On Finite Blaschke Products Sharing Preimages of Sets

The role of complex analysis in complex dynamics in ICMS

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23 May 2013

Problem

Given two compact sets $E_1, E_2 \subset \mathbb{D}$, how to characterize all the finite Blaschke products B_1, B_2 satisfying

$$B_1^{-1}(E_1) = B_2^{-1}(E_2)?$$

Will solve this problem when E_1 and E_2 are connected compact sets of positive hyperbolic capacity.

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Background

B is a finite Blaschke product of degree n if

$$B(z) = e^{i\theta} \frac{z - z_1}{1 - \overline{z_1} z} \cdot \frac{z - z_2}{1 - \overline{z_2} z} \cdot \cdots \cdot \frac{z - z_n}{1 - \overline{z_n} z},$$

where $z_i \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

- Fatou (1923) proved that $B : \mathbb{D} \to \mathbb{D}$ is analytic and n-valent (i.e., every point in $\mathbb{D}$ has precisely n preimages in $\mathbb{D}$ counted with multiplicity) iff B is a finite Blaschke product of degree n.
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Finite Blaschke Products vs Polynomials

Dictionary

These two kinds of finite maps share many similar properties and hence we can establish a dictionary between these two kinds of finite maps.

Let $f : X \to X$ be a finite map with $\deg f > 1, X = \mathbb{C}, \mathbb{D}$.

Definition

A polynomial/finite Blaschke product f is said to be *prime* if there do not exist two polynomials/finite Blaschke products f_1, f_2 with $\deg f_1, \deg f_2 \geq 2$ s.t.

$$f(z) = f_1[f_2(z)].$$

Otherwise, f is called *composite*.

Given a polynomial/finite Blaschke product f, we can factorize it as a composition of prime polynomials/finite Blaschke products only, and this factorization will be called a *prime factorization*.

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A nonlinear polynomial P is composite if and only if the monodromy group of P is imprimitive.

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Any two prime factorizations of P have the same number of prime polynomial factors.

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1) $\tilde{P} \circ \hat{P} = (\tilde{P} \circ L) \circ (L^{-1} \circ \hat{P})$, *with polynomials* $\tilde{P}, \hat{P}$ *and linear polynomial* L;
2) $T_m \circ T_n = T_n \circ T_m$, *where* T_n *is the degree* n *Chebyshev polynomial;*
3) $z^r[P_0(z)]^k \circ z^k = z^k \circ [z^r P_0(z^k)]$, *with* $r, k \in \mathbb{Z}^+$ *and* $P_0 \in \mathbb{C}[z]$.

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A finite Blaschke product B $(\deg B > 1)$ is composite if and only if the monodromy group of B is imprimitive.

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Chebyshev Blaschke Products

Definition

The Chebyshev polynomial $T_n(z)$ is a polynomial of degree n, defined by

$$T_n(\cos\theta) = \cos n\theta.$$

To define Chebyshev Blaschke products:

- $\cos\theta$ is replaced by $\mathrm{cd}(u,\tau) := \frac{\mathrm{cn}(u,\tau)}{\mathrm{dn}(u,\tau)}$ for $\tau \in \mathbb{R}_+ i$.
- Note that $\mathrm{cd}(u,\tau) = \mathrm{sn}(u + \frac{\omega_1}{2}, \tau)$.
  $\therefore$ cd is an elliptic function with the periods $2\omega_1$ and ω_2, where

$$\begin{aligned} \omega_1(\tau) &= \pi\vartheta_3^2(0,\tau) = \pi(1 + 2q + 2q^4 + \cdots)^2, q = e^{\pi i\tau} \\ \omega_2(\tau) &= \tau\omega_1(\tau). \end{aligned}$$

- The elliptic modulus $k(\tau) = \frac{\vartheta_2^2(0,\tau)}{\vartheta_3^2(0,\tau)}$, $\sqrt{k(\tau)} := \frac{\vartheta_2(0,\tau)}{\vartheta_3(0,\tau)}$.

