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Hayman's classical conjecture on some nonlinear second order algebraic ODEs

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In this paper, we study the growth, in terms of the Nevanlinna characteristic function, of meromorphic solutions of three types of second order nonlinear algebraic ordinary differential equations. We give all their meromorphic solutions explicitly, and hence show that all of these ODEs satisfy the *classical conjecture* proposed by Hayman in 1996.

Keywords: meromorphic solutions; complex differential equations; Nevannlina theory; Wiman-Valiron theory

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1. Introduction

One important aspect of the studies of complex ordinary differential equations (ODEs) is to investigate the growth of their solutions which are meromorphic on the whole complex plane $\mathbb{C}$. A well known problem in this direction is the following *classical conjecture* [1, 2, p. 344] proposed by Hayman in [3].

Conjecture 1.1 (Hayman) If w is a meromorphic solution of

$$P(z, w, w', \cdots, w^{(n)}) = 0, \tag{1}$$

where P is a polynomial in all its arguments, then there exist $a, b, c \in \mathbb{R}^+$ such that

$$T(r, w) < a \exp_{n-1}(br^c), 0 \leq r < \infty, \tag{2}$$

where $T(r, w)$ is the Nevanlinna characteristic of $w(z)$ and $\exp_l(x)$ is the l times iterated exponential, i.e.,

$$\exp_0(x) = x, \exp_1(x) = e^x, \exp_l(x) = \exp\{\exp_{l-1}(x)\}.$$

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The conjecture for the case $n=2$ was proposed by Bank in [4]. When $n=1$, (2) reduces to

$$T(r,w) < ar^c, 0 \leq r < \infty, \tag{3}$$

and we say that a meromorphic function w has finite order if it satisfies (3). The infimum σ of all possible numbers c is called the order of w. For example, $\sin z, \cos z, \tan z, e^z$ and the gamma function $\Gamma(z)$ have order 1. The Weierstrass ellptic function $\wp(z)$, which satisfies $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$, has order 2. The function e^{e^z} has infinite order but satisfies (2) with $n=2, c=1$. Throughout the paper, we use the standard notations and results of Nevanlinna theory [5, 6] and we shall consider solutions meromorphic on $\mathbb{C}$.

In this paper, we consider the following three types of second order ordinary differential equations (ODEs)

$$\begin{aligned} ww'' - w'^2 + P(w) &= 0, && (4)\\ w'' + cw'^2 + P(w) &= 0, && (5)\\ w'' + cw' + P(w) &= 0, && (6) \end{aligned}$$

where $c \in \mathbb{C}$ and P is a polynomial. We prove (see Theorems 3.1, 3.2, 3.4) that the *classical conjecture* holds for above equations and moreover we find all their meromorphic solutions in closed form.

2. Existing results

The *classical conjecture* holds for any linear algebraic differential equation. In fact, if f is a meromorphic solution of $a_n(z)w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_1(z)w' + a_0(z)w = h(z)$, where all the coefficients $a_0, a_1, \ldots, a_n, h$ are polynomials, then w is of finite order [7, 8]. One class of nonlinear ODEs which supports the *classical conjecture* is the higher order Briot-Bouquet differential equation: $Q(w^{(n)}, w) = 0, n \in \mathbb{N}$, where Q is a polynomial in two variables, as Eremenko, Liao and Ng [9] proved that all their non-entire meromorphic solutions belong to the class W, which consists of elliptic functions and their successive degeneracies, i.e., elliptic functions, rational functions of one exponential $\exp(kz), k \in \mathbb{C}$ and rational functions of z.

Some other positive results toward the *classical conjecture* are as follows.

Theorem 2.1 *(Gol'dberg [10] or [6, 11, p. 223]) Suppose* (1) *is a first order differential equation, then all its meromorphic solutions have finite order.*

The *classical conjecture* for $n=2$ remains open and only some partial results are known. Steinmetz [12] proved that (2) holds for any second order differential equation $P(z, w, w', w'') = 0$ provided that P is homogeneous in w, w' and w''. He further proved that each meromorphic solution of such an ODE can be expressed in terms of entire functions of finite order.

