

Bounded Synchronization of A Heterogeneous Complex Switched Network [★]

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Abstract

This paper investigates synchronization issues of a heterogeneous complex network with a general switching topology in the sense of boundedness, when no complete synchronization manifold exists. Several sufficient conditions are established with the Lyapunov method and the differential analysis of convergence to determine the existence and estimate the convergence domain for the local and global bounded synchronization of a heterogeneous complex network. By using the consensus convergence of a switched linear system associated with the switching topology, explicit bounds of the maximum deviation between nodes are obtained in the form of a scalar inequality involving the property of the consensus convergence, the homogeneous and heterogeneous dynamics of individual nodes for the local and global cases. These analytical results are simple yet generic, which can be used to explore synchronization issues of various complex networks. Finally, a numerical simulation illustrates their effectiveness.

Key words: Dynamical networks; heterogeneity; synchronization; switching topology.

1 Introduction

Synchronization of populations of locally interacting units is an active field of research with applications in science and engineering, see [Pikovsky *et al.*, 2001], [Osipov, Kurths, & Zhou, 2007], [Arenas *et al.*, 2008] and references therein. From the viewpoint of complex network, synchronization of complex systems is usually determined by the dynamics of individual nodes and the coupling configuration between nodes. This result has been established by mainly assuming that all the node dynamics are identical, see [Wu, 2005], [Belykh *et al.*, 2006], [Yu *et al.*, 2009], [Stilwell *et al.* 200]), [Zhao *et al.* 2009]. However, significant differences commonly exist within the relevant individual nodes. Motivated by this, we investigate synchronization issues of complex heterogeneous networks in this paper.

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The behavior of complex dynamical networks with non-identical nodes is much more complicated than that of the identical-node case since synchronization manifold, guaranteed by the diffusive condition in identical-node networks, disappears due to the heterogeneity of individual nodes. And the ultimate synchronous trajectory, in general, has to be confined to some particular solution. Also, decompositions into a few of lower dimensional subsystems are no longer possible even for local synchronization of heterogeneous networks. Thus, it is quite difficult to explore synchronization of complex heterogeneous networks, and very few results have been reported to date. A simple case for all nonidentical nodes shared with a common equilibrium has been studied in [Xiang & Chen, 2007], [Zhao *et al.*, 2011]. Besides, synchronization of coupled nonidentical chaotic systems have also been discussed in [Femat *et al.*, 2005], [Li, Chen, & Aihara, 2006], [Duan & Chen, 2009].

There is no doubt that a complex network of coupled nonidentical systems may still exhibit some kind of synchronous behaviors that need to be understood. Bounded synchronization is a typical weaker form of synchronization when complete synchronization is impossible. Examples include clock synchronization in mobile robot-

s, or task coordination of swarming animals, or the appearance of synchronized bulk oscillations in suspension of yeast cells etc. Several investigations have discussed the bounded synchronization issues, e.g., in stochastic complex networks [Shen *et al.*, 2011] and consensus control of multi-agent systems [Zhong *et al.*, 2012]. A very recent result in [Zhao *et al.*, 2012] has addressed synchronization of a general dynamical network with nonidentical nodes and symmetric coupling matrix. Wang *et al.* [Wang *et al.*, 2015] introduced a generalized connection graph stability method to avoid the calculating eigenvalues of an asymmetry coupling matrix. It is noted that the above mentioned topology has dealt with a “slow-varying” structure, i.e., a continuous function is simply taken as the varying connection for a time-varying network. Such a description has limitations in handling the varying topologies that change very quickly such as switches due to connection failures and new creations. In general, the relevant time-varying results exhibit strong conservativeness, particularly in analyzing an unconnected topology. A few recent results [Frasca *et al.*, 2008], [Wang *et al.*, 2010] have discussed switching synchronization issues of a mobile agent network under the fast-switching constraint. More results on switching networks have focused on the consensus of multi-agent systems from the viewpoint of control science, see [Olfati-Saber & Murray, 2004], [(Ren & Beard, 2005)] for example. This paper investigates bounded synchronization of a heterogeneous complex switched network. With the Lyapunov function approach as well as differential analysis of convergence, we derive several bounded synchronization conditions for the heterogeneous network. In particular, the problem of bounded synchronization under the general switching topology is solved by partially using the techniques of the consensus problem because of the similarity of synchronization in complex coupled networks and consensus in multi-agent systems.

The rest of this paper is organized as follows. Section 2 presents a heterogeneous complex switched network and some mathematical preliminaries. In Section 3, we derive several sufficient conditions to guarantee the local and global bounded synchronization of the considered network, respectively. A numerical example is given to elucidate the effectiveness of the presented results in Section 4. Section 5 concludes the investigation.

2 A Heterogeneous Network Model

Consider a nonlinear system of N linearly and diffusively coupled nonidentical nodes which are represented by

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}_i(t, \mathbf{x}_i) - c \sum_{j=1}^N L_{\sigma_{ij}} \Gamma \mathbf{x}_j(t), \quad i = 1, 2, \dots, N, \quad (1)$$

where $\mathbf{x}_i = [x_{i1}, x_{i2}, \dots, x_{in}]^T \in \mathbb{R}^n$ is the state vector of node i , $\mathbf{f}_i : [0, \infty) \times D \rightarrow \mathbb{R}^n$ is continuously differentiable with $D \subseteq \mathbb{R}^n$, governing the dynamics of each node, $c > 0$ is the overall coupling strength, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^{n \times n}$ is the inner-coupling matrix, the switching signal $\sigma(t) : [0, \infty) \rightarrow \mathcal{P} = \{1, 2, \dots, p\}$ with $p < \infty$ is a piecewise constant function with successive times to describe the topology switches between subintervals. In network (1), the communication topology is represented by digraph \mathcal{G}^σ and described in a matrix form by the Laplacian $L_{\sigma(t)} = (L_{\sigma_{ij}}) \in \mathbb{R}^{N \times N}$. The Laplacian of digraph \mathcal{G}^σ is defined as follows: If there is a directed connection from node j to node i ($i \neq j$) at time t , then $L_{\sigma_{ij}} < 0$; otherwise $L_{\sigma_{ij}} = 0$, and its diagonal entries satisfy $L_{\sigma_{ii}} = -\sum_{j=1, j \neq i}^N L_{\sigma_{ij}}, \forall i$, and $\sigma(t)$.

