Reliable Location-Routing Design under Probabilistic Facility Disruptions

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This study considers an integer programming formulation of a reliable location routing problem in which built facilities are subject to the risk of probabilistic disruptions. The objective of the study is to determine facility locations, outbound delivery routing and backup plans during disruptions to minimize the facility setup, routing and customer penalty costs (if no delivery is possible). A variety of approaches (i.e., Lagrangian relaxation with embedded column generation and local search) to solving the problem are assessed. Numerical case studies are conducted to test the performance of the proposed model and solution algorithms. The findings offer managerial insights into how various system parameters affect the optimal solution.

Key words: Location routing; reliable facility location; set covering; Lagrangian relaxation; column generation.

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1. Introduction

Facility location and vehicle routing are two crucial issues associated with logistics systems design. These two problems are typically solved separately due to their high complexity and difficulty. However, efforts have been made to integrate them into so-called "location routing" problems (LRP) (Perl and Daskin 1985, Nagy and Salhi 2007), in which a logistics service provider chooses the vehicle depot locations from which to dispatch routing trucks to serve spatially scattered customers. This work was motivated by the railroad industry where resources and corresponding facilities are deployed to serve locomotive maintenance demand (customers) across the network. These facilities store necessary maintenance materials and heavy equipment, while the trucks pick up and carry them to demand points for on-site services. The railroad company typically contracts trucks from a third party vendor since locomotive maintenance needs are usually periodic (e.g., every 90 days). These service facilities are often disrupted by adverse weather (e.g., snow storms in northern U.S.) or disasters (e.g., flooding in the Midwest). Once the disruptions take place, trucks must be dispatched via other depots to complete the tasks.

Many similar problems can be found in different application contexts. For example, in concrete dispatching, one needs to strategically set up concretion production plants in the region to serve multiple construction sites that demand concrete for the construction work. The concrete mixer trucks need to collect the concrete mix from the plant, and then deliver to a group of construction sites. Reliability is an important concern, because any disruption of concrete delivery will significantly affect the construction work. If any of the production plants are disrupted due to machine breakdown, unacceptable material quality, power disruption, etc., the construction sites will have to be served through backup sites. In newspaper delivery systems, in dawn, each printing factory (i.e., depot) needs to print the newspapers in a timely manner. The delivery trucks will collect the newspapers and deliver them to the customers. Reliability is also an important concern, because failing to distribute the newspapers on time is unacceptable. If a printing facility is disrupted due to machine failure, Internet breakdown (failing to receive the soft copy for printing), power disruption, etc., the delivery trucks from other surviving depots will need to serve as backups for newspapers distribution. In these example applications, it is very common to out-source the delivery service to an outside trucking company which serves as a “third-party vendor”.

Earlier research focused on the deterministic version of the LRP. Laporte and Nobert (1981) formulated a mixed-integer program and developed a branch-and-bound algorithm to find the optimal location of a single depot in addition to the vehicle routing plan. Later, Laporte et al. (1988) used graph augmentation to transform LRP into a travelling salesman problem that could be addressed using a branch-and-bound algorithm. Belenguer et al. (2011) recently strengthened the LRP with valid inequalities and solved it to near optimality using a branch-and-cut procedure. Many studies have also been devoted to developing heuristic methods (Min et al. 1998). The LRP can usually be decomposed into two parts, i.e., location and routing decisions and solved sequentially (Wu et al. 2002). Albareda-Sambola et al. (2005) incorporated standard tabu-search heuristics and derived a lower bound to verify the quality of the solution. Prins et al. (2007) arrived at depot location decisions via a Lagrangian relaxation (LR) heuristic and vehicle routing decisions via a granular tabu-search method. Other classes of heuristic methods have included those based on customer clustering (Barreto et al. 2007).

Attention has also been given to the LRP under stochasticity. Laporte et al. (1989) proposed a class of stochastic LRP models with uncertain customer demand that could violate vehicle capacity limits and cause route failures. They derived a chance constrained mixed-integer program model to minimize the facility and routing costs and used a branch-and-bound approach to solve it. Simchi-Levi (1991) solved the LRP using a single location choice while generating demand from a given probabilistic distribution. Liu and Lee (2003) integrated inventory decisions into the LRP under stochastic customer demand and solved it using a two-phase heuristic method. Zarandi et al. (2011) solved the LRP with random travel times using a mixture of simulated annealing and simulation. Ahmadi-Javid and Seddighi (2013) recently considered random reductions in depot capacity (and the number of vehicle visits) and introduced several risk measurement functions in a two-stage heuristic solution method based on simulated annealing.

In practice, facilities/depots risk being completely disabled by probabilistic disruptions (e.g., due to natural or human-induced disasters such as hurricanes, earthquakes or labor action). Drezner (1987) first considered reliable p-median and p-center problems under probabilistic facility failure. Many researchers have recently recognized and incorporated such effects into their facility location models. Snyder and Daskin (2005) introduced independent and equal probability facility failures into an uncapacitated facility location model and a p-median model. An LR algorithm was proposed to solve these models. Later, Berman et al. (2007) proposed a reliable p-median facility location model whose disruption is dependent on the facility design rather than their locations and they revealed that the facilities tend to be centralized or co-located with a high facility disruption probability. Lim et al. (2010) studied a reliable facility location model where reliable and unreliable facilities are considered (with and without disruptions) and they also showed that under some conditions, the optimal solution will always deploy the reliable facilities. Cui et al. (2010) further addressed site-dependent probabilities and also proposed an alternative continuum approximation (CA) model. Several extensions on the reliable facility location problems have been proposed under various contexts (Chen et al. 2011, Peng et al. 2011, O'Hanley et al. 2013). Some researchers have also investigated correlated failure mechanisms. For example, Berman and Krass (2011) studied a reliable facility location problem with correlated disruptions when the facilities are located in a line segment, while Berman et al. (2013) further extended the results when the customer may get access to only partial information on facility disruptions. Li and Ouyang (2010) formulated a CA model to address spatial failure correlations and Li et al. (2013) further proposed a support station model to capture more general interdependent facility failures. Despite these efforts, however, all of these studies assumed direct outbound delivery to the customer. To the best of our knowledge, no studies have considered reliable location routing design under probabilistic service disruptions to the vehicle depots.