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$$f_{n,\tau}(\sqrt{k(\tau)}\,\mathrm{cd}(u\omega_1(\tau),\tau)) = \sqrt{k(n\tau)}\,\mathrm{cd}(nu\omega_1(n\tau),n\tau).$$

For example,

$$\begin{aligned} f_{1,\tau}(z) &= z; \\ f_{2,\tau}(z) &= \frac{z^2-a}{1-az^2}, \text{ where } a=\sqrt{k(2\tau)}=\frac{\vartheta_2(0,2\tau)}{\vartheta_3(0,2\tau)}; \\ f_{3,\tau}(z) &= z\frac{z^2-a}{1-az^2}, \text{ where } a=k(\tau)\mathrm{cd}^2(\frac{\omega_1(\tau)}{6},\tau)=\frac{\vartheta_2^2(1/6,\tau)}{\vartheta_3^2(1/6,\tau)}. \end{aligned}$$

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The Correspondence Table

	T_n	$f_{n,\tau}$
Definition	$T_n(z) = \cos n\theta$, where $z = \cos\theta$	$f_{n,\tau}(z) = \sqrt{k(n\tau)}\,\mathrm{cd}(nu\omega_1(n\tau), n\tau)$, where $z = \sqrt{k(\tau)}\,\mathrm{cd}(u\omega_1(\tau), \tau)$
Zeros	$z_p = \cos\frac{(2p-1)\pi}{2n}$ $(p = 1, \ldots, n)$	$z_p = \sqrt{k(\tau)}\,\mathrm{cd}(\frac{(2p-1)\omega_1(\tau)}{2n}, \tau)$ $(p = 1, \ldots, n)$
Critical points	$w_p = \cos\frac{p\pi}{n}$ $(p = 1, \ldots, n-1)$	$w_p = \sqrt{k(\tau)}\,\mathrm{cd}(\frac{p\omega_1(\tau)}{n}, \tau)$ $(p = 1, \ldots, n-1)$
Critical values	± 1 in $\mathbb{C}$	$\pm\sqrt{k(n\tau)}$ in $\mathbb{D}$
Preimage	$T_n^{-1}([-1, 1]) = [-1, 1]$	$f_{n,\tau}^{-1}([-\sqrt{k(n\tau)}, \sqrt{k(n\tau)}]) = [-\sqrt{k(\tau)}, \sqrt{k(\tau)}]$
Nesting property	$T_{mn} = T_m \circ T_n$	$f_{mn,\tau} = f_{m,n\tau} \circ f_{n,\tau}$
Julia set	$J(T_n) = [-1, 1]$	$J(f_{n,\tau}) = \partial\mathbb{D}$

Objective

Given two compact sets $E_1, E_2 \subset \mathbb{D}$, try to characterize all finite Blaschke products B_1, B_2 satisfying $B_1^{-1}(E_1) = B_2^{-1}(E_2)$.

Know how to do it for polynomials p_1, p_2 sharing compact $K_1, K_2 \subset \mathbb{C}$:

$$p_1^{-1}(K_1) = p_2^{-1}(K_2) = K.$$

The case $K_1 = K_2 = K =$ Julia set of p_1 and p_2 has been studied by, Baker & Eremenko(1987), Beardon (1992), Schmidt & Steinmetz (1995), Atela & Hu (1996),etc.

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Known Results for Polynomials

Sharing set problems $f_1^{-1}(K_1) = f_2^{-1}(K_2)$ for $f_1, f_2 \in \mathbb{C}[z]$ were studied:

- If $K_1 = K_2 = \{-1, 1\}$ and $\deg f_1 = \deg f_2$, Pakovitch (1995) proved that $f_1 = \pm f_2$ which solved a problem of C.C. Yang (1978)
- Pakovitch made use of the uniqueness property of the least deviations from zero.
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Theorem (N. & C.Y. Tsang)