For simplicity, let $\mathcal{G} = \{\mathcal{G}^i | i \in \mathcal{P}\}$ denote the set of all possible communication graphs of network (1) in the process of switching, each of which represents a digraph with the Laplacian L_i for $i \in \mathcal{P}$. Also, consider an infinite sequence of nonempty, bounded and contiguous time intervals $[t_k, t_{k+1})$, $k = 0, 1, \dots$, with $t_0 = 0$ and $t_{k+1} - t_k \leq T_{\max}$ for some positive constant T_{\max} . In each interval $[t_k, t_{k+1})$, there is a sequence of non-overlapping subintervals $[t_{k_0}, t_{k_1}), [t_{k_1}, t_{k_2}), \dots, [t_{k_{m_{k-1}}}, t_{k_{m_k}})$ with $t_{k_0} = t_k$, $t_{k_{m_k}} = t_{k+1}$ satisfying $t_{k_{j+1}} - t_{k_j} \geq T_{\min}$, $0 \leq j \leq m - 1$, for some integer $m \geq 1$ and a given positive constant T_{\min} . In particular, the digraph \mathcal{G}^σ with the Laplacian L_σ switches at t_{k_l} and it does not change during each subinterval $[t_{k_l}, t_{k_{l+1}})$. Throughout this paper, notations for graphs and their corresponding Laplacian matrices are not differentiated unless stated otherwise.

Assumption 1 (A1): The dynamics of each isolated node can be expressed in the form of $\mathbf{f}_i(t, \mathbf{x}_i) = \mathbf{f}(t, \mathbf{x}_i, \mathbf{X}) + \mathbf{g}_i(t, \mathbf{X})$ and $\|\mathbf{g}_i\| \leq \delta$ holds uniformly for all nodes with constant δ as the heterogeneity parameter, where $\mathbf{f} : [0, \infty) \times D \times \dots \times D \rightarrow \mathbb{R}^n$, $\mathbf{g}_i : [0, \infty) \times D \times \dots \times D \rightarrow \mathbb{R}^n$, $\mathbf{X} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]^T \in \mathbb{R}^{nN}$, and $\|\cdot\|$ denotes the Euclidean norm.

Note that the heterogeneous dynamics $\mathbf{g}_i(t, \mathbf{X})$ represents the differences arising from the individual nodal dynamics. A common choice of \mathbf{f} is $\mathbf{f}(t, \mathbf{x}_i, \mathbf{X}) = \sum_{i=1}^N \xi_i \mathbf{f}_i(t, \mathbf{x}_i)$, where $\xi_i \geq 0$ for all i and $\sum_{i=1}^N \xi_i = 1$. Sometimes, one can simply select $\mathbf{f}(t, \mathbf{x}_i, \mathbf{X}) = \mathbf{f}(t, \mathbf{x}_i)$ according to the nodal dynamics for some i . Then, $\mathbf{g}_i(t, \mathbf{X}) = \mathbf{f}_i(t, \mathbf{x}_i) - \mathbf{f}(t, \mathbf{x}_i)$, and the estimation δ associates with the states of nodes. It is noted that δ can be analytically calculated for many coupled limit-cycle or chaotic systems as if the bound of nodal states is known as *a priori*. Besides, the function $\mathbf{g}_i(t, \mathbf{X})$ can also take the noise or external disturbances into account.

Now, let $\mathbf{F}(t, \mathbf{X}) = [\mathbf{f}^T(t, \mathbf{x}_1, \mathbf{X}), \dots, \mathbf{f}^T(t, \mathbf{x}_N, \mathbf{X})]^T \in \mathbb{R}^{nN}$, $\mathbf{G}(t) = [\mathbf{g}_1^T(t, \mathbf{X}), \dots, \mathbf{g}_N^T(t, \mathbf{X})]^T \in \mathbb{R}^{nN}$. Then,

network (1) can be rewritten in a block form as

$$\dot{\mathbf{X}}(t) = \mathbf{F}(t, \mathbf{X}) - c(L_\sigma \otimes \Gamma)\mathbf{X}(t) + \mathbf{G}(t), \quad (2)$$

where \otimes is the Kronecker product. From **A1**, $\mathbf{G}(t) \equiv 0$ means that the heterogeneity among the node dynamics disappears. Then, network (2) reduces to a complex network of coupled identical nodes

$$\dot{\mathbf{X}}(t) = \mathbf{F}(t, \mathbf{X}) - c(L_\sigma \otimes \Gamma)\mathbf{X}(t). \quad (3)$$

For network (3), there always exists an invariant synchronization manifold $\mathbf{S} = \{(\mathbf{x}_1^T, \dots, \mathbf{x}_N^T)^T \in \mathbb{R}^{nN} : \mathbf{x}_i = \mathbf{x}_j, \forall i, j\}$. We denote an orthonormal basis of \mathbf{S} by $\tilde{\mathbf{v}} \triangleq (\mathbf{v}^T \otimes I_n) \in \mathbb{R}^{n \times nN}$ and a basis of orthocomplement space of \mathbf{S} (denoted by \mathbf{S}^+) by $\tilde{\mathbf{v}}_+ \triangleq (\mathbf{v}_+^T \otimes I_n) \in \mathbb{R}^{n(N-1) \times nN}$, where $\mathbf{v} = \frac{1}{\sqrt{N}}[1, 1, \dots, 1]^T \in \mathbb{R}^n$, and $\mathbf{v}_+ \in \mathbb{R}^{n \times (N-1)}$. Then, it is easy to verify that $\mathbf{X} \in \mathbf{S} \iff \tilde{\mathbf{v}}_+ \mathbf{X} = 0$. Left-multiplying Eq. (2) by $\tilde{\mathbf{v}}_+$ yields