In this paper, we consider a reliable version of the LRP in which every built depot is subject to independent disruption with an identical probability (known *a priori*). A fleet of vehicles with identical capacity can be dispatched from a third party to any built depot to serve the customers. The disruption of a depot makes it unsuitable for vehicle dispatches and hence its customers must be served by extra vehicles from other operating depots. Such backup operations usually incur higher costs due to the longer travel distances involved. Each customer is assigned a number of backup depots. If all of the assigned backup depots are disrupted simultaneously, the customer incurs a penalty for losing service. For the sake of simplicity, we further assume that customers receive delivery service in fixed sequences, regardless of the depot from which the service vehicle is dispatched. Given this setting, it is important to choose the depot locations and plan the customer delivery sequences (aka delivery strings) carefully according to the possibility of disruptions, thus minimizing the facility setup and expected vehicle routing costs and the expected service loss penalties, across every possible depot disruption scenario.

We attempt to develop suitable model formulations and customized algorithms to effectively address the proposed reliable LRP (RLRP). We begin by formulating it into an integer linear program, building on a set-covering based formulation for the vehicle routing part of the problem. Our customized solution algorithm uses an LR framework to decompose the problem into location and routing subproblems and then uses a column generation (CG) algorithm to solve the subproblems. We conduct numerical studies to test the performance of the proposed model formulation and solution approaches and gain managerial insights into how the system parameters affect the optimal solution.

The remainder of this paper is organized as follows. Section 2 formulates the RLRP model in the form of an integer linear program and Section 3 presents the solution procedure based on LR and CG. Section 4 presents the numerical experiments to test the model and algorithm. Section 5 concludes the paper and discusses future research directions.

2. Formulation of RLRP Model

In this section, we present an integer linear program model that minimizes the fixed setup cost for depot facilities, expected transportation cost for customer delivery and expected penalty cost for service loss. Depots may be built among a set of candidate locations J for a fixed construction cost f_j associated with each built depot $j \in J$. These depots collectively deliver products to a set of customers I, with each customer $i \in I$ bearing a demand $\bar{h}_i$. Identical vehicles from a third party vendor ship the products[1]. Suppose there are at most M vehicles available, each vehicle is responsible for a single delivery trip and that the capacity of each vehicle is C. We assume that the total demand from all customers $\sum_{i\in I}\bar{h}_i$ is no larger than the total vehicle capacity MC and there is no capacity limit within each depot $j \in J$. We also assume that each built depot, once built, is subject to independent failure with an identical probability q. After depot disruptions, the vehicles can be routed through any of the surviving depots to serve customers. Each vehicle travelling

[1] This model can be also applied to situations where (i) the rail company owns these trucks and (ii) the trucks are stationed at the depots.

through the depot at location j incurs an inbound transportation fee $\bar{d}_j$. The transportation cost between any two nodes $i, j \in I \cup J$ in the network is denoted by d_{ij} and the costs among any three nodes satisfy the triangular inequality.

A (feasible) string $g \in G$ is defined as a subset of customers with a given sequence of truck visits, whose total demand $\sum_{i\in g} \bar{h}_i$ is no larger than C. There is an exponential number of possible ordered customer strings G and $|G| = O(|I|!)$. We define parameter $a_{ig} = 1$ if the customer string $g \in G$ contains customer $i \in I$; otherwise, $a_{ig} = 0$. Meanwhile, the total demand of the customer string $g \in G$ is denoted by $h_g = \sum_{i\in g} \bar{h}_i$ and the corresponding vehicle delivery distance to/from a depot $j \in J$ is denoted by d_{gj}, i.e., the sum of the local travel distances within the string (with given visiting sequence), the sum of the line-haul distance between customer string g and depot j and the inbound fee $\bar{d}_j$.

Once a depot fails, its customers are serviced by the vehicles from other depots or subjected to service loss penalties. For each customer string, we define its rth choice of service as its level-r depot. We further assume that each customer string is assigned to up to $R \geq 1$ depots and seeks service from a level-r ($r \leq R-1$) depot if its level-$0, \cdots, r-1$ depots have all failed. When all of its R assigned depots have failed, each customer within the string incurs a penalty ϕ per unit of demand. According to our assumption, the probability of a sequential string being assigned to a level-r ($0 \leq r \leq R-1$) depot is $(1-q)q^r$ and the probability of a customer incurring a penalty is q^R.

We let $X_j = 1$ if a depot is built at location $j \in J$; otherwise, $X_j = 0$. We also let $Z_g = 1$ if a feasible customer string $g \in G$ (i.e., the total demand h_g less than or equal to vehicle capacity C) is formed; otherwise, $Z_g = 0$. Furthermore, for a customer string $g \in G$, let $Y_{gj}^r = 1$ if it is assigned to a depot $j \in J$ as the level r choice; otherwise, $Y_{gj}^r = 0$. For notation simplicity, we use $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ to represent the sets of decision variables $\{X_j\}_{\forall j\in J}, \{Y_{gj}^r\}_{\forall g\in G, j\in J, r=\{0,1,\cdots,R-1\}}, \{Z_g\}_{\forall g\in G}$, respectively.

It should be noted that due to the assumption that each customer must be assigned to up to R different facilities and the penalty cost will not be incurred until all of these facilities are disrupted, the total expected penalty cost across all customers $\phi \sum_{i\in I} \bar{h}_i q^R$ is a constant and can be dropped from the objective function. Hence, the simplified reliable location routing problem can be formulated into an integer linear program as follows:

$$\min_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \sum_{j\in J} f_j X_j + \sum_{g\in G}\sum_{j\in J}\sum_{r=0}^{R-1} (1-q)q^r d_{gj} Y_{gj}^r, \tag{1a}$$

$$\text{s.t.} \sum_{g\in G}\sum_{j\in J} a_{ig} Y_{gj}^r = 1, \forall r = 0, \cdots, R-1, \forall i \in I, \tag{1b}$$

$$\sum_{g\in G}\sum_{r=0}^{R-1} a_{ig} Y_{gj}^r \leq X_j, \forall i \in I, \forall j \in J, \tag{1c}$$

$$\sum_{g \in G} a_{ig} Z_g = 1, \forall i \in I, \tag{1d}$$

$$\sum_{j \in J} Y_{gj}^r \le Z_g, \forall r = 0, \cdots, R-1, \forall g \in G, \tag{1e}$$

$$\sum_{g \in G} Z_g \le M, \tag{1f}$$

$$X_j \in \{0,1\}, Y_{gj}^r \in \{0,1\}, Z_g \in \{0,1\}, \forall j \in J, \forall g \in G, \forall r = 0, \cdots, R-1. \tag{1g}$$