Theorem 2.2 *(Steinmetz [6, 12, p. 248]) Suppose* $P(z, w, w', w'')$ *is homogeneous* *in w, w' and w''. Then all meromorphic solutions of the ODE* (1) *are of the form*

$$G(z) = \frac{g_1(z)}{g_2(z)} \exp g_3(z),$$

where g_1, g_2 and g_3 are entire functions of finite order.

For higher order cases, little is known. If we restrict ourselves to the study of entire solutions of (1), by making use of Wiman-Valiron theory, Hayman [3] obtained a positive result to certain subclass of the algebraic differential equation

$$P = \sum_{\lambda \in I} a_\lambda(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n} = 0, \tag{7}$$

where I consists of finite multi-indices of the form $\lambda = (i_0, i_1 \cdots, i_n)$, $i_k \in \mathbb{N}$ and a_λ are polynomials in z. To state Hayman's result, we recall the definition of degree and weight of the ODE (7).

The degree of each term in (7) is defined to be $|\lambda| = i_0 + i_1 + \cdots + i_n$ and the weight $||\lambda||$ is defined by $||\lambda|| = i_0 + 2i_1 + \cdots + (n+1)i_n$. We shall consider the terms with the highest weight among all those with the highest degree in (7). Let $\Lambda = \{\lambda | |\lambda| = \max_{\lambda' \in I} |\lambda'|\}$ and Ω be the subset of Λ such that it consists of those terms with the highest weight, then we have

Theorem 2.3 *(Hayman [3]) Suppose Ω is defined as above for ODE* (7)*. Let d be the maximum degree of all the polynomials $a_\lambda(z)$ and suppose that*

$$\sum_{\lambda \in \Omega} a_\lambda(z) \neq 0.$$

Then all entire solutions of (1) *have finite order $\sigma \leq \max\{2d, 1+d\}$.*

Remark 2.4 The upper bound in Theorem 2.3 is sharp but the same result cannot be extended to meromorphic solutions [3].

Remark 2.5 Theorem 2.3 includes Theorem 2.1 if we restrict ourselves to entire solutions of first order algebraic ODEs, since the assumption in Theorem 2.3 always holds in this case (there is only one term with highest weight and highest degree). However, it is not the case for second order ODEs. One example for which Theorem 2.3 is inappliable is $ww'' - w'^2 - ww' = 0$, but it satisfies the *classical conjecture* according to Theorem 2.2.

Among the second order differential equations which are not covered by Theorems 2.2 and 2.3, the simplest one [1, 3] is perhaps

$$ww'' - w'^2 = a_2 w'' + a_1 w' + a_0 w + b, \tag{8}$$

where a_j and b are rational in z (or even constants) and are not all identically zero. If all the a_j and b are constants, Chiang and Halburd [1] proved that all meromorphic solutions of (8) satisfy (2) by explicitly giving all its meromorphic solutions which are either polynomials or rational functions of one exponential.

Their result is obtained mainly by combining Wiman-Valiron theory [6, 13, Chapter 3], local series analysis and reduction of order. Using a different approach, Liao [14] obtained the same result. Recently, Halburd and Wang [15] verified the *classical conjecture* for a_j and b rational.

3. Main Results

We will show that the *classical conjecture* is true for the ODEs (4)-(6) which are not covered by any of the above results. The main results can now be stated as follows.