$$\begin{aligned} \dot{\mathbf{Y}}(t) &= \tilde{\mathbf{v}}_+ \mathbf{F}(t, \tilde{\mathbf{v}}_+^T \mathbf{Y}(t) + \bar{\mathbf{X}}(t)) \\ &\quad - c(\tilde{L}_\sigma \otimes \Gamma)\mathbf{Y}(t) + \tilde{\mathbf{v}}_+ \mathbf{G}(t), \end{aligned} \quad (4)$$

where I_n is an $n \times n$ identity matrix, $\mathbf{Y} = \tilde{\mathbf{v}}_+ \mathbf{X} \in \mathbb{R}^{n(N-1)}$, $\bar{\mathbf{X}} = [\bar{\mathbf{x}}^T, \bar{\mathbf{x}}^T, \dots, \bar{\mathbf{x}}^T]^T \in \mathbb{R}^{nN}$, $\bar{\mathbf{x}} = \frac{1}{\sqrt{N}}\tilde{\mathbf{v}}\mathbf{X}$ is the average state trajectory, and $\tilde{L}_\sigma = \mathbf{v}_+^T L_\sigma \mathbf{v}_+$. Obviously, $\mathbf{Y}(t) = 0$ is an equilibrium point of system

$$\dot{\mathbf{Y}}(t) = \tilde{\mathbf{v}}_+ \mathbf{F}(t, \tilde{\mathbf{v}}_+ \mathbf{Y}(t) + \bar{\mathbf{X}}(t)) - c(\tilde{L}_\sigma \otimes \Gamma)\mathbf{Y}(t). \quad (5)$$

The exponential stability of system (5) is equivalent to the exponential synchronization of network (3) [Wang & Wang, 2013]. Correspondingly, the bounded synchronization of network (2) can be assessed by the ultimate boundedness of the solution $\mathbf{Y}(t)$ of system (4).

Definition 1. Network (1) is said to achieve bounded synchronization to the convergence domain \mathcal{M} if $\forall i, j = 1, \dots, N$, $\mathbf{X}_{ij}(t)$ approaches to \mathcal{M} , i.e., $\lim_{t \rightarrow \infty} \text{dist}(\mathbf{X}_{ij}(t), \mathcal{M}) = 0$, where $\mathbf{X}_{ij}(t) = \mathbf{x}_i(t) - \mathbf{x}_j(t)$, $\text{dist}(\mathbf{x}^*, \mathcal{M})$ denotes the distance from a point \mathbf{x}^* to a set \mathcal{M} , that is, the smallest distance from \mathbf{x}^* to any point in \mathcal{M} .

3 Bounded Synchronization Analysis

3.1 Analysis of local bounded synchronization

Theorem 1. Suppose that **A1** holds. If there exist a uniformly symmetric positive definite matrix $P(t) \in \mathbb{R}^{n(N-1) \times n(N-1)}$ and positive scalars a, b , and η such that

$$a\|\mathbf{Y}(t)\|^2 \leq \mathbf{Y}^T(t)P(t)\mathbf{Y}(t) \leq b\|\mathbf{Y}(t)\|^2, \quad \forall \mathbf{Y}(t), \quad (6)$$

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + 2\eta I_{n(N-1)} \leq 0, \quad (7)$$

then network (1) with the initial condition $\mathbf{X}_{ij}(t_0) \in B_{\bar{\delta}} = \{\mathbf{Y} \in \mathbb{R}^{n(N-1)} \mid \|\mathbf{Y}\| \leq \bar{\delta}\}$ for any i and j synchronizes to the set $\mathcal{M}_1 = \{\mathbf{X} \in \mathbb{R}^{nN} \mid \|\mathbf{X} - \bar{\mathbf{X}}\| \leq \underline{\delta}\sqrt{a/b}\}$, where $A(t) = I_{N-1} \otimes J_f(t, \bar{\mathbf{X}}) - c\tilde{L}_\sigma \otimes \Gamma$, $J_f(t, \bar{\mathbf{X}})$ is the Jacobian matrix of $\mathbf{f}(t, \mathbf{X}(t))$ evaluated at $\bar{\mathbf{X}}(t)$, $\phi = \sqrt{\sum_{i=1}^n \phi_i^2}$ with the Hessian matrix Φ_j of function f_j satisfying $\|\Phi_j\| \leq \phi_j$ for all j , $\underline{\delta} = 4\sqrt{N}\delta/(\eta + \sqrt{\eta^2 - 8\sqrt{nN}\phi\delta})$ and $\bar{\delta} = 4\sqrt{N}\delta/(\eta - \sqrt{\eta^2 - 8\sqrt{nN}\phi\delta})$.

Proof. Use the Taylor formula with the integral remainder at $\bar{\mathbf{x}}$ and write it in a compact form as $\mathbf{F}(t, \mathbf{X}) = \sqrt{N}\mathbf{v} \otimes \mathbf{f}(t, \bar{\mathbf{X}}) + (I_N \otimes J_f(t, \bar{\mathbf{X}}))(\mathbf{X}(t) - \bar{\mathbf{X}}(t)) + \mathbf{H}(t)$, where $\mathbf{h}_i(t) = [h_{i1}(t), \dots, h_{in}(t)]^T \in \mathbb{R}^n$ with $h_{ij}(t) = \int_0^1 (1-s)(\mathbf{x}_i - \bar{\mathbf{x}})^T \Phi_j((1-s)\mathbf{x}_i + s\bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}}) ds$, $\mathbf{H}(t) = [\mathbf{h}_1^T(t), \dots, \mathbf{h}_N^T(t)]^T$. Then, system (4) can be further rewritten as

$$\dot{\mathbf{Y}}(t) = A(t)\mathbf{Y}(t) + \tilde{\mathbf{v}}_+(\mathbf{H}(t) + \mathbf{G}(t)). \quad (8)$$

