

Bridging the gap between general probabilistic theories and the device-independent framework for nonlocality and contextuality

Giulio Chiribella¹

Department of Computer Science, The University of Hong Kong, Pokfulam Road, Hong Kong

Xiao Yuan

Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing, 100084, China

Abstract

Characterizing quantum correlations in terms of information-theoretic principles is a popular chapter of quantum foundations. Traditionally, the principles used for this scope have been expressed in terms of conditional probability distributions, specifying the probability that a black box produces a certain output upon receiving a certain input. This approach is known as *device-independent*. Another major chapter of quantum foundations is the information-theoretic characterization of quantum theory as a whole, with its sets of states and measurements, and with its allowed dynamics. The different frameworks adopted for this scope are known under the umbrella term of *general probabilistic theories*. With only a few exceptions, the two research programmes on characterizing quantum correlations and characterizing quantum theory have so far proceeded on separate tracks, each one developing its own methods and its own agenda. Still, both programmes share the same basic goal: a new and better understanding of quantum mechanics in information-theoretic terms. This considered, it is quite striking that the connections between the two programmes are still largely undeveloped. This paper aims at bridging the gap, by presenting a “Rosetta stone” for the two frameworks and by illustrating how the two programmes can benefit each other.

As a case study, we focus on two device-independent features known as Local Orthogonality (LO) and Consistent Exclusivity (CE). In a recent work [1], we showed that CE and LO can be derived from the basic idea that, at the fundamental level, measurements are repeatable and minimally disturbing. In this paper we provide a new, alternative derivation based on a different set of principles, revolving around the notion of *pure orthogonal measurement*—a mea-

¹Corresponding author: giulio@cs.hku.hk

surement that cannot be further refined and that identifies states without error. The first principle, Measurement Purification, states that every measurement can be reduced to a pure orthogonal measurement by adding an auxiliary system and by coarse-graining over some outcomes. The second principle, Locality of Pure Orthogonal Measurements, states that two pure orthogonal measurements performed independently on two systems yield a pure orthogonal measurement on the composite system. The third principle, Strong No Disturbance Without Information, states that every measurement that does not extract information about a source can be realized without disturbing the states in that source and without disturbing the pure orthogonal measurements that identify states in that source. These three principles together imply LO. CE is then derived by adding a fourth principle, called Pure State Identification, stating that every outcome of a pure orthogonal measurement identifies a pure state.

1. Introduction

One of the most profound mysteries of quantum theory is nonlocality [2, 3], namely the fact that experiments performed at spacelike separated locations can exhibit stronger correlations than those allowed by any local realistic model. Still, quantum correlations are not the strongest correlations one can imagine: the assumption that correlations cannot be used to communicate at unbounded speed, known as No-Signalling, is compatible with a larger set of exotic, non-quantum correlations [4, 5]. This observation stimulated the search for other principles, of similar information-theoretic flavour, aimed at achieving a complete characterization. Up to now, several principles for quantum correlations have been proposed, such as Non-Trivial Communication Complexity [6, 7], No-Advantage in Nonlocal Computation [8], Information Causality [9], Macroscopic Locality [10], and Local Orthogonality (LO) [11]. These principles have been spectacularly successful in constraining the allowed correlations, narrowing them to a set that is close to the quantum set. However, no combination of the presently known principles is sufficient to characterize the quantum set completely [12]. Similar considerations apply to the study of quantum contextuality [13, 14], where the principles of Consistent Exclusivity (CE) [15, 16, 17, 18] and Macroscopic Non-Contextuality [19] have been proposed in order to characterize the degree of contextuality exhibited by projective measurements in quantum theory. Also in this case, a complete information-theoretic characterization of the contextuality bounds satisfied by projective quantum measurements is still missing.

On the other hand, several reconstructions of quantum theory from information-theoretic principles have been proposed in recent years [20, 21, 22, 23, 24, 25, 26, 27, 28]. With different background assumptions and slightly different goals, these works single out the Hilbert space formalism of quantum theory: in particular, they imply that physical systems are associated to Hilbert spaces, that states are described by density matrices, and that the probabilities of measurement outcomes are computed with the Born rule. As a byproduct, they also

characterize the particular sets of correlations arising in quantum theory. Is this a solution to the long sought-after characterization of quantum nonlocality and contextuality? Yes and no. *Yes*, because every set of information-theoretic principles that singles out quantum theory provides also an information-theoretic justification of quantum nonlocality and contextuality. *And no*, because such a justification may not be as satisfactory as one may desire: ideally, one would like to have principles that *directly* imply bounds on correlations, without the detour of a full derivation of the Hilbert space framework.

The importance of a direct characterization is reflected in the nature of the principles used for quantum correlations. Principles like those in Refs. [29, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18] refer only to the conditional probabilities of obtaining output data from input data, without making any assumption on the process generating the output from the input. The framework in which these principles are formulated has been aptly named *device-independent* (see [30] and [31] for an introduction). In stark contrast, the framework used for reconstructing quantum theory is not device-independent. And for good reasons: a full-fledged physical theory is not only about input-output probability distributions, but also about physical systems and how they can interact through physical processes [32]. Naturally, the principles used for reconstructing quantum theory presuppose that experimental data have already been organized in the basic structure of a physical theory, which has systems, states, transformations, and measurements at its backbone. For example, principles like Local Tomography [24, 33, 34] or Ideal Compression [20] explicitly refer to the “states of a given physical system”, to the “measurements performed on a composite system”, and to the “processes that transform a system into another”. The formulation of these principles is based on the framework of *general probabilistic theories* [24, 34, 33, 35, 36, 37, 20, 38, 21, 39, 40], which describes on the same footing classical and quantum theory, as well as many hypothetical, post-quantum theories.

Up to now, the research programmes on reconstructing quantum theory and that on characterizing quantum correlations have proceeded along separate tracks. However, it is clear that the interaction between these two programmes has a potential for understanding the picture of reality (if any) at which quantum theory is hinting. In a recent work [1], we started exploring the relations between principles for correlations and principles for general probabilistic theories. We first defined a class of ideal measurements, called *sharp measurements*, which represent an ideal standard of measurements that are repeatable and cause minimal disturbance on future observations. Then we postulated that all measurements are fundamentally sharp, i.e. they can be obtained by performing a joint sharp measurement on the system together and on an environment. Combining this requirement with two additional requests about the compositional properties of sharp measurements we have been able to derive the validity of Local Orthogonality and Consistent Exclusivity.

In this paper we present an alternative derivation of LO and CE, based on a different notion of “ideal measurement” and on a different set of physical principles. We take LO and CE as the subject of this further study because

they provide the simplest testbed for investigating the interplay between device-independent and device-dependent notions. We will first discuss the relations between the device-independent framework and the framework of general probabilistic theories. Then we will set up the scene for the derivation of LO and CE, defining a privileged class of measurements, here called *spiky measurements*. Spiky measurements are obtained by coarse-graining pure orthogonal measurements, i.e. measurements that cannot be further refined and that identify some states without error. In quantum theory, spiky measurements coincide with projective measurements, which in turn coincide with the sharp measurements defined in Ref. [1]. In a general theory, however, spiky and sharp measurements can potentially differ. Using the notion of pure orthogonal measurement, we then formulate three requirements which allow one to derive LO and, under an additional assumption, CE. Interestingly, our requirements do not include Causality [37], which is instead *derived* as a byproduct. Nevertheless, one of the requirements, called Sufficient Orthogonality, has no immediate operational interpretation. To address this issue, we show how Sufficient Orthogonality can be reduced to a strong version of the No Information Without Disturbance property discussed in Refs. [20, 41]. Such a reduction, alas, assumes Causality.

The paper is structured as follows: in Sections 2 and 3 we present the device-independent framework and the framework of general probabilistic theories, respectively. The bridge between the two frameworks is provided in Section 4, where we specify the physical model in which the input-output probabilities are generated. The relation between No-Signalling at the level of probability distributions and Causality at the level of physical processes is discussed in Section 5. Sections 6 and 7 provide the definition of spiky measurements and state three axioms about their structure. The three axioms imply the validity of Local Orthogonality (briefly recalled in Section 8) and Causality, as shown by the derivation in Section 9. Since one of the three axioms has no compelling operational interpretation, in Section 10 we show one way to reduce it to more fundamental physical statements—in this case, Causality and a strong version of the No Information Without Disturbance principle. The analysis carried out for nonlocality is then applied to the study of contextuality: we first review the device-independent framework and illustrate CE as an example of device-independent principle (Section 11) and then bridge it with the framework of general probabilistic theories (Section 12). In Section 13 we show different formulations of CE as a physical principle regarding a privileged class of measurements, which generalize projective measurements in quantum theory. Choosing spiky measurements as our privileged class of measurements, we then provide a derivation of CE (Section 14). Finally, in Section 15 we compare the notion of spiky measurement, used in this paper, with other potential generalizations of the notion of projective measurement in quantum theory. The conclusions are drawn in Section 16. The Appendices present some technical proofs that are not of immediate interest for the comprehension of the main points of the paper.

2. The device-independent framework for nonlocality

In this section we briefly review the device-independent framework for nonlocality [5, 42, 43], pointing the reader to [30] for a more extended discussion. The framework describes games where a group of players respond to a set of possible questions posed by a referee. The strategy of the players is described by the conditional probability distribution of the answers given the questions. Regarding the questions as inputs and the answers as outputs, the limitations on the physical theory that describes the players' strategies are encoded into limitations on the allowed input-output probability distributions.

2.1. Non-local games

A non-local game is a game involving N players and a referee, where the referee gives to the i -th player an input x_i in some input alphabet X_i and the player returns an output y_i in some output alphabet Y_i . For brevity, let us denote by $\mathbf{x} = (x_1, x_2, \dots, x_N) \in X_1 \times X_2 \times \dots \times X_N =: \prod_{i=1}^N X_i$ the string of all inputs given by the referee and by $\mathbf{y} = (y_1, y_2, \dots, y_N) \in Y_1 \times Y_2 \times \dots \times Y_N =: \prod_{i=1}^N Y_i$ the string of all outputs returned by the players. In each run of the game, the referee chooses the input string \mathbf{x} at random according to a probability distribution $q(\mathbf{x})$ and assigns a payoff $\omega(\mathbf{x}, \mathbf{y})$ to the output string \mathbf{y} . The goal of the players is to maximize their expected payoff, given by

$$\omega = \sum_{\mathbf{x}} q(\mathbf{x}) \left[\sum_{\mathbf{y}} \omega(\mathbf{x}, \mathbf{y}) p(\mathbf{y}|\mathbf{x}) \right], \quad (1)$$

where $p(\mathbf{y}|\mathbf{x})$ is the conditional probability that the players produce the output \mathbf{y} upon receiving the input \mathbf{x} .

2.2. Principles about input-output distributions

The input-output probability distribution $p(\mathbf{y}|\mathbf{x})$ describes the strategy of the players in a black box fashion, disregarding the specific details of the devices used to generate the outputs. Such a description, called *device-independent*, is particularly suited for cryptographic applications [44, 45, 42, 43, 46, 47, 48]. In this context, the constraints on the allowed strategies are expressed as constraints on the allowed probability distributions. For example, the most common constraint in the literature is the No-Signalling principle [29, 4, 5], which imposes that the correlations in the probability distribution $p(\mathbf{y}|\mathbf{x})$ cannot be used to simulate classical communication among the players. For $N = 2$, No-Signalling amounts to the set of linear constraints

$$\begin{aligned} \sum_{y_2 \in Y_2} p(y_1, y_2 | x_1, x_2) &= \sum_{y_2 \in Y_2} p(y_1, y_2 | x_1, x'_2) \quad \forall x_2, x'_2 \in X_2 \\ \sum_{y_1 \in Y_1} p(y_1, y_2 | x_1, x_2) &= \sum_{y_1 \in Y_1} p(y_1, y_2 | x'_1, x_2) \quad \forall x_1, x'_1 \in X_1. \end{aligned} \quad (2)$$

For $N \geq 2$, no-signalling is imposed by partitioning the N players into two disjoint groups and by imposing the above equations for all possible bipartitions.

Principles like Non-Trivial Communication Complexity [6, 7], No-Advantage in Nonlocal Computation [8], Information Causality [9], Macroscopic Locality [10] and Local Orthogonality [11] are also examples of restrictions about input-output probability distributions. For example, Non-Trivial Communication Complexity is the requirement that the probability distribution $p(\mathbf{y}|\mathbf{x})$ should not allow two players to compute arbitrary Boolean functions with a single bit of classical communication.

Treating $p(\mathbf{y}|\mathbf{x})$ as a black box also allows for an interesting connection with the framework of interactive proof systems [49], as highlighted in Ref. [50, 51]. In short, one regards the N players as N untrusted provers and the referee as a verifier, with the communication between provers and verifier restricted to be classical. In this context, different physical principles represent different constraints on the power of the provers. Starting from the seminal work by Raz [52], the no-signalling constraint has been studied extensively [53, 54, 55, 56]. It is natural to expect that also other information-theoretic principles, such as Non-Trivial Communication Complexity or Information Causality, may have interesting consequences for interactive proof systems.

2.3. Characterizing quantum correlations

The original scope of nonlocal games is the study of quantum correlations. Here one imagines a scenario where N parties prepare N quantum systems in a joint quantum state and the i -th party generates the output y_i by performing a measurement on the i -th system, choosing the measurement settings according to the input x_i . In this scenario, the probability distribution $p(\mathbf{y}|\mathbf{x})$ has the form

$$p(\mathbf{y}|\mathbf{x}) = \text{Tr} \left[\left(P_{y_1}^{(1,x_1)} \otimes P_{y_2}^{(2,x_2)} \otimes \dots \otimes P_{y_N}^{(N,x_N)} \right) \rho \right], \quad (3)$$

where ρ is the quantum state of the N systems and $\left\{ P_{y_i}^{(i,x_i)} \right\}_{y_i \in \mathcal{Y}_i}$ is the Positive Operator-Valued Measure² (POVM) describing the measurement performed by the i -th party upon receiving the input x_i .

Input-output distributions that are generated as in Eq. (3) are called *quantum*. For given N and given input/output alphabets, the set of quantum input-output distributions is convex. Hence, characterizing it is equivalent to characterizing the maximum payoffs achieved by strategies of the form (3) in all possible games. Since the payoff ω in Eq. (1) can be viewed as a *correlation*, the problem of characterizing the maximum payoffs is often referred to as the problem of *characterizing the set of quantum correlations*.

²We recall that a POVM with outcomes in \mathcal{Y} is defined as a collection of non-negative operators $\{P_y\}_{y \in \mathcal{Y}}$ satisfying the normalization condition $\sum_{y \in \mathcal{Y}} P_y = I$, I being the identity on the system's Hilbert space.

Clearly, the definition of “quantum input-output distribution” does refer to the way the probability distribution is generated. It does in two ways:

1. it prescribes that the $p(\mathbf{y}|\mathbf{x})$ is generated by a specific operational procedure (preparing a multipartite state and performing local measurements)
2. it specifies the physical theory (quantum theory) in which the procedure is implemented.

Now the question is: Can we characterize the set of quantum correlations through device-independent principles? The principles proposed so far [5, 6, 7, 8, 9, 10, 11] are important milestones towards the achievement of this goal. However, a complete characterization of the quantum set solely in terms of conditional probability distributions appears to be challenging [12].

3. The framework of operational-probabilistic theories

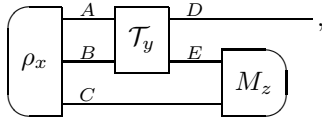
Ultimately, the maximum payoff that the players can win in a non-local game depends on the physical theory that underlies its implementation. Constraints on the physical theory imply constraints on the conditional probability distributions $p(\mathbf{y}|\mathbf{x})$ that the players can generate. For example, the no-signalling conditions of Eq. (2) are often motivated by a space-time scenario where physical systems travel at a bounded speed and the players are far enough from one another that no signal can be exchanged among them during a run of the game.

Among all possible theories, classical and quantum theories are the two prominent examples, due to their central role in physics. However, in order to understand what is specific about these two theories and to explore future generalizations, it is convenient to step back from their specific details and to place them in the wider context of general probabilistic theories [24, 34, 33, 35, 36, 37, 20, 38, 21, 39, 40]—see also the contributed volume [57] for an introduction to the different frameworks. Among the available frameworks, here we adopt the framework of *operational-probabilistic theories (OPTs)* [37, 20, 40, 58], which extends the language of quantum circuits [59, 60] to arbitrary physical theories, combining the categorical framework initiated by Abramsky and Coecke [61, 62, 32] with the toolbox of elementary probability theory. An informal summary of the OPT framework is provided in the following subsections. For a more formal exposition we direct the reader to Ref. [40]. For more discussion on the physical assumptions at the basis of the OPT framework we recommend Hardy’s recent works [21, 39, 63], which adopt a closely related framework and provide a number of enlightening comments on the relation between the operational and the theoretical level.