THEOREM 3.1 *Suppose the differential equation (4)*

$$ww'' - w'^2 + P(w) = 0,$$

where $P(w) = \sum_{n=0}^{k} a_n w^n, a_k \neq 0$ *is a polynomial with constant coefficients, has non-constant meromorphic solutions, then we have* $k \leq 4$ *and its meromorphic solutions are characterized as follows:*

1) *if* $k = 0$ *or* 1, *then* (4) *is included in* (8)*;*
2) *if* $k = 2$, *then* $a_0 = a_1 = 0$ *and the non-constant meromorphic solutions, which are actually the general solution, of* (4) *are zero-free entire functions given by*

$$w(z) = c_1 e^{-\frac{a_2}{2}z^2 + c_2 z}, \quad c_1, c_2 \in \mathbb{C};$$

3) *if* $k = 3$ *or* 4, *then we must have* $a_2 = 0$ *and any meromorphic solution* w *of* (4) *satisfies*

$$w'^2 + a_4 w^4 + 2a_3 w^3 + 2Cw^2 - 2a_1 w - a_0 = 0, C \in \mathbb{C},$$

whose general solution is meromorphic [16, Chapter 11] and given in the Appendix A.

THEOREM 3.2 *Consider the differential equation* (5)

$$w'' + cw'^2 + P(w) = 0,$$

where $c \in \mathbb{C}$ *and* $P(w) = \sum_{n=0}^{k} a_n w^n$ *is a polynomial. If the ODE* (5) *has non-constant meromorphic solutions, then we have* $k \leq 4$ *and its meromorphic solutions are characterized as follows:*

1) *for* $c = 0$, *we have* $k \leq 3$ *and*
   i) *non-entire meromorphic solutions of* (5) *exist only for* $k = 2$ *or* 3 *and they are given in the Appendix A as ODE* (5) *can then be reduced to ODE* (A1).
   ii) *entire solutions of* (5) *exist only for* $k = 0$ *or* 1 *and they are given by*

$$w(z) = \begin{cases} c_1 \sin\left(\sqrt{a_1} z\right) + c_2 \cos\left(\sqrt{a_1} z\right) - \frac{a_0}{a_1}, & k = 1, \\ c_1 + c_2 z - \dfrac{a_0}{2} z^2, & k = 0, \end{cases} \tag{9}$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants.

2) for $c \neq 0$, any meromorphic solution w of (5) satisfies

$$4c^5 w'^2 + 4a_4c^4w^4 + \left(4a_3c^4 - 8a_4c^3\right) w^3 + \left(4a_2c^4 - 6a_3c^3 + 12a_4c^2\right) w^2 + \left(4a_1c^4 - 4a_2c^3 + 6a_3c^2 - 12a_4c\right) w + 4a_0c^4 - 2a_1c^3 + 2a_2c^2 - 3a_3c + 6a_4 = 0,$$

whose general solution is meromorphic and given in the Appendix A.

The estimate of order of meromorphic solutions of (4) and (5) comes as an immediate corollary to Theorem 3.1 and Theorem 3.2.

Corollary 3.3 *All meromorphic solutions of (4) and (5) are of finite order and hence classical conjecture holds for ODEs (4) and (5).*

Finally, using the results in [17, 18] and Lemma 4.5, we have

Theorem 3.4 *The classical conjecture holds for the second order differential equation (6)*

$$w'' + cw' + P(w) = 0,$$

where $c \in \mathbb{C}$ and $P(w)$ is a polynomial with constant coefficients in w of degree k.

Most results of this paper are contained in the third author's thesis [19].

4. Proof of main results

Before proving our main results, let us recall two theorems which will be used later.

Theorem 4.1 *[20, p. 43] Let $f_j(z)$ and $g_j(z) (j = 1, 2, \ldots, n)(n \geq 2)$ be two systems of entire functions satisfying the following conditions:*

1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.

2) For $1 \leq j, k \leq n, j \neq k, g_j(z) - g_k(z)$ is non-constant.

3) For $1 \leq j \leq n, 1 \leq h, k \leq n, h \neq k$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}.$$

Then $f_j(z) \equiv 0\,(j = 1, 2, \ldots, n)$.

Theorem 4.2 *[20, p. 210] Let f and g be two transcendental entire functions. Then*

$$\lim_{r\to\infty} \frac{\log M(r, f \circ g)}{\log M(r, f)} = \infty, \quad \lim_{r\to\infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty,$$
$$\lim_{r\to\infty} \frac{\log M(r, f \circ g)}{\log M(r, g)} = \infty, \quad \lim_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty.$$

4.1. *Proof of Theorem 3.1*

If $k = 0$ or 1, the ODE (4) is just a special case of the ODE (8) and thus we only need to consider the case $k \geq 2$.