We use $V(t, \mathbf{Y}(t)) = \frac{1}{2}\mathbf{Y}^T(t)P(t)\mathbf{Y}(t)$ as a Lyapunov function candidate for system (8), where $P(t)$ is solved by Eq. (7). Then, we calculate the derivative of $V(t, \mathbf{Y}(t))$ along the trajectories of system (8) and recall the constraints in Eqs. (6)–(7) as well as $\|\mathbf{H}(t)\| \leq \frac{\sqrt{n}}{2}\phi\|\mathbf{Y}(t)\|^2$ to derive

$$\dot{V}(t) \leq \delta b\|\mathbf{Y}(t)\| - \eta\|\mathbf{Y}(t)\|^2 + \frac{\sqrt{n}}{2}\phi b\|\mathbf{Y}(t)\|^3. \quad (9)$$

It follows from Eq. (9) that there exists a K -class function $\alpha(\|\mathbf{Y}(t)\|)$ such that $\dot{V}(t, \mathbf{Y}(t)) \leq -\alpha(\|\mathbf{Y}(t)\|)$, $\forall \underline{\delta} \leq \|\mathbf{Y}(t)\| \leq \bar{\delta}$. Taking $\epsilon = b\underline{\delta}^2$ ensures that $B_{\underline{\delta}} \subset \Omega_{t, \epsilon} = \{\mathbf{Y} \in \mathbb{R}^{n(N-1)} \mid V(t, \mathbf{Y}(t)) \leq \epsilon\}$. Since $\dot{V}(t, \mathbf{Y}(t)) \leq 0$ on $B_{\bar{\delta}}$, for any t_0 and any $\mathbf{Y}(t_0) \in \Omega_{t_0, \epsilon}$, the solution starting at $(t_0, \mathbf{Y}(t_0))$ stays in $\Omega_{t, \epsilon}$. Therefore, any solution starting in $B_{\bar{\delta}}$ will stay in $\Omega_{t, \epsilon}$ for all future time. To calculate the ultimate bound on $\mathbf{Y}(t)$, we have

$$V(t, \mathbf{Y}(t)) \leq \epsilon \implies a\|\mathbf{Y}(t)\|^2 \leq \epsilon \iff \|\mathbf{Y}(t)\| \leq \underline{\delta}\sqrt{a/b}.$$

Therefore, the ultimate boundedness of system (8) can be taken as $\underline{\delta}\sqrt{a/b}$. The proof is thus completed. \square

Remark 1. As shown in [Zhao *et al.*, 2012], the linear matrix inequality (7) is a sufficient condition for local exponential synchronization of network (3). Actually, the nonautonomous system (5) can be rewritten as a linear time-varying system by using the Mean Value Theorem

$$\dot{\mathbf{Y}}(t) = A(t)\mathbf{Y}(t) + \tilde{\mathbf{v}}_+\Sigma(t)\tilde{\mathbf{v}}_+^T\mathbf{Y}(t), \quad (10)$$

where $\Sigma(t) = \text{diag}(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \mathbf{Z}_1) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \bar{\mathbf{X}}), \dots, \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \mathbf{Z}_N) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t, \bar{\mathbf{X}})) \in \mathbb{R}^{nN \times nN}$, and \mathbf{Z}_i is a vector on the line seg-

ment connecting \mathbf{X} to $\bar{\mathbf{X}}$. Then, according to the converse Lyapunov theorem, if the origin of system (10) is exponentially stable, there always exists a time-varying quadratic Lyapunov function $V(t)$ defined for system (10). Then, the exponential stability of the synchronization manifold in network (3) guarantees the existence of function $V(t)$. Thus, one can easily estimate the boundedness of synchronization from Theorem 1 as if network (3) achieves exponential synchronization.

Remark 2. Consider the solvability of inequality (7). If $P(t) = P_1 \otimes P_2(t)$, then inequality (7) can be guaranteed by the following two inequalities:

$$\dot{P}_2(t) + J_f^T(t, \bar{\mathbf{X}})P_2 + P_2J_f(t, \bar{\mathbf{X}}) - \beta_1P_2\Gamma \leq 0, \quad (11)$$

$$\beta_1P_1 - c\tilde{L}_\sigma^T P_1 - cP_1\tilde{L}_\sigma + \beta_2I_N \leq 0, \quad (12)$$

where constant matrix $P_1 \in \mathbb{R}^{(N-1) \times (N-1)}$ is positive definite, diagonal matrix $P_2(t) \in \mathbb{R}^{n \times n}$ is uniformly positive definite, β_1 and β_2 are arbitrary positive constants. It is noted that Eq. (11) implies the local stability of linear system

$$\dot{\mathbf{x}}(t) = (J_f(t, \bar{\mathbf{X}}) - \beta_1\Gamma)\mathbf{x}(t), \forall \mathbf{x}(t) \in \mathbb{R}^n. \quad (13)$$

As shown in the connection graph stability method [Belykh *et al.*, 2004], the existence of P_2 has been widely investigated when P_2 is a constant matrix. And the inequality (12) implies the exponential stability of the linear switched system written as

$$\dot{\mathbf{y}}(t) = -(c\tilde{L}_\sigma - \beta_1I_{N-1})\mathbf{y}(t), \forall \mathbf{y}(t) \in \mathbb{R}^{N-1}. \quad (14)$$

The following lemma provides a simple convergence domain when the inner-coupling matrix is an identity matrix and its proof is given in Appendix A.