200 3.1. Operational structure

An *OPT* [37] describes the operations that an agent can perform on physical systems. The theory specifies a catalog of (generally non-deterministic) devices that the agent can compose with each other: each device transforms an input system into an output system, generally in a stochastic way, producing a random

represents a state of the composite system $A \otimes B$. Devices can be connected in parallel and in series, giving rise to circuits, such as



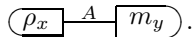
or, in equation $(\mathcal{I}_D \otimes M_z)(\mathcal{T}_y \otimes \mathcal{I}_C)\rho_x$. Circuits in an operational-probabilistic theory obey the same rules as circuits in quantum information. In fact, these rules are already encapsulated in the graphical language used to represent them, whose foundation lies in the theory of strict symmetric monoidal categories [65, 66]. The idea that the definition of a physical theory should be based on (strict) symmetric monoidal categories was introduced by Abramsky and Coecke [61, 62]. A discussion of this idea, along with a comprehensive exposition of the categorical framework can be found in Coecke’s review [32].

3.2. Probabilistic structure

When a preparation device with ensemble $\{\rho_x\}_{x \in X}$ is connected to a measurement device $\{m_y\}_{y \in Y}$, the joint probability distribution of the outcomes is written as

$$p(x, y) = (m_y | \rho_x), \quad (4)$$

and is identified with the diagram



Note that this is the *joint* probability distribution that the preparation device gives the random outcome x and the measurement device gives the outcome y . Accordingly, it is normalized as

$$\sum_{x \in X} \sum_{y \in Y} p(x, y) = 1.$$

Once probabilities are introduced, the sets of states, effects, and transformations inherit a linear structure⁵, so that we can think of each state, effect, or transformation as an element of a suitable vector space [37, 40]. By construction, the action of a transformation on states and effects is linear and, in particular, states (effects) are linear functionals on effects (states) (see paragraph II.F of Ref. [37] and paragraph 2.3.3 of Ref. [40] for details).

Quantum theory can be cast in the framework of OPTs as a special example. Here systems are described by Hilbert spaces. A preparation device is described

⁵The linear structure is obtained through an operation of quotient, which consists in identifying transformations that give the same probabilities in all possible circuits. Note that the assumption of convexity is not made here: the OPT framework can also be used to describe theories with non-convex state spaces, such as Spekkens’ toy theory [67].

by an ensemble $\{\rho_x\}_{x \in X}$ of unnormalized density matrices, acting on the system's Hilbert space and satisfying the condition $\sum_x \text{Tr}[\rho_x] = 1$. A measurement device is described by a positive operator-valued measure (POVM), namely a collection $\{P_y\}_{y \in Y}$ of non-negative operators satisfying the condition

$$\sum_y P_y = I, \quad (5)$$

where I is the identity operator on the system's Hilbert space. The pairing between states and effects is given by the Born rule

$$(P_y | \rho_x) := \text{Tr}[P_y \rho_x]. \quad (6)$$

A test with non-trivial input and output is a *quantum instrument* [68], i. e. a collection of completely positive, trace non-increasing linear maps $\{\mathcal{T}_y\}_{y \in Y}$, transforming operators on the input system's Hilbert space into operators on the output system's Hilbert space and satisfying the condition that the map $\sum_{y \in Y} \mathcal{T}_y$ is trace-preserving. Classical theory can also be represented in this way, by choosing density matrices and POVM operators that are diagonal in a fixed basis, and quantum instruments that transform diagonal operators into diagonal operators.

3.3. Coarse-graining

A key notion that comes with the probabilistic structure is the notion of *coarse-graining*: for a test $\mathcal{T} = \{\mathcal{T}_y\}_{y \in Y}$ one can decide to identify some outcomes, thus obtaining another, coarse-grained test. Mathematically, a coarse-graining is defined by a partition of the outcome set Y into mutually disjoint subsets $\{Y_z\}_{z \in Z}$. The coarse-grained test is the test $\mathcal{T}' = \{\mathcal{T}'_z\}_{z \in Z}$ defined by

$$\mathcal{T}'_z := \sum_{y \in Y_z} \mathcal{T}_y. \quad (7)$$

Note that the summation is well-defined because transformations are elements of a vector space (cf. paragraph II.F of Ref. [37] and paragraph 2.3.3 of Ref. [40]).

4. Physical modelling of non-local games

The OPT framework can be naturally applied to the study of nonlocal games. A strategy in a nonlocal game can be modelled as follows:

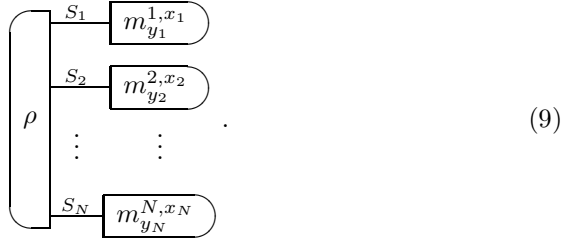
1. The correlations shared by the N players are modelled by a joint state ρ of N systems S_i , $i = 1, \dots, N$, with system S_i in possession of the i -th player. Here we restrict the attention to states ρ that can be prepared deterministically, that is, to states generated by a preparation device with only one possible outcome.

2. Upon receiving the input x_i from the referee, the i -th player will produce an output by performing a measurement on system S_i . Note that in this broad context, “measurement” can be any process that produces a classical output y_i given the input x_i and the state of the system. Even evaluating a function of x_i on a computer and reading the result on the screen would count as a “measurement”.

Let us denote by $\mathbf{m}^{i,x_i} := \{m_{y_i}^{i,x_i}\}_{y_i \in Y_i}$ the measurement performed by the i -th player upon receiving input x_i . The conditional probability distribution $p(\mathbf{y}|\mathbf{x})$, generated by the measurements of all players is then given by

$$p(\mathbf{y}|\mathbf{x}) = (m_{y_1}^{1,x_1} \otimes m_{y_2}^{2,x_2} \otimes \cdots \otimes m_{y_N}^{N,x_N} | \rho), \quad (8)$$

and corresponds to the diagram



For brevity, we will often use the notation $m_{\mathbf{y}}^{\mathbf{x}}$ to denote the product effect

$$m_{\mathbf{y}}^{\mathbf{x}} := m_{y_1}^{1,x_1} \otimes m_{y_2}^{2,x_2} \otimes \cdots \otimes m_{y_N}^{N,x_N}. \quad (10)$$

Accordingly, we will write Eq. (8) in the compact form

$$p(\mathbf{y}|\mathbf{x}) = (m_{\mathbf{y}}^{\mathbf{x}} | \rho), \quad (11)$$

Once a physical theory has been specified, the goal of the players is to find the best state ρ and the best measurements that maximize the expected payoff ω , given by Eq. (1). For a given theory \mathbb{T} , we denote by $\omega_{\mathbb{T}}$ the maximum payoff that can be obtained by optimizing over all possible states and measurements allowed in \mathbb{T} .

5. Causality, no-signalling, and conditional tests

In general, the probability distribution $p(\mathbf{y}|\mathbf{x})$ in Eq. (8) can allow for signalling. In the framework of operational-probabilistic theories, No-Signalling is imposed by the Causality principle, stating that the probability of an outcome at a given step in a circuit is independent of the choice of tests performed at later steps. Precisely, the principle can be stated as follows:

Definition 1 (Causality [37, 20]). *A theory satisfies causality iff for every system S , every preparation-test $\rho = \{\rho_x\}_{x \in X}$ for system S , and every two*

measurements $\mathbf{m}^0 = \{m_{y_0}^0\}_{y_0 \in Y_0}$ and $\mathbf{m}^1 = \{m_{y_1}^1\}_{y_1 \in Y_1}$ on system S the conditional probability distributions $p(x, y_z | z) := (m_{y_z}^z | \rho_x)$ satisfy the condition

$$\sum_{y_0 \in Y_0} p(x, y_0 | 0) = \sum_{y_1 \in Y_1} p(x, y_1 | 1) \quad \forall x \in X. \quad (12)$$

Informally, Eq. (12) expresses a condition of No-Signalling from the future: the (marginal) probability of a preparation does not depend on the choice of measurement.

Causality is equivalent to the requirement that for every system S there exists a unique effect u_S , called the *unit effect*⁶, such that

$$\sum_{y \in Y} m_y = u_S \quad (13)$$

for every measurement $\{m_y\}_{y \in Y}$ on S . When there is no ambiguity, we will drop the subscript from u_S . In quantum theory, u_S is the identity operator on the Hilbert space of the system and Eq. (13) expresses the fact that quantum measurements are resolutions of the identity [cf. Eq. (5)].

Causality is equivalent to the statement that for every system S there exists a unique *deterministic* effect $u_S \in \text{Eff}_1(S)$ [37]. In categorical terms, this condition is the terminality of the tensor unit (the trivial system, in our language) and defines a special class of categories called *causal categories* [69, 70].

5.1. Causality and No-Signalling

Causality implies that the probability distributions $p(\mathbf{y}|\mathbf{x})$ generated by local measurements as in Eq. (8) satisfy the no-signalling condition (cf. theorem 1 of Ref. [37] and theorem 5.1 of Ref. [70]). In fact, under a minimalistic assumption, Causality is *equivalent* to the request that all the probability distributions of the form of Eq. (9) are no-signalling. The assumption is that every ensemble of states can be generated by performing a measurement on one side of a bipartite state:

Assumption 1 (cf. Axiom 2 of [40]). *For every system A and for every ensemble $\{\rho_x\}_{x \in X}$, describing a random preparation of A , there exists a system B , a deterministic state $\sigma \in \text{St}_1(A \otimes B)$ and a measurement $\{b_x\}_{x \in X}$ such that*

$$\rho_x \text{---} A = \left(\begin{array}{c} \sigma \\ \text{---} B \end{array} \right) \text{---} b_x \quad \forall x \in X. \quad (14)$$

⁶We adopt this terminology to facilitate the comparison of our framework with the convex set framework [24, 34, 38], where the existence and uniqueness of the unit effect—and therefore the validity of Causality—is built in.

This assumption is so natural that could even be included in the definition of OPT: indeed, one can think of system B in Eq. (14) as the physical support that carries the classical information about the outcome x —information which is read-out by performing the measurement $\{b_x\}_{x \in \mathbf{X}}$. If Eq. (14) were not to hold, we could not represent the outcome x as information carried by an actual physical system. Note that this observation applies not only to ensembles of states, but also to generic tests with non-trivial input and non-trivial output.

Under the validity of Assumption 1, Causality and No-Signalling are equivalent:

Proposition 1. *For every theory satisfying Assumption 1, the following conditions are equivalent*

1. *the theory is causal*
2. *every input-out probability distribution $p(\mathbf{y}|\mathbf{x})$ generated as in Eq. (8) is no-signalling.*

The proof is rather elementary and is provided in Appendix A. As a consequence, maximizing the payoff of a nonlocal game over all possible theories that satisfy Causality is equivalent to maximizing the payoff ω in Eq. (1) over all possible conditional distributions that satisfy No-Signalling. The relation between Causality and No-Signalling has recently played an important role in the study of network scenarios inspired by Pearl’s notion of causal networks [71, 72] and of the entropic relations implied by causal networks in operational-probabilistic theories [73].

5.2. Causality and conditional tests

Thanks to Causality, one can use the information gained in the past to decide which tests are performed in the future, thus implementing *conditional tests*. Conditional tests are defined as follows: If $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbf{X}}$ is a test with input A and output B and, for every x , $\mathcal{S}^x = \{\mathcal{S}_y^x\}_{y \in \mathbf{Y}}$ is a test with input B and output C for every x , then the conditional test $\{\mathcal{S}_y^x \mathcal{T}_x\}_{x \in \mathbf{X}, y \in \mathbf{Y}}$ is the test that results from performing the test \mathcal{T} and, conditionally on outcome x , the test \mathcal{S}^x , as in the the diagram

$$-A \boxed{\mathcal{T}_x} B \boxed{\mathcal{S}_y^x} C-$$

Causality guarantees that the collection of transformations $\{\mathcal{S}_y^x \mathcal{T}_x\}_{x \in \mathbf{X}, y \in \mathbf{Y}}$ can be included among the tests allowed by the theory without generating contradictions [37]. Since they *can* be included, one may as well assume that they *are* included, which amounts to the following

Assumption 2. *For every test $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbf{X}}$ of type $A \rightarrow B$ and for every set of tests $\mathcal{S}^x = \{\mathcal{S}_y^x\}_{y \in \mathbf{Y}}$, $x \in \mathbf{X}$, of type $B \rightarrow C$, the collection of transformations $\{\mathcal{S}_y^x \mathcal{T}_x\}_{x \in \mathbf{X}, y \in \mathbf{Y}}$ is a test of type $A \rightarrow C$.*

Quite importantly, Assumption 2 *implies* Causality (cf. lemma 7 of Ref. [37]): only in a causal world the agent can freely choose future tests depending on the outcomes of previous ones. From now on, we will assume Assumption 2 and convexity as part of the Causality package coming: by Causal theory, we will mean a theory satisfying Assumption 2.

6. Spiky measurements

In this section we define a privileged class of measurements, which we call *spiky measurements*. In Quantum Theory, spiky measurements coincide with projective measurements, i.e. measurements consisting of projectors on a complete set of orthogonal subspaces.

6.1. Purity

A *pure transformation* \mathcal{P} is a transformation that cannot be obtained from the coarse-graining of two different transformations \mathcal{P}_1 and \mathcal{P}_2 : precisely, the transformation \mathcal{P} is pure iff one has

$$\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 \implies \mathcal{P}_1 = p\mathcal{P}, \quad \mathcal{P}_2 = (1-p)\mathcal{P}, \quad p \in [0, 1].$$

Intuitively, the pure transformations are those for which the evolution of the system is known with the maximal accuracy allowed by the theory. In quantum theory, the pure transformations are those with a single Kraus operator, i.e. those of the form $\mathcal{P}(\rho) = M\rho M^\dagger$, for some operator M satisfying $M^\dagger M \leq I_S$, I_S being the identity on the system's Hilbert space.

As a particular case of pure transformations, one can consider pure states and pure effects. A pure state is just a pure transformation with trivial input. A pure effect is a pure transformation with trivial output. In quantum theory, pure states and pure effects are proportional to rank-one projectors. Using the notion of pure effect, it is natural to define pure measurements:

Definition 2. *We say that a measurement \mathbf{m} is pure iff it consists of pure effects.*

Intuitively, a pure measurement extracts information in a way that cannot be further refined. For example, for a three-level quantum system, the computational basis measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|, |2\rangle\langle 2|\}$ is pure, while the two-outcome projective measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1| + |2\rangle\langle 2|\}$ is not pure, since it can be obtained from the former by coarse-graining.

6.2. Orthogonality

In addition to purity, another desirable feature of measurements is orthogonality. We say that a measurement is orthogonal if it can perfectly distinguish among the states in a given set:

Definition 3. A measurement on system S , say $\mathbf{m} = \{m_y\}_{y \in \mathcal{Y}}$, is orthogonal iff there exists a set of states, say $\{\rho_y\}_{y \in \mathcal{Y}}$, such that

$$(m_y | \rho_{y'}) = \delta_{y,y'} \quad \forall y, y' \in \mathcal{Y}. \quad (15)$$

This notion of orthogonality can be easily extended to sets of effects that do not necessarily form a measurement:

Definition 4 (Orthogonality of states and effects). A set of effects $\{m_y\}_{y \in \mathcal{Y}} \subset \text{Eff}(S)$ and a set of states $\{\rho_y\}_{y \in \mathcal{Y}} \subset \text{St}(S)$ are biorthogonal iff

$$(m_y | \rho_{y'}) = \delta_{y,y'}$$

for every $y, y' \in \mathcal{Y}$. A set of effects $\{m_y\}$ is orthogonal iff there exists a set of states $\{\rho_y\}$ such that the two sets are biorthogonal. A set of states $\{\rho_y\}$ is orthogonal, iff there exists a set of effects $\{m_y\}$ such that the two sets are biorthogonal.

The familiar example of Quantum Theory should not mislead the reader. In this paper we do *not* define orthogonal states as states that can be perfectly distinguished by a measurement. Distinguishability implies orthogonality, but in general the converse does not hold: if the states $\{\rho_y\}_{y \in \mathcal{Y}}$ are orthogonal, this only means that there exist effects $\{m_y\}_{y \in \mathcal{Y}}$ such that $(m_y | \rho_{y'}) = \delta_{y,y'}$, but in general the effects $\{m_y\}_{y \in \mathcal{Y}}$ may not form a measurement⁷. Nevertheless, orthogonality and distinguishability are equivalent notions for *pairs* of states:

Proposition 2. Two states ρ_0 and ρ_1 are orthogonal if and only if they are perfectly distinguishable.