Let B_1, B_2 be finite Blaschke products, $\deg B_1 = d_1$, $\deg B_2 = d_2$, $d_1 \leq d_2$, *and $E_1, E_2 \subset \mathbb{D}$ be connected compact sets of positive hyperbolic capacity s.t.*

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Furthermore there exist finite Blaschke products $\tilde{B}_1, \tilde{B}_2, W$, *with* $\deg W = d$, *s.t.*

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Hyperbolic Capacity

For a compact subset $E \subset \mathbb{D}$, the hyperbolic capacity $\mathrm{cap}_h(E)$ can be defined in a similar way of the logarithmic capacity (by replacing the Euclidean metric $|z - \zeta|$ by the pseudohyperbolic metric

$$\rho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|, \ z, \zeta \in \mathbb{D}.)$$

Let $\mathcal{P}(E)$ be the class of all probability measures on a compact set $E \subset \mathbb{D}$.

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Let $E \subset \mathbb{D}$ be compact and $\mu \in \mathcal{P}(E)$. The *hyperbolic potential* of μ is the function $u_\mu^h : \overline{\mathbb{D}} \to (-\infty, +\infty]$ defined by

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Hyperbolic Equilibrium Measure

Theorem (M.Tsuji, 1947)

For each compact set $E \subset \mathbb{D}$ with $\operatorname{cap}_h(E) > 0$, there exists a unique measure μ_E^h s.t. $V_E^h = I_h(\mu_E^h)$.

Such a measure μ_E^h is called the *hyperbolic equilibrium measure* for E.

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Let $E \subset \mathbb{D}$ be compact and let μ_E^h be the hyperbolic equilibrium measure for E. Then its potential $u^h_{\mu_E^h}$ has the following properties:

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Application of the Hyperbolic Equilibrium Measure

Theorem (A)

Let B_1 and B_2 be finite Blaschke products of degrees $d_1 \geq 1$ and $d_2 \geq 1$ respectively, and let $E_1, E_2 \subset \mathbb{D}$ be compact.
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$$\frac{u^h_{\mu^h_{E_1}} \circ B_1(z)}{d_1} = \frac{u^h_{\mu^h_{E_2}} \circ B_2(z)}{d_2}, \quad \text{for all } z \in \overline{\mathbb{D}}.$$

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Pullback Measure

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Given a probability measure μ on E, the *pullback measure* $B^*\mu$ is the probability measure on $B^{-1}(E)$ s.t. for all holomorphic functions f on $B^{-1}(E)$,

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where the summation is over all the roots of $B(\xi) - \zeta$ and a root of multiplicity m is repeated m times. Indeed,

$$B^*\mu(B^{-1}(E_0)) = \int_{E_0} \sum_{\xi \in B^{-1}(\{\zeta\})} 1 d\mu(\zeta) = d \cdot \mu(E_0), \; E_0 \subset E.$$

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Equilibrium Measure on $B^{-1}(E)$

Proposition

Let B be a finite Blaschke product of degrre d and $\Omega = B^{-1}(E)$. Suppose $\mathrm{cap}_h(E) > 0$. If μ_E^h is the equilibrium measure on E, then the equilibrium measure μ_Ω^h on Ω is

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Recall that

Theorem (A)

Let B_1 and B_2 be finite Blaschke products of degrees $d_1 \geq 1$ and $d_2 \geq 1$ respectively, and let $E_1, E_2 \subset \mathbb{D}$ be compact. Suppose that $\mathrm{cap}_h(E_1), \mathrm{cap}_h(E_2) > 0$ and $\Omega := B_1^{-1}(E_1) = B_2^{-1}(E_2)$. Then

$$\frac{u^h_{\mu^h_{E_1}} \circ B_1(z)}{d_1} = \frac{u^h_{\mu^h_{E_2}} \circ B_2(z)}{d_2}, \quad \text{for all } z \in \overline{\mathbb{D}}.$$

To prove this theorem, we need the following lemma.