Objective function (1a) minimizes the sum of the depot setup and expected outbound delivery costs. Constraints (1b) enforce that each customer $i \in I$ is assigned to a string and receives a level-r service from an open depot. Constraints (1c) ensure that customer strings are served by only open facilities. Constraints (1d) indicate that each customer should belong to one string. Constraints (1e) enforce that the assignment is valid only if the string is formed. Constraints (1f) enforce that the number of formed customer strings should not exceed vehicle availability. Constraints (1g) require the binary decision variables.

Some optimality properties of the preceding problem are readily available. For example, if string g uses depot j_1 as a level-r_1 choice and depot j_2 as its r_2th choice, then $d_{gj_1} \le d_{gj_2}$ if $0 \le r_1 < r_2 \le R-1$. Otherwise, we could swap the r_1th and r_2th string-to-depot assignments to decrease the total cost. The following proposition summarizes this discussion.

Proposition 1. *Any optimal solution to the RLRP should satisfy the following property:* $d_{gj_1} \le d_{gj_2}, \forall g \in G, \forall j_1, j_2 \in J (j_1 \ne j_2)$, *if* $Y_{gj_1}^{r_1} = Y_{gj_2}^{r_2} = 1, 0 \le r_1 < r_2 < R$.

3. Solution Technique

The RLRP is NP-hard since a standard fixed-charge facility location problem is a special case (i.e., when $R = 1, |G| = M = |I|$ and $\sum_{i \in I} a_{ig} = 1, \forall g \in G$). Moreover, it is easy to see that our formulation is of exponential size; i.e., $|G| = \Omega(|I|!)$ for a sufficiently large vehicle capacity C, where $\Omega(\cdot)$ indicates the asymptotic lower bound. Commercial solvers are hardly able to solve even small to moderate sized instances of such problems (e.g., $R = 1, |I| = |J| = 10$, there will be more than 50 million binary variables). This difficulty motivates us to develop a customized algorithm based on LR and CG.

We relax constraints (1b) and (1e) with Lagrangian multipliers $\mu = \{\mu_i^r\}$ and $\lambda = \{\lambda_{gj}^r\}$ to obtain the following relaxed problem:

$$V(\mu, \lambda) = \min_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}} \sum_{j \in J} \sum_{g \in G} \sum_{r=0}^{R-1} \left((1-q) q^r d_{gj} - \sum_{i \in I} a_{ig} \mu_i^r + \lambda_g^r \right) Y_{gj}^r + \sum_{j \in J} f_j X_j - \sum_{g \in G} \sum_{r=0}^{R-1} \lambda_g^r Z_g + \sum_{i \in I} \sum_{r=0}^{R-1} \mu_i^r. \tag{2}$$

s.t. (1c), (1d), (1f), (1g).

It is known that function $V(\mu,\lambda)$ in the relaxed problem (2) is concave over (μ,λ) since $V(\mu,\lambda)$ is a point-wise minimum over a function which is affine in (μ,λ). This indicates that the subgradient method may be applied to solve the relaxed problem (2).

Note that because relaxed problem (2) no longer contains any connections between the $\mathbf{Z}$ variables and the $\mathbf{X}/\mathbf{Y}$ variables, we may decompose it into two subproblems as discussed in the following subsections.

3.1. Subproblem 1

The first subproblem **(SP1)** contains only integer variables $\mathbf{Z}$ as follows:

$$\textbf{(SP1)} \quad \min_{\mathbf{Z}} -\sum_{g\in G}\sum_{r=0}^{R-1}\lambda_g^r Z_g, \tag{3a}$$

$$\text{s.t. } Z_g \in \{0,1\}, \forall g \in G, \tag{3b}$$

$$\text{and (1d), (1f).}$$

(SP1) takes the form of a set-covering problem, which selects a set of feasible customer strings with minimum costs such that each customer is included in one of the customer strings. Such a problem is known to be difficult due to the exponential number of candidate strings. We propose a CG based relaxation method to obtain a tight lower bound to **(SP1)**. We begin with a small portion of strings and then iteratively add new strings into the relaxed **(SP1)** until an optimal solution is found. To illustrate this procedure, observe that the dual of the **(SP1)** relaxation is

$$\textbf{(DSP1)} \quad \max \sum_{i\in I}\pi_i - M\theta, \tag{4a}$$

$$\text{s.t. } \sum_{i\in I} a_{ig}\pi_i - \theta \le -\sum_{r=0}^{R-1}\lambda_g^r, \ \forall g \in G, \tag{4b}$$

$$\theta \ge 0. \tag{4c}$$

Given the dual solution $\{\bar{\pi}_i\}, \bar{\theta}$, the reduced cost of column g is

$$\bar{c}_g = -\sum_{r=0}^{R-1}\lambda_g^r + \bar{\theta} - \sum_{i\in I} a_{ig}\bar{\pi}_i, \forall g \in G.$$

We note that if $\bar{c}_g \le 0, \forall g \in G$, the current dual solution is optimal to **(DSP1)**. However, because the dual solution to constraints (4a)–(4c) is achieved with a subset of customer strings, it is necessary to check whether $\bar{c}_g \le 0$ holds for the entire set of strings. Hence, we define a new variable, $v_i = 1$, to indicate that a customer $i \in I$ is selected in the column with the smallest reduced cost;

otherwise, $v_i = 0$. For the sake of simplicity, we denote $\mathbf{v} = \{v_i\}$. We now try to find the column with the smallest reduced cost, i.e., the optimal solution to the following minimization problem:

$$\textbf{(P1)} \quad \min_{\mathbf{v}} \ -\sum_{r=0}^{R-1} \lambda_g^r + \bar{\theta} - \sum_{i\in I} \bar{\pi}_i v_i, \tag{5a}$$

$$\text{s.t.} \ \sum_{i\in I} \bar{h}_i v_i \le C, \tag{5b}$$

$$v_i \in \{0,1\}, \forall i \in I. \tag{5c}$$

Constraint (5b) defines the total demand limit of a customer string. The objective value of **(P1)** is clearly non-negative according to constraints (4b) for the current set of strings. Thus, if the optimal value of **(P1)** is negative, a new string is found. Note that for any given λ, **(P1)** can be solved as a Knapsack problem through the use of dynamic programming in pseudo-polynomial time.