The proof is elementary and is provided in Appendix B. The above proposition shows that orthogonality for pairs of states is a very special notion.

Note that pairwise orthogonality does not imply orthogonality: The condition that two states ρ_y and $\rho_{y'}$ are orthogonal for every $y, y' \in \mathcal{Y}$, $y \neq y'$ is not enough to guarantee that the states $\{\rho_y\}_{y \in \mathcal{Y}}$ are orthogonal. The canonical counterexample is the *square bit* [74, 75], discussed in the following:

Example 1 (The square bit). Consider a physical system whose deterministic states form a square. Suppose that the measurements are represented as positive affine functionals summing up to the functional that gives 1 on every point of the square⁸.

⁷Recall that the set of measurements is part of the specification of the theory.

⁸For the purpose of this example, we only need to declare the states and the effects of the system. We omit the specification of the full OPT in which the square bit lives. As a matter of fact, there are many different OPTs that contain “square bits” among their systems. For example, consider an OPT where the systems are composite systems of square bits, the states are convex combinations of product states, the measurements are those that can be implemented by (coarse-grainings of) measurements on individual square bits, and the general tests are those of the “measure-and-prepare” form, i.e. those consisting on measuring the input system and preparing an output state depending on the outcome of the measurement.

The square bit has four pure states and four pure effects, given by the vectors

$$|\varphi_y\rangle = \begin{pmatrix} r \cos(2\pi y/4) \\ r \sin(2\pi y/4) \\ 1 \end{pmatrix}$$

$$(a_y| = \frac{1}{2} \begin{pmatrix} r \cos[(2y-1)\pi/4] \\ r \sin[(2y-1)\pi/4] \\ 1 \end{pmatrix}$$

$$x = 1, 2, 3, 4 \quad r = 2^{1/4},$$

respectively. The probabilities are given by the scalar product of vectors, yielding

$$(a_y|\varphi_y\rangle = (a_{y\oplus 1}|\varphi_y\rangle = 1$$

$$(a_{y\oplus 2}|\varphi_y\rangle = (a_{y\oplus 3}|\varphi_y\rangle = 0 \quad \forall y = 1, 2, 3, 4 \quad (16)$$

where \oplus denotes the addition modulo 4. Here there are two pure measurements, namely $\{a_1, a_3\}$ and $\{a_2, a_4\}$. Indeed, it is easy to check that $a_1 + a_3 = a_2 + a_4 = u$, where u is the deterministic effect

$$(u| \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

giving probability 1 on every pure state. It is not hard to see that the four pure states $\{\varphi_y\}_{y=1}^4$ are pairwise orthogonal, but not orthogonal (ad, therefore, not perfectly distinguishable). Similarly, the four effects $\{a_y\}_{y=1}^4$ are pairwise orthogonal, but not orthogonal.

Finally, note that two orthogonal effects, as defined in Definition 4, may not coexist in a measurement. An easy counterexample can be found in quantum theory. Consider the two projectors $P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $P_2 = |0\rangle\langle 0| + |2\rangle\langle 2|$. The two projectors correspond to orthogonal effects in the sense of Definition 4: indeed, there exist two states $\rho_1 = |1\rangle\langle 1|$ and $\rho_2 = |2\rangle\langle 2|$ such that $\text{Tr}[P_i \rho_j] = \delta_{i,j}$, for every i and j in $\{1, 2\}$. However, P_1 and P_2 cannot coexist in the same measurement, because for $\rho_0 = |0\rangle\langle 0|$ one has $\text{Tr}[P_1 \rho_0] + \text{Tr}[P_2 \rho_0] = 2$, in contradiction with the normalization of probabilities.

6.3. Purity plus orthogonality

We are now ready to define the notion of *pure orthogonal measurement*:

Definition 5. A pure and orthogonal measurement is an orthogonal measurement consisting of pure effects.

In Quantum Theory, the pure orthogonal measurements are the measurements consisting on rank-one projectors on the vectors of an orthonormal basis. Pure orthogonal measurements featured in a recent work [76], where the authors explored different inequivalent notions of dimension of a physical system. In this

work, the (maximum) number of outcomes in a pure orthogonal measurement was called the *measurement dimension* of the system.

To appreciate the meaning of Definition 5 outside the quantum context, it is worth having a look at the square bit of Example 1. Here, each of two pure measurements $\{a_1, a_3\}$ and $\{a_2, a_4\}$ is orthogonal: for example, $\{a_1, a_3\}$ allows one to distinguish perfectly between the two states ρ_1 and ρ_3 defined as

$$\rho_1 = p\varphi_1 + (1-p)\varphi_4 \quad \rho_3 = q\varphi_3 + (1-q)\varphi_2,$$

where p and q are arbitrary probabilities. Note that here a pure effect can give probability 1 on a mixed state. In this respect, the square bit differs radically from the quantum bit, where a pure effect can give probability 1 on one and only one pure state. The one-to-one correspondence between pure states and effects is a non-trivial property, which played an important role in several reconstructions of Quantum Theory [77, 21, 20, 23, 22, 28, 27] and will also play a role in the present paper.

6.4. Spiky measurements

We are now ready to define the set of spiky measurements:

Definition 6 (Spiky measurement). *A measurement $\mathbf{m} = \{m_y\}_{y \in Y}$ is spiky⁹ iff it is the coarse-graining of a pure orthogonal measurement, i.e. iff there exists a pure orthogonal measurement $\mathbf{a} = \{a_z\}_{z \in Z}$ and a partition of the outcome set Z into disjoint subsets $\{Z_y\}_{y \in Y}$ such that*

$$(m_y| = \sum_{z \in Z_y} (a_z|.$$

The above definition of “spiky” measurements is equivalent to the definition of “sharp” measurements by Barnum, Müller, and Ududec [78]. In this paper we prefer to avoid the term “sharp”, because we would like to reserve it for measurements that are repeatable and minimally disturbing [1], this being a property traditionally associated to sharp measurements in quantum theory [79, 80]. Admittedly, the choice of terminology is mostly a matter of taste here, since in quantum theory the two definitions coincide and single out set of projective quantum measurements.

7. Axioms

Here we present three requirements about spiky measurements. These three requirements are satisfied by both classical and quantum theory and imply the validity of Causality and Local Orthogonality.

⁹Here the “spikes” are the pure orthogonal effects a_z , $z \in Z$.

7.1. Measurement Purification

Measurement Purification is the statement that every measurement can be reduced to a spiky measurement performed jointly on the system and on an environment:

Axiom 1 (Measurement Purification). *For every system S and for every measurement on system S —say $\mathbf{m} = \{m_y\}_{y \in Y}$ —there is another system E , a state $\sigma \in \text{St}(E)$, and a spiky measurement $\mathbf{M} = \{M_y\}_{y \in Y}$ such that*

$$\text{---}^S \boxed{m_y} = \begin{array}{c} \text{---}^S \\ \boxed{\sigma} \text{---}^E \end{array} \boxed{M_y} \quad \forall y \in Y.$$

Roughly speaking, one can think of the above axiom as an operational version of Naimark’s theorem for finite dimensional quantum systems [81, 82], which states that every quantum measurement can be dilated to a projective measurement performed jointly on the system and on an environment¹⁰.

The idea that arbitrary measurements can be reduced to ideal measurements by introducing an environment immediately reminds of the Purification Principle [37, 20, 83, 84, 40], which states that arbitrary states can be reduced to pure states by adding an environment. In this sense, the spirit of this paper is akin to the “purification philosophy” of Refs. [37, 20, 83, 84, 40] namely the idea that all physical processes can be reduced to ideal processes by including additional systems into the description. The interaction with an environment is a powerful structure also in the abstract framework of categorical quantum mechanics [85, 86], where it leads to an axiomatization of Selinger’s CPM construction [87].

We now give an elementary consequence of measurement purification that will be useful later:

Lemma 1. *Let $\{\mathbf{m}^x\}_{x \in X}$ be a finite set of measurements labelled by a parameter $x \in X$. If measurement purification holds, then there exists a system E , a state $\sigma \in \text{St}(E)$, and set of spiky measurements $\{\mathbf{M}^x\}_{x \in X}$ such that*

$$\text{---}^S \boxed{m_y^x} = \begin{array}{c} \text{---}^S \\ \boxed{\sigma} \text{---}^E \end{array} \boxed{M_y^x} \quad \forall y \in Y \forall x \in X.$$

The proof is provided in Appendix C. Compared to the Measurement Purification axiom, the above lemma only adds the fact that the system E and the state $\sigma \in \text{St}(E)$ can be chosen to be independent of the setting $x \in X$, the dependence on the setting being only in the orthogonal measurement \mathbf{M}^x .

7.2. Locality of Pure Orthogonal Measurements

Measurement Purification can be interpreted as the statement that, at the most fundamental level, measurements are generated by the coarse-graining of

¹⁰Note that Naimark’s theorem also includes the fact that the state of the environment is *pure* and that the dilation is *unique*, up to partial isometries. These two additional facts are also important, but not for the purpose of the present paper.

pure orthogonal measurements. If one pushes this requirement further, it is natural to ask that the product of two pure orthogonal measurements is a pure orthogonal measurement on the composite system:

Axiom 2 (Locality of Pure Orthogonal Measurements). *If $\mathbf{m} = \{m_x\}$ and $\mathbf{n} = \{n_y\}$ are pure orthogonal measurements on two systems A and B , respectively, then their product $\mathbf{m} \otimes \mathbf{n} = \{m_x \otimes n_y\}$ is a pure orthogonal measurement on the composite system $A \otimes B$.*

Two comments are in order:

1. Locality of Pure Orthogonal Measurement may superficially look as a consequence of the definition. And in part it is: clearly, the product of two orthogonal measurements is orthogonal. The part that does not follow from the definition is the purity of the product effects. This condition would be guaranteed by Local Tomography [24, 33, 34], which is *not* assumed here. The Locality of Pure Orthogonal Measurements is much weaker condition than Local Tomography: for example, it is satisfied by Quantum Theory on real Hilbert spaces [88, 89, 90], a well-known example of theory wherein Local Tomography fails to hold.
2. If one postulates Measurement Purification, then it is natural to assume the Locality of Pure Orthogonal Measurements as well. Indeed, suppose that two distant parties, Alice and Bob, perform two pure orthogonal measurements \mathbf{m} and \mathbf{n} on their systems. By Measurement Purification, we know that the product measurement $\mathbf{m} \otimes \mathbf{n}$ can be reduced to a pure orthogonal measurement—call it \mathbf{M} —performed jointly on A , B , and an environment E . If the measurement \mathbf{M} could not be chosen of the product form, it would mean that the measurements that are performed independently by Alice and Bob would require some nonlocal interaction at the fundamental level.

7.3. Sufficient Orthogonality

Here we introduce *Sufficient Orthogonality (SO)*, a structural property of the measurements allowed by the theory. We do not attach a particular operational meaning to this property, e.g. we do not argue that this should be a fundamental principle of physics. Nevertheless, we show that SO plays a key role, allowing one to derive LO and CE. We view SO as an intermediate step, which can be used to reduce LO and CE to other, more fundamental features of physical processes, such as the *Strong No Disturbance Without Information* principle discussed in Section 10.

Axiom 3 (Sufficient Orthogonality). *Every set of pure orthogonal effects can coexist in a measurement, i.e. for every set of pure orthogonal effects $\{a_y\}_{y \in Y}$ there exists a measurement \mathbf{m} such that $\{a_y\}_{y \in Y} \subseteq \mathbf{m}$.*

In the statement of SO it is essential that the effects $\{p_y\}$ are *pure*, otherwise one can find counterexamples even in classical and quantum theory. For example, the non-pure effects $P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $P_2 = |0\rangle\langle 0| + |2\rangle\langle 2|$ are

500 orthogonal, because $\text{Tr}[P_i \rho_j] = \delta_{i,j}$ for $i, j = 1, 2$ and $\rho_j = |j\rangle\langle j|$. However, they cannot coexist in a quantum measurement, since $P_1 + P_2 > I$.

SO is satisfied by both classical and quantum probability theory, where a set of pure orthogonal effects is a set of rank-one projectors $\{P_y\}_{y \in Y}$ satisfying $\text{Tr}[P_y P_{y'}] = \delta_{y,y'}$. Interestingly, SO is violated by the square bit of Example 1. More generally, SO is violated by all systems whose state space is a regular polygon of $n > 3$ vertices:

Example 2 (Regular polygons [75]). *Consider an hypothetical physical system S_n whose deterministic states $\text{St}_1(S_n)$ form a regular polygon of n vertices. The vertices of the polygon are the pure states and can be represented by the real vectors*

$$|\varphi_y\rangle = \begin{pmatrix} r_n \cos \frac{2\pi y}{n} \\ r_n \sin \frac{2\pi y}{n} \\ 1 \end{pmatrix} \quad r_n := \sqrt{\frac{1}{\cos(\pi/n)}},$$

for $y = 0, 1, \dots, n-1$. Effects are also represented as real vectors, and the probability of an effect on a state is given by the scalar product. The unit effect, which has probability 1 on every pure state, is represented as

$$(u| = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For the measurements, one typically assumes the no-restriction hypothesis (cf. Ref. [37], definition 16, and Ref. [91], section III), which states that all collections of positive affine functionals summing up to the unit represent allowed measurements¹¹. Under this hypothesis, one has the pure effects

$$(a_y| = \frac{1}{2} \begin{pmatrix} r_n \cos \frac{(2y-1)\pi}{n} \\ r_n \sin \frac{(2y-1)\pi}{n} \\ 1 \end{pmatrix}$$

¹¹We do not specify here the full OPT in which the regular polygon is included. In general, it is easy to include a given system S , its state space, and its set of allowed measurements into a full-blown OPT. For example, one can consider the OPT where all systems consists of multiple copies of S , the states are product states (or convex combination thereof), the measurements are product measurements (or convex combinations thereof), and the general tests are of the measure-and-prepare form. Unless one imposes additional physical constraints, the specification of the OPT in which a given system can be embedded is highly non unique. Still, such a specification is irrelevant for the scopes of the present example.

for even n and

$$(a_y| = \frac{1}{r_n^2 + 1} \begin{pmatrix} r_n \cos \frac{2y\pi}{n} \\ r_n \sin \frac{2y\pi}{n} \\ 1 \end{pmatrix}$$

for odd n . Note that for every y one has $(a_y|\varphi_y) = 1$.

For $n = 3$, the effects $\{a_y\}_{y=1}^3$ can coexist in a measurement, and, therefore, all the three pure states are perfectly distinguishable. This is not surprising, because the triangle is a simplex, simplices represent the states of classical systems, and classical systems satisfy SO.

We now show that the triangle is the only regular polygon satisfying SO. Let us start from the case of even n . Here, the inner product between a pure effect and a pure state is given by

$$(a_y|\varphi_{y'}) = \frac{1}{2} \left\{ r_n^2 \cos \left[\frac{(2y - 2y' - 1)\pi}{n} \right] + 1 \right\}$$

and it is immediate to check that one has

$$\begin{aligned} (a_y|\varphi_y) &= (a_{y \oplus 1}|\varphi_y) = 1 \\ (a_{y \oplus \frac{n}{2}}|\varphi_y) &= (a_{y \oplus \frac{n}{2} \oplus 1}|\varphi_y) = 0 \quad \forall y \in \{0, 1, \dots, n-1\} \end{aligned}$$

where \oplus denotes the addition modulo n . Clearly, the pure effects $\{a_j, a_{j \oplus \frac{n}{2} \oplus 1}\}$ are orthogonal, since they are biorthogonal to the pure states $\{\varphi_j, \varphi_{j \oplus \frac{n}{2}}\}$. However, they cannot coexist in a measurement: by absurd, if they coexisted in a measurement, the total probability of the measurement outcomes on the state $|\varphi_{j \oplus \frac{n}{2} \oplus 1}\rangle$ would exceed one:

$$(a_j|\varphi_{j \oplus \frac{n}{2} \oplus 1}) + (a_{j \oplus \frac{n}{2} \oplus 1}|\varphi_{j \oplus \frac{n}{2} \oplus 1}) = (a_j|\varphi_{j \oplus \frac{n}{2} \oplus 1}) + 1 > 1.$$

Hence, every polygon with even number of vertices violates SO.

