# Identification of critical combination of vulnerable links in transportation networks – a global optimization approach

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# Identification of critical combination of vulnerable links in transportation networks – a global optimization approach

**Abstract**: This paper presents a global optimization framework for identifying the most critical combination of vulnerable links in a transportation network. The problem is formulated as a mixed-integer nonlinear program with equilibrium constraints, aiming to determine the combination of links whose deterioration would induce the most increase in total travel cost in the network. A global optimization solution method applying a piecewise linearization approach and range reduction technique is developed to solve the model. From the numerical results, it is interesting and counterintuitive to note that the set of most vulnerable links when simultaneous multiple-link failure occurs are not simply the combination of the most vulnerable links with single-link failure, and the links in the critical combination of vulnerable links are not necessarily connected or even in the neighbourhood of each other. The numerical results also show that the ranking of vulnerable links will be significantly affected by certain input parameters.

**Keywords**: resilience; vulnerability; transport network; MPEC; global optimization

## 1. Introduction

The prosperity of a society is closely linked to the performance of the supporting transportation infrastructure, with consequences to economic growth and quality of life. Unfortunately, it is known that the operation of transportation infrastructure is vulnerable to unexpected disruptions. A key issue in the vulnerability analysis and resilience transport system design is to identify the critical elements (e.g., set of links or nodes) of a network, where the failure of these elements would bring the most serious impacts on the entire system (Chen et al., 2012). The robustness of the transportation network can then be enhanced through reinforcing these critical elements subject to budgetary constraints.

Traditionally, critical elements in a network can be identified by using a brute-force simulation based approach (Taylor et al., 2006). Each link or node is iteratively removed from the network and the corresponding consequence is estimated as the changes in the network performance (e.g., increase / decrease in travel times) through repeated simulations. Nevertheless, there are two weaknesses in such simulation-based approach. First, the travellers' responses are not captured. As it is understood, travellers will respond (e.g., reroute or reschedule their travel) when the conditions of the transportation network change. Second, the simulation-based approach does not guarantee the global optimality of the solution unless all possible combinations are scanned. However, the number of possible combinations can be prohibitively huge such that it may be time-consuming to simulate the outcomes of all possible combinations, and therefore some combinations may be excluded in advance to reduce the computation time. As a result, the 'most' critical combination of elements identified from the simulation approach may not be the 'true' one and hence may be misleading for resilient network planning.

This study aims to narrow the research gap in the literature by proposing a rigorous mathematical program model formulation to identify the critical elements in a transportation network that will bring the most severe impact under disruptions. Following Scott et al. (2006) and Nagurney and Qiang (2009), the impact of losing a network element (e.g. link, node) is reflected by the changes in the corresponding cost (e.g. travel distance, cost, or time) between all origin-destination pairs in the network subject to disruptions. We adopt an optimization approach which is formulated as a transport network design problem (NDP) that involves the decision on removal of a link or a set of links from an existing transportation system that will cause the most deterioration (i.e., increase in travel costs) in the network. The optimization is subject to equilibrium constraints which represent travellers' routing decisions following Wardrop's first principle (Wardrop, 1952). Assuming travellers always choose the route with the least travel time, which is calculated through BPR function, the equilibrium constraints can guarantee that the network equilibrium condition is reached when no user can unilaterally lower his route travel time. With the equilibrium constraints, the optimization problem hence is a mathematical program with equilibrium constraints (MPEC) or a mathematical program with complementarity constraints (MPCC). The MPEC or MPCC problems are special cases of bi-level programs (e.g., Meng et al., 2001; Meng and Yang, 2002; Ban et al., 2006). In this study, we have an upper-level problem seeking a link or a combination of links that "maximizes" the total network cost while the lower-level problem is a minimization problem whose solution satisfies the Wardrop equilibrium conditions at optimality (Beckmann et al., 1956). Nevertheless, it is known that MPEC is inherently non-convex, and global optimal solutions are generally difficult to solve (Luo et al., 1996). (Here, to claim a mathematical program is non-convex, we are not focusing on the non-convex or non-cave property of the objective function, but highlighting that the problem constraints form a non-convex set.) Many global optimization solution methods have been developed to solve the transportation network design problems (Wang and Lo, 2010; Luathep et al., 2011a; Wang et al., 2013; Ekström et al., 2014; Liu and Wang, 2015; Wang et al. 2015; Riemann et al. 2015; Liu et al. 2015). A comprehensive review and discussion on network design problems and bi-level programming can be found in Magnanti and Wong (1984), Yang and Bell (1998), and Farahani et al. (2013).

It should be noted that, the model formulation in this study is to *maximize* the total network cost, rather than the minimization of the total cost as in the traditional network design problem. Many existing solution algorithms for solving conventional transportation network design problems highly rely on the formulation property that the objective function is to be minimized and the objective function is convex (e.g., the convex BPR travel time function). The problem of interest − identification of the most critical combination of vulnerable links − does not have this nice property. Therefore, most of the existing algorithms will not be applicable any more to solve this problem. Based on the idea of the global optimization solution method proposed in Wang and Lo (2010), this study applies linearization techniques to develop a single-level mixed-integer *linear* programming (MILP) to approximate the formulated NDP. Then, a global optimal solution of the transformed MILP can be guaranteed and obtained by applying general solution algorithms like the branch and bound method. The model results could provide the decision makers important assisting tools to generate insights on how one should spend the limited budget on transportation infrastructure construction and maintenance to maximize its resilience.

Compared to existing literatures (e.g., Chen et al., 2012), the main contributions of this study are:

i) A rigorous mathematical program model formulation which identifies the most critical combination of vulnerable links in vulnerability analysis when simultaneous multiple-link disruptions and travellers' route choice are considered.

ii) A global optimization solution method which ensures the solution quality and ascertain the true most critical elements in the transportation network, and

iii) A methodology which determines most critical *sets* of *vulnerable* links and their *ranking* in terms of vulnerability when only single-link disruption is considered.