The following pseudo-code briefly summarizes the CG used to solve the linear relaxation of **(SP1)**.

Step 1 Generate an initial set of customer strings $G' \subset G$ (partition the customers based on the nearest depots and find the routing distance within each string using the ring-sweep heuristic (Daganzo 1984));

Step 2 Solve **(DSP1)** in terms of set G' and obtain the optimal dual variables $\{\bar{\pi}_i\}, \bar{\theta}$;

Step 3 Solve **(P1)** and check its optimal value $\bar{c}_g^{\min}$:

Case 1: If $\bar{c}_g^{\min} < 0$, find a string g that achieves $\bar{c}_g^{\min}$ and let $G' = G' \bigcup \{g\}$. Return **Step** 2;

Case 2: If $\bar{c}_g^{\min} \ge 0$, solve relaxed **(SP1)**, obtain the optimal solution and terminate.

Suppose that $V_{\textbf{(SP1)}}(\lambda)$ denotes the optimal integer solution of **(SP1)**. We also define a "reasonable partition" to a set of strings that cover all of the customers and each customer is involved in only one of the strings. A reasonable partition is *feasible* if each customer is covered by only one string and the string number is no larger than M. The following proposition reveals some properties of the relaxed problem **(SP1)**.

PROPOSITION 2. *For any given Lagrangian multiplier λ, the subset of customer strings which are (or partially) chosen in the optimal solution of the relaxed problem **(SP1)** has the following properties.*

(1) There are at most two distinct reasonable partitions of customers within this subset;

*(2) Suppose there is a feasible reasonable partition within this subset. As such, either this partition is the optimal solution to **(SP1)** or the optimal solution can be bounded by*

$$\frac{M}{2} V^f_{\textbf{\textit{(SP1)}}}(\lambda) \le V_{\textbf{\textit{(SP1)}}}(\lambda) \le V^f_{\textbf{\textit{(SP1)}}}(\lambda),$$

where $V^f_{\textbf{\textit{(SP1)}}}(\lambda)$ denotes the objective value of the feasible reasonable partition.

Proof: see Appendix A.

3.2. Subproblem 2

It is obvious that the second subproblem **(SP2)** can be decomposed into a set of subproblems **(SP2-j)** for each $j \in J$ as follows:

$$\textbf{(SP2-j)} \quad \min_{\mathbf{X},\mathbf{Y}} \sum_{g\in G}\sum_{r=0}^{R-1}\left((1-q)q^r d_{gj} - \sum_{i\in I} a_{ig}\mu_i^r + \lambda_g^r\right) Y_{gj}^r + f_j X_j, \tag{6a}$$

$$\text{s.t. } X_j \in \{0,1\}, Y_{gj}^r \in \{0,1\}, \forall g \in G, \forall r = 0, \cdots, R-1; \tag{6b}$$

$$\text{and (1c).}$$

Due to constraints (1c), the objective function of **(SP2-j)** is clearly zero when $X_j = 0$. Hence, we only need to consider the remaining case with $X_j = 1$, which is

$$\textbf{(SP2-j')} \quad \min_{\mathbf{X},\mathbf{Y}} \sum_{g\in G}\sum_{r=0}^{R-1}\left((1-q)q^r d_{gj} - \sum_{i\in I} a_{ig}\mu_i^r + \lambda_g^r\right) Y_{gj}^r + f_j, \tag{7a}$$

$$\text{s.t. } \sum_{g\in G}\sum_{r=0}^{R-1} a_{ig} Y_{gj}^r \le 1, \forall i \in I, \tag{7b}$$

$$Y_{gj}^r \in \{0,1\}, \forall g \in G, \forall r = 0, \cdots, R-1. \tag{7c}$$

The dual problem of **(SP2-j')** is

$$\textbf{(DSP2-j')} \quad \max -\sum_{i\in I} \eta_i, \tag{8a}$$

$$\text{s.t. } -\sum_{i\in I} a_{ig}\eta_i \le (1-q)q^r d_{gj} - \sum_{i\in I} a_{ig}\mu_i^r + \lambda_g^r, \forall g \in G, \forall r = 0, \cdots, R-1, \tag{8b}$$

$$\eta_i \ge 0, \forall i \in I. \tag{8c}$$

and the dual solution $\{\bar{\eta}_i\}$ yields the reduced cost of column g for each $r = 0, \cdots, R-1$; i.e.,

$$\bar{c}_{gj}^r = (1-q)q^r d_{gj} - \sum_{i\in I} a_{ig}\mu_i^r + \lambda_g^r + \sum_{i\in I} a_{ig}\bar{\eta}_i.$$

Again, if $\bar{c}_{gj}^r \le 0, \forall g \in G, \forall r = 0, \cdots, R-1$, the current dual solution is optimal to **(DSP2-j')**. However, as the dual solution is achieved with a subset of customer strings, it is necessary to check whether $\bar{c}_{gj}^r \le 0$ holds for the entire set of strings. Hence, we let $\nu_i = 1$ if a customer $i \in I$ is selected in the column with the smallest reduced cost; otherwise, $\nu_i = 0$. For the sake of simplicity, we denote $\nu = \{\nu_i\}$. We are now ready to find the column with the smallest reduced cost for each $r = 0, \cdots, R-1$; i.e., the optimal solution to the following optimization problem for each $r = 0, \cdots, R-1$:

$$\textbf{(P2-jr)} \quad \min_{\nu} (1-q)q^r F(\nu) + \sum_{i\in I} (\bar{\eta}_i - \mu_i^r)\nu_i + \lambda_g^r, \tag{9a}$$

$$\text{s.t.} \sum_{i\in I} \bar{h}_i \nu_i \le C, \tag{9b}$$

$$\nu_i \in \{0,1\}, \forall i \in I, \tag{9c}$$

where $F(\nu)$ is the optimal tour length of the travelling salesman problem (TSP) for the selected customer set $\{i \in I : \nu_i = 1\} \cup \{j\}$ and constraint (9b) enforces the maximum demand of a customer string.