For odd number of vertices, the inner products of a pure effect with a pure state is

$$(a_y|\varphi_{y'}) = \frac{1}{r_n^2 + 1} \left\{ r_n^2 \cos \left[\frac{(2y - 2y')\pi}{n} \right] + 1 \right\} \quad (17)$$

and one has

$$\begin{aligned} (a_y|\varphi_y) &= 1 \\ (a_{y \oplus \frac{n-1}{2}}|\varphi_y) &= (a_{y \oplus \frac{n+1}{2}}|\varphi_y) = 0 \quad \forall y \in \{0, 1, \dots, n-1\}. \end{aligned}$$

Clearly, the two effects $\{a_y, a_{y \oplus \frac{n+1}{2}}\}$ are orthogonal, as they are biorthogonal to the states $\{\varphi_y, \varphi_{y \oplus \frac{n+1}{2}}\}$. However, they cannot coexist in a measurement, because the sum of their probabilities on the state $|\varphi_{y \oplus 1}\rangle$ exceeds one, as shown in Appendix D. In summary, the only regular polygon compatible with SO is the triangle, representing a three-level classical system.

8. Local Orthogonality

Here we briefly discuss LO, a requirement on the conditional probability distributions $p(\mathbf{y}|\mathbf{x})$ generated by N players of a non-local game. To state the requirement, it is handy to introduce a notation for the output/input pairs $\mathbf{e} = (\mathbf{x}, \mathbf{y})$, which will be called *events*.

Definition 7 (Locally orthogonal events). *Two events $\mathbf{e} = (\mathbf{x}, \mathbf{y})$ and $\mathbf{e}' = (\mathbf{x}', \mathbf{y}')$ are locally orthogonal, denoted as $\mathbf{e} \perp \mathbf{e}'$, if there exists at least one party i such that $x_i = x'_i$ and $y_i \neq y'_i$. A set of events \mathcal{O} is locally orthogonal if every two elements in \mathcal{O} are locally orthogonal.*

For an event $\mathbf{e} = (\mathbf{x}, \mathbf{y})$, we use the notation $p(\mathbf{e}) := p(\mathbf{y}|\mathbf{x})$. With this notation, LO is defined as follows

Definition 8 (Local Orthogonality [11, 17]). *A conditional probability distribution $p(\mathbf{y}|\mathbf{x})$ satisfies Local Orthogonality iff one has*

$$\sum_{\mathbf{e} \in \mathcal{O}} p(\mathbf{e}) \leq 1 \tag{18}$$

for every locally orthogonal set \mathcal{O} . A theory satisfies LO iff every probability distribution generated as in Eq. (11) satisfies LO.

In a bipartite setting, LO is equivalent to No-Signalling [11, 17]. LO comes to its own in the multipartite setting, where Eq. (18) is more restrictive than the No-Signalling condition. In a theory satisfying LO, the maximum payoff achievable by the players of a generic game is upper bounded as

$$\omega_{\mathbb{T}} \leq \omega_{LO}, \tag{19}$$

where ω_{LO} denotes the maximum of the payoff ω in Eq. (1) over *all* probability distributions $p(\mathbf{y}|\mathbf{x})$ satisfying LO.

Note that LO has a slightly different flavour from other device-independent principles. Indeed, principles like Nontrivial Communication Complexity, No Advantage in Nonlocal Computation and Information Causality are expressed as limitations about some distinguished information-theoretic task. Such limitations are subsequently used to derive upper bounds on the payoffs of nonlocal games. Instead, LO is *defined* as an upper bound on a payoff, as one can see by comparing the l.h.s. of Eq. (18) with the r.h.s. of Eq. (1). The particular games that define the LO constraint have been characterized in Ref. [17] and have been therein named *maximally difficult guessing games*. In this sense, Eq. (19) represents the upper bound on the payoff of a generic game under the condition that the payoff in some privileged class of games is upper bounded as in Eq. (18).

LO can be generalized to an infinite hierarchy of constraints [11, 17]. This is done as follows: Suppose that the N parties are given k copies of the black box

generating outputs according to the conditional probability distribution $p(\mathbf{y}|\mathbf{x})$. As a result, the overall input-output distribution will be given by

$$p^{\otimes k}(\mathbf{y}_1\mathbf{y}_2 \dots \mathbf{y}_k|\mathbf{x}_1\mathbf{x}_2 \dots \mathbf{x}_k) = p(\mathbf{y}_1|\mathbf{x}_1)p(\mathbf{y}_2|\mathbf{x}_2) \dots p(\mathbf{y}_k|\mathbf{x}_k).$$

Defining the event $\mathbf{e}_k = (\mathbf{y}_1\mathbf{y}_2 \dots \mathbf{y}_k|\mathbf{x}_1\mathbf{x}_2 \dots \mathbf{x}_k)$ and its probability $p^{\otimes k}(\mathbf{e}_k) := p^{\otimes k}(\mathbf{y}_1\mathbf{y}_2 \dots \mathbf{y}_k|\mathbf{x}_1\mathbf{x}_2 \dots \mathbf{x}_k)$, one can formulate the k -th level of the LO hierarchy as

Definition 9 (Local Orthogonality at the k -th level [11, 17]). *A conditional probability distribution $p(\mathbf{y}|\mathbf{x})$ satisfies LO at the k -th level iff*

$$\sum_{\mathbf{e}_k \in \mathcal{S}_k} p^{\otimes k}(\mathbf{e}_k) \leq 1 \quad (20)$$

for every locally orthogonal set \mathcal{S}_k . A theory satisfies LO at the k -th level iff every probability distribution generated as in Eq. (11) satisfies LO at the k -th level.

By increasing k , one gets more and more restrictive conditions on the probability distribution $p(\mathbf{y}|\mathbf{x})$. For example, PR box correlations satisfy LO for $k = 1$, but violate it for $k \geq 2$ [11].

9. Deriving Local Orthogonality and Causality

We now provide a derivation of LO from Measurement Purification, Locality of Pure Orthogonal Measurements, and Sufficient Orthogonality. Since LO implies No-Signalling [11, 17] and in our framework No-Signalling is equivalent to Causality (proposition 1 of this paper), our derivation of LO also amounts to a derivation of Causality. The derivation consists of a few steps, discussed in the following paragraphs.

9.1. Local Orthogonality for pure orthogonal measurements

We start by showing the validity of LO for probability distributions generated by pure orthogonal measurements:

Lemma 2. *Let $p(\mathbf{y}|\mathbf{x}) = (m_{\mathbf{y}}^{\mathbf{x}}|\rho)$ be a set of probability distributions defined as in Eq. (11) with the product effect $m_{\mathbf{y}}^{\mathbf{x}}$ arising from a set of pure orthogonal measurements. If the theory satisfies Locality of Pure Orthogonal Measurements and Sufficient Orthogonality, then it $p(\mathbf{y}|\mathbf{x})$ satisfies Local Orthogonality at all levels of the hierarchy.*

Proof. Firstly, we consider the proof for the first level. For every locally orthogonal set of events \mathcal{O} , we have to prove the relation $\sum_{(\mathbf{x},\mathbf{y}) \in \mathcal{O}} p(\mathbf{y}|\mathbf{x}) \leq 1$. The proof runs as follows: First, consider an arbitrary party i and a fixed (but otherwise arbitrary) input value x_i . By hypothesis, the effects $\{m_{y_i}^{x_i}\}_{y_i \in Y_i}$ are

orthogonal, which means that there exists a set of states $\{\rho_{y_i}^{x_i}\}_{y_i \in Y_i} \subset \text{St}(S_i)$ such that

$$\left(m_{y_i}^{x_i} | \rho_{y'_i}^{x_i}\right) = \delta_{y_i, y'_i} \quad \forall y_i, y'_i \in Y_i. \quad (21)$$

Now, define the product states

$$|\rho_{\mathbf{y}}^{\mathbf{x}}\rangle := |\rho_{y_1}^{x_1}\rangle \cdots |\rho_{y_N}^{x_N}\rangle$$

and the product effects

$$(m_{\mathbf{y}}^{\mathbf{x}} | := (m_{y_1}^{x_1} | \cdots (m_{y_N}^{x_N} | .$$

By the Locality of Pure Orthogonal Measurements, the effects $m_{\mathbf{y}}^{\mathbf{x}}$ are pure. With this definition, if two events (\mathbf{x}, \mathbf{y}) and $(\mathbf{x}', \mathbf{y}')$ are locally orthogonal, then one has

$$(m_{\mathbf{y}}^{\mathbf{x}} | \rho_{\mathbf{y}'}^{\mathbf{x}'}\rangle) = 0.$$

By definition, this means that the pure effects $\{m_{\mathbf{y}}^{\mathbf{x}}\}_{(\mathbf{x}, \mathbf{y}) \in \mathcal{O}}$ are orthogonal. Invoking Sufficient Orthogonality, we have that there exists a measurement \mathbf{m} such that $\{m_{\mathbf{y}}^{\mathbf{x}}\}_{(\mathbf{x}, \mathbf{y}) \in \mathcal{O}} \subseteq \mathbf{m}$. Using this fact we obtain

$$\begin{aligned} \sum_{(\mathbf{x}, \mathbf{y}) \in \mathcal{O}} p(\mathbf{y} | \mathbf{x}) &= \sum_{(\mathbf{x}, \mathbf{y}) \in \mathcal{O}} (m_{\mathbf{y}}^{\mathbf{x}} | \rho) \\ &\leq \sum_{\mathbf{e} \in \tilde{\mathcal{O}}} (m_{\mathbf{e}} | \rho) \\ &= 1, \end{aligned}$$

where $\tilde{\mathcal{O}}$ denotes the set of all outcomes of the measurement \mathbf{m} . The above inequality concludes the proof of LO in the case when each party performs a pure orthogonal measurement on one subsystem of a composite system. The argument can be easily extended to prove LO at every level: in this case, one has simply to replace \mathbf{x} and \mathbf{y} with the strings $\mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_k$ and $\mathbf{y}_1 \mathbf{y}_2 \dots \mathbf{y}_k$, respectively. \square

9.2. Local Orthogonality for generic measurements

Having derived LO for pure orthogonal measurements, it is easy to extend the derivation to arbitrary measurements. The strategy is to extend the proof first to spiky measurements (by coarse-graining) and then to arbitrary measurements (by measurement purification). The first step is achieved by the following

Lemma 3. *Let $p(\mathbf{z} | \mathbf{x})$ be a conditional probability distribution of the variable $\mathbf{z} \in \prod_{i=1}^N Z_i$ conditional to the variable $\mathbf{x} \in \prod_{i=1}^N X_i$. Let $p(\mathbf{y} | \mathbf{x})$ be the probability distribution resulting from local coarse-grainings of $p(\mathbf{z} | \mathbf{x})$, that is,*

$$p(\mathbf{y} | \mathbf{x}) = \sum_{\mathbf{z} \in \prod_{i=1}^N Z_{y_i}} p(\mathbf{z} | \mathbf{x}) \quad \forall \mathbf{y} \in \prod_{i=1}^N Y_i,$$

where, for each i , $\{Z_{y_i}\}_{y_i \in \mathcal{Y}_i}$ is a partition of Z_i into disjoint subsets. If the distributions $p(\mathbf{z}|\mathbf{x})$ satisfy LO, then also coarse-grained distributions $p(\mathbf{y}|\mathbf{x})$ satisfy LO.

We omit the proof of the lemma, which can be found in Section V of Ref. [92].

An immediate corollary is the following:

Corollary 1. *Let $p(\mathbf{y}|\mathbf{x}) = (m_{\mathbf{y}}^{\mathbf{x}}|\rho)$ be a set of probability distributions defined as in Eq. (11) with a set of spiky measurements. If the theory satisfies Locality of Pure Orthogonal Measurements and Sufficient Orthogonality, then $p(\mathbf{y}|\mathbf{x})$ satisfies LO at all levels of the hierarchy.*

Combining this observation with Measurement Purification, one can prove the desired result:

Theorem 1. *Every theory that satisfies Axioms 1, 2, and 3 must satisfy LO at every level of the hierarchy.*

Proof. Let us start from the first level of the hierarchy. Let $p(\mathbf{y}|\mathbf{x})$ be an arbitrary probability distribution arising from local measurements \mathbf{m}^{x_i} as in Eq. (11). For every party i , use lemma 1 to represent the measurement \mathbf{m}^{x_i} as

$$(m_{y_i}^{x_i} | = (M_{y_i}^{x_i} | [\mathcal{I}_S \otimes |\sigma_i\rangle])$$

for some spiky measurement \mathbf{M}^{x_i} on $S_i \otimes E_i$ and for some state $\sigma_i \in \text{St}(E_i)$. Now, by construction the conditional probability distribution is equal to

$$p(\mathbf{y}|\mathbf{x}) = (M_{\mathbf{y}}^{\mathbf{x}}|\sigma)$$

where

$$\begin{aligned} (M_{\mathbf{y}}^{\mathbf{x}} | &:= (M_{y_1}^{x_1} | \dots (M_{y_N}^{x_N} | \\ |\sigma) &:= |\rho\rangle|\sigma_1\rangle \dots |\sigma_N\rangle \end{aligned}$$

(with a little abuse of notation, consisting in the fact that the systems are ordered as $S_1 E_1 S_2 E_2 \dots S_N E_N$ in the expression of the effect $M_{\mathbf{y}}^{\mathbf{x}}$ and as $S_1 S_2 \dots S_N E_1 E_2, \dots E_N$ in the expression of the state σ). By Corollary 1 we conclude that $p(\mathbf{y}|\mathbf{x})$ satisfies LO at every level of the hierarchy. □

9.3. Deriving Causality

Since LO implies No-Signalling [11, 17], we have just shown that every probability distribution generated by measurements in a theory satisfying Axioms 1-3 satisfies No-Signalling. Under the minimalistic Assumption 1, Proposition 1 tells us that the theory must satisfy Causality:

Corollary 2. *If a theory satisfies Axioms 1-3 and Assumption 1, then the theory satisfies Causality.*

The fact that Causality follows from the axioms, rather than being assumed from the outset is a pretty remarkable fact. Up to now, the only axiomatization of quantum theory that does not assume Causality from the outset is Hardy’s 2011 axiomatization [21]. There, Causality is derived from an axiom called Sharpness, stating that for every pure state there exists a unique effect that gives probability one on that state and only on that state¹².

It is worth stressing that, as per today, only a few works acknowledge Causality explicitly as an axiom [37, 20, 21, 69, 1, 70, 72], while most works assume Causality implicitly as part of the framework¹³, see e. g. [34, 38, 23, 22, 27, 41]. Recognizing Causality as an axiom is a good starting point to explore deviations from it, thus developing an operational approach to quantum gravity and indefinite causal structure [93, 94, 37, 95, 96, 97].

10. Deriving Sufficient Orthogonality

In the previous section we showed that Local Orthogonality and Causality can be obtained from three requirements on the structure of measurements. While the first two requirements (Measurement Purification and Locality of Pure Orthogonal Measurements) are physically well motivated, the third (Sufficient Orthogonality) sounds rather *ad hoc*. Can one reduce it to some other, better motivated axiom? In this section we give a possible answer, which however, requires us to *assume* Causality.

10.1. No Disturbance Without Information

Informally, the *No Disturbance Without Information* (NDWI) principle states that if a measurement extracts no information about a source, then the measurement can be implemented without disturbing the states in that source. NDWI appeared originally in the axiomatization work of Ref. [20], where it was obtained as a consequence of the axioms (cf. Corollary 10 of [20]). Recently, NDWI has been promoted to the rank of an axiom by Pfister and Wehner [41], who showed that every discrete theory satisfying this requirement must be classical.

In order to give the precise statement of NDWI, it is useful to give some definitions. Here by *source* we mean a deterministic state ρ , considered as the average state of an ensemble of signal states. A *state in the source* ρ is a state that can be contained in a convex decomposition of ρ :

Definition 10. *Let ρ and τ be two deterministic states of system S . We say that τ is in the source ρ iff there exists a nonzero probability $p > 0$ and a state $\tau' \in \text{St}_1(S)$ such that $\rho = p\tau + (1 - p)\tau'$.*

With this definition, a non-informative measurement is one that gives the same statistics for all possible states in the source:

¹² Note that the Sharpness axiom by Hardy is slightly different from the Sharpness axiom used by Wilce in Ref. [28]

¹³For example, Causality enters in the convex set framework [38] in the moment when measurements are defined as decompositions of the order unit.

Definition 11 (Non-informative measurements). *Let \mathbf{m} be a measurement on system S , with outcomes in the set Y . We say that the measurement \mathbf{m} does not extract information about the source $\rho \in \text{St}_1(S)$ iff there exists a set of probabilities $\{p_y\}_{y \in Y}$ such that*

$$(m_y|\tau) = p_y \quad \forall y \in Y$$

for every state τ in the source.

In other words, a measurement extracts no information about the state ρ iff the probability of the outcome y is the same for every state in a convex decomposition of ρ . In Ref. [41] Pfister and Wehner consider the special case of non-informative measurements where the measurement gives an outcome with certainty, i.e. $p_{y_0} = 1$ for a particular outcome y_0 .