First of all, one can check immediately that (4) does not admit polynomial solutions for $k > 2$. By Wiman-Valiron theory [6, 13, Chapter 3], we can show that the ODE (4) does not have any transcendental entire solution when $k > 2$ as there is only one top degree term in (4). Now for a meromorphic solution w of (4) with a pole at some $z = z_0 \in \mathbb{C}$, we consider its Laurent series $w(z) = \sum_{n=0}^{\infty} w_n(z - z_0)^{n+p}, p < 0, p \in \mathbb{Z}, w_0 \neq 0$. If $k = 2$, we have $w_0 = 0$, which is impossible and hence for $k = 2$ there is no nonentire meromorphic solution. For $k > 2$, w_0 is determined by $ww'' - w'^2 + a_k w^k$. Comparing the terms determining w_0 yields $(k - 2)p = -2$. Since $p \in \mathbb{Z}$ and $k > 2$, there are only two choices 3 and 4 for k. In conclusion, meromorphic solutions of the ODE (4) exist only in the cases $k = 2, 3$ or 4 and they are entire for $k = 2$ and nonentire for $k = 3, 4$. We will divide them into the following two cases.

Case 1: $k = 3$ or 4.

By considering the Laurent series expansion around a pole of w (if it exists), a necessary condition for the existence of non-entire meromorphic solution of (4) is $a_2 = 0$, otherwise logarithmic branch singularity appears in the solution. Then with the integration factor $w^{-3}w'$, the ODE (4) becomes

$$w'^2 + a_4 w^4 + 2a_3 w^3 + 2Cw^2 - 2a_1 w - a_0 = 0, \quad C \in \mathbb{C}, \tag{10}$$

whose general solution is meromorphic and is given in the Appendix A.

Case 2: $k = 2$.

We consider the following two subcases.

Subcase 2a: If a nonconstant entire solution w of the ODE (4) is zero-free on $\mathbb{C}$, then there exists a nonconstant entire function $h(z)$ such that $w(z) = e^{h(z)}$. Substituting $w(z) = e^{h(z)}$ into (4) yields

$$a_0 e^{-2h} + a_1 e^{-h} + a_2 + h'' = 0.$$

If h is a transcendental entire function, then by Theorem 4.2 and the properties $T(r, a+f) = T(r, f) + O(1), a \in \mathbb{C}$ and $T(r, f^{(n)}) = O(T(r, f))$ for $r \in (0, \infty)$ outside a possible exceptional set of finite linear measure, we have $T(r, a_2 + h'') = o\{T(r, e^h)\}$. If $h(z)$ is a nonconstant polynomial, then $T(r, a_2 + h'') = o\{T(r, e^h)\}$ holds obviously. Therefore, according to Theorem 4.1, we have

$$a_0 = a_1 = a_2 + h'' = 0.$$

Thus (4) has a meromorphic solution in this case only if $a_0 = a_1 = 0$ and it can be explicitly solved with the general solution given by $w(z) = c_1 e^{-\frac{a_2}{2}z^2 + c_2 z}$, where $c_1, c_2 \in \mathbb{C}$.

Subcase 2b: The entire solution w of the ODE (4) has at least one zero in $\mathbb{C}$. Then $v = \dfrac{1}{w}$ is a meromorphic function with at least one pole in $\mathbb{C}$ and it satisfies

$$vv'' - v'^2 - a_0v^4 - a_1v^3 - a_2v^2 = 0. \tag{11}$$

If a_0 and a_1 do not vanish simultaneously, from *Case 1*, we know that (11) does not have any meromorphic solution since $a_2 \neq 0$. If $a_0 = a_1 = 0$, then *Subcase 2a* implies $v(z)$ is an entire function which is a contradiction. Therefore in *Case 2*, all the nonconstant entire solutions of (4) are zero-free.