If the optimal value of **(P2-jr)** is negative, a new string is similarly identified. We note that **(P2-jr)** takes the form of a well-known prize–collecting travelling salesman problem (Bixby 1997) that is itself NP -hard. However, several approximate algorithms can provide lower bounds (Toth and Vigo 2002). Among them, we choose a pseudo-polynomial algorithm based on dynamic programming (Desrochers et al. 1992). Let function $F_j(i,c)$ denote the minimum reduced cost of a tour (allowing node repetition) that begins at depot j and visits customer i as the last node, subject to total service demand c. Let $prev(i,c)$ be the customer preceding i in the path generated from $F_j(i,c)$. Function $\bar{F}_j(i,c)$ is the lowest cost path from depot j to customer i subject to the condition that the customer preceding i is not $prev(i,c)$. The relationships between these functions are based on the Bellman equation type of arguments. For all $i \in I$ and $1 \le c \le C$, we have the following (Desrochers et al. 1992, Toth and Vigo 2002):

$$\begin{aligned}
F_j(i,c) &= \begin{cases} (1-q)q^r d_{ji} + (\bar{\eta}_i - \mu_i^r) & \text{if } c = \bar{h}_i, \\ \infty & \text{otherwise;} \end{cases} \\
F_j(i,c) &= \min_{k\neq i} \left\{ \left[F_j\left(k, c-\bar{h}_i\right) + (1-q)q^r d_{ki} + (\bar{\eta}_i - \mu_i^r) : i \ne prev\left(k, c-\bar{h}_i\right)\right], \right. \\
&\qquad \left. \left[\bar{F}_j\left(k, c-\bar{h}_i\right) + (1-q)q^r d_{ki} + (\bar{\eta}_i - \mu_i^r) : i = prev\left(k, c-\bar{h}_i\right)\right] \right\}; \\
\bar{F}_j(i,c) &= \min_{k\neq i} \left\{ \left[F_j\left(k, c-\bar{h}_i\right) + (1-q)q^r d_{ki} + (\bar{\eta}_i - \mu_i^r) : i \ne prev\left(k, c-\bar{h}_i\right) \text{ and } k \ne prev(i,c)\right], \right. \\
&\qquad \left. \left[\bar{F}_j\left(k, c-\bar{h}_i\right) + (1-q)q^r d_{ki} + (\bar{\eta}_i - \mu_i^r) : i = prev\left(k, c-\bar{h}_i\right) \text{ and } k \ne prev(i,c)\right] \right\}.
\end{aligned}$$

To form a tour, we must add back the distance from the last node to depot j when the above procedure terminates; i.e., $F_j(i,c) := F_j(i,c) + d_{ij}, \forall i \in I, 1 \le c \le C$.

The following briefly summarizes the procedure for solving the relaxed problem **(SP2-j)** for each $j \in J$.

Step 1 Set $X_j = 1$ and generate an initial set of customer strings $G' \subset G$ (similar to **Step** 1 in the solution procedure of **(SP1)**).

Step 2 Solve **(DSP2-j')** in terms of set G' and obtain the optimal dual variables $\{\bar{\eta}_i\}$.

Step 3 Obtain a lower bound $\bar{c}_{gj}^{r,\text{LB}}$ for **(P2-jr)** and check its value.

Case 1: If $\bar{c}_{gj}^{r,\text{LB}} < 0$ for some $0 \le r \le R-1$, identify a small string set G_0 (e.g., less than 10) with the most negative reduced costs and let $G' = G' \bigcup G_0$. Return to **Step** 2;

Case 2: If $\bar{c}_{gj}^{r,\text{LB}} \geq 0, \forall 0 \leq r \leq R-1$, solve the relaxed subproblem **(SP2-j')** and obtain the optimal solution. If the optimal value is less than or equal to 0, $X_j = 1$; otherwise $X_j = 0$ and $Y_{gj}^r = 0, \forall g \in G', \forall 0 \leq r \leq R-1$.

3.3. Feasible Solution and Multiplier Updates

Although the results from the relaxed subproblems 1 and 2 attribute a lower bound to the original problem, the decision variables $\{\bar{X}_j\}, \{\bar{Y}_{gj}^r\}, \{\bar{Z}_g\}$ may violate the constraints in the original problem. Hence, we perturb the solution from the relaxed problem to generate a feasible solution to the original problem, which simultaneously provides an upper bound. This can be done as follows.

i) Select a subset of strings $\bar{G}$ such that $\{\bar{Z}_g\}$ from **(SP1)** is greater than zero (i.e., $\bar{G} = \{g \in G' : \bar{Z}_g > 0\}$). Obtain a feasible solution $\{Z_g\}$ with only set $\bar{G}$, i.e., partition the customers with some strings from set $\bar{G}$. With the feasible solution $\{Z_g\}$, compute feasible solutions $\{X_j\}, \{Y_{gj}^r\}$ from the original problem in constraints (1a) - (1g);

ii) Improve the current solution using a local search. This is discussed in Section 3.4.