Lemma 1. The solution of the linear switched system

$$\dot{\tilde{\mathbf{Z}}}(t) = (I_{N-1} \otimes J_f(t, \bar{\mathbf{x}}) - c(\tilde{L}_\sigma \otimes I_n))\tilde{\mathbf{Z}}(t) \quad (15)$$

satisfies $\|\tilde{\mathbf{Z}}(t)\| \leq e^{(-\mu + \nu_{\max} + \epsilon)(t-t_0)}\|\tilde{\mathbf{Z}}(t_0)\|$ for a sufficiently small scalar ϵ .

Theorem 2. Suppose that **A1** holds. If $\nu_{\max} - \mu < 0$, then network (1) with $\Gamma = I_n$ locally synchronizes to the set $\mathcal{M}_2 = \{\mathbf{X} \in \mathbb{R}^{nN} \mid \|\mathbf{X} - \bar{\mathbf{X}}\| \leq \frac{\delta}{\mu - \nu_{\max}}\}$, where ν_{\max} is the largest Lyapunov exponent of system

$$\dot{\xi}(t) = J_f(t, \bar{\mathbf{X}})\xi(t), \quad (16)$$

and μ is the convergence rate of switched system

$$\dot{\mathbf{w}}(t) = -c\tilde{L}_\sigma\mathbf{w}(t). \quad (17)$$

Proof. It is well-known that for the local synchronization issue of system (2), we can investigate the linearized system (4) written as

$$\dot{\mathbf{Z}} = (I_{N-1} \otimes J_f(t, \bar{\mathbf{X}}) - c\tilde{L}_\sigma \otimes I_n)\mathbf{Z} + \tilde{\mathbf{v}}_+\mathbf{G}, \quad (18)$$

where the solution of system (18) can be expressed by

$$\mathbf{Z}(t) = \tilde{\Phi}(t, t_0)\mathbf{Z}(0) + \int_{t_0}^t \tilde{\Phi}(t, \tau)\tilde{\mathbf{v}}_+\mathbf{G}(\tau)d\tau, \quad (19)$$

state transition matrix $\tilde{\Phi}(t, \tau) = \Phi(t)\Phi^{-1}(\tau)$, and $\Phi(t) \in \mathbb{R}^{n \times n}$ is a fundamental solution of system (15).

From the definition of an induced matrix norm, we have $\|\tilde{\Phi}(t, t_0)\| = \max_{\|\tilde{\mathbf{Z}}\|=1} \|\tilde{\Phi}(t, t_0)\tilde{\mathbf{Z}}\|$. It follows from Lemma 1 that

$$\|\tilde{\Phi}(t, t_0)\| \leq \max_{\|\tilde{\mathbf{Z}}\|=1} e^{(-\mu + \nu_{\max} + \epsilon)(t-t_0)}\|\tilde{\mathbf{Z}}\|. \quad (20)$$

Combining it with Eq. (19) and taking the limit give the ultimate boundedness of $\mathbf{Y}(t)$. \square

Remark 3. Theorem 2 provides a reasonably simple boundedness criterion of local synchronization, relating to the heterogeneity of individual nodes, the largest Lyapunov exponent ν_{\max} of an isolated homogeneous system (16), and the convergence rate μ of a linear switched system (17). In particular, if $\delta = 0$, then Theorem 2 reduces to a criterion of local synchronization of a linearly coupled complex network. A similar result can also be found in Refs. [Wang & Wang, 2013, Xiang & Wei, 2011].

3.2 Analysis of global bounded synchronization

Assumption 2 (A2): Suppose that there exist a time-varying matrix $B(t) \in \mathbb{R}^{n \times n}$ and a vector field $\tilde{\mathbf{g}}_i(t)$ for $i = 1, \dots, N$ such that $\mathbf{f}_i(t, \mathbf{x}_i) = B(t)\mathbf{x}_i + \tilde{\mathbf{g}}_i(t)$, where $\|\tilde{\mathbf{v}}_+\tilde{\mathbf{G}}(t)\| \leq \tilde{\delta}$ for some positive scalar $\tilde{\delta}$ with $\tilde{\mathbf{G}}(t) = [\tilde{\mathbf{g}}_1^T(t), \dots, \tilde{\mathbf{g}}_N^T(t)]^T$, and the fundamental solution of the linear system $\dot{\mathbf{x}} = B(t)\mathbf{x}$, denoted by $\tilde{\Psi}(t)$, satisfying $\|\tilde{\Psi}(t)\| \leq e^{\nu_b t}$ for some constant ν_b .

Theorem 3. Suppose that **A2** holds and system (17) achieves global exponential stability, i.e., there exists a positive scalar μ such that $\|\mathbf{w}(t)\| \leq \|\mathbf{w}(t_0)\|e^{-\mu t}$, $\forall \mathbf{w}(t_0) \in \mathbb{R}^{N-1}$. If $\mu > \nu_b$, then network (1) globally synchronizes to the set $\mathcal{M}_3 = \{\mathbf{X} \in \mathbb{R}^{nN} \mid \|\mathbf{X} - \bar{\mathbf{X}}\| \leq \frac{\tilde{\delta}}{\mu - \nu_b}\}$.

The detailed proof is omitted here since it can be easily deduced by following the proof of Theorem 2.

Remark 4. **A2** is quite reasonable since, for a general nonlinear vector function $\mathbf{f}_i(t, \mathbf{x})$, it can be easily decomposed into a linear part $B\mathbf{x}$ with constant matrix B and

a nonlinear part $\tilde{\mathbf{g}}_i(t)$. Then, **A2** is consistent with **A1**. However, such a simple decomposition is sometimes not a good choice to obtain smaller scalars ν_b and $\tilde{\delta}$. By using the Taylor formula, we may decompose $\mathbf{f}_i(t, \mathbf{x}_i)$ into

$$\mathbf{f}_i(t, \mathbf{x}_i) = \mathbf{f}(t, \bar{\mathbf{X}}) + J_f(t, \bar{\mathbf{X}})(\mathbf{x}_i - \bar{\mathbf{x}}) + \mathbf{g}_i(t) + \mathbf{r}(t, \mathbf{x}_i),$$

where $\mathbf{r}(t, \mathbf{x}_i) \in \mathbb{R}^n$ is the integral form of the reminder, $B(t) = J_f(t, \bar{\mathbf{X}})$, and $\tilde{\mathbf{g}}_i = \mathbf{f}(t, \bar{\mathbf{X}}) - J_f(t, \bar{\mathbf{X}})\bar{\mathbf{x}} + \mathbf{g}_i(t) + \mathbf{r}(t, \mathbf{x}_i)$. Then a conservative condition that guarantees **A2** can be given as $\tilde{\delta} \triangleq \delta + \|\mathbf{R}(t, \mathbf{x}_i)\|$.