4.2. *Proof of Theorem 3.2*

If $c = 0$, then for $k \geq 4$, by Wiman-Valiron theory and from the local series expansion around a pole of w (if it exists), one can easily see that (5) does not have any entire and non-entire meromorphic solution, respectively. Hence we must have $k \leq 3$. Since $c = 0$, (5) is a second order Briot-Bouquet differential equation and hence by [9, Theorem 1] *all* the non-entire meromorphic solutions of (5) belong to class W. If $P(w)$ is linear in w or vanishes identically, then the general solution of the ODE (5) is given by (9). Suppose $k = 2$ or 3, then the equation (5) has neither transcendental entire solutions (by Wiman-Valiron theory) nor polynomial solutions. For the non-entire meromorphic solutions, they are given in the Appendix A as ODE (5) can be reduced to ODE (A1).

In the following, we shall consider the case $c \neq 0$. Similarly, we have $k \leq 4$ otherwise (5) has no meromorphic solutions.

Assume that z_0 is neither a pole nor a critical point of the meromorphic solution w of (5), i.e., $w(z_0) \neq \infty, w'(z_0) \neq 0$, then there exist a neighborhood $\mathcal{N}'$ of w_0 and a neighborhood $\mathcal{N}$ of z_0 such that $w : \mathcal{N} \to \mathcal{N}'$ is univalent. Since w is a nonconstant univalent function from $\mathcal{N}$ to $\mathcal{N}'$, it has an inverse univalent function $z = \varphi(w)$. We define $y : \mathcal{N}' \to \mathbb{C}$ to be

$$y(w) := w'(\varphi(w)). \tag{12}$$

Therefore $y(w)$ is an analytic function in $\mathcal{N}'$. By using (12), the ODE (5) reduces to

$$\left(\frac{y^2}{2}\right)' + cy^2 + P(w) = 0, \tag{13}$$

where $y(w)$ is defined by (12) and is analytic in w in some domain of $\mathbb{C}$. Solving the linear ODE (13) yields

$$\begin{gathered}
4a_4c^4w^4 + \left(4a_3c^4 - 8a_4c^3\right)w^3 + \left(4a_2c^4 - 6a_3c^3 + 12a_4c^2\right)w^2 \\
+ \left(4a_1c^4 - 4a_2c^3 + 6a_3c^2 - 12a_4c\right)w \\
+4a_0c^4 - 2a_1c^3 + 2a_2c^2 - 3a_3c + 6a_4 + 4c^5w'^2 = Ce^{-2cw}, C \in \mathbb{C}.
\end{gathered} \tag{14}$$

Notice that for nonzero C, the ODE (14) has neither non-constant polynomial solution nor transcendental entire solution (by Theorem 4.2). Next, assume w is a meromorphic solution of (14) with a pole at $z = z_1$, then z_1 is either a pole or a removable singularity of the l.h.s of (14) while z_1 is an essential singularity of

Ce^{-2cw} for nonzero C. Hence, in order for (14) to hold for w meromorphic, we must have $C = 0$ for which the general solution of the equation (14) is meromorphic and given in the Appendix A.

Remark 4.3 One may also apply the integration factor $e^{2cw}w'$ for (5) to obtain the equation (14).

4.3. *Proof of Theorem 3.4*

To prove Theorem 3.4, let us recall some lemmas that we will need.

Lemma 4.4 *The equation*

$$w''(z) + cw'(z) - \frac{6}{\lambda}(w(z) - e_1)(w(z) - e_2) = 0, \lambda \neq 0 \tag{15}$$

has meromorphic solutions if and only if $c\left(c^2\lambda + 25e_1 - 25e_2\right)\left(c^2\lambda - 25e_1 + 25e_2\right) = 0$ *and they are given respectively as follows*

(1) if $c = 0$, then the general solution to the equation (15) is meromorphic and given in the Appendix A as ODE (15) can be reduced to ODE (A1).
(2) for $c^2\lambda = 25(e_i - e_j) \neq 0, i, j \in \{1, 2\}$, then the general solution to the equation (15) [17, 18] is

$$w_2(z) \;=\; (e_i - e_j)e^{\frac{-2c}{5}z}\wp\left(e^{\frac{-c}{5}z} - \zeta_0; 0, g_3\right) + e_j, \tag{16}$$

where $\zeta_0, g_3 \in \mathbb{C}$ are arbitrary.