Let us specify what it means to realize a measurement without disturbing the states in a source:

Definition 12 (Realization of a measurement). *Let $\mathcal{T} = \{\mathcal{T}_y\}_{y \in Y}$ be a test of type $A \rightarrow B$ and let $u \in \text{Eff}_1(B)$ be a deterministic effect on system B . The pair (\mathcal{T}, u) is a realization of the measurement $\mathbf{m} = \{m_y\}_{y \in Y}$ iff one has*

$$(m_y| = (u|\mathcal{T}_y \quad \forall y \in Y. \quad (22)$$

Definition 13 (Non-disturbing test). *A test \mathcal{T} of type $S \rightarrow S$ is non-disturbing for the source ρ iff one has*

$$\sum_{y \in Y} \mathcal{T}_y |\tau) = |\tau),$$

for every state τ in the source.

Definition 14 (Non-disturbing realization). *A measurement \mathbf{m} on system S admits a non-disturbing realization for the source ρ iff there exists a realization of \mathbf{m} , call it (\mathcal{T}, u) , such that \mathcal{T} is non-disturbing for the source ρ .*

Using the above definitions, we can give the precise statement of NDWI:

Definition 15. *A theory satisfies No Disturbance Without Information (NDWI) iff every measurement \mathbf{m} that does not extract information about the source ρ has a realization (\mathcal{T}, u) that is non-disturbing for this source.*

10.2. From Causality and NDWI to the joint distinguishability of orthogonal states

Here we show that Causality and NDWI imply that orthogonal states can be perfectly distinguished. Although this fact may sound obvious (it is trivially true in Quantum Theory), its validity is far from obvious in a general physical theory.

Theorem 2 (Orthogonal states are perfectly distinguishable). *In a convex theory¹⁴ satisfying Causality and NDWI orthogonal states are perfectly distinguishable.*

Proof. Let $\{\rho_y\}_{y \in \mathsf{Y}}$ be a set of orthogonal states. By definition 4, there exists a set of effects $\{m_y\}_{y \in \mathsf{Y}}$ such that $(m_y | \rho_{y'}) = \delta_{y,y'}$ for every $y, y' \in \mathsf{Y}$. As a consequence, the measurement $\mathbf{m}^{(y)} = \{m_y, m_y^\perp\}$, $m_y^\perp := u - m_y$ does not extract information about the source

$$\rho_y^\perp := \frac{1}{|\mathsf{Y}| - 1} \sum_{y' \neq y} \rho_{y'}.$$

Indeed, the relations $(m_y | \rho_y^\perp) = 0$ and $(m_y^\perp | \rho_y^\perp) = 1$ imply analog relations

$$\begin{aligned} (m_y | \tau) &= 0 \\ (m_y^\perp | \tau) &= 1 \end{aligned} \tag{23}$$

for every state τ in the source ρ_y^\perp .

By the NDWI axiom, $\mathbf{m}^{(y)}$ has a non-disturbing realization, given by two transformations $\{\mathcal{T}_y, \mathcal{T}_y^\perp\}$ such that

$$\begin{aligned} (u | \mathcal{T}_y) &= m_y \\ (u | \mathcal{T}_y^\perp) &= m_y^\perp \end{aligned} \tag{24}$$

and $(\mathcal{T}_y + \mathcal{T}_y^\perp | \tau) = |\tau\rangle$ for every state τ in the source ρ_y^\perp . In particular, we have

$$(\mathcal{T}_y + \mathcal{T}_y^\perp | \rho_{y'}) = |\rho_{y'}\rangle \quad \forall y' \neq y. \tag{25}$$

Applying the unit effect on both sides of Eq. (25) and using Eqs. (24) and (23) one obtains $(u | \mathcal{T}_y | \rho_{y'}) = 0$. By Causality, this relation implies $\mathcal{T}_y | \rho_{y'} = 0$ ¹⁵. Hence, Eq. (25) becomes

$$\mathcal{T}_y | \rho_{y'} = 0 \tag{26}$$

$$\mathcal{T}_y^\perp | \rho_{y'} = |\rho_{y'}\rangle \quad \forall y' \in \mathsf{Y}, y' \neq y \tag{27}$$

Note also that by construction we have $(u | \mathcal{T}_y^\perp | \rho_y) = (m_y^\perp | \rho_y) = 0$, which implies

$$\mathcal{T}_y^\perp | \rho_y = 0. \tag{28}$$

¹⁴A “convex theory” is defined as a theory where all the set of states, effects, and transformations are convex. So far, we never assumed convexity, and indeed such assumption is not part of the basic framework of OPTs. Even in the present theorem, we will use convexity only in a minor way, just to guarantee that we can mix with non-zero probabilities the states in a given set.

¹⁵Causality implies that for every effect $a \in \text{Eff}(S)$, the two effects $\{a, u - a\}$ form a legitimate measurement. Hence, the condition $(u | \rho) = 0$ implies $(a | \rho) = 0$ for every effect, i. e. $\rho = 0$.

In addition, we can also assume without loss of generality

$$\mathcal{T}_y|\rho_y) = |\rho_y), \quad (29)$$

because we can always replace \mathcal{T}_y with $\mathcal{T}'_y := |\rho_y)(u|\mathcal{T}_y$ ¹⁶.

Summarizing Eqs. (26), (27), (28) and (29) we have

$$\begin{aligned} \mathcal{T}_y|\rho_{y'}) &= \delta_{y,y'}|\rho_y) \\ \mathcal{T}_y^\perp|\rho_{y'}) &= (1 - \delta_{y,y'})|\rho_{y'}) \quad \forall y, y' \in \mathcal{Y}. \end{aligned} \quad (30)$$

Now, the test $\{\mathcal{T}_y, \mathcal{T}_{y^\perp}\}$ allows one to discriminate between the state ρ_y and all the other states $\{\rho_{y'}\}_{y' \in \mathcal{S}, y' \neq y}$ without introducing any disturbance. Hence, one way to distinguish perfectly the states $\{\rho_y\}_{y \in \mathcal{Y}}$ is to enumerate the elements of \mathcal{Y} , say $\mathcal{Y} = (y_1, \dots, y_N)$ and to apply the tests $\{\mathcal{T}_{y_n}, \mathcal{T}_{y_n}^\perp\}$ one after the other. The resulting test, denoted by $\{\mathcal{S}_y\}_{y \in \mathcal{Y}}$ will consist of the transformations

$$\begin{aligned} \mathcal{S}_{y_1} &:= \mathcal{T}_{y_1} \\ \mathcal{S}_{y_2} &:= \mathcal{T}_{y_2} \mathcal{T}_{y_1}^\perp \\ \mathcal{S}_{y_3} &:= \mathcal{T}_{y_3} \mathcal{T}_{y_2}^\perp \mathcal{T}_{y_1}^\perp \\ &\vdots \\ \mathcal{S}_{y_{N-1}} &:= \mathcal{T}_{y_{N-1}} \mathcal{T}_{y_{N-2}}^\perp \cdots \mathcal{T}_{y_1}^\perp \\ \mathcal{S}_{y_N} &:= \mathcal{T}_{y_N}^\perp \mathcal{T}_{y_{N-2}}^\perp \cdots \mathcal{T}_{y_1}^\perp \end{aligned}$$

Clearly, Eq. (30) implies $\mathcal{S}_{y_m}|\rho_{y_n}) = \delta_{m,n}|\rho_{y_n})$. Hence, the states $\{\rho_y\}_{y \in \mathcal{Y}}$ can be perfectly distinguished using the measurement \mathbf{m} defined by $(m_y| := (u|\mathcal{S}_y, \forall y \in \mathcal{Y}$. □

Note the difference between the statement theorem 2 and the statement that a set of pairwise distinguishable states are jointly distinguishable. As we already observed, orthogonality [cf. definition (4)] and pairwise distinguishability are different notions. In general, it is not clear whether the joint distinguishability of pairwise distinguishable states follows from NDWI.

10.3. Strong No Disturbance Without Information

We now present a strengthened version of the NDWI axiom, stating that, in addition to not disturbing the states in a given source, a non-informative measurement does not disturb the pure effects that occur with unit probability on those states:

¹⁶ The existence of the transformation \mathcal{T}'_y is guaranteed by Causality along with Assumption 2. Indeed, one can perform the test $\{\mathcal{T}_y, \mathcal{T}_y^\perp\}$ and, conditionally on outcome y , re-prepare the state ρ_y .

Definition 16 (Strongly non-disturbing test). *The test $\mathcal{T} = \{\mathcal{T}_y\}_{y \in Y}$ is strongly non-disturbing for the source $\rho \in \text{St}_1(S)$ iff*

$$\sum_{y \in Y} \mathcal{T}_y | \tau) = | \tau)$$

for every state τ in the source and

$$\sum_{y \in Y} (a | \mathcal{T}_y = (a |$$

for every pure effect a such that $(a | \tau) = 1$.

A strongly non-disturbing realization of a measurement is defined in the obvious way, as a realization in terms of a strongly non-disturbing test. Using this definition, we can now state the strong version of the NDWI principle:

Axiom 3' (StrongNDWI). *Every measurement that does not extract information about the source ρ has a strongly non-disturbing realization for this source.*

10.4. Derivation of SO

It is easy to prove that Causality and StrongNDWI imply Sufficient Orthogonality:

Theorem 3. *Every convex theory satisfying Causality and StrongNDWI must satisfy SO.*

Proof. Let $\{a_y\}_{y \in Y}$ be a set of pure orthogonal effects and let $\{\rho_y\}_{y \in Y}$ a set of states biorthogonal to it. For these two sets, we follow the construction of theorem 2: we consider the measurement $\mathbf{m}^y = \{a_y, a_y^\perp\}$, $a_y^\perp := u - a_y$ and note that it is non-informative for the state

$$\rho_y^\perp = \frac{1}{|Y| - 1} \sum_{y' \neq y} \rho_{y'}.$$

By the StrongNDWI principle, \mathbf{m}^y will have a strongly non-disturbing realization, given by a binary test $\{\mathcal{T}_y, \mathcal{T}_y^\perp\}$ such that

$$\begin{aligned} (u | \mathcal{T}_y &= a_y \\ (u | \mathcal{T}_y^\perp &= a_y^\perp \end{aligned} \tag{31}$$

and

$$(a_{y'} | \mathcal{T}_y + (a_{y'} | \mathcal{T}_y^\perp = (a_{y'} | \quad \forall y' \neq y,$$

the last equation coming from the StrongNDWI condition. Now, since $a_{y'}$ is a pure effect, the above equation implies $(a_{y'} | \mathcal{T}_y = p(a_{y'})$ and $(a_{y'} | \mathcal{T}_y^\perp = (1 -$

$p)(a_{y'}|$ for some probability $p \in [0, 1]$. Now, it is easy to show that $p = 0$: indeed, we have

$$\begin{aligned}
p &= p(a_{y'}|\rho_{y'}) \\
&= (a_{y'}|\mathcal{T}_y|\rho_{y'}) \\
&\leq (u|\mathcal{T}_y|\rho_{y'}) \\
&= (a_y|\rho_{y'}) \\
&= 0.
\end{aligned}$$

Hence, we conclude that

$$(a_{y'}|\mathcal{T}_y^\perp = (a_{y'}| \quad \forall y' \neq y. \quad (32)$$

Like in the proof of Theorem 2, we now enumerate the elements of Y , say $Y = (y_1, \dots, y_N)$, and apply the tests $\{\mathcal{T}_{y_n}, \mathcal{T}_{y_n}^\perp\}$ one after the other. As a result, we obtain a test $\{\mathcal{S}_y\}_{y \in \tilde{Y}}$ with outcomes in the set $\tilde{Y} := Y \cup \{rest\}$, defined by

$$\begin{aligned}
\mathcal{S}_{y_1} &:= \mathcal{T}_{y_1} \\
\mathcal{S}_{y_2} &:= \mathcal{T}_{y_2} \mathcal{T}_{y_1}^\perp \\
\mathcal{S}_{y_3} &:= \mathcal{T}_{y_3} \mathcal{T}_{y_2}^\perp \mathcal{T}_{y_1}^\perp \\
&\vdots \\
\mathcal{S}_{y_N} &:= \mathcal{T}_{y_N} \mathcal{T}_{y_{N-1}}^\perp \dots \mathcal{T}_{y_1}^\perp \\
\mathcal{S}_{rest} &:= \mathcal{T}_{y_N}^\perp \mathcal{T}_{y_{N-1}}^\perp \dots \mathcal{T}_{y_1}^\perp
\end{aligned}$$

To conclude the proof, we consider the measurement \mathbf{m} defined by $(m_y| := (u|\mathcal{S}_y, \forall y \in \tilde{Y}$: for this measurement we have

$$\begin{aligned}
(m_{y_1}| &= (u|\mathcal{T}_{y_1} = (a_{y_1}| \\
(m_{y_2}| &= (u|\mathcal{T}_{y_2} \mathcal{T}_{y_1}^\perp = (a_{y_2}|\mathcal{T}_{y_1}^\perp = (a_{y_2}| \\
(m_{y_3}| &= (u|\mathcal{T}_{y_3} \mathcal{T}_{y_2}^\perp \mathcal{T}_{y_1}^\perp = (a_{y_3}|\mathcal{T}_{y_2}^\perp \mathcal{T}_{y_1}^\perp = (a_{y_3}| \\
&\vdots \\
(m_{y_N}| &= (u|\mathcal{T}_{y_N} \mathcal{T}_{y_{N-1}}^\perp \dots \mathcal{T}_{y_1}^\perp = (a_{y_N}|\mathcal{T}_{y_{N-1}}^\perp \dots \mathcal{T}_{y_1}^\perp = (a_{y_N}| \\
(m_{rest}| &= (u|\mathcal{T}_{y_N}^\perp \mathcal{T}_{y_{N-1}}^\perp \dots \mathcal{T}_{y_1}^\perp \equiv (u| - \sum_{y \in Y} (a_y|,
\end{aligned}$$

where we used Eqs. (31) and (32). Hence, we have proven that an arbitrary set of pure orthogonal effects $\{a_y\}_{y \in Y}$ can coexist in a single measurement, as required by SO. \square

11. The device-independent framework for contextuality

In this section we review the device-independent framework for studying contextuality. We also review the principle known as *Consistent Exclusivity (CE)* [13, 15, 16, 17, 18].

11.1. Contextual games

Consider a game featuring a referee, who asks a question $x \in \mathbf{X}$, and a player, who responds with an answer $y \in \mathbf{Y}_x$. In general, two different questions may have overlapping sets of answers, i. e. one can have $\mathbf{Y}_x \cap \mathbf{Y}_{x'} \neq \emptyset$ for some $x \neq x'$. At each round of the game, the referee chooses a question x at random with probability $q(x)$ and assigns a payoff $\omega(x, y)$ to the answer y . The goal of the player is to maximize her expected payoff, given by

$$\omega = \sum_x q(x) \left[\sum_y \omega(x, y) p(y|x) \right], \quad (33)$$

where $p(y|x)$ is the conditional probability of producing the output y upon receiving the input x . We call a game of the above form a *contextual game*, by analogy with the non-local games discussed before.

Without further restrictions, the maximization of the payoff is trivial: for every given question x , the player only needs to respond deterministically with the answer $y(x)$ that maximizes $\omega(x, y)$. The problem becomes non-trivial if the player is forced to assign to each answer y a probability that does not depend on which question—among the questions that have y as possible answer—is asked by the referee. Mathematically, this amounts to the *response non-contextuality condition*¹⁷

$$p(y|x) \equiv p(y|x') \quad \forall x, x' \in \mathbf{X}, \quad \forall y \in \mathbf{Y}_x \cup \mathbf{Y}_{x'}. \quad (36)$$

A strategy satisfying Eq. (36) is a strategy where the player partly disregards the question x . *Partly*, because she will still make use of her knowledge of x ,

¹⁷“Non-contextuality” here refers to the fact that the context x that gives rise to an answer does not influence the probability of its occurrence. The reader should not confuse the response non-contextuality condition of Eq. (36) with the statement that quantum mechanics is “contextual”. The latter is just a shorthand for the statement that some of the conditional probability distributions $p(y|x) = \text{Tr}[P_y^x \rho]$, generated by measurements on quantum states, cannot be reproduced by a *non-contextual ontological model* [14], i. e. cannot be written in the form

$$p(y|x) = \sum_{\lambda \in \Lambda} p(\lambda) r_\lambda(y|x) \quad (34)$$

where λ is a random variable with probability distribution $p(\lambda)$, and, for every $\lambda \in \Lambda$, $r_\lambda(y|x)$ is a deterministic, non-contextual response function, that is, a conditional probability distribution of the form

$$r_\lambda(y|x) = \begin{cases} 1 & y = f_\lambda(x) \\ 0 & y \neq f_\lambda(x) \end{cases} \quad (35)$$

for some suitable function $f_\lambda : \mathbf{X} \rightarrow \mathbf{Y}$ satisfying the conditions

1. *normalization*: for every question $x \in \mathbf{X}$, there exists only one answer $y \in \mathbf{Y}$ such that $y = f_\lambda(x)$
2. *non-contextuality*: for every pair of questions $x, x' \in \mathbf{X}$ and every answer $y \in \mathbf{Y}_x \cap \mathbf{Y}_{x'}$, one has $f_\lambda(x) = f_\lambda(x')$.

by restricting the range of her answers to the set Y_x . Note that response non-contextuality is different from no-signalling, in that the choice of question x can affect the conditional probability distribution of the answer y , albeit in a highly constrained way.