Unlike Wang and Lo (2010), this paper addresses a different network design problem with a maximization of the total network cost of the objective, besides, a more advanced linearization model and a range reduction technique is adopted to enhance the computational efficiency. While Liu and Wang (2015) considered a continuous network design problem with stochastic user equilibrium, this paper deals with a discrete network design problem with deterministic user equilibrium. The proposed methodology is expected to be very effective when the number of the combination of links considered is large. It should also be noted that, though single link failure occurs more frequently in real life, it is indeed a very special case of the combination of multiple link failures, which is more general and thus deserves more research attention. More importantly, although the probability of occurrence of simultaneous multiple failures (due to, for example, earthquakes) may be low, the consequence/impact can be extremely high (e.g., immediate needs of affected people such as food and clean water cannot be transported to affected areas). Therefore, this paper carries the study of identification of vulnerable links one step forward by extending from the special case of single vulnerable link identification to a more general problem with consideration of multiple link failures.

This paper is organized as follows: Section 2 presents the network design model formulation and the transformed mixed-integer linear model reformulation for seeking a global optimal solution. Section 3 presents numerical examples showing how the proposed method is applied for network vulnerability analysis. Finally, Section 4 provides some concluding remarks and future extensions.

## 2. Problem formulation

This section presents the bi-level formulation for determining the most critical links in a transportation network whose closure or failure would incur the worst deterioration of network performance. The performance of a network is measured as the sum of travel costs associated with all flows on it. Consequently, the most critical link or combination of links refers to those without which could induce the most increase in the total travel cost.

### *2.1 Model formulation*

The following notation is used in the formulation.

*Sets:*

$A$ set of road links in the network

$W$ set of origin-destination (OD) pairs

$R_w$ set of paths between OD pair $w \in W$

*Parameters:*

$d^w$ OD travel demand of OD pair $w \in W$, which is given as a constant

$\Delta$ the link-path incidence matrix, $\Delta = [\delta_{ap}^w]$, $\forall w \in W, a \in A, p \in R_w$, where $\delta_{ap}^w = 1$ if link $a$ belongs to path $p$ between OD pair $w$, and $\delta_{ap}^w = 0$ otherwise

$N$ total number of links in a given network, $N = |A|$

$k$ total number of links assumed to be disrupted simultaneously

$t_{a,0}$ free flow travel time of link $a \in A$

$y_a$ capacity of link $a \in A$

$M$ a large enough positive number

$L$ a negative number whose absolute value is large enough

$\varepsilon$ a small enough positive number

$m$ power to $x_a / y_a$, usually set to an integer number

*Variables:*

$x_a$ traffic flow of link $a \in A$

$u_a$ binary decision variable indicating which link is disrupted: link $a \in A$ is disrupted if $u_a = 0$; otherwise it is in the normal condition.

$t_a$ travel time on link $a \in A$

$f_p^w$ flow on path $p$ between OD pair $w \in W$

$c_p^w$ travel time on path $p$ between OD pair $w \in W$

$\pi^w$ minimum path travel cost between OD pair $w \in W$

$\sigma_p^w$ binary variable indicating whether a path is used or not: path $p$ is used if $\sigma_p^w = 0$ and is not used if $\sigma_p^w = 1$

The problem of finding the most critical link(s) in the network is formulated as a bi-level mixed-integer nonlinear program with user equilibrium constraints shown as follows:

**[MINLP]**

$$\text{Max } Z = \sum_{a \in A} x_a \cdot t_a \tag{1}$$

Subject to

$$\begin{cases} \sum_{a \in A} u_a = N - k \\ u_a \in \{0,1\}, \quad a \in A \end{cases} \tag{2}$$

$$t_a = t_{a,0}\left[1 + b\left(x_a / y_a\right)^m\right] + \left(1 - u_a\right)M, \quad a \in A \tag{3}$$

$$x_a \le u_a M, \quad a \in A \tag{4}$$

$$\begin{cases} L \cdot \sigma_p^w + \varepsilon \le f_p^w \le M \cdot \left(1 - \sigma_p^w\right) \\ L \cdot \sigma_p^w \le c_p^w - \pi^w \le M \cdot \sigma_p^w \\ c_p^w - \pi^w \ge 0 \\ \sigma_p^w \in \{0,1\} \\ \forall p \in R_w, \quad w \in W \end{cases} \tag{5}$$

$$\begin{cases} d^w = \sum_{p \in R_w} f_p^w, \quad w \in W \\ x_a = \sum_{w \in W} \sum_{p \in R_w} \delta_{ap}^w \cdot f_p^w, \quad a \in A \\ c_p^w = \sum_{a \in A} \delta_{ap}^w \cdot t_a, \quad p \in R_w, \quad w \in W \\ x_a \geq 0, \quad a \in A \\ f_p^w \geq 0, \quad \forall p \in R_w, \quad w \in W \end{cases} \tag{6}$$

The optimization problem above seeks the critical link(s) whose disruption(s) would maximize the increase in total travel cost of the network, $Y = \sum_{a \in A} x_a \cdot t_a - \sum_{a \in A} x_a^* \cdot \tau_a$. (As was proposed in Scott et al. (2006), the difference between the total network travel time costs before and after the link disruptions is the network robustness index.) In the expression of Y, $x_a^*$ represents the equilibrium traffic flow and $\tau_a$ is the link travel time respectively for each link a in the network when no link disruptions occur. As the values of $\sum_{a \in A} x_a^* \cdot \tau_a$ for all links are fixed as base condition and do not change after the link disruptions, it can be ignored in the objective function of the maximization formulation. Consequently, the objective $\max Y = \sum_{a \in A} x_a \cdot t_a - \sum_{a \in A} x_a^* \cdot \tau_a$ can be reduced to $\max Z = \sum_{a \in A} x_a \cdot t_a$ as (1).

