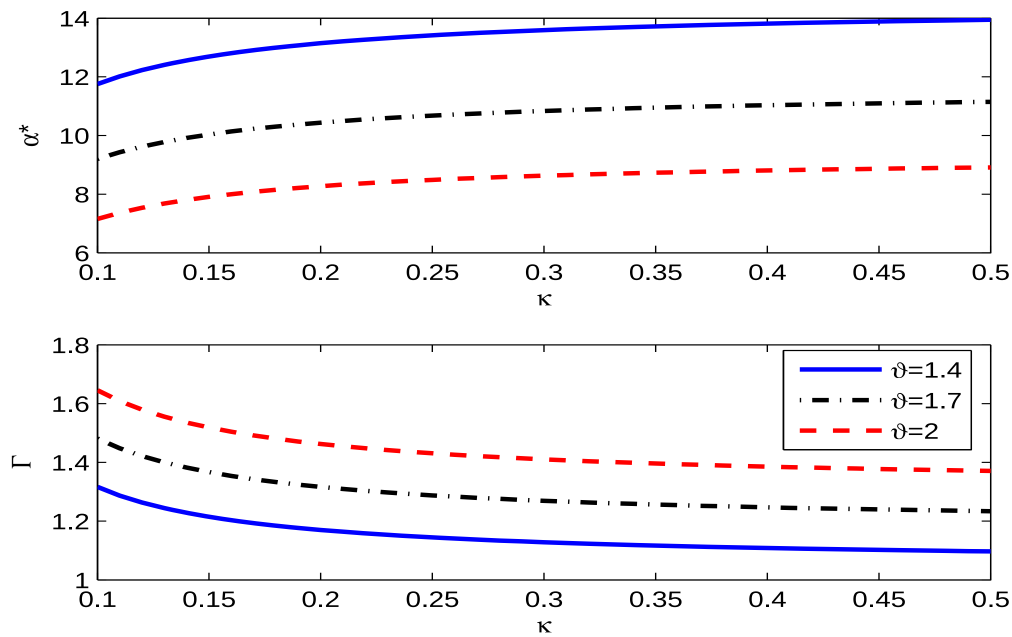

Figure 3: Robust Portfolio Rule and Precautionary Savings under RI

Table 1: Mean value of stock holdings for families with holdings in the 2013 SCF (thousands of 2013 dollars)

Year	(1) Less than high school	(2) High school degree	(3) Some college	(4) College degree or more
1992	37.1	51.5	71.5	147.2
1995	38.6	68.6	115.6	199.3
1998	45.9	102.7	141.0	326.8
2001	77.3	95.3	158.5	427.7
2004	59.0	82.7	135.0	360.9
2007	74.2	94.4	112.2	408.6
2010	42.4	71.3	84.9	360.0
2013	44.0	79.7	129.7	399.1

Note: Stock holdings here includes: (1) directly-held stock; (2) stock mutual funds: full value if described as stock mutual fund and 1/2 value of combination mutual funds; (3) IRAs/Keoghs invested in stock: full value if mostly invested in stock, 1/2 value if split between stocks/bonds or stocks/money market, and 1/3 value if split between stocks/bonds/money market; (4) other managed assets w/equity interest (annuities, trusts, MIAs): full value if mostly invested in stock, 1/2 value if split between stocks/MFs & bonds/CDs, or "mixed/diversified," 1/3 value if "other"; and (5) thrift-type retirement accounts invested in stock full value if mostly invested in stock, 1/2 value if split between stocks and interest earning assets.

Table 2: Mean value of stock holdings for families with holdings in the 2013 SCF (thousands of 2013 dollars)

Income percentile	(1) Less than high school	(2) High school degree	(3) Some college	(4) College degree or more
0 – 20%	0.6	3.9	6.1	33.4
20% – 39.9%	3.9	10.4	15.3	29.8
40% – 59.9%	12.9	21.7	27.2	45.3
60% – 79.9%	18.8	39.3	56.2	90.3
80% – 89.9%	38.7	65.3	86.3	155.6
90% – 100%	186.8	258.0	396.8	765.4

Networth percentile	(1) Less than high school	(2) High school degree	(3) Some college	(4) College degree or more
0 – 24.9%	0.1	0.5	0.8	2.2
25% – 49.9%	1.6	4.0	5.5	9.4
50% – 74.9%	6.4	17.7	23.3	35.4
75% – 89.9%	26.4	71.4	85.9	122.4
90% – 100%	187.3	341.5	474.5	966.3

Table 3: Implications of the correlation on α^* and Γ under MU and SU ($p = 10\%, \sigma_s = 5$)

		$\rho_{ye} = 0$	$\rho_{ye} = 0.18$	$\rho_{ye} = 0.35$
$\kappa = 0.1$	ϑ	1.455	1.475	1.536
	α^*	22.46	16.51	10.52
	α_s^*	22.46	22.28	21.74
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.259	0.516
	Γ	1.534	1.497	1.391
$\kappa = 0.2$	ϑ	1.418	1.437	1.495
	α^*	24.18	18.23	12.22
	α_s^*	24.18	23.99	23.44
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.240	0.479
	Γ	1.364	1. 31	1.216
$\kappa = 0.5$	ϑ	1.396	1.415	1.47
	α^*	25.21	19.24	13.23
	α_s^*	25.21	25.01	24.45
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.231	0.459
	Γ	1.279	1.217	1.129
$\kappa = 1$	ϑ	1.389	1.407	1.462
	α^*	25.54	19.58	13.57
	α_s^*	25.54	25.35	24.78
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.228	0.453
	Γ	1.253	1.188	1.102

Table 4: Implications of income uncertainty on α^* and Γ under MU and SU ($p = 10\%$, $\rho_{ye} = 0.18$)

		$\sigma_s = 4$	$\sigma_s = 5$	$\sigma_s = 6$
$\kappa = 0.1$	ϑ	1.770	1.475	1.265
	α^*	15.29	16.51	17.42
	α_s^*	19.90	22.28	24.34
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.232	0.259	0.284
	Γ	1.675	1.497	1.370
$\kappa = 0.2$	ϑ	1.715	1.437	1.238
	α^*	16.92	18.23	19.21
	α_s^*	21.54	23.99	26.13
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.214	0.240	0.265
	Γ	1.460	1.310	1.203
$\kappa = 0.5$	ϑ	1.682	1.415	1.222
	α^*	17.90	19.24	20.26
	α_s^*	22.52	25.01	27.18
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.205	0.231	0.255
	Γ	1.352	1.217	1.120
$\kappa = 1$	ϑ	1.671	1.407	1.216
	α^*	18.23	19.58	20.61
	α_s^*	22.84	25.35	27.54
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.202	0.228	0.251
	Γ	1.319	1.188	1.094