Remark 5. Theorem 2 and Theorem 3 require the convergence rate μ . For system (17), the convergence rate μ is equal to the consensus convergence rate of multi-agent system written as

$$\dot{\mathbf{x}}_i(t) = -c \sum_{j=1}^N L_{\sigma ij} \mathbf{x}_j(t), \quad i = 1, \dots, N. \quad (21)$$

A simple estimation of the convergence rate μ can be calculated as shown in [Xiang & Wei, 2011].

For a general global synchronization issue of network (1), we here give a relatively conservative result by constructing a quadratic Lyapunov function.

Theorem 4. Suppose that **A1** holds. If there exist a uniformly symmetric positive definite matrix $P(t) \in \mathbb{R}^{n(N-1) \times n(N-1)}$, positive scalars a, b, α and θ such that

$$a\|\mathbf{Y}(t)\|^2 \leq \mathbf{Y}^T(t)P(t)\mathbf{Y}(t) \leq b\|\mathbf{Y}(t)\|^2, \quad \forall \mathbf{Y}(t), \quad (22)$$

$$\frac{1}{2}(P(t)\Gamma + \Gamma P(t)) - \theta P(t) \geq 0, \quad (23)$$

$$\begin{aligned} & \mathbf{X}_{ij}^T(t) \left(\frac{1}{2} \dot{P}(t) - \alpha P(t) \right) \mathbf{X}_{ij}(t) \\ & + (\mathbf{f}(t, \mathbf{x}_i, \mathbf{X}) - \mathbf{f}(t, \mathbf{x}_j, \mathbf{X}))^T P(t) \mathbf{X}_{ij} \leq 0, \quad (24) \end{aligned}$$

and there exists an infinite sequence $\{t_k, k = 0, 1, \dots\}$ such that

$$(t_{k+1} - t_k)^{-1} \int_{t_k}^{t_{k+1}} \gamma_{\sigma(t)} dt \geq \gamma, \quad (25)$$

then network (1) globally synchronizes to the set $\mathcal{M}_4 = \left\{ \mathbf{X} \in \mathbb{R}^{nN} \mid \|\mathbf{X} - \bar{\mathbf{X}}\| \leq \frac{b\delta\tilde{\gamma}(1-\gamma^{*m+1})}{a(1-e^{-\frac{1}{2}\gamma T_{\min}})(1-\gamma^*)} \right\}$, where $\gamma_{\sigma} = c\lambda_2^{\sigma}\theta - \alpha$, λ_2^{σ} is the smallest eigenvalue of matrix \tilde{L}_{σ} except zero, γ is a positive constant, $\tilde{\gamma} = \max_{\forall l \in \mathbb{N}} \left\{ \frac{1}{\gamma_{\sigma}} (1 - e^{-\frac{1}{2}\gamma_{\sigma}(t_{k_{l+1}} - t_{k_l}})} \right\}$, and $\gamma^* = \max_{\forall l \in \mathbb{N}} \left\{ e^{-\frac{1}{2}\gamma_{\sigma}(t_{k_{l+1}} - t_{k_l})} \right\}$.

Proof. Choose the Lyapunov function as

$$V(t, \mathbf{X}(t)) = \frac{1}{2} \mathbf{X}^T(t) (\Pi \otimes P(t)) \mathbf{X}(t), \quad (26)$$

where $P(t)$ is solved by Eq. (22)–(24). Differentiating $V(t, \mathbf{X}(t))$ along the trajectory of network (2) gives

$$\begin{aligned} \dot{V}(t) &= \frac{1}{4N} \sum_{i,j=1}^N \mathbf{X}_{ij}^T(t) \left(\dot{P}(t) \mathbf{X}_{ij}(t) + 2P(t) (\mathbf{f}(t, \mathbf{x}_i, \mathbf{X}) \right. \\ &\quad \left. - \mathbf{f}(t, \mathbf{x}_j, \mathbf{X})) \right) + \mathbf{Y}^T(t) (I_N \otimes P(t)) \mathbf{G}(t) \\ &\quad - c \mathbf{Y}^T(t) (\tilde{L}_{\sigma} \otimes P(t)) \mathbf{Y}(t) \\ &\leq -\frac{1}{2N} \sum_{i,j=1}^N \gamma_{\sigma} \mathbf{X}_{ij}^T(t) P(t) \mathbf{X}_{ij}(t) + b\delta \|\mathbf{Y}(t)\|. \end{aligned}$$

As we know, the Laplacian L_{σ} in each time subinterval is constant. Then, in the subinterval $[t_{k_l}, t_{k_{l+1}})$, the derivative of $V(t)$ satisfies

$$\dot{V}(t) \leq -\gamma_{\sigma} V(t) + b\delta \|\mathbf{Y}(t)\|. \quad (27)$$