We use the standard subgradient method (Fisher 1981) to iteratively solve the relaxed problem and obtain the best multipliers (i.e., those that yield the tightest lower bound). The initial values of the multipliers are all set to zero. At the end of iteration k, the multipliers (μ^k, λ^k) are updated as follows:

$$(\mu_i^r)^{k+1} = \max\left\{0, (\mu_i^r)^k + \alpha^k \left(1 - \sum_{g \in G} \sum_{j \in J} a_{ig} \bar{Y}_{gj}^r\right)\right\}, \forall i \in I, 0 \leq r \leq R-1,$$

$$\left(\lambda_g^r\right)^{k+1} = \max\left\{0, \left(\lambda_g^r\right)^k + \alpha^k \left(\sum_{j \in J} \sum_{r=0}^{R-1} \bar{Y}_{gj}^r - \bar{Z}_g\right)\right\}, \forall g \in G', 0 \leq r \leq R-1,$$

where the step size α^k is updated as

$$\alpha^k = \frac{\gamma^k \left(V_{UB}^k - V_{LB}^k\right)}{\sum_{i \in I} \sum_{r=0}^{R-1} \left(1 - \sum_{g \in G} \sum_{j \in J} a_{ig} \bar{Y}_{gj}^r\right)^2 + \sum_{j \in J} \sum_{g \in G'} \left(\sum_{j \in J} \sum_{r=0}^{R-1} \bar{Y}_{gj}^r - \bar{Z}_g\right)^2}.$$

Here, γ^k is a control parameter, V_{UB}^k is the upper bound from the best solution so far and V_{LB}^k is the optimal solution to the relaxed problem. The initial value of γ^k is set to 2.0 and then decreases by a constant fractional factor (less than 1) whenever the best lower bound does not improve in several iterations.

The following briefly summarizes the overall RLRP solution algorithm.

Step 1. Let $k = 0$. Initialize $V_{UB}^0 = \infty$, $V_{LB}^0 = -\infty$ and μ^0, λ^0 as zero vectors. Choose the depot locations based on the facility location model without routing (i.e., let $|G| = M = |I|, \sum_{i \in I} a_{ig} = 1, \forall g \in G$). Set $G' = \emptyset$;

Step 2. Let $k = k + 1$. Solve **(SP1)** and **(SP2)**, update G' and get relaxed solutions $\{\bar{X}_j\}, \{\bar{Y}^r_{gj}\}, \{\bar{Z}_g\}$;

Step 3. Get feasible solutions and update the Lagrangian multipliers;

Step 4. Conduct optimality test. Stop if $\left|\frac{V^k_{UB} - V^k_{LB}}{V^k_{UB}}\right| \leq \varepsilon$ for some tolerance $\varepsilon > 0$ or if any other stopping criterion is reached. Otherwise, continue on to the next step;

Step 5. For any string $g \in G'$, if $\bar{Y}^r_{gj} = 0, \forall j \in J, 0 \leq r \leq R - 1$ and $\bar{Z}_g = 0$, remove the string from G';

Step 6. If it is the first time the size of G' (i.e., $|G'|$) is greater than a threshold number (e.g., 500), fix G' and run LR procedure without CG subroutine until it converges, update the current Lagrangian multipliers and return to **Step 2**.

3.4. Local Search Algorithm

Any feasible solution obtained during the RLRP solution procedure, could be further improved using the following local search algorithm.

Step 1. Obtain a feasible solution from Section 3.3;

Step 2. Fix the facility locations and perform the following local searches among the strings. If the cost can be improved, update the string and assignment decisions and continue on to the next step;

(a) Swap two customer positions within each customer string;

(b) Remove one customer from a string and insert it into another string;

(c) Swap two customers from two distinct customer strings;

Step 3. Fix the customer strings and perform the following local searches among the facilities. If the cost can be improved, update the facility locations and string assignment decisions and continue on to the next step;

(a) Replace each chosen facility with a non-chosen facility.

(b) For each chosen facility, remove it from the chosen facility list.

(c) For each non-chosen facility, add it to the chosen facility list;

Step 4. Return to **Step** 2. if any cost improvement occurs. Stop if there is no cost improvement.

4. Numerical Study

In this section, we present several numerical examples. All of the proposed solution algorithms were coded in VC++ with calls to ILOG CPLEX 12.2 on a personal computer with a 2.67 GHz CPU and 2.0 GB of memory. We began by testing the proposed LR and CG algorithms using a 30-node data set generated by Augerat et al. (1995). We subsequently applied the model to an empirical case study of a full-scale railroad network.

4.1. Model Testing

We tested the discrete model and solution algorithm using a 30-node dataset from Augerat et al. (1995) (available online at `http://www.coin-or.org/SYMPHONY/branchandcut/VRP/data/B/`). Each of the 30 nodes represents demand point. There is a homogeneous fixed depot construction cost of 20 at each location and the vehicle inbound delivery cost ratio is set at zero; i.e., $\bar{d}_j = 0, \forall j \in J$. The number and capacity of vehicles are set at $M = 6$ and $C = 100$, respectively. We calculated the distance based on the Euclidean metric. We tested our model for $q \in \{0.05, 0.10, 0.15\}$ and $R \in \{3, 5\}$. The maximum computation time of the LR and CG algorithm is limited to 4 hours. We also presented the results of two other algorithms. First, we obtained the initial depot location solution using a simple location model (without considering the outbound vehicle routing) and then found the vehicle routes using a ring sweep algorithm (Daganzo 1984). We obtained another result with the local search algorithm, using the initial solution as a starting point. Table 1 shows the results.

Table 1 Computational results for 30-node test cases.

Algorithm	q	R	# depots	Total costs	Gap (%)	Time (s)
LR+CG	0.05	3	4	194.29	1.01	14074
	0.1	3	4	208.86	1.19	7935
	0.15	3	4	220.61	0.94	4405
	0.05	5	5	204.33	4.19	7756
	0.1	5	5	215.95	3.11	9900
	0.15	5	5	226.67	4.37	8924
Ring Sweep	0.05	3	4	211.55	9.09	<1
	0.1	3	4	227.90	9.44	<1
	0.15	3	4	243.80	10.36	<1
	0.05	5	5	219.10	9.86	<1
	0.1	5	5	232.25	9.91	<1
	0.15	5	5	241.30	10.16	<1
Local Search	0.05	3	4	202.49	5.02	1
	0.1	3	4	218.89	5.72	1
	0.15	3	4	229.72	4.87	2
	0.05	5	5	205.23	3.76	2
	0.1	5	5	218.17	4.09	2
	0.15	5	5	229.31	5.46	1

According to Table 1, the proposed LR and CG algorithm solved every instance with a small optimality gap (less than 5%) within 4 hours. The initial solution from the ring sweep approach gives us a near 10% gap with a computation time of less than a second. The local search method further improves the solutions by about 5% within 1 or 2 seconds. The local search method for these moderately sized instances performs well.