Lemma

Let μ be a finite Borel measure on $\mathbb{D}$ with compact support. Then

$$\Delta u^h_\mu = -2\pi\mu,$$

where Δ is the generalized Laplacian.

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Proof of Theorem A

Let $\mu^h_{E_j}$ be the hyperbolic equilibrium measure of E_j.

- $d_1^{-1}B_1^*\mu^h_{E_1}$ and $d_2^{-1}B_2^*\mu^h_{E_2}$ are hyperbolic equilibrium measures of Ω.
- By the uniqueness, we get $d_1^{-1}B_1^*\mu^h_{E_1} = d_2^{-1}B_2^*\mu^h_{E_2}$.
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- Let $\Psi = u^h_{\frac{B_1^*\mu^h_{E_1}}{d_1}} - u^h_{\frac{B_2^*\mu^h_{E_2}}{d_2}}$ on $\overline{\mathbb{D}}$. As $\Delta\Psi = 0$, Ψ is harmonic on $\mathbb{D}$. As $u^h_{\frac{B_i^*\mu^h_{E_i}}{d_i}} \equiv 0$ on $\partial\mathbb{D}$, $\Psi \equiv 0$ on $\partial\mathbb{D}$.
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Invariants of a Finite Blaschke Product near $\partial\mathbb{D}$

- Study the continuous function $u : \partial\mathbb{D} \to \partial\mathbb{D}$ which is invariant for a finite Blaschke product B, i.e.,

$$B \circ u = B \text{ on } \partial\mathbb{D}.$$

- In fact, these functions form a cyclic group.

Theorem (Cassier & Chalendar (2000))

Let B be a finite Blaschke product of degree $d \geq 1$. The set of the continuous functions $u : \partial\mathbb{D} \to \partial\mathbb{D}$ s.t. $B \circ u = B$ is a cyclic group (for the composition) of order d, say $\{u_1, \cdots, u_d\}$.

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Analytic Extension of u_k

Moreover, each u_k can be extended analytically to a neighborhood of $\partial\mathbb{D}$.

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Denote the extension $\tilde{u}_1$ by u_B.

Then u_B is a conformal map in a small neighborhood of $\partial\mathbb{D}$ s.t.

1. $B \circ u_B^{\circ k} = B$ $(1 \leq k \leq d)$,
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The u_B will give information about the factorizations of B

Theorem (B)

Let B_1 and B_2 be two finite Blaschke products of degrees $d_1 \geq 1$ and $d_2 \geq 1$, and let $d = \gcd(d_1, d_2)$.

(a) *If Φ is a finite Blaschke product s.t. $\Phi \circ u_{B_1} = \Phi$ in the neighborhood of $\partial\mathbb{D}$, then there exists a finite Blaschke product B s.t.*

$$\Phi = B \circ B_1.$$

(b) *If $u_{B_1}^{\circ k_1 d_1/d} = u_{B_2}^{\circ k_2 d_2/m}$ ($\gcd(k_j, d) = 1$), then there exist finite Blaschke products B, $\tilde{B}_1$, $\tilde{B}_2$ ($\deg B = d$) s.t.*

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How to get $g_1 \circ B_1 = g_2 \circ B_2$?

By Theorem A, we have

$$\frac{u^h_{\mu^h_{E_1}} \circ B_1(z)}{d_1} = \frac{u^h_{\mu^h_{E_2}} \circ B_2(z)}{d_2}, \quad \text{for all } z \in \overline{\mathbb{D}}.$$

- For $i = 1, 2$, let K_i be the component of $\mathbb{D} \setminus E_i$ which borders on $\partial\mathbb{D}$. Since E_i is connected, K_i is doubly connected and there exists a biholomorphic function φ_i from K_i onto $\{\rho_i < |w| < 1\}$ s.t. $\varphi_i(\partial\mathbb{D}) = \partial\mathbb{D}$.
- Note that $u^h_{\mu^h_{E_i}}(z) = -\log|\varphi_i(z)|$ for all $z \in K_i$.
- $\frac{\log|\varphi_1 \circ B_1(z)|}{d_1} = \frac{\log|\varphi_2 \circ B_2(z)|}{d_2}$ and $|z^{d_2} \circ \varphi_1 \circ B_1(z)| = |z^{d_1} \circ \varphi_2 \circ B_2(z)|$ for any $z \in \mathbb{D}$ sufficiently close to $\partial\mathbb{D}$.