Recalling Eq. (22) and taking $U(t) = \sqrt{V(t)}$, we use $\dot{V}(t) = 2U(t)\dot{U}(t)$ to obtain $\dot{U}(t) \leq -\frac{1}{2}\gamma_{\sigma}U(t) + \frac{b\delta}{2\sqrt{a}}$. Moreover, by applying the comparison lemma, we have

$$U(t_{k_{l+1}}) \leq e^{-\frac{1}{2}\gamma_{\sigma}\Delta t_{k_l}} U(t_{k_l}) + \frac{b}{\sqrt{a}} \delta \tilde{\gamma}, \quad (28)$$

where $\Delta t_{k_l} = t_{k_{l+1}} - t_{k_l}$. Note that the common Lyapunov function $V(t)$ is a continuous function in the interval $[t_k, t_{k+1})$. Then, according to Eq. (28), we deduce a relation by recursion

$$\begin{aligned} U(t_k) &\leq e^{-\frac{1}{2} \int_0^{t_k} \gamma_{\sigma} dt} U(0) \\ &\quad + \frac{b\delta\tilde{\gamma}(1 - e^{-\frac{1}{2}\gamma T_{\min}(k+1)})(1 - \gamma^{*m+1})}{\sqrt{a}(1 - e^{-\frac{1}{2}\gamma T_{\min}})(1 - \gamma^*)}. \quad (29) \end{aligned}$$

Taking the limit on both sides of Eq. (29) indicates that all trajectories $\mathbf{Y}(t)$ converge to the set \mathcal{M}_4 . \square

Remark 6. The inequality (24) in Theorem 4 implies the exponential stability of the time-varying systems $\forall i, j$,

$$\dot{\mathbf{X}}_{ij} = \left(\int_0^1 J_f(t, s\mathbf{x}_i + (1-s)\mathbf{x}_j) ds - \alpha I_n \right) \mathbf{X}_{ij}. \quad (30)$$

The detailed proof can be easily deduced by constructing the Lyapunov functions $V_{ij}(t) = \frac{1}{2} \mathbf{X}_{ij}^T(t) Q(t) \mathbf{X}_{ij}(t)$ for each system in Eq. (30). In synchronization analysis and control issues of complex dynamical networks, see [Wang & Chen, 2002], [Li *et al.*, 2004], [Pham & Slotine, 2007], [Chen *et al.* 2007], a sufficiently large α can ensure the existence of $P(t)$. And inequality (22) is solvable by setting $P(t)$ as a constant matrix for the trivial case that all nodes are identical in many coupled limit-cycle or chaotic systems [Belykh *et al.*, 2006].

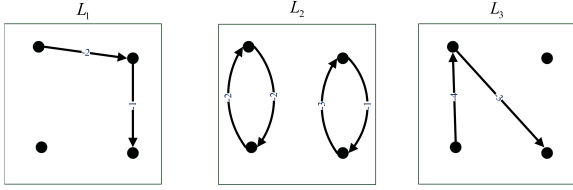


Fig. 1. The switching topologies.

Remark 7. It is observed that from inequality (25), $\bar{\lambda}_2^\sigma \triangleq \frac{1}{\Delta t_k} \int_{t_k}^{t_{k+1}} \lambda_2^\sigma dt$ is the time average of the second smallest eigenvalues corresponding to the Laplacians of switching topology, and α is a constant, which is determined by the dynamics of individual node. As a matter of fact, the inequality (25) indicates the exponential synchronization of the considered switched network (3). A recent result can also be derived from Theorem 4 by setting the Laplacians to be symmetric [Wang & Wang, 2011] or asymmetric [Wang & Wang, 2012], which indicates that synchronization for a linearly coupled network with a switching topology can be evaluated by the time average of second smallest eigenvalues corresponding to the Laplacians of switching topology.

4 A Numerical Simulation

Our simulation model is built on the laboratory devices of coupled pendulums [Fradkov *et al.*, 2005], where each node is a similar pendulum, and every two pendulums are connected by an elastic link. The dynamic equations of the nodes are described by $\dot{x}_{i1}(t) = x_{i2}(t) + u_{i1}(t)$, $\dot{x}_{i2}(t) = -r_i \sin(x_{i1}) - qx_{i2}(t) + \phi_i(t) + u_{i2}(t)$, where $i = 1, \dots, 4$, $q = 3.15$ and $r_i = 0.02i$ are system parameters, $[\phi_1, \phi_2, \phi_3, \phi_4]^T = [0.2 \sin t, -0.15 \sin t, 0.1 \sin t, -0.05 \sin t]^T$ are disturbance inputs, $\mathbf{u}_i(t) = [u_{i1}(t), u_{i2}(t)]^T$ are control signals that are selected as the distributed consensus protocol: $\mathbf{u}_i(t) = -c \sum_{j=1}^4 L_{ij\sigma} \mathbf{x}_j(t)$. The collection of switching topologies is shown in Fig. 1.

Every second, $\sigma(t)$ updates its value in the order of $1, 2, 3, r, 1, 2, 3, r, \dots$, where r is a random valuable randomly chosen from the set $\{1, 2, 3\}$. For $T = 4$, the transition matrix $\Xi_\sigma(kT, (k-1)T)$ corresponding to linear system (15) must be one of the three matrices written as $\Xi_i = \mathbf{v}_+^T e^{-cL_i} e^{-cL_3} e^{-cL_2} e^{-cL_1} \mathbf{v}_+$ with $i = 1, 2, 3$. According to [Xiang & Wei, 2011], $\mu = 2.7784$ by setting the overall coupling strength $c = 4$. On the other hand, one can select $B = [0, 1; 0, -q]$ and $\tilde{\mathbf{g}}_i = [0, -r_i \sin(x_{i1}) + \phi_i]^T$. We then derive $\nu_b = 0$ and $\tilde{\delta} = 0.295$.

Thus, by Theorem 3, it can be concluded that system (1) with such a randomly switching topology achieves global bounded synchronization, and the ultimate bounded regions are $\mathcal{M}_4 = \{\mathbf{X} \mid \|\mathbf{X} - \bar{\mathbf{X}}\| < 0.106\}$. The simulation results are shown in Fig. 2, which shows that the global bounded synchronization of the nonlinear networked

system can be guaranteed.