The table indicates that the optimal system cost increases with the probability of depot disruption. This is intuitive, as the larger the failure probability, the higher likelihood that customers will

Table 2 Computational results for 33-node and 40-node test cases with CG and LR algorithm

Instances	f	C	M	$\bar{d}$	q	R	# of depots	Total costs	Gap (%)	Time (s)
33-node	10	100	7	0	0.05	3	5	233.00	5.22	6460
	10	100	7	0	0.10	3	6	242.59	4.13	14400
	10	100	7	0	0.15	3	8	256.22	3.69	14400
	20	100	7	0	0.05	3	5	279.40	2.02	9157
	20	100	7	0	0.10	3	5	300.55	2.92	14400
	20	100	7	0	0.15	3	6	321.48	4.25	14400
	30	100	7	0	0.05	3	5	329.40	1.05	14400
	30	100	7	0	0.10	3	5	350.55	1.16	8060
	30	100	7	0	0.15	3	5	372.55	2.48	14400
	40	100	7	0	0.05	3	4	379.73	1.63	9862
	40	100	7	0	0.10	3	5	400.55	1.49	5924
	40	100	7	0	0.15	3	5	417.94	2.21	14400
Average							5	323.66	2.69	11689
40-node	20	100	8	0	0.05	3	6	372.41	1.46	14400
	20	100	9	0	0.05	3	7	371.99	1.42	14400
	20	100	10	0	0.05	3	7	371.35	1.31	14400
	20	100	8	5	0.05	3	6	542.05	3.73	10823
	20	100	9	5	0.05	3	7	541.63	3.53	9876
	20	100	10	5	0.05	3	7	542.52	2.87	7982
	20	100	8	10	0.05	3	6	712.36	3.59	14400
	20	100	9	10	0.05	3	7	711.81	4.37	14400
	20	100	10	10	0.05	3	7	711.56	3.91	14400
	20	100	8	15	0.05	3	6	885.01	2.95	8423
	20	100	9	15	0.05	3	7	883.40	3.86	14400
	20	100	10	15	0.05	3	7	881.29	5.22	14400
Average							7	627.28	3.18	12692

receive service from more distant depots at higher travel costs. In addition, the optimal system cost increases only slightly with the number of permitted backup depots R, which explains that limiting the use of backup facilities does not significantly influence the optimal system cost.

We further test our proposed method with CG and LR procedure on two datasets with 33 nodes and 40 nodes from the same website. The results are shown in the Table 2, where we denote $f := f_j, \forall j \in J, \bar{d} := \bar{d}_i, \forall i \in I$. We observe that the proposed algorithm works quite well on those instances with relatively small optimality gap but the running time tends to be longer if more candidate facilities or demand points are involved, which is possibly because of the slow convergence of the subgradient method. Meanwhile, we also conduct the sensitivity analyses on the influence of the facility fixed cost f and the probability q of facility disruption with the 33-node instance and on the number of available trucks M and the inbound delivery ratio $\bar{d}$ with the 40-node instance. We note that the number of depot to be built increases to balance the total costs when the facility fixed cost increases or the disruption probability increases. Another interesting observation is that the increase of the number of available trucks somehow increases the number of depots to be built, since it could be more cost effective to distribute these trucks to different places. However, from the inbound transportation cost does not have much influence on this system, for example, when

$M = 8$, the solution always suggests building 6 facilities regardless of the value of $\bar{d}$, which might be because each demand point will bear this same inbound transportation cost.