We call a strategy *response-non-contextual* iff it satisfies the response non-contextuality constraint of Eq. (36). One way to enforce response non-contextuality is by imagining that the game is played a large number of times, allowing the referee to estimate the conditional probability distribution $p(y|x)$ and to penalize deviations from Eq. (36).

11.2. The graph-theoretic framework

Contextual games can be conveniently cast in a graph-theoretic framework [98, 17], which has its roots in the framework *test spaces*¹⁸ [100, 101, 99] (see also [102, 103]).

To a given game, one can associate a hypergraph¹⁹ $H = (Y, X)$ as follows:

1. the vertices are the answers in $Y := \bigcup_{x \in X} Y_x$
2. the hyperedges are the questions in X , with the question $x \in X$ being identified with the subset of its possible answers Y_x [accordingly, we will write $y \in x$ in place of $y \in Y_x$].

A conditional probability distribution $p(y|x)$ obeying the response non-contextuality condition (36) can be completely described by the function $w : Y \rightarrow [0, 1]$ defined by

$$w(y) := p(y|x) \quad \forall x \in X : y \in x. \quad (37)$$

The function w is a *probability weight* on the hypergraph H , in the following sense:

Definition 17. *A function $w : Y \rightarrow [0, 1]$ is a probability weight (or a state [105, 106]) on the hypergraph $H = (Y, X)$ iff it satisfies the condition*

$$\sum_{y \in x} w(y) = 1 \quad \forall x \in X. \quad (38)$$

In terms of the probability weight w , the payoff (33) can be re-written as

$$\omega = \sum_{y \in Y} c(y) w(y), \quad (39)$$

with $c(y) := \sum_{x \in X} q(x) \omega(x, y)$.

The maximization of the payoff over all possible response-non-contextual strategies is then equivalent to the maximization of the payoff over all possible probability weights. We denote such maximum by ω_{RNC} .

¹⁸Test spaces (modulo minor variations) have been also called *manuals*, *spaces*, *hypergraphs*, *cover spaces*, and *generalized sample spaces*, see [99] for references to the original articles.

¹⁹We recall that a hypergraph $H = (Y, X)$ consists of a collection of vertices Y , along with a collection X of subsets of Y , called hyperedges [104].

11.3. Principles about probability weights: the example of Consistent Exclusivity

The physical limitations affecting the player’s strategy will result into constraints on the probability weight $w(y)$. Consistent Exclusivity is one such requirement: it states that the sum of the probabilities associated to a set of mutually exclusive vertices of the hypergraph H should be smaller than 1. The precise definitions are given as follows:

Definition 18. *Two vertices $\{y, y'\} \subseteq Y$ are exclusive iff there exists an hyperedge x such that $\{y, y'\} \subseteq x$. A subset of outcomes $E \subseteq Y$ is called mutually exclusive iff every pair of distinct outcomes $\{y, y'\} \subseteq E$ are exclusive.*

Definition 19 (Consistent Exclusivity ([13, 15, 16, 17, 18])). *A probability weight $w : Y \rightarrow [0, 1]$ satisfies Consistent Exclusivity (CE) iff one has*

$$\sum_{y \in E} w(y) \leq 1 \tag{40}$$

for every mutually exclusive set $E \subseteq Y$.

The above formulation of CE is “device-independent”, in that it makes reference to the probability weight w , but not to the specific set of measurements $\{\mathbf{m}^x\}_{x \in X}$ that generate w . Nevertheless, the request that a probability weight satisfies CE is hard to motivate on physical grounds²⁰. To find such motivation, we argue that one should look outside the device-independent framework. This point will be discussed in section 13.

11.4. The CE hierarchy

Like LO, CE can be generalized to an infinite hierarchy of constraints [17]. The k -th level of the hierarchy is defined by considering k identical copies of a black box, the i -th copy generating an answer y_i with probability weight $w(y_i)$. In this setting, the string of answers $\mathbf{y} = (y_1, y_2, \dots, y_k) \in Y^{\times k}$ has probability weight

$$w^{\otimes k}(\mathbf{y}) := w(y_1) w(y_2) \dots w(y_k)$$

and the k -th level of the hierarchy is defined as follows:

Definition 20 (Consistent Exclusivity at the k -th level [17, 15]). *A probability weight $w(y)$ satisfies CE at the k -th level iff one has*

$$\sum_{\mathbf{y} \in E^k} w^{\otimes k}(\mathbf{y}) \leq 1 \tag{41}$$

for every mutually exclusive set $E_k \subseteq Y^{\times k}$.

²⁰Ref. [18] reports the following comment by Simon Kochen on the background role of CE in the Kochen-Specker paper: “Ernst and I spent many hours discussing the principle. [...] The difficulty lays in trying to justify it on general physical grounds, without already assuming the Hilbert space formalism of quantum mechanics.”

11.5. Characterizing the degree of contextuality of projective quantum measurements

Since the pioneering work of Kochen and Specker, projective measurements have played a privileged role in the study of contextuality in quantum mechanics (see Spekkens [14] for a discussion). Following this tradition, a number of recent works ([98, 17, 15]) have attempted a device-independent characterization of the input-output probability distributions of the form

$$p(y|x) = \text{Tr} [P_y^x \rho] , \quad (42)$$

where

1. ρ is a quantum state, and
2. for every $x \in \mathbf{X}$, $\mathbf{P}^x := \{P_y^x\}_{y \in Y_x}$ is a projective quantum measurement satisfying the non-contextuality condition

$$P_y^x = P_y^{x'} \quad \forall x, x' \in \mathbf{X}, \quad \forall y \in Y_x \cap Y_{x'} . \quad (43)$$

We call an input-output probability distribution of the form (42) *projective quantum (PQ)*.

It is not hard to see that, for given input/output alphabets, the set of PQ input-output distributions is convex. Hence, characterizing it is equivalent to characterizing the maximum payoffs achievable in all possible contextual games. Since the maximum payoff is an indicator of the degree of contextuality, we refer to the problem as “characterizing the degree of contextuality of projective quantum measurements”. Just like in the case of nonlocality, finding a device-independent characterization is a spectacularly hard problem. CE provides remarkable results in this direction, but provenly [12] not a complete characterization.

12. Physical implementation of contextual games

Like in the case of non-locality, the framework of operational-probabilistic theories can be applied to the study of contextual games. For the physical implementation of a given contextual game, we propose the following model:

Definition 21. *A physical implementation of a contextual game is a protocol where*

1. *the referee sends to the player a physical system S , prepared in a state ρ , and a classical input $x \in \mathbf{X}$, chosen at random with probability $q(x)$*
2. *The player performs a measurement $\mathbf{m}^x = \{m_y^x\}_{y \in Y_x}$ on system S and communicates to the referee the measurement outcome y*
3. *The referee assigns the payoff $\omega(x, y)$ to the answer.*

In the implementation of the protocol, the player’s measurements are subject to the following constraints

1. they should satisfy the effect non-contextuality condition

$$m_y^x = m_y^{x'} \quad \forall x, x' \in X, \forall y \in Y_x \cap Y_{x'}. \quad (44)$$

2. they have to be performed on the input system provided by the referee, (not on some other system of the same type prepared in the player's laboratory). In other words, the conditional probability distribution of the answer y must satisfy

$$p(y|x) = (m_y^x | \rho). \quad (45)$$

The above physical implementation departs radically from the device-independent scenario. This can be observed in the following points:

1. While the original game had only a classical input x and a classical output y , its physical implementation involves also the communication of a specific physical system S , known to the player.
2. The effect non-contextuality condition (44) is device-dependent. In order to check its validity, one needs to make a full tomography of the measurement devices $\{\mathbf{m}^x\}_{x \in X}$. Indeed, effect non-contextuality is a stronger condition than response non-contextuality (36): it is equivalent to response non-contextuality for every possible state $\rho \in \text{St}_1(S)$.
3. Imagining that the game is played a large number of times, the effect non-contextuality condition can be enforced by the referee by randomly switching from the “game-playing mode” to a “constraint-checking mode”, which consists in sending, instead of the state ρ , a state chosen at random from a tomographically complete set of states. By collecting enough statistics, the referee will be able to identify (up to statistical errors) the measurements $\{\mathbf{m}^x\}_{x \in X}$ and to check (up to statistical errors) whether they satisfy the effect non-contextuality condition.
4. The constraint that the player's measurement are performed on the state ρ can also be checked once a tomographic estimate of the measurements $\{\mathbf{m}^x\}_{x \in X}$ is available. For this purpose, the referee only needs to compare the empirical distribution of the player's answers with the desired distribution $p(y|x) = (m_y^x | \rho)$.

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The physical implementation of a contextual game can also be phrased in graph theoretic terms. Given the hypergraph $H = (X, Y)$ associated to the original game, the player's strategy is completely specified by the function $\hat{w} : Y \rightarrow \text{Eff}(S)$ defined by

$$\hat{w}(y) := m_y^x \quad \forall x \in X : y \in x \quad (46)$$

[recall that the label x is identified with the subset $Y_x \subseteq Y$]. We refer to the function $\hat{w}(y)$ as an *effect-valued weight* on the hypergraph H :

Definition 22. A function $\hat{w} : Y \rightarrow \text{Eff}(S)$ is an effect-valued weight on the hypergraph $H = (Y, X)$ iff the collection of effects $\{\hat{w}(y)\}_{y \in X}$ is a measurement for every $x \in X$.

Given an effect-value weight $\hat{w}(y)$ and a state ρ , one obtains a probability weight $w(y)$, defined as

$$w(y) := (\hat{w}(y)|\rho) \quad \forall y \in \mathcal{Y}. \quad (47)$$

Once a physical theory has been specified, the goal of the player is to find the best measurements that maximize her expected payoff ω . Among all possible physical implementations with a system S and a state $\rho \in \text{St}_1(S)$, it is interesting to consider the ones that lead to the highest payoff. For a given theory \mathbb{T} , we denote by $\omega_{\mathbb{T}}$ the maximum payoff that can be obtained by optimizing over all systems, states, and measurements.

13. Reformulating Consistent Exclusivity as a (device-dependent) physical principle

In its original formulation, CE is a principle about probability weights. To interpret it as a physical principle, one needs to specify what physical situations give rise to probability weights satisfying Eq. (40). This specification, however, is far from straightforward. The naive formulation “*All the probability weights arising in Nature satisfy CE*” is ultimately wrong, since one can easily construct examples of quantum measurements giving rise to probability weights violating the CE property: in fact, for every contextual game, the maximum payoff achievable in quantum mechanics is *equal* to the maximum payoff achievable with arbitrary response-non-contextual strategies²¹, namely

$$\omega_{\text{QUANTUM}} = \omega_{\text{RNC}}. \quad (48)$$

The fix for this problem is to restrict the validity of CE to probability weights generated by *projective* measurements: the correct condition satisfied by quantum mechanics is “*All the probability weights arising from projective measurements satisfy the CE property*”. Note that this is by no means a device-independent statement, as it refers explicitly to a property of the devices used to generate the probability weight.

In order to formulate CE as physical principle, one has first to define the analogue of the “projective measurements”. This can be done in different ways, depending on which aspect of projective quantum measurements is chosen as distinctive. Every definition will lead to a different “CE principle”, potentially encompassing a different picture of the physical world. For example, in Ref. [1] we proposed a formulation of the CE principle in terms of sharp measurements:

²¹Trivially, every probability weight $w(y)$ defines a set of quantum measurements, the x -th measurement described by the POVM $\mathbf{P}^x := \{P_y^x\}_{y \in \mathcal{X}}$ defined by $P_y^x := w(y) I_S$, where I_S is the identity operator on the system’s Hilbert space. For every system and for every density matrix ρ one then has $p(y|x) = \text{Tr}[P_y^x \rho] \equiv w(y)$. No matter what the system’s dimensionality is, the maximum of the payoff over all quantum measurements coincides with the maximum over all response-non-contextual strategies.

Definition 23 (SharpCE). *A theory satisfies SharpCE (at the k -th level of the hierarchy) iff every probability weight generated by sharp measurements according to Eq. (47) satisfies CE (at the k -th level of the hierarchy).*

This formulation of the CE principle has been adopted by Cabello in Refs. [107, 108], as capturing the intuition at the basis of the formulation of CE in the graph-theoretic framework. We now explore an alternative formulation, in terms of spiky measurements:

Definition 24 (SpikyCE). *A theory satisfies SpikyCE (at the k -th level of the hierarchy) iff every probability weight generated by spiky measurements according to Eq. (47) satisfies CE (at the k -th level of the hierarchy).*

Interesting, the two formulations turn out to be equivalent if we restrict our attention to *pure* measurements, because in this case “sharp” and “spiky” are equivalent notions. The equivalence is discussed in Section 15, which also lists other alternative generalizations of the notion of projective measurement in quantum theory.

14. Deriving Consistent Exclusivity for Spiky Measurements

Here we provide a derivation of SpikyCE from the following three principles:

1. Causality
2. Strong No Disturbance Without Information
3. Pure State Identification,

the last of which will be defined precisely later in this section.

14.1. Reduction to Coexistence of Mutually Exclusive Spiky Effects

Our derivation of SpikyCE proceeds through a sequence of reductions. The first reduction is based on the following notions:

Definition 25 (Mutually exclusive effects). *Two effects m and m' are exclusive iff there exists a measurement \mathbf{m} such that $\{m, m'\} \subseteq \mathbf{m}$. A set of effects $\{m_y\}_{y \in E}$ are mutually exclusive iff every pair of effects $\{m, m'\} \subseteq \{m_y\}_{y \in E}$ are exclusive.*

Definition 26 (SpikyCMEE). *A theory satisfies Coexistence of Mutually Exclusive Spiky Effects (SpikyCMEE) iff every set of mutually exclusive spiky effects can coexist in a measurement.*

SpikyCMEE coincides with the formulation of the CE principle used by Barnum, Müller, and Ududec in Ref. [78]. Their choice of name was motivated by the following observation

Proposition 3. *SpikyCMEE implies SpikyCE.*

The proof is elementary and can be found in Appendix E.

14.2. *Deriving SpikyCMEE from Sufficient Orthogonality and Pure State Identification*

We now reduce SpikyCMEE to Sufficient Orthogonality combined with a principle of Pure State Identification. In order to phrase the latter, we need the following definitions:

Definition 27. *An effect $m \in \text{Eff}(S)$ is normalized iff there exists a state $\rho \in \text{St}(S)$ such that $(m|\rho) = 1$.*

Definition 28. *Let m and φ be an effect and a pure state of system S , respectively. We say that m identifies φ iff*

1. $(m|\varphi) = 1$
2. $(m|\rho) < 1$ for every state $\rho \neq \varphi$.

For example, in Quantum Theory every rank-one projector, considered as a measurement effect, identifies a pure state. In a general theory, the fact that every normalized pure effect identifies a state is a nontrivial property²². We refer to it as *Pure State Identification*²³:

Axiom 4 (Pure State Identification). *A theory satisfies Pure State Identification (PSI) iff every normalized pure effect identifies a pure state.*

In a general theory, one has the following

Proposition 4. *Pure State Identification and Sufficient Orthogonality imply SpikyCMEE.*

Proof. We first prove SpikyCMEE for pure effects. Let $\{a_i\}_{i=1}^N$ be a set of mutually exclusive pure effects. By PSI, each pure effect a_i identifies a pure state φ_i . Since the effects are mutually exclusive, for every pair $\{a_i, a_j\}$ there exists a measurement \mathbf{m}^{ij} such that $\{a_i, a_j\} \subseteq \mathbf{m}^{ij}$. Hence, the condition $(a_j|\varphi_j) = 1$ implies $(a_i|\varphi_j) = \delta_{ij}$. Since i and j are arbitrary, this means that the effects $\{a_i\}_{i=1}^N$ are orthogonal. SO then implies that the effects $\{a_i\}_{i=1}^N$ coexist in a measurement. This argument proves the validity SpikyCMEE for pure effects. The extension to arbitrary spiky measurements is immediate, since spiky measurements are coarse-graining of pure orthogonal measurements. \square

²²Think, e. g. of the square bit, where every pure effect happens with probability 1 on all the states on one of the four sides of the square (cf. example 1).