Constraint (2) states that $k$ links are simultaneously disrupted at a certain time, wherein $k$ is a given fixed integer and the binary decision variable $u_a$ is used to indicate whether a specific link is disrupted. For example, supposing the situation of finding a combination set of three links, one only needs to set $k$ equal to three. It should be noted that one can also adjust constraint (2) to model the special situation if only one link fails by only letting $k = 1$. Since setting $k$ does not affect model property, the proposed solution algorithm is still applicable to solve the model. Link travel time function is given in constraint (3), which is the only nonlinear constraint in the model. The first term of the link travel time function follows the form of traditional Bureau of Public Roads (BPR) function, while the second term ensures that link travel time is a very large value if the link is disrupted, which will make this link unlikely be chosen in travellers' routing process. Constraint (4) relates link flow with the binary decision variable and entails that there is no flow on a disrupted or failed link. Then, Wardrop's principle is applied to describe travellers' routing choice behaviour and the resultant user equilibrium traffic assignment is depicted via the linear set of constraints (5). In (5), the binary variable $\sigma_p^w$ is introduced to ensure that, if a path is used, the travel time must be equal to other used path and smaller than the unused one. The details of the verification of the equivalence between this constraint and user equilibrium conditions can be referred to Wang and Lo (2010). The flow conservation constraints in the network and nonnegative constraints are stated in constraints (6). Here, for simplicity and illustration purpose, it is assumed that the demand does not change after a disruption of link(s). However, if considering the impact of social media, the travel demand should be reduced because some travellers will change their minds and make different decisions on trip-making. In this case, if we assume the demand can be determined through a pre-processing module as was done in Chen et al. (2012), the developed model formulation and solution method would be still applicable.

The BPR function calculates link travel time under a certain flow for each link and impacts the route choice of network users, while network users' route choice in turn impacts the traffic flow on a specific link. Following the Wardrop's first principle, a new equilibrium condition is reached after a certain link or a combination of link failures when no user can reduce one's own travel time by changing route unilaterally. (Here, for simplicity, no other routing behaviours like bounded rationality (e.g., Wu et al. 2013) is considered.) Thus, the total network travel time in this situation can be obtained. The presented model is indeed to find the link/combination of links, whose failure will result in the largest network travel time.

One can notice that the objective of this formulation is to maximize the total travel time of the transportation network, which is rather different from the traditional transportation network design problem formulation, wherein the total system travel time is to be minimized. Therefore, many existing solution algorithms in the literature for solving transportation network design problems, taking advantage of the minimization of convex objective function (i.e., the convex BPR travel time function), will not be applicable. To solve for a global optimal solution of this formulated model, we present a solution approach that linearizes the model into a mixed-integer linear program (MILP), whose global optimization solution can be obtained with standard solution algorithms for MILPs such as the branch and bound method.

### *2.2 Linearization scheme*

The optimization model formulation in Section 2.1 is a nonlinear program which is inherently non-convex. Following Wang and Lo (2010), we adopt a linearization for seeking a global optimal solution.

In the constraint sets of the model formulation, the only nonlinear terms are contained in (3), i.e., the link travel time function. It will be transformed into an equivalent set of linear constraints, and thus the model constraints will become completely in linear form.

An efficient linearization technique is applied here following the Log model introduced in Vielma et al. (2010) to ensure the convex combination is always between at most two adjacent breakpoints.

Letting $l_a = \left(x_a / y_a\right)^m$, the link travel time function (3) can be expressed as:

$$t_a = t_{a,0}\left(1 + bl_a\right) + \left(1 - u_a\right)\mathrm{M}, \quad a \in A \tag{7}$$

which is linear with respect to $l_a$ and $u_a$. It should be noted that in $l_a = \left(x_a / y_a\right)^m$, $x_a$ is the only decision variable while the link capacity $y_a$ is fixed and given. Therefore, following the Log model (Vielma et al., 2010), we have (8)-(10) to approximate the single-variable nonlinear term $l_a$:

$$\begin{aligned}&\sum_{v \in V(P)} \alpha_a^v \cdot x_a^v = x_a \\ &\sum_{v \in V(P)} \alpha_a^v \cdot \left( x_a^v / y_a \right)^4 = \tilde{l}_a\end{aligned} \tag{8}$$

$$\begin{aligned}&\alpha_a^v \geq 0 \quad \forall v \in V(P) \\ &\sum_{v \in V(P)} \alpha_a^v = 1\end{aligned} \tag{9}$$

$$\begin{aligned}&\sum_{v \in L_k} \alpha_a^v \leq \lambda_a^k \\ &\sum_{v \in R_k} \alpha_a^v \leq 1 - \lambda_a^k \\ &\lambda_a^k \in \{0,1\} \quad \forall k \in K_a\end{aligned} \tag{10}$$

where $\tilde{l}_a$ is the linear approximation of $l_a$. $x_a^v$ is the ($v$+1)th breakpoint in the feasible region of $x_a$, in which $x_a^0$ is the first break point with the minimum value of $x_a$ in the region. $\alpha_a^v$ is the convex combination factor of the associated breakpoint $x_a^v$. $P$ is the set of partitioned intervals and $V(P)$ is the set of breakpoints. $L_k$ and $R_k$ are two types of breakpoint sets based on the concept of Gray codes. $\lambda_a^k$ is a binary variable. $K_a$ is the index set of binary variables for link $a \in A$.