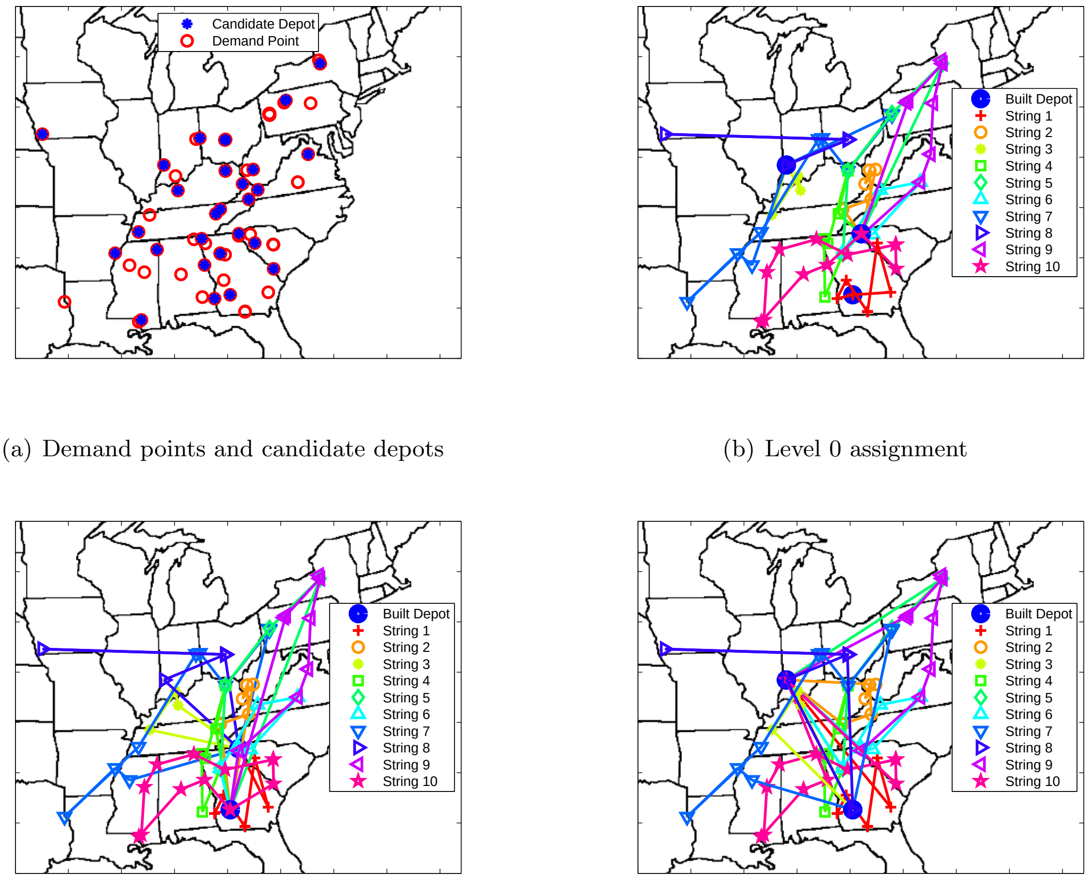
4.2. Empirical Case Study

The locomotives of a major U.S. railroad company must receive periodic maintenance from movable service trucks at selected rail yards. Each service truck can visit multiple yards, but must return to the service centers (i.e., depots) before a maximum service time is reached. Thus, it is of interest to find the optimal number and location of the service centers in addition to the routing schedules of the service trucks to minimize the overall operating cost.

The railroad network contains 70 selected rail yards at which locomotives receive service and 30 service center candidate locations. These locations are shown in Figure 1(a). To protect data confidentiality, we report only the approximate values of the system parameters. Within the periodic planning horizon, the prorated setup cost for the service depots is somewhere between \$30,000 and \$60,000 and the hourly travel cost for the service trucks is between \$15 and \$25. There is no fee to deliver the trucks to the rail yards, i.e., $\bar{d}_j = 0, \forall j \in J$. The number of trucks M is between 10 and 12 and the service capacity C is set between 200 and 400 locomotive-hours. The locomotive service demand of each yard is set according to the periodic arrivals and expected service times of trains. The distance matrix is calculated based on the shortest path along the railroad network. The failure probability of each depot (due to operational and technical challenges) is assumed to be $q = 0.1$ and each yard has $R = 3$ backup depots.

This empirical case is solved using the LR and CG algorithm, which achieves a 4% optimality gap after around 12 hours of computation. The best solution is to build 3 depots and employ 10 trucks. The solution is shown in detail in Figures 1(b) - 1(d). Although each string of yards is basically a cluster based on proximity, the distances among these yards may not include the shortest Euclidean distance, mainly because the shortest network path between two yards may detour away from the straight line. Also note that we choose three relatively close depots to avoid significantly changing the line-haul distances from each string to the chosen depot.

The current practice of the company is to provide locomotive service via direct truck visits (i.e., no truck routing is involved). For the sake of comparison, we also run a benchmark scenario with direct shipments (i.e, $|G| = M = 70$ and $\sum_{i \in I} a_{ig} = 1, \forall g \in G$). The results show that permitting truck routing saves 50% of the depot setup costs (by constructing three rather than six depots) and 41.5% of the truck travel costs for this case study. We further conduct a benchmark scenario without any disruption (i.e., $q = 0$). It shows that the total cost of the proposed model is only 6% more than that of the benchmark model, which demonstrates that our model can hedge the disruption risks with a small increase of the financial budget. In contrast, if we design the system



(a) Demand points and candidate depots

(b) Level 0 assignment

(c) Level 1 assignment

(d) Level 2 assignment

Figure 1 Illustration of the railroad case study solution

without taking disruption risks into account, the penalty is notably greater. In this case, the total expected cost of the benchmark solution (i.e., designed under the assumption that $q = 0$) under actual disruption risks (i.e., $q = 0.10$) is 17% higher than that of the proposed model.

5. Conclusion

This study considered a reliable location routing problem in which facilities were subject to probabilistic disruptions. The objective was to minimize the fixed setup costs and the expected routing and penalty costs (due to loss of service). We began by formulating the problem into an integer linear program and proposed a variety of solution algorithms including an LR framework with embedded CG, ring sweep heuristics and local search. We also conducted a set of numerical case studies to test the performance of the proposed solution algorithms. We found that LR and CG algorithms were able to solve small or medium-sized instances to a small optimality gap. Furthermore, we applied our solution approaches to a locomotive maintenance service planning problem in a full-scale railroad network. Compared with current industry practice, our model results significantly decreased the vehicle travel and fixed facility costs.

Future research may be conducted in several directions. Additional uncertainties may be introduced into our model to account for stochastic travel costs and random customer demand. Meanwhile, the capacity limit and capacity expansion option within each depot Xie and Ouyang (2013) can be further incorporated. Moreover, site-dependent disruption probabilities (e.g., see in Cui et al. (2010)) and correlated disruptions (e.g., see Li and Ouyang (2010), Li et al. (2013)) may be introduced to further generalize the model. Another possible extension would be to consider a dynamic problem in which the possible opening and closure of depots are allowed over a time horizon. Finally, although we assumed in this study that customer strings remained unchanged across every depot assignment level, it would be more realistic to allow customers to be regrouped when vehicles are routed from backup depots.

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Appendix A: Proof of Proposition 2

Proof:

1. Proposition 2 is proved by construction. It is obvious that all of the constraints are satisfied if the partitions are reasonable such that the total number of strings is no larger than M. However, if there exists another reasonable partition with a lower average cost per customer but the total number of strings is larger than M, we will choose both two partitions.

   Suppose there is a third reasonable partition. If the average cost per customer with this partition is less than that of one of the previous two partitions, we could replace it with the third one, which would lead to a lower total cost. Otherwise, there would be no reason to add the third reasonable partition into the relaxed solution.

2. Suppose that this feasible partition is not the optimal solution to **(SP1)**. Let n_f, n_o denote the number of strings within the feasible partition and optimal solution. From the previous argument, we know that the following inequality must hold:

$$\frac{V^f_{\mathbf{(SP1)}}(\lambda)}{2} \le \frac{V^f_{\mathbf{(SP1)}}(\lambda)}{n_f} \le \frac{V_{\mathbf{(SP1)}}(\lambda)}{n_o} \le \frac{V^f_{\mathbf{(SP1)}}(\lambda)}{n_0}.$$

The first inequality holds because there are at least two strings. The second inequality is based on the argument in part 1 and the third inequality simply states that a feasible solution is an upper bound to the optimal solution. We therefore arrive at the conclusion immediately.