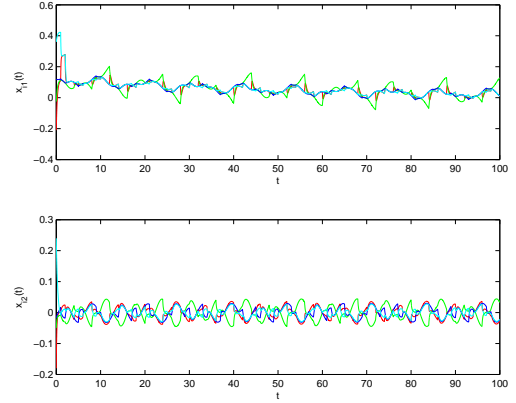


Fig. 2. Trajectories of the complex network.

5 Conclusions

Bounded synchronization has been studied for a heterogeneous complex network with a general switching topology in this paper. By using the Lyapunov function approach and the differential analysis of convergence, several convergence domain estimates are derived for the local and global bounded synchronization of a heterogeneous network. In particular, we have shown that the ultimate boundedness of synchronization can be easily evaluated by the consensus convergence rate of a switched linear network associated with the switching topology, the homogeneous and heterogeneous dynamics of individual nodes. Our results are quite simple and powerful, without assumption of a symmetric or connective Laplacian matrix, which can be widely used to investigate various topologies, no matter they are undirected or directed, weighted or unweighted, time-invariant or switching. All these results provide an insight into the analysis and regulation of collective behavior of complex dynamical systems.

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A The proof of Lemma 1

Proof. Let $\Psi(t)$ be a fundamental solution of system (16), then $\dot{\Psi}(t) = J_f(t, \bar{x})\Psi(t)$ and $\frac{d}{dt}\Psi^{-1}(t) = -\Psi^{-1}(t)J_f(\bar{x})$. Introducing a time-dependent coordinate transformation $\mathbf{W}(t) = (I_{N-1} \otimes \Psi^{-1}(t))\tilde{\mathbf{Z}}(t)$ to system (15) yields

$$\dot{\mathbf{W}}(t) = -c(\tilde{L}_\sigma \otimes I_n)\mathbf{W}(t). \quad (\text{A.1})$$

According to the matrix theory, there must exist a permutation matrix $A \in \mathbb{R}^{n(N-1) \times n(N-1)}$ such that $A^T(\tilde{L}_\sigma \otimes I_n)A = I_n \otimes \tilde{L}_\sigma$. We then introduce a coordinate transformation $\tilde{\mathbf{W}}(t) = A^T\mathbf{W}(t)$ to system (A.1)

to obtain

$$\dot{\bar{\mathbf{W}}}(t) = -c(I_n \otimes \tilde{L}_\sigma)\bar{\mathbf{W}}(t). \quad (\text{A.2})$$

Obviously, system (A.2) is the combination of n independent associated systems (17). Following the previous arguments, the solution of system (15) can be expressed by $\tilde{\mathbf{Z}}(t) = \sqrt{N}(I_{N-1} \otimes \Psi(t))A^{-T}(\mathbf{v} \otimes \mathbf{w}(t))$. Correspondingly, we obtain

$$\|\tilde{\mathbf{Z}}(t)\| \leq \sqrt{N}\|\Psi(t)\| \cdot \|\mathbf{w}(t)\|. \quad (\text{A.3})$$

The following provides how to estimate the bound of $\|\mathbf{w}(t)\|$ and $\|\Psi(t)\|$. Denote the convergence rate of system (17) as μ , which is defined as

$$\mu = - \sup_{\mathbf{w}(t_0) \neq 0} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\mathbf{w}(t)\|}{\|\mathbf{w}(t_0)\|}. \quad (\text{A.4})$$

It is easy to verify from [Adrianova, 1995] that the equation $\mu \neq \pm\infty$ holds if and only if, for any $\epsilon > 0$, the following conditions hold simultaneously

$$\lim_{t \rightarrow \infty} e^{(\mu+\epsilon)t}\|\mathbf{w}(t)\| = 0, \quad (\text{A.5})$$

$$\lim_{t \rightarrow \infty} e^{(\mu-\epsilon)t}\|\mathbf{w}(t)\| = \infty. \quad (\text{A.6})$$

Apparently, if $\mu > 0$, $\lim_{t \rightarrow \infty} \|\mathbf{w}(t)\| = 0$, which further implies that $\lim_{t \rightarrow \infty} \|\bar{\mathbf{W}}(t)\| = \lim_{t \rightarrow \infty} \|\mathbf{Y}(t)\| = 0$. Thus, the synchronization will be achieved. And if $\mu \leq 0$ for any bounded initial state $\mathbf{w}(0)$, synchronization cannot be achieved for system (17). By the definition of convergence rate μ and Eqs. (A.5)–(A.6), there exists a positive constant k_1 , dependent on $\epsilon_1 > 0$, such that

$$\|\mathbf{w}(t)\| \leq k_1 e^{(-\mu+\epsilon_1)(t-t_0)}\|\mathbf{w}(t_0)\| \quad (\text{A.7})$$

for any positive scalar ϵ_1 .

And $\|\Psi(t)\|$ can be evaluated by the the largest Lyapunov exponent ν_{\max} of system (16) satisfying

$$\|\Psi(t)\| \leq k_2 e^{(\nu_{\max}+\epsilon_2)(t-t_0)}, \quad (\text{A.8})$$

where ϵ_2 is an arbitrary positive scalar and k_2 is a positive constant, dependent on ϵ_2 .

Therefore, it follows from Eqs. (A.3)–(A.8) that

$$\|\tilde{\mathbf{Z}}(t)\| \leq k e^{(-\mu+\nu_{\max}+\epsilon)(t-t_0)}\|\tilde{\mathbf{Z}}(t_0)\|, \quad (\text{A.9})$$

where ϵ is an arbitrary positive scalar and k is a positive constant, dependent on ϵ . There is no doubt that one can select a sufficiently small ϵ such that $k \leq 1$. The proof is thus completed.