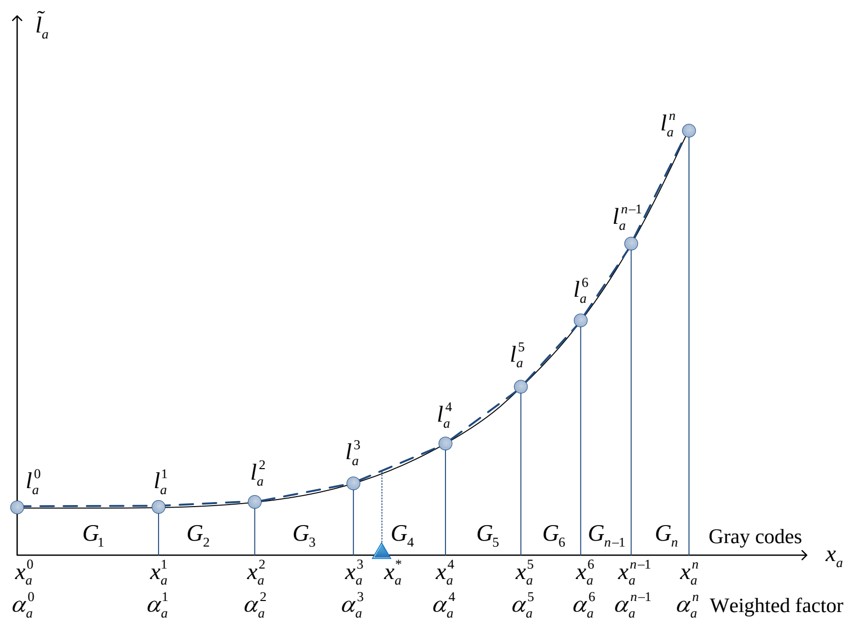

**Fig. 1.** Piecewise-linear approximation of $\left(x_a / y_a\right)^m$.

As is shown in Fig. 1, the feasible region of link flow $x_a$ is partitioned into a number of intervals, and the original nonlinear function is approximated by a linear interpolation, that is, a piecewise-linear function. It is clear that the piecewise-linear function is much closer to the original function with a higher number of intervals. However, the linearization scheme greatly affects the performance of solution process, especially when the number of intervals is large. In the literature, many model formulations are proposed for developing piecewise-linear functions, for example the model employed in Wang and Lo (2010) and Luathep et al. (2011a). Nevertheless, the number of binary variables needed in the linearization is equal to the number of linearization intervals. When more intervals are applied to improve the solution accuracy, more binary variables need to be introduced in the formulation, which reduces the solution efficiency significantly. Here, in this study, we employ the Log model as our linearization scheme to achieve higher computational efficiency because only a logarithmic number of binary variables and constraints are required. The mechanism is explained below.

Specifically, linear constraints (8)-(10) are used to describe the piecewise-linear approximation. Constraints (8) state that $x_a$ is a convex combination of the breakpoints and $\tilde{l}_a$ is the value of the piecewise-linear function evaluated at $x_a$. Constraints (9) ensure that all weighted factors are non-negative and the sum of them is equal to one. A number of binary variables $\lambda_a^k$ is employed in constraints (10) to guarantee that at most a pair of $\alpha_a^v$, which are associated to two adjacent breakpoints, is strictly positive, making the interval between the two breakpoints active in determining the linear approximation. In other words, constraints (10)ensure that the convex combination is always between two adjacent breakpoints.

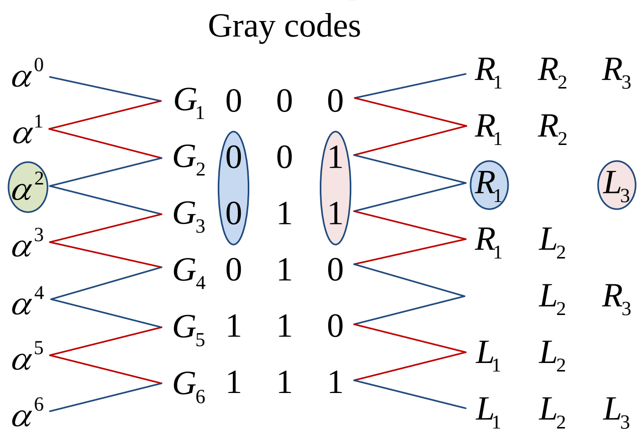


**Fig. 2.** Example of Gray codes and definition of sets $L_k$ and $R_k$.

It should be noted that the number of binary variables required for the partition scheme with $n$ breakpoints (including two end points) is only $k_{\max} = \lceil \log_2(n-1) \rceil$. Next, for one link $a \in A$, we demonstrate how constraints (10) work as a linear approximation by an illustrative example, as shown in Fig. 2. Since all variables in this small example are specifically for link $a$, thus for simplification purpose, the subscripts $a$ are omitted. A series of Gray codes are utilized to describe how the intervals between adjacent breakpoints are activated. Gray code, also known as the reflective binary code, is a binary numeral system with two adjacent numbers differing by only one bit. In this example, seven breakpoints are used and therefore the length of each of the Gray codes is $\lceil \log_2(6) \rceil = 3$. A Gray code is pre-specified for each interval as shown in Fig. 2, ensuring that two adjacent codes differ by only one bit. For example, the second and third codes differ only in the second bit and have the same values in the other bits. Indeed, two consecutive Gray codes can be used to represent one factor $\alpha^k$, as is schematically shown in Fig. 2. In this example, two consecutive codes with both the first bit equal to zero and the third bit equal to one are used to define $\alpha^2$. Since each Gray code is unique and the sequence is pre-determined, the factor can be determined if the same bits and values of two consecutive codes are specified and vice versa.

The variable $v$ is included in the set $L_k$ if the values on the $k$th bit of two consecutive codes ($v$ and $v$+1) are both equal to one, and included in the set $R_k$ if both are equal to zero. It is noted that the first and last weighted factors are only related to one Gray code. In this case, the sets $L_k$ and $R_k$ in constraints (10) are determined. The mathematical form of $L_k$ and $R_k$ can be given by

$$L_k = \left\{ v \in V \mid \left( G_v^k = 1 \text{ and } G_{v+1}^k = 1 \right) \cup \left( v = 0 \text{ and } G_1^k = 1 \right) \cup \left( v = n \text{ and } G_n^k = 1 \right) \right\}$$

and

$$R_k = \left\{ v \in V \mid \left( G_v^k = 0 \text{ and } G_{v+1}^k = 0 \right) \cup \left( v = 0 \text{ and } G_1^k = 0 \right) \cup \left( v = n \text{ and } G_n^k = 0 \right) \right\},$$

where $G_v^k$ represents the $k$th bit of the $v$th Gray code. For any given value of $\lambda^k$, constraints (10) will enforce that only one interval between two adjacent breakpoints is active, i.e., the associated $\alpha^k$ and $\alpha^{k+1}$ are positive and all the other $\alpha^i, i \neq k, k+1$ are zero. One can verify an arbitrarily selected case, for example, when $\lambda^1 = \lambda^2 = 0; \lambda^3 = 1$. Based on the combination of $L_k$ and $R_k$ as shown in Fig. 2, we have

$$\begin{cases} \alpha^0 + \alpha^1 + \alpha^2 + \alpha^3 \leq 1 - \lambda^1 = 1 \\ \alpha^5 + \alpha^6 \leq \lambda^1 = 0 \\ \alpha^0 + \alpha^1 \leq 1 - \lambda^2 = 1 \\ \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 \leq \lambda^2 = 0 \\ \alpha^0 + \alpha^4 \leq 1 - \lambda^3 = 0 \\ \alpha^2 + \alpha^6 \leq \lambda^3 = 1 \end{cases} \tag{11}$$

From (11), we have $\alpha^0 = \alpha^3 = \alpha^4 = \alpha^5 = \alpha^6 = 0$ and $\alpha^2$ and $\alpha^3$ are positive, which means the interval between breakpoints 2 and 3 is active for this given value of $\lambda^k$. Similarly, one can verify other specific cases as well.