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$$\frac{u^h_{\mu^h_{E_1}} \circ B_1(z)}{d_1} = \frac{u^h_{\mu^h_{E_2}} \circ B_2(z)}{d_2}, \quad \text{for all } z \in \overline{\mathbb{D}}.$$

- For $i = 1, 2$, let K_i be the component of $\mathbb{D} \setminus E_i$ which borders on $\partial\mathbb{D}$. Since E_i is connected, K_i is doubly connected and there exists a biholomorphic function φ_i from K_i onto $\{\rho_i < |w| < 1\}$ s.t. $\varphi_i(\partial\mathbb{D}) = \partial\mathbb{D}$.
- Note that $u^h_{\mu^h_{E_i}}(z) = -\log|\varphi_i(z)|$ for all $z \in K_i$.
- $\frac{\log|\varphi_1 \circ B_1(z)|}{d_1} = \frac{\log|\varphi_2 \circ B_2(z)|}{d_2}$ and $|z^{d_2} \circ \varphi_1 \circ B_1(z)| = |z^{d_1} \circ \varphi_2 \circ B_2(z)|$ for any $z \in \mathbb{D}$ sufficiently close to $\partial\mathbb{D}$.

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- For any $z \in \mathbb{D}$ sufficiently close to $\partial\mathbb{D}$,

$$\Psi(z) := z^{d_2} \circ \varphi_1 \circ B_1(z) = e^{i\theta} z^{d_1} \circ \varphi_2 \circ B_2(z).$$

- φ_i can extend continuously and homeomorphically to $\partial\mathbb{D}$ and

$$\Psi = z^{d_2} \circ \varphi_1 \circ B_1 = e^{i\theta} z^{d_1} \circ \varphi_2 \circ B_2$$

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- Try to show that u_{B_1} and u_{B_2} satisfy conditions in (b) or (c) of Theorem B.

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Pakovich's approach for $\Omega = B_1^{-1}(E_1) = B_2^{-1}(E_2)$

Let $\mathcal{B}_n$ denote the set of all finite Blaschke products of degree n and let $E \subset \mathbb{D}$ be compact.

Definition

A finite Blaschke product $\tilde{B} \in \mathcal{B}_n$ is called a minimal Blaschke product of degree n for E if $\|\tilde{B}\|_E = \min_{B \in \mathcal{B}_n} \|B\|_E$.

Theorem (Walsh (1952))

(Existence and location of zeros) A minimal Blaschke product $\tilde{B}$ exists and its zeros lie in the convex hull of E with respect to the hyperbolic geometry in $\mathbb{D}$.

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The counter-part of the two conjectures below are known to be true for ploynomials.

Conjecture (A)

Such a minimal Blaschke product of degree n is unique up to multiplication by $e^{i\theta}$ when $|E| \geq n$.

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Let T be a minimal Blaschke product of degree m for E. Then for any finite Blaschke product B of degree n, $T \circ B$ is a minimal Blaschke product of degree mn for $B^{-1}(E)$.

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- For $i = 1, 2$, let C_i be a minimal Blaschke product of degree d_i/d for E_i, where $d = \gcd(d_1, d_2)$.
- By conjecture B, $C_i \circ B_i$ is a minimal Blaschke product of degree $d_1 d_2/d$ for $\Omega = B_i^{-1}(E_i)$.
- Suppose $|\Omega| \geq \operatorname{lcm}(d_1, d_2) = d_1 d_2/d$, by Conjecture A, we have

$$C_1 \circ B_1 = e^{i\theta} C_2 \circ B_2$$

- Then apply Ritt's theorem for finite Blaschke products.

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