²³A very similar axiom appeared in Hardy's 2011 axiomatization [21, 109], under the name *Logical Sharpness*. Hardy's axiom is slightly stronger than Pure State Identification, in that it requires that every pure state is identified by some pure effect. Another, closely related axiom was put forward by Wilce [28], who considered a privileged set of measurements with the property that each outcome identifies a pure state. The privileged measurements are not assumed to be pure from the start, but turn out to be so as a consequence of the axioms.

14.3. Derivation of SpikyCE

Combining proposition 4 with theorem 3, we get the desired result:

Theorem 4. *If a theory satisfies Causality, Strong No Disturbance Without Information, and Pure State Identification, then it also satisfies SpikyCE.*

900 A derivation of SpikyCE from completely different axioms is provided in Ref. [78], where SpikyCE is obtained from the requirement that *i)* every state can be represented as a mixture of perfectly distinguishable pure states and *ii)* all sets of perfectly distinguishable pure states of a given cardinality can be transformed into one another by reversible transformations. It is also interesting to compare Proposition 4 with the results of Ref. [1], where we formulated SharpCE and derived it from a *single* axiom about sharp measurements. Yet another way of deriving CE was found by Wilce [110], who interestingly obtained it from a requirement about *bipartite* systems²⁴ and from a requirement about coarse-graining of tests, very similar in spirit to the axiom used in [1]. The existence of different, alternative ways to obtain the CE principle provides a good illustration of the fact that a device-independent feature can arise from different features of the underlying physical theory.

14.4. Derivation of SpikyCE at all levels of the hierarchy

Deriving SpikyCE at higher levels of the hierarchy is easy if we assume the Locality of Pure Orthogonal Measurements. This principle guarantees that, for every pure orthogonal measurement $\mathbf{a}^x := \{a_y^x\}_{y \in Y_x}$, the product measurement with effects

$$a_{\mathbf{y}}^{\mathbf{x}} := a_{y_1}^{x_1} \otimes a_{y_2}^{x_2} \otimes \cdots \otimes a_{y_k}^{x_k}$$

is also a pure orthogonal measurement. The k -th level of the hierarchy just follows from the application of the SpikyCMEE. In summary, we have proven the following

Corollary 3. *If a theory satisfies Causality, Locality of Pure Orthogonal Measurements, Strong No Disturbance Without Information, and Pure State Identification, then it also satisfies SpikyCE at all levels of the hierarchy.*

15. Different generalizations of the notion of projective quantum measurement

We have seen that CE, as a physical principle, can be formulated in different ways, depending on how the notion of “projective quantum measurement” is generalized to arbitrary physical theories. In this section we discuss four different generalizations and establish a number of relations between them.

²⁴More specifically, Wilce requires the existence of a *conjugate system*, in the sense of [28]. Roughly speaking, a system S is said to have a conjugate \bar{S} if there exists a suitable state of $S \otimes \bar{S}$ that exhibits perfect correlations for all measurements in a suitable class of privileged measurements, which we can identify e. g. with the spiky measurements of this article, or with the sharp measurements of Ref. [1].

15.1. Sharp measurements

Definition 29 (Sharp measurement [1]). A measurement $\mathbf{m} = \{m_x\}_{x \in \mathbf{X}}$ is sharp iff it can be implemented by a repeatable and minimally disturbing test $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbf{X}}$, i. e. a test satisfying the repeatability condition

$$(m_x | \mathcal{T}_x = (m_x | \quad \forall x \in \mathbf{X} \quad (49)$$

and the minimal disturbance condition

$$(n_y | \left(\sum_{x \in \mathbf{X}} \mathcal{T}_x \right) = (n_y | \quad \forall y \in \mathbf{Y}, \quad (50)$$

for every measurement $\mathbf{n} = \{n_y\}_{y \in \mathbf{Y}}$ that is compatible²⁵.

An equivalent characterization of sharp measurements is provided by the following

Proposition 5. A measurement \mathbf{m} is sharp iff there exists a test \mathcal{T} such that

$$(n_{xy} | \mathcal{T}_x = (n_{xy} | \quad \forall x \in \mathbf{X}, \forall y \in \mathbf{Y} \quad (51)$$

for every measurement $\{n_{xy}\}_{x \in \mathbf{X}, y \in \mathbf{Y}}$ that refines \mathbf{m} , i.e. $\sum_y n_{xy} = m_x$.

The proof can be found in section III of [1]. Eq. (51) is closely related to the notion of *coherent Lüders rule* introduced by Kleinmann in Ref. [111]. Roughly speaking, the sharp measurements of definition 29 are the measurements that can be implemented by tests in which each transformation is a coherent Lüders rule for the corresponding effect²⁶.

²⁵We recall that two measurements $\mathbf{m} = \{m_x\}_{x \in \mathbf{X}}$ and $\mathbf{n} = \{n_y\}_{y \in \mathbf{Y}}$ are said to be *compatible* iff there exists a third measurement $\mathbf{o} = \{o_z\}_{z \in \mathbf{Z}}$ and two disjoint partitions of \mathbf{Z} , denoted by $\{\mathbf{Z}_x^{\mathbf{m}}\}_{x \in \mathbf{X}}$ and $\{\mathbf{Z}_y^{\mathbf{n}}\}_{y \in \mathbf{Y}}$, respectively, such that

$$\begin{aligned} m_x &= \sum_{z \in \mathbf{Z}_x^{\mathbf{m}}} o_z & \forall x \in \mathbf{X} \\ n_y &= \sum_{z \in \mathbf{Z}_y^{\mathbf{n}}} o_z & \forall y \in \mathbf{Y}. \end{aligned}$$

²⁶Some care is required with such an identification, which sometimes turns out to be incorrect. The main differences between Refs. [1] and [111] can be summarized as follows:

1. *Framework.* Ref. [111] associates to a physical system S an order unit vector space V_S , making the following

Assumption 3. $\text{Eff}(S) = \{m \in V_S \mid 0 \leq m \leq u_S\}$.

Not every OPT satisfies such an assumption, which is strictly stronger than convexity of the space of effects and is partly related to the so-called No-Restriction Hypothesis [37, 91] (see Appendix F for details).

2. *Positive maps vs physical transformations.* A *coherent Lüders rule (CLR)* is defined

15.2. Maximally discriminating measurements

A third generalization of projective quantum measurements appeared often in the literature on the reconstructions of quantum theory [24, 20, 22, 21, 27, 23]. In this context it is often convenient to consider measurements that distinguish perfectly among a maximal set of states, i. e. sets of states $\{\rho_n\}_{n=1}^N$ with the property that there is no state ρ_{N+1} such that the states $\{\rho_n\}_{n=1}^{N+1}$ are jointly distinguishable. Here we call a measurement that distinguish among a maximal set of states a *maximally discriminating measurement*. In the case of quantum theory it is easy to see that the maximally discriminating measurements coincide with the projective measurements.

as a positive linear map $\phi : V_S \rightarrow V_S$ satisfying the conditions

$$\phi(u_S) = m \quad (m\text{-compatibility}) \quad (52)$$

$$\phi(n) = n \quad \forall n \in V_S : 0 \leq n \leq m \quad (\text{coherence}). \quad (53)$$

Each positive map is regarded as a potential candidate for a physical transformation, leaving the actual choice of physical transformations open. We argue that the most sensible way to make such a choice is to start from a full OPT, where the composition of transformations in parallel and sequence is built in the operational structure, thanks to the adoption of the categorical framework [61, 62, 32]. This allows one to bypass problems like the difference between positivity and complete positivity, and the problem that the correspondence between positive maps and physical transformations may not be uniquely defined if the axiom of Local Tomography is not satisfied [37, 40].

3. *Sharpness vs coherence.* The sharpness condition (51) is generally not equivalent to the coherence condition (53). The two conditions become equivalent under the validity of the following

Assumption 4. *Every two effects $m, n \in V_S$ satisfying $n \leq m$ are compatible, that is, the three effects $n, m - n$ and $u_S - m$ can coexist in a measurement allowed by the theory.*

Assumption 4 holds for theories satisfying the Purification axiom (see Corollary 36 of Ref. [37]) and for theories with a Jordan-algebraic structure [28, 78]. Sharpness and coherence are potentially different notions for OPTs that do not satisfy assumption 4.

4. *Effects vs measurements.* While Ref. [1] focusses on measurements, Ref. [111] focusses on individual effects. As a result, an effect with a CLR may not be a sharp effect (i. e. an effect belonging to a sharp measurement): indeed, an effect m can have a CLR even if there exists no CLR for the complementary effect $u_S - m$ [112].

We have seen that defining a privileged set of measurements is important for the study of contextuality. Hence, one may want to define measurements starting from CLR. There is a tricky issue here: the most obvious definition “CLR measurement := measurement $\mathbf{m} = \{m_x\}_{x \in X}$ where each effect has a CLR rule ϕ_x ” does not have a clear operational meaning, because the collection of maps $\{\phi_x\}_{x \in X}$ may not correspond to any test allowed by the theory. For this reason, we suggest to define “CR measurement := measurement that can be implemented by a test $\mathcal{T} = \{\mathcal{T}_x\}_{x \in X}$ wherein each transformation induces a CLR for the corresponding effect.” Adopting this definition, we have the following

Proposition 6. *Sharp measurements coincide with CLR measurements in causal OPTs satisfying Assumptions 3 and 4.*

The proof follows from the discussion presented in Appendix F, which also contains more details on the relations between sharp measurements and CLR measurements.

15.3. Measurements consisting of extremal effects

Yet another possible generalization of projective quantum measurement is the one in terms of measurements that consist of extremal effects [98], i.e. effects that are extreme points of the set of effects associated to a given system²⁷. Note the difference between extremal effects and the notion of pure effects used in this paper: in quantum theory, the effect $p|0\rangle\langle 0|$ is pure in our sense, but is not extremal in the convex set of effects, because it is a mixture of the effect $|0\rangle\langle 0|$ with the zero effect. On the other hand, the projector $|0\rangle\langle 0| + |1\rangle\langle 1|$ is an extremal effect, but is not pure, because it can be obtained by coarse-graining the pure effects $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. In general, it is easy to see that the extremal effects in quantum theory are the projectors, while the pure effects are the rank-one positive operators upper bounded by the identity matrix.

15.4. Relations among the different definitions

A summary of the possible generalizations of projective quantum measurement is as follows:

1. Maximally discriminating measurements. Definition based on the notion of distinguishability of a maximal set of states.
2. Spiky measurements. Definition based on purity and orthogonality.
3. Sharp measurements. Definition based on the dynamical features of the measurement process, which is required to be repeatable and minimally disturbing
4. Measurements consisting of extremal effects. Definition based on the convex structure of the set of effects.

Not much is known about the relations among these four definitions, except for a few observations that one can readily make. First, if a pure measurement is maximal, one has the following implications:

Proposition 7. *Let \mathbf{m} be a pure measurement in a causal theory. Then,*

1. *if \mathbf{m} is maximal, then it is also spiky*
2. *\mathbf{m} is spiky iff it is sharp*
3. *if \mathbf{m} is spiky, then it consists of extremal effects.*

The proof is presented in Appendix G. One interesting question here is under which conditions the implication 1 can be reversed, i.e. under which conditions a pure spiky measurement is maximal. Here is a possible answer: Suppose that

1. the theory satisfies Pure State Identification, and
2. the set of effects that give probability 1 on a given state has a non-trivial lower bound, that is, for every system S and for every state $\rho \in \mathbf{St}(S)$, there exists an effect $a_\rho \in \mathbf{Eff}(S)$, $a_\rho \neq 0$ such that $a_\rho \leq m$ for every effect m such that $(m|\rho) = 1$.

²⁷This definition presupposes the mild assumption that such effects form a convex set.

These two conditions are sufficient to guarantee that all spiky pure measurements are maximal:

Proposition 8. *In a causal theory satisfying Conditions 1-2 every spiky pure measurement is maximal.*

The proof is provided in the Appendix H. It is worth stressing that the simple equivalences presented here hold for *pure* measurements, while the situation is much more involved for generic measurements. This fact prevents a direct comparison of the results of this paper with those of Ref. [1], where the main arguments were based on the properties of *non-pure* sharp measurements.

16. Conclusions

1000 In this paper we reviewed the device-independent framework for nonlocality/contextuality and the framework of general probabilistic theories, with the aim of bridging the gap between the two approaches. We see a high payoff in the transfer of results from one paradigm to the other. From the point of view of quantum axiomatizations, being able to reconstruct a device-independent principle from the axioms provides a direct access to many fundamental features of quantum nonlocality and contextuality. From the point of view of quantum nonlocality/contextuality, the approach of general probabilistic theories offers the possibility to find a deeper understanding of the device-independent features, which may help overcoming the current difficulties in finding a complete device-independent characterization.

In this paper we explored both directions. Following Ref. [1], we focussed on the principles of Local Orthogonality and Consistent Exclusivity and derived them from principles about the structure of the measurement process. The derivation presented here differs significantly from the one presented in Ref. [1], both in the requirements and in the notions used to formulate them. Essentially, the two papers investigate two different notions of “ideal measurement”, providing two different and potentially inequivalent generalizations of the notion of projective measurement in quantum theory. The two generalizations, called sharp and spiky measurements, respectively, refer to different operational properties of measurements: repeatability and minimal disturbance for the former, purity and orthogonality in the latter.

How should we interpret the fact that the same device-independent features—LO and CE in this case—can be reduced to two different physical pictures? Several answers are possible: On the one hand, one could argue that correlations are only a partial aspect of a physical theory and that, in fact, it is even possible that two different physical theories lead to the same set of correlations. In this sense, it is no surprise that inequivalent sets of physical principles entail the same device-independent bounds. On the other hand, one could argue that the framework of general probabilistic theories is *too* general, in that it allows for more theories than those that are actually worth studying. In general, the inequivalence of two sets of axioms could be due to some artificial and uninteresting counterexample. Inequivalent axioms could turn out to be equivalent

under some reasonable assumption—the only problem being that the right assumptions have yet to be pinpointed. As a matter of fact, we believe that both answers contain elements of truth.

In the present work, a partial simplification was achieved at the level of pure measurements, where the difference between sharp and spiky measurements disappears. It is remarkable that, once more [37, 20, 40], bringing the analysis to the level of pure processes simplifies proofs and unites different notions. This fact could be taken as a clue that the core of Quantum Theory is encoded in the peculiar interaction between the operational level and an underlying world of pure processes. From this point of view, the most natural continuation of the research initiated in this paper is to combine the Measurement Purification axiom with the State Purification axiom of Ref. [37], seeking for a new axiomatization of Quantum Theory only in terms of Purification features.

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Appendix A. Equivalence between causality and no-signalling

Proof. The implication $1 \implies 2$ is an immediate consequence of the uniqueness of the deterministic effect. To prove the implication $2 \implies 1$, let us assume that u_0 and u_1 are two deterministic effects for some system A . Consider a the following two-party scenario:

- Party 1 holds system A and party 2 holds system B . Systems A and B are prepared in some joint deterministic state $\sigma \in \text{St}_1(A \otimes B)$
- Party 1 applies either the effect u_0 or the effect u_1 depending on the value of her input $x_1 \in \{0, 1\}$. Since both measurements have a single outcome, in both cases the output y_1 can take a single value, say $y_1 = 0$
- Party 2 has a single measurement setting, say $x_2 = 0$, which consists in performing a measurement $\{b_{y_2}\}_{y_2 \in Y}$ on her system, getting the outcome y_2 .