By doing so, only $\lceil \log_2(n-1) \rceil$ number of binary variables is needed in this linearized model formulation if a partition scheme with $n$ number of breakpoints is used. Thus, comparing with other linearization model formulation that usually needs $n-1$ number of binary variables, this method involves much fewer binary variables and number of constraints.

For the objective function, the current form of $\sum_{a \in A} x_a \cdot t_a$ is nonlinear. However, we can rewrite the total system travel cost to this equivalent form, $\sum_{w \in W} d^w \cdot \pi^w$. Not using link travel time, we use the minimum OD travel time to calculate the total system travel time cost. According to Wardrop's first principle, all used paths for the same OD pair take on this minimum OD travel time, $\pi^w$. Thus, the product between $\pi^w$ and the OD demand $d^w$ describes exactly the total system travel time. Noting the fact that the OD demand $d^w$ is fixed and is a given constant, the total system travel cost function $\sum_{w \in W} d^w \cdot \pi^w$ is now linear in form.

In summary, the final linearized model formulation can be expressed as follows:

**[MILP]**

$$\text{Max } F = \sum_{w \in W} d^w \cdot \pi^w \tag{12}$$

Subject to

$$\begin{cases} \sum_{a \in A} u_a = N - k \\ u_a \in \{0,1\}, \quad a \in A \end{cases} \tag{13}$$

$$t_a = t_{a,0}\left(1 + b\tilde{l}_a\right) + \left(1 - u_a\right)\mathrm{M}, \quad a \in A \tag{14}$$

$$\begin{cases} \sum_{v \in V(P)} \alpha_a^v \cdot x_a^v = x_a \\ \sum_{v \in V(P)} \alpha_a^v \cdot \left(x_a^v / y_a\right)^m = \tilde{l}_a \\ \alpha_a^v \geq 0 \quad \forall v \in V(P) \\ \sum_{v \in V(P)} \alpha_a^v = 1 \qquad\qquad , \quad a \in A \\ \sum_{v \in L_k} \alpha_a^v \leq \lambda_a^k \\ \sum_{v \in R_k} \alpha_a^v \leq 1 - \lambda_a^k \\ \lambda_a^k \in \{0,1\} \quad \forall k \in K \end{cases} \tag{15}$$

$$x_a \leq u_a M, \quad a \in A \tag{16}$$

$$\begin{cases} L \cdot \sigma_p^w + \varepsilon \le f_p^w \le M \cdot \left(1-\sigma_p^w\right) \\ L \cdot \sigma_p^w \le c_p^w - \pi^w \le M \cdot \sigma_p^w \\ c_p^w - \pi^w \ge 0 \\ \sigma_p^w \in \{0,1\} \\ \forall p \in R_w, \;\; w \in W \end{cases} \tag{17}$$

$$\begin{aligned} & d^w = \sum_{p \in R_w} f_p^w, \;\; w \in W \\ & x_a = \sum_{w \in W} \sum_{p \in R_w} \delta_{ap}^w \cdot f_p^w, \;\; a \in A \\ & c_p^w = \sum_{a \in A} \delta_{ap}^w \cdot t_a, \;\; p \in R_w, \;\; w \in W \\ & x_a \ge 0, \;\; a \in A \\ & f_p^w \ge 0, \;\; \forall p \in R_w, \;\; w \in W \end{aligned} \tag{18}$$

By the application of linearization, the original model, which is a mixed-integer nonlinear programming, is now transformed into an MILP. The original model, if solved by a traditional solution algorithm, can only obtain a local optimal solution; whereas the transformed MILP can now be solved by standard solvers for global optimal solutions, which is also the reason of applying linearization.

### *2.3 Range-reduction technique*

To further improve the computational efficiency, an optimization-based range reduction technique is applied to reduce the solution space, i.e., the feasible link flow variable $x_a$. The partition scheme used in the solution method affects the solution accuracy and solution computational time significantly: more refined partition scheme with more breakpoints will make the linear approximation more accurate, while at cost of introducing more binary variables and thus more calculation time. It is ideal that the variable feasible region is reduced, meanwhile ensuring the optimal solution is not ruled out, so that using the same number of partition breakpoints can achieve higher solution accuracy. Besides, the fact that Range-reduction technique is able to significantly improve the solution efficiency has been tested and proved in many previous studies in the literature. Readers who are interested in this technique can refer to Lin and Tsai (2012, 2013).

For a link flow variable $x_a$, the improved new bounds $\overline{x}_a^{new}$ and $\underline{x}_a^{new}$ can be obtained through the following mixed integer linear programming model:

$\overline{x}_a^{new} = \text{maximize } x_a$

Subject to

$$F = \sum_{w \in W} d^w \cdot \pi^w \geq \underline{F}$$

and all the constraints in MILP: (13) - (18)

$$\underline{x}_a^{new} = \text{minimize } x_a$$

Subject to

$$F = \sum_{w \in W} d^w \cdot \pi^w \geq \underline{F}$$

and all the constraints in MILP: (13) - (18)

We further add the following two remarks:

**Remark 1**: The key to apply the range reduction technique is to find a reasonable lower bound $\underline{F}$ to the solution of the original maximization problem. Many different measures can be devised to obtain this lower bound. One way is to construct a linear outer-approximation to each of the nonlinear travel time functions, so that the nonlinear constraints are relaxed and cast into a linear form. By solving the relaxed linear program, a lower bound of the solution of the original maximization problem is achieved. One may also simply solve the traffic assignment of the original network assuming no link failure, whose resultant total travel time could be regarded as a lower bound to the maximum solution of this model formulation, which indeed describes the worst-case scenario if any road link is disrupted in the network.

**Remark 2**: This model formulation is designed to find out the most critical combination of vulnerable links in the network. However, it can also be easily extended to find i) the most critical link in the network when only single link failure occurs, and ii) the most vulnerable set of links as well as their ranking in terms of vulnerability. To do so, we can just assume $k = 1$ in the model formulation. The model solution will provide the information of the most critical link in the network if only single link is disrupted. For the second case, the already calculated most critical link(s) should be ruled out from the set of possible failed links and constraint (13) should be adjusted accordingly. The mathematical property of the modified MILP model remains unchanged and thus can be solved by the proposed solution method.

## 3. Numerical examples

In this section, the performance of the proposed model and the linearization solution method is tested on two networks: one is the 16-link network used in Suwansirikul et al. (1987) as a medium-size test network and the other is the well-known Sioux Falls network as a large-scale network for finding a global optimal solution to the MILP model. A personal computer with an Intel(R) Core(TM) i7 860 @ 2.80GHz CPU, an 8GB RAM and Windows 7 Enterprise operating system (64-bit) is used for solving the numerical test. The model was coded with a free MATLAB toolbox YALMIP-R20131220 (Löfberg, 2004), calling an external commercial optimization solver CPLEX optimization studio 12.3 (IBM ILOG, 2009) to solve the MILP model.

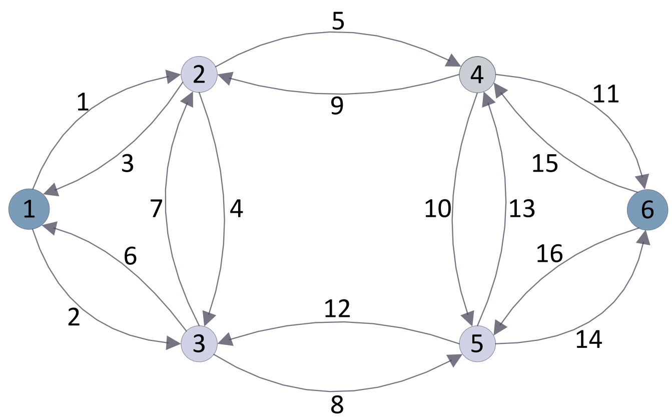


Fig. 3. The 16-link test network

### *3.1 Case 1: the 16-link test network*

#### *3.1.1 Effect of demand under single link failure*

The 16-link network as shown in Fig. 3 has 6 nodes. Two OD pairs (1,6) and (6,1) are used for all the tests. To consider the effect of travel demand, this problem is solved under three levels of demand, a low level, a medium level and a high level demand, as listed in Table 1. All the other input data for this test network is as the same as those in Suwansirikul et al. (1987). Each link flow variable is partitioned by 16 intervals.

**Table 1** Three different levels of demand.

| OD pair | Low level demand | Medium level demand | High level demand |
|---|---|---|---|
| (1,6) | 2.5 | 5 | 10 |
| (6,1) | 5 | 10 | 20 |

Table 2 shows the five most vulnerable links under the three cases. The most critical link in all these cases is link 16. Once it is disrupted, the whole network travel time cost will greatly increase by 172 times (=15689.01/91.07), 1486 times (=500161.48/336.57) or 2780 (=16001205.19/5756.59) times of the original travel time cost under the three demand levels, respectively. Hence, transportation network planners or managers should take more measures to reinforce this critical link to enhance the network robustness. Though demand level and total system travel cost are quite different, the five most vulnerable links and their orders are exactly the same. However, this result may be specific to the network layout and problem setting of this example and it cannot be concluded that the demand level does not affect the most vulnerable links. In practice, the identification of vulnerable links should be made based on the proposed model and the specific network layout and travel demand characteristics.

**Table 2** The five most vulnerable links in the 16-link network.

| Vulnerability order (ranking) | Low level demand | | Medium level demand | | High level demand | |
|---|---|---|---|---|---|---|
| | Link No. | Total cost | Link No. | Total cost | Link No. | Total cost |
| 1 | 16 | 15689.01 | 16 | 500161.48 | 16 | 16001205.19 |
| 2 | 3 | 3994.54 | 3 | 125436.02 | 3 | 4009047.40 |
| 3 | 9 | 712.22 | 9 | 20402.86 | 9 | 648218.43 |
| 4 | 13 | 216.47 | 13 | 4763.19 | 13 | 148085.41 |
| 5 | 8 | 177.30 | 8 | 2442.00 | 8 | 71818.40 |

*3.1.2 Effects of the power of link performance functions*

**Table 3** Sensitivity of parameter *m* in the link travel time function

| Power | Vulnerability order | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $m$=1 | Link No. | 3 | 9 | 8 | 16 | 13 |
| | Total cost | 5.1086E+03 | 3.2895E+03 | 2.8643E+03 | 2.6428E+03 | 1.7515E+03 |
| $m$=2 | Link No. | 3 | 16 | 9 | 8 | 13 |
| | Total cost | 4.1802E+04 | 4.0660E+04 | 1.6733E+04 | 7.3763E+03 | 6.5716E+03 |
| $m$=3 | Link No. | 16 | 3 | 9 | 13 | 8 |
| | Total cost | 8.0083E+05 | 4.0371E+05 | 1.0141E+05 | 3.0387E+04 | 2.2419E+04 |
| $m$=4 | Link No. | 16 | 3 | 9 | 13 | 8 |
| | Total cost | 1.6001E+07 | 4.0090E+06 | 6.4822E+05 | 1.4809E+05 | 7.1818E+04 |
| $m$=5 | Link No. | 16 | 3 | 9 | 13 | 8 |
| | Total cost | 3.2000E+08 | 4.0025E+07 | 4.2349E+06 | 7.3338E+05 | 2.3479E+05 |
| $m$=6 | Link No. | 16 | 3 | 9 | 13 | 8 |
| | Total cost | 6.4000E+09 | 4.0008E+08 | 2.7957E+07 | 3.6542E+06 | 7.7537E+05 |

We also test the effects of different values of power $m$ used in the link travel time function. By separately setting $m$ to be different integer values, the five most vulnerable links and the corresponding system total cost are calculated. The high demand level is adopted in this test. From the results in Table 3, one can find that, in all the six cases, links 3, 8, 9, 13 and 16 are always the five most vulnerable links; however, the rankings are different with various values of $m$. When $m = 1$, links 3 and 9 are the two most vulnerable links, whereas the second most vulnerable link changes to link 16 when $m = 2$. Rankings of the other four cases ($m = 3, \ldots, 6$) are the same, but very different from those cases when $m < 3$. Thus, one can find that the power $m$, i.e., the travel time function, significantly affects the ranking of vulnerable links in the network. In practice, accurately calibrated travel time function is required in identifying the most vulnerable links in the network.