Defining the probability distributions

$$p(y_1, y_2 | x_1, x_2) := (u_{x_1} \otimes b_{y_2} | \sigma),$$

we have that the no-signalling condition becomes

$$(u_0 \otimes b_{y_2} | \sigma) = (u_1 \otimes b_{y_2} | \sigma) \quad \forall y_2 \in Y,$$

or, equivalently

$$(u_0 | \rho_{y_2}) = (u_1 | \rho_{y_2}) \quad \forall y_2 \in Y,$$

where ρ_{y_2} is the state $\rho_{y_2} := (\mathcal{I}_A \otimes b_{y_2})\sigma$. Now, Assumption 1 guarantees that every state of system A is of the form $\rho_{y_2} = (\mathcal{I}_A \otimes b_{y_2})\sigma$ for some suitable state σ and some suitable measurement $\{b_{y_2}\}$. Hence, u_0 and u_1 give the same probability on every input state. By the very definition of effect (cf. paragraph II.F of Ref. [37]), this means $u_0 = u_1$. \square

Appendix B. Proof of proposition 2

Proof. Let m_0 and m_1 be two effects such that $(m_i | \rho_j) = \delta_{i,j}$ for arbitrary $i, j, \in \{0, 1\}$. Since every effect belongs to a measurement, there must exist a measurement $\{\tilde{m}_y\}_{y \in Y}$ such that $m_0 \equiv \tilde{m}_{y_0}$ for some outcome $y_0 \in Y$. By coarse-graining, the measurement $\{\tilde{m}_y\}_{y \in Y}$ can be turned into a binary measurement $\{m_0, m_{-0}\}$, defined by

$$m_{-0} := \sum_{y \in Y, y \neq y_0} \tilde{m}_y.$$

By construction one has $(m_0 | \rho_0) = (m_{-0} | \rho_1) = 1$ and $(m_0 | \rho_1) = (m_{-0} | \rho_0) = 0$. In other words, the states ρ_1 and ρ_0 can be perfectly distinguished using the measurement $\{m_0, m_{-0}\}$. \square

Appendix C. Proof of lemma 1

Proof. For every setting x , use the measurement purification axiom to represent the measurement \mathbf{m}^x as $(m_y^x | = (M_y^x | [\mathcal{I}_S \otimes |\sigma_x\rangle])$ for some orthogonal measurement \mathbf{M}^x on $S \otimes E_x$ and for some state $\sigma_x \in \text{St}(E_x)$, where E_x is a suitable environment. Since there is a finite number of settings, one can always define $E := \bigotimes_{x \in X} E_x$, $\sigma := \bigotimes_{x \in X} \sigma_x$ and replace the measurement \mathbf{M}^x with a new spiky measurement \mathbf{N}^x given by

$$(N_y^x | := (M_y^x | \otimes \left[\bigotimes_{x' \in X, x' \neq x} (u_{x'} | \right],$$

where u_x denotes a unit effect on system E_x (note that, since Causality is not assumed in the hypothesis, the unit effect may not be unique). In this way, the probability distribution can be expressed as

$$\begin{aligned} p(y|x) &= (m_y^x | \rho) \\ &= (M_y^x | \rho \otimes \sigma_x) \\ &= (N_y^x | \rho \otimes \sigma). \end{aligned}$$

□

Appendix D. Violation of SO for polygons with odd number of vertices

To prove that the effects $\{a_y, a_{y \oplus \frac{n+1}{2}}\}$ cannot coexist in a measurement, we show that the sum of their probabilities on the state $|\varphi_{y \ominus 1}\rangle$ exceed one. Indeed, define

$$s := (a_y | \varphi_{y \ominus 1}) + (a_{y \oplus \frac{n+1}{2}} | \varphi_{y \ominus 1}).$$

Then, Eq. (17) yields

$$s = \frac{1}{r_n^2 + 1} \left[r_n^2 \cos\left(\frac{2\pi}{n}\right) - r_n^2 \cos\left(\frac{3\pi}{n}\right) + 2 \right].$$

Now, the condition $s > 1$ is equivalent to

$$r_n^2 \cos\left(\frac{2\pi}{n}\right) - r_n^2 \cos\left(\frac{3\pi}{n}\right) - r_n^2 + 1 > 0.$$

Inserting the definition $r_n := \sqrt{1/\cos(\pi/n)}$ into this inequality, one obtains

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{\pi}{n}\right) > \cos\left(\frac{3\pi}{n}\right) + 1,$$

which is equivalent to

$$2 \cos\left(\frac{3\pi}{2n}\right) \cos\left(\frac{\pi}{2n}\right) > 2 \left[\cos\left(\frac{3\pi}{2n}\right) \right]^2, \quad (\text{D.1})$$

1100 having used the relation $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$. Clearly, the inequality (D.1) is satisfied for every $n \geq 5$. Hence, all the polygons with $n > 3$ odd vertices violate SO.

Appendix E. Proof of proposition 3

Proof. Let $H = (Y, X)$ be the hypergraph associated to a given contextual game and let $\hat{w} : Y \rightarrow \text{Eff}(S)$ be the effect-valued weight describing the player's strategy in a physical implementation of the game. Clearly, for every exclusive set of vertices $E \subset Y$ (in the sense of definition 18), the effects $\{\hat{w}(y)\}_{y \in E}$ are mutually exclusive (in the sense of definition 25). It is also clear that, if the mutually exclusive effects $\{\hat{w}(y)\}_{y \in E}$ coexist in a single measurement, then one has the inequality

$$\sum_{y \in E} (\hat{w}(y) | \rho) \leq 1, \quad (\text{E.1})$$

for every state $\rho \in \text{St}_1(\rho)$. This means that every probability weight w generated by the effect-valued weight \hat{w} satisfies CE. \square

Appendix F. Sharp measurements and coherent Lüders rules

In the following we provide a more detailed discussion of the difference between sharp measurements and CLR's.

Appendix F.1. Framework

The framework of Ref. [111] differs from the OPT framework in a number of significant ways. Ref. [111] associates a physical system S with an order unit vector space (OUVS), which we denote by V_S . The unit in V_S corresponds to the deterministic effect u_S , and every positive element $m \in V_S$ satisfying the condition $m \leq u_S$ is assumed to be an effect, physically realizable in some test. In the language of OPTs, this amounts to the assumption

Assumption 5. $\text{Eff}(S) = \{m \in V_S \mid 0 \leq m \leq u_S\}$.

Such a condition means that

1. system S has a unique deterministic effect
2. the set of effects $\text{Eff}(S)$ is convex
3. effects can be "scaled up": for every effect $m \in \text{Eff}(S)$ and for every scaling coefficient $\lambda \geq 1$ satisfying $\lambda m \leq u_S$, one has that λm belongs to $\text{Eff}(S)$.

For theories that are not deterministic, the first two conditions can be operationally motivated as part of the “Causality package”—in particular, see Corollary 5 of Ref. [37] for the convexity of $\text{Eff}(S)$. Although Causality is a very natural assumption (one that we also wanted to make in this paper), it is worth noting that the operational definition of sharp measurement (definition 29) in terms of repeatability and minimal disturbance can be applied even in exotic non-causal scenarios, like those arising in Refs.[93, 94, 37, 95, 96, 97].

The condition that effects can be scaled up is more specific. It would follow if one assumed the No Restriction Hypothesis [37, 91], which guarantees both the validity of Assumption (3) and the fact that $\text{Eff}(S)$ is the full dual cone associated to the set of states $\text{St}(S)$. One way to motivate the No-Restriction Hypothesis on operational grounds is provided by Barnum, Müller, and Ududec [78], who showed that the No-Restriction Hypothesis holds if *i*) all states of system S can be decomposed into convex combinations of perfectly distinguishable pure states and *ii*) every two sets of perfectly distinguishable pure states can be connected by a reversible transformation. Interesting, one way to bypass the No-Restriction hypothesis is to assume the Purification axiom [37], which also guaranteed that effects can be scaled up (for the proof, see Corollary 36 of Ref. [37]).

Appendix F.2. Positive maps vs physical transformations

A direct comparison between Refs. [1] and [111] can be made only for the restricted set of OPTs that satisfy the conditions 1-3. From now on we restrict our attention to such theories.

Given an effect $m \in V_S$, Kleinmann defines a *coherent Lüders rule (CLR)* for m as a positive linear map $\phi : V_S \rightarrow V_S$ satisfying the conditions

$$\phi(u_S) = m \quad (m\text{-compatibility}) \quad (\text{F.1})$$

$$\phi(n) = n \quad \forall n \in V_S : 0 \leq n \leq m \quad (\text{coherence}). \quad (\text{F.2})$$

The map ϕ is interpreted as a *potential candidate* for a physical transformation. Whether or not a given map ϕ really represents a physical transformation, however, is another issue: the definition of CLR only refers to positive maps satisfying conditions (F.1) and (F.2).

In order to compare Refs. [1] and [111] we need to restrict our attention to those effects m that admit a CLR with the extra property that the map ϕ represents a physical transformation. While Ref. [111] does not specify how physical transformations are defined, for the sake of comparison we now assume that *some* choice has been made. In our opinion, the most sensible way to make such a choice is to start from a full OPT, where the composition of transformations in parallel and sequence is built in the operational structure (thanks to the adoption of the categorical framework by Abramsky and Coecke [61, 62, 32]). The advantage of using the categorical approach with respect to the single-system approach is that in this way one can bypass problems like

1. the difference between positivity and complete positivity (maps that send effects into effect for individual systems may not do so when applied locally on composite systems), and

2. the fact that the action of a transformation on a composite system may not be uniquely determined by the linear map ϕ (this is often the case when the axiom of Local Tomography is not satisfied [37, 40]).

Let us denote by $V_{S \rightarrow S}^{\text{phys}}$ the set of positive maps induced by physical transformations, in the following sense:

Definition 30 (cf. Eq. (22) of [37]). *The map $\phi : V_S \rightarrow V_S$ is induced by the physical transformation $\mathcal{T} \in \text{Transf}(S \rightarrow S)$ iff for every effect $m \in V_S$ one has $\phi(m) = m'$ where m' is the effect defined by $(m'| := (m|\mathcal{T}$.*

A *physical CLR* is one where the map ϕ belongs to $V_{S \rightarrow S}^{\text{phys}}$.

Appendix F.3. Sharpness vs coherence

Equipped with the definition of physical CLR, we can now compare the coherence condition of Eq. (F.2) with the sharpness condition of Eq. (51). In general, these two conditions express different operational requirements: Kleinmann’s coherence condition requires that ϕ do not disturb all the effects n that are “less likely to be triggered” than m , in the following sense

Definition 31. *The effect n is less likely to be triggered than m iff $n \leq m$.*

Our sharpness condition requires fact that ϕ do not disturb all the effects that are “compatible with” m . This means that the three effects

$$m_1 = n, \quad m_2 = m - n, \quad m_3 = u_S - m$$

coexist in a measurement. Such a condition is stronger than just $n \leq m$. Due to this fact, an effect m may satisfy the sharpness condition (51), and still fail to satisfy the coherence condition (53). The two conditions become equivalent under the following

Assumption 6. *Every two effects $m, n \in V_S$ satisfying $n \leq m$ are compatible.*

At present, we (the authors) only know that the assumption holds for theories satisfying for theories satisfying the Purification axiom (for the latter, see again Corollary 36 of Ref. [37]) and for theories with a Jordan-algebraic structure [28, 78].

Appendix F.4. Effects vs measurements.

While we have so far we discussed about individual effects m , it is eventually interesting to bring the comparison to the level of measurements. In this there is some ambiguity, since Ref. [111] does not give an explicit definition of *CLR measurement*. One might be tempted to define it as a measurement $\mathbf{m} = \{m_x\}_{x \in X}$ for which each every effect m_x admits a physical CLR ϕ_x . This definition, however, does not have a clear operational meaning, because it is not a priori clear if the collection of maps $\{\phi_x\}_{x \in X}$ is a measurement allowed by the theory. As per our present knowledge, such a condition is met by theories

satisfying Local Tomography and Purification, and possibly for some theories with Jordan algebraic structure. In general, the most reasonable approach is to define a CLR measurement $\mathbf{m} = \{m_x\}_{x \in \mathsf{X}}$ as a measurement that can be implemented by test $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathsf{X}}$ for which each transformation induces a CLR. When this definition is adopted, Proposition 6 follows from the discussion presented in the previous points.

Appendix G. Proof of Proposition 7

Proof. The implication 1 is an immediate consequence of the definitions: by definition, a maximal measurement is orthogonal. Hence, a pure maximal measurement is a pure orthogonal measurement. By definition, pure orthogonal measurements are a special case of spiky measurements (with “only one spike per outcome”). The equivalence between spiky and sharp measurements at point 2 is proven as follows: Since the refinements of a pure measurement are trivial, Proposition 5 implies that a pure measurement $\mathbf{m} = \{m_y\}_{y \in \mathsf{Y}}$ is sharp iff there exists a test $\{\mathcal{M}_y\}_{y \in \mathsf{Y}}$ such that

$$(m_y | \mathcal{M}_y = (m_y | \quad \forall y \in \mathsf{Y}. \quad (\text{G.1})$$

In other words, a pure measurement is sharp iff it is repeatable.

We first prove the implication “ \mathbf{m} is spiky” \implies “ \mathbf{m} is sharp”. If \mathbf{m} is spiky, then there exists a set of states $\{\rho_y\}$ such that $(m_y | \rho_{y'}) = \delta_{y,y'}$. Since the theory satisfies Assumption 2, we can define the measure-and-prepare test $\{\mathcal{M}_y\}$ with $\mathcal{M}_y := |\rho_y\rangle\langle m_y|$, which, by construction satisfies the condition (G.1). Hence, we proved that \mathbf{m} is sharp. Let us prove the converse implication “ \mathbf{m} is sharp” \implies “ \mathbf{m} is spiky”. If \mathbf{m} is sharp, one can take a state ρ such that $(m_y | \rho) > 0$ for every y . In a convex theory, one can always find one such state by mixing sufficiently many states in the state space of the system²⁸. Note that the possibility of mixing states is guaranteed by Assumption 2. Since \mathbf{m} is a sharp measurement, there exists a test $\mathcal{M} = \{\mathcal{M}_y\}_{y \in \mathsf{Y}}$ such that $(m_y | = (u | \mathcal{M}_y$ for every outcome y . Thus, one can define the state

$$\rho_y := \frac{\mathcal{M}_y | \rho}{(m_y | \rho)}.$$

By construction, one has $(m_y | \rho_y) = 1$ for every y , which, by the normalization of probabilities implies the orthogonality condition $(m_y | \rho_{y'}) = \delta_{y,y'}$ for every y, y' . This proves that \mathbf{m} is a (pure) orthogonal measurement. Hence, \mathbf{m} is spiky. Finally, we prove the implication at point 3: “ \mathbf{m} is spiky” \implies “ \mathbf{m} is extremal”. For a given outcome $y \in \mathsf{Y}$, suppose that

$$m_y = p n_y + (1 - p) o_y, \quad (\text{G.2})$$

²⁸In the whole proof, this is the only point invoking convexity.

where $p \in (0, 1)$ is a probability and n_y and o_y are two effects. Since \mathbf{m} is orthogonal, there exists a state ρ_y such that $(m_y|\rho_y) = 1$. Hence, Eq. (G.2) implies the relation $1 = p(n_y|\rho_y) + (1-p)(o_y|\rho_y)$ and, therefore

$$(n_y|\rho_y) = (o_y|\rho_y) = 1. \quad (\text{G.3})$$

Since m_y is pure, Eq. (G.2) implies $n_y = \alpha_y m_y$ and $o_y = \beta_y m_y$ for two suitable constants $\alpha_y \geq 0$ and $\beta_y \geq 0$. Using Eq. (G.3) one finally obtains

$$1 = (n_y|\rho_y) = \alpha_y (m_y|\rho_y) = \alpha_y$$

and

$$1 = (o_y|\rho_y) = \beta_y (m_y|\rho_y) = \beta_y.$$

In conclusion, one has $n_y \equiv o_y \equiv m_y$. This means that the effect m_y is extremal. Since the outcome y is generic, we obtained that the whole measurement \mathbf{m} is extremal. \square

Appendix H. Proof of Proposition 8

Proof. We have to prove that a pure spiky measurement is maximal. To this purpose, observe that for pure measurements “spiky” is synonymous of “orthogonal”. Now, suppose that $\mathbf{a} = \{a_y\}_{y \in Y}$ is a pure orthogonal measurement and let $\{\rho_y\}_{y \in Y}$ be the set of states such that $(a_y|\rho'_y) = \delta_{y,y'}$. By the Pure State Identification Property, each state ρ_y must be pure—let us denote it as $\rho_y =: \varphi_y$. Now, we prove that the set $\{\varphi_y\}$ is maximal. Indeed, suppose by absurd that the states $\{\varphi_y\} \cup \rho$ are perfectly distinguishable for some ρ , and let $\{m_y\} \cup m_\rho$ the measurement that distinguishes among them. Since $(m_y|\varphi_y) = 1$, we must have $m_y \geq l_y$, where $l_y \neq 0$ is the lower bound to the set of effects that have probability 1 on φ_y (the existence of the lower bound is guaranteed by Condition 2). But since $(a_y|\varphi_y) = 1$, we must also have $a_y \geq l_y$. By the purity of a_y , this implies $l_y = p_y a_y$, for some probability $p_y > 0$. This condition implies the relation

$$\begin{aligned} (a_y|\rho) &= p_y^{-1}(l_y|\rho) \\ &\leq p_y^{-1}(m_y|\rho) \\ &= 0 \quad \forall y \in Y. \end{aligned}$$

Moreover, since the effects $\{a_y\}_{y \in Y}$ form a measurement, we have $(u|\rho) = \sum_y (a_y|\rho) = 0$, which implies that ρ is the zero state, $\rho = 0$. Hence, the set of pure states $\{\varphi_y\}$ is maximal. \square

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