### *3.1.3 Effect of demand under simultaneous link failures*

By applying the developed model formulation and solution algorithm, we also test the scenarios wherein two links or three links in the network fail at the same time under different demand levels. In this medium size network, it is possible that two simultaneous link failures may result in certain OD pair disconnection. One can apply many existing methods particularly designed for identification of critical links leading to a disconnected network to figure out what combinations of link failures may incur disconnected OD pairs. As the vulnerability evaluation and analysis for disconnected network are unique and beyond the scope of this study, we exclude these link combinations first. Indeed, we formulate and solve a simple linear programming to find out that there are six link combinations that result in OD pair disconnection, i.e., (1, 2), (3, 6), (5, 8), (9, 12), (11, 14) and (15, 16). Except the above six combinations, the five most vulnerable link combinations are listed in Table 4. One can find that the ranking is affected by the demand level, as the fifth vulnerable link combination changes from (14, 16) with the low level demand to (2, 16) with the medium and high level demand. Besides, we also notice that the most vulnerable two-link combination under the three cases is always (6, 9), which are far apart from each other and do not connect to the same node. More interestingly, link 9 is the third most vulnerable link in Table 2 and link 6 does not include in the five most vulnerable links when only one link failure is considered. However, the combination of these two links is much more critical than the combination of the first two most vulnerable links, i.e., (3, 16), as obtained in Table 2. Computational time for this test is about 12 seconds. This numerical example demonstrates that the results of most vulnerable link combination with multiple-link failure could be way different from the results of most vulnerable single link.

**Table 4** The five most vulnerable two-link combinations when $m$=4.

| Vulnerability order | Low level demand | | Medium level demand | | High level demand | |
|---|---|---|---|---|---|---|
| | Link combination | Total cost | Link combination | Total cost | Link combination | Total cost |
| 1 | (6 9) | 3.1366E+04 | (6 9) | 1.0011E+06 | (6 9) | 3.2031E+07 |
| 2 | (3 16) | 1.9623E+04 | (3 16) | 6.2553E+05 | (3 16) | 2.0012E+07 |
| 3 | (9 16) | 1.6455E+04 | (9 16) | 5.2398E+05 | (9 16) | 1.6762E+07 |
| 4 | (8 16) | 1.5774E+04 | (8 16) | 5.0227E+05 | (8 16) | 1.6067E+07 |
| 5 | (14 16) | 1.5729E+04 | (2 16) | 5.0062E+05 | (2 16) | 1.6015E+07 |

Results of the critical three-link combination in Table 5 also show the similar conclusion that the most critical link combination may not be the first three most vulnerable links obtained in Table 2 and their locations may be far apart from each other. Some links that included in the critical combination do not even appear in Table 2, e.g., links 2, 6, 14, and 15. In this case, the demand level does not greatly impact the ranking of the critical link combinations. Computational time for the critical three-link combinations test is about 40 seconds.

**Table 5** The five most vulnerable three-link combinations when $m$=4.

| Vulnerability order | Low level demand | | Medium level demand | | High level demand | |
|---|---|---|---|---|---|---|
| | Link combination | Total cost | Link combination | Total cost | Link combination | Total cost |
| 1 | (6 9 16) | 4.711E+04 | (6 9 16) | 1.505E+06 | (6 9 16) | 4.815E+07 |
| 2 | (6 8 9) | 3.146E+04 | (6 8 9) | 1.004E+06 | (6 8 9) | 3.211E+07 |
| 3 | (6 9 14) | 3.141E+04 | (2 6 9) | 1.002E+06 | (2 6 9) | 3.204E+07 |
| 4 | (2 6 9) | 3.139E+04 | (6 9 14) | 1.002E+06 | (6 9 14) | 3.204E+07 |
| 5 | (5 6 9) | 3.137E+04 | (6 9 15) | 1.001E+06 | (6 9 15) | 3.203E+07 |

### *3.2 Case 2: the Sioux-Falls network*

To test the proposed model formulation and solution algorithm on a larger scale network, the well-known Sioux Falls network, as shown in Fig. 4, is adopted. The numerical test was conducted to find out the most vulnerable two-link combination in the network. In this large-size network, if the traditional network vulnerability scanning method is applied by removing each link from the network and then formulating and solving the total travel time loss, one can imagine how mechanically tedious and time-consuming this solution process could be. However, applying the model formulation and solution method proposed in this study, only one mixed-integer linear programming is needed to be formulated and solved. The travel demand between 24 nodes and free flow travel time of each link are the same as input data in Leblanc (1975). The parameter $b$ in the link travel time function is set equal to 0.15 and the given link capacity is listed in the Appendix. Each link flow variable is partitioned into ten even segments. In the numerical example, the total travel time cost from the original network with no disrupted links is used to initiate the range-reduction process.

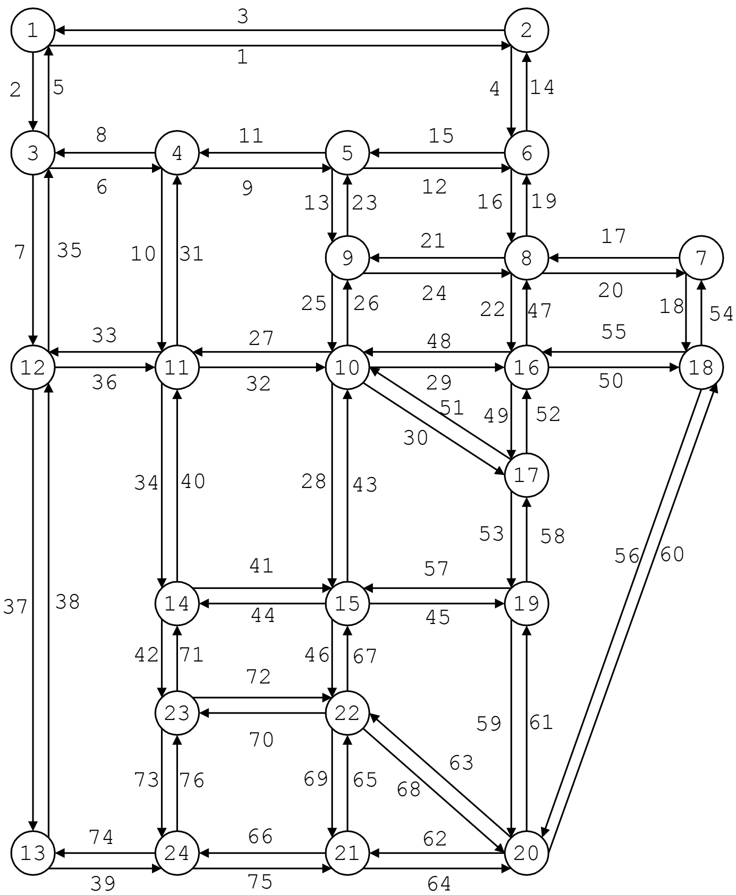


**Fig. 4.** Sioux-Falls test network

When calculating the most vulnerable two-link combination, those link pairs which will result in OD pair connection failure are excluded. In this network, this class of two-link combinations includes (20, 54), (17, 18), (38, 39), (37, 74), (1, 2), (3, 5), (5,14), (3, 4), (1, 14), and (2, 4). Except the above ten combinations, the five most vulnerable two-link combinations are shown in Table 6. The computational time is about 16 hours for the two-link failure case. However, if an enumeration method is adopted to solve the same problem, the computational time exceeds 40 hours.

**Table 6 The five most vulnerable two-link combinations in the Sioux-Falls network.**

| Vulnerability order | Link combination | Total cost |
|---|---|---|
| 1 | (43 60) | 2.55E+09 |
| 2 | (28 56) | 2.54E+09 |
| 3 | (7 74) | 2.33E+09 |
| 4 | (35 39) | 2.33E+09 |
| 5 | (23 27) | 1.92E+09 |

## 4. Conclusion

In this study, we propose a bi-level formulation for identifying the most critical combination of vulnerable links in a transportation network. The bi-level problem here is indeed a mixed integer nonlinear program with equilibrium constraints. Unlike the traditional network design problem formulation which aims to minimize the total system travel costs with convex objective functions, the problem of identifying critical vulnerable link(s) studied in this problem is formulated as a maximization problem. Therefore, many existing solution algorithms for solving traditional network design problems (e.g., Szeto et al. 2014; Li et al. 2013) are not applicable for the model solution in this study. To this end, a global optimization solution method applying piecewise linearization approach is developed to approximate the model formulation into a mixed integer linear program so that general branch and bound method could be used to guarantee the global optimization solution of the model. The global optimization ensures that the solution can be used to identify the true globally most critical link(s), rather than a local solution that is not exactly the most critical one we are seeking. The model and method can also be used to determine the ranks of vulnerable links and the impacts of their failures in monetary terms. The results of the numerical tests demonstrate that the power value of the underlying travel time function, as well as the network travel demand level, will significantly affect the ranking of vulnerable links in the network. Moreover, it is interesting to note that the set of most vulnerable links when multiple-link failure occurs are not simply the combination of the most vulnerable links for single-link failure scenario, and the links in the critical combination of vulnerable links are not necessarily connected or even in the neighbourhood of each other. In the numerical examples, we have limited to combinations of two failed links considering the computational effort required for solving combinations of more links. Nevertheless, the proposed model can indeed be extended to cover combinations of more simultaneous link failures with sufficient computing power. To overcome the limitation of the proposed model and solution algorithm in solving large-scale network problem, more efficient solution method would be developed in the future study. Besides, in the future, we will develop a model and solution method to handle the case of node failures. Moreover, we will extend the current methodology to handle other vulnerability measures (e.g., Balijepalli and Oppong, 2014; Chen et al., 2007; Ho et al., 2013; Luathep et al., 2011b) and demand uncertainty (e.g., Chen and Yang, 2004; Chen et al., 2010; Ng and Waller, 2009a,b; Chen et al. 2011; Li et al., 2012, 2014) to find the critical combination of vulnerable links.

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**Appendix. Link capacity of the Sioux Falls network**

**Table A Capacity of each link in the Sioux Falls network.**

| Link | $y_a$ | Link | $y_a$ | Link | $y_a$ | Link | $y_a$ |
|---|---|---|---|---|---|---|---|
| 1 | 25.9002 | 20 | 7.8418 | 39 | 5.0913 | 58 | 4.8240 |
| 2 | 23.4035 | 21 | 5.0502 | 40 | 4.8765 | 59 | 5.0026 |
| 3 | 25.9002 | 22 | 5.0458 | 41 | 5.1275 | 60 | 23.4035 |
| 4 | 4.9582 | 23 | 10.0000 | 42 | 4.9248 | 61 | 5.0026 |
| 5 | 23.4035 | 24 | 5.0502 | 43 | 13.5120 | 62 | 5.0599 |
| 6 | 17.1105 | 25 | 13.9158 | 44 | 5.1275 | 63 | 5.0757 |
| 7 | 23.4035 | 26 | 13.9158 | 45 | 15.6508 | 64 | 5.0599 |
| 8 | 17.1105 | 27 | 10.0000 | 46 | 10.3150 | 65 | 5.2299 |
| 9 | 17.7828 | 28 | 13.5120 | 47 | 5.0458 | 66 | 4.8854 |
| 10 | 4.9088 | 29 | 5.1335 | 48 | 5.1335 | 67 | 10.3150 |
| 11 | 17.7828 | 30 | 4.9935 | 49 | 5.2299 | 68 | 5.0757 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12 | 4.9480 | 31 | 4.9088 | 50 | 19.6799 | 69 | 5.2299 |
| 13 | 10.0000 | 32 | 10.0000 | 51 | 4.9935 | 70 | 5.0000 |
| 14 | 4.9582 | 33 | 4.9088 | 52 | 5.2299 | 71 | 4.9248 |
| 15 | 4.9480 | 34 | 4.8765 | 53 | 4.8240 | 72 | 5.0000 |
| 16 | 4.8986 | 35 | 23.4035 | 54 | 23.4035 | 73 | 5.0785 |
| 17 | 7.8418 | 36 | 4.9088 | 55 | 19.6799 | 74 | 5.0913 |
| 18 | 23.4035 | 37 | 25.9002 | 56 | 23.4035 | 75 | 4.8854 |
| 19 | 4.8986 | 38 | 25.9002 | 57 | 15.6508 | 76 | 5.0785 |