Lemma 4.5 *The ODE*

$$w'' + cw' - \frac{2}{\lambda^2}(w - q_1)(w - q_2)(w - q_3) = 0, \lambda(\neq 0), c, q_1, q_2, q_3 \in \mathbb{C} \tag{17}$$

has nonconstant meromorphic solutions if and only if c *satisfies*

$$c\prod(c\lambda + q_i + q_j - 2q_k)(-c\lambda + q_i + q_j - 2q_k) = 0, \tag{18}$$

where (ijk) *is any permutation of* (123)*, and we further have*

1) for $c = 0$*, the general solution of* (17) *is meromorphic and given in the Appendix A as ODE* (17) *can be reduced to ODE* (A1).
2) for $c \neq 0$ *satisfying* (18)*, we can classify the nonconstant meromorphic solutions of* (17) *into two families*
   i) for $c = \dfrac{2q_i - q_j - q_k}{\lambda} = \dfrac{-q_i + 2q_j - q_k}{-\lambda}$,

$$w_6(z) = q_k - \frac{q_i - q_k}{2}e^{-\frac{q_i - q_k}{\lambda}z}\frac{\wp'(e^{-\frac{q_i - q_k}{\lambda}z} - \zeta_0; g_2, 0)}{\wp(e^{-\frac{q_i - q_k}{\lambda}z} - \zeta_0; g_2, 0)},\ \zeta_0, g_2 \textit{ arbitrary.} \tag{19}$$

   ii) if $c = \dfrac{2q_i - q_j - q_k}{\pm\lambda}$,

$$w_7(z) = \frac{q_j e^{\frac{q_j(z-z_0)}{\pm\lambda}} - q_k e^{\frac{q_k(z-z_0)}{\pm\lambda}}}{e^{\frac{q_j(z-z_0)}{\pm\lambda}} - e^{\frac{q_k(z-z_0)}{\pm\lambda}}},\ z_0 \ \textit{arbitrary}, \tag{20}$$

which for $q_j = q_k$ *degenerates to*

$$w_8(z) = \frac{\pm\lambda}{z-z_0} + q_j,\ z_0 \ \textit{arbitrary}. \tag{21}$$

Remark 4.6 For $c \neq 0$, all the meromorphic solutions of the equation (17) are given by (19)-(21) and the solution (19) is the general solution.

Proof of Lemma 4.5. One can easily see that constant solutions of the ODE (17) are $w = q_n, n = 1, 2, 3$. Next we consider nonconstant meromorphic solutions of (17). By making use of Wiman-Valiron theory, it can be proven immediately that the ODE (17) does not have any nonconstant transcendental entire solution. Meanwhile, the ODE (17) does not admit any nonconstant polynomial solution. Consequently, each nonconstant meromorphic solution of the equation (17) should have at least one pole on $\mathbb{C}$.

Suppose w is a meromorphic solution of (17) with a pole at $z = z_0$. Without loss of generality, we may assume $z_0 = 0$ then $w(z) = \sum_{j=p}^{+\infty} w_j z^j, -p \in \mathbb{N}, w_p \neq 0$. Substituting the series expansion of w into the ODE (17) gives $p = -1, w_{-1} = \pm\lambda$. The ODE (17) has Fuchs indices $-1, 4$ and the corresponding compatibility conditions regarding the existence of meromorphic solution are

$$\begin{cases} c\prod(c\lambda + q_i + q_j - 2q_k) = 0, \text{ if } w_{-1} = \lambda, \\ c\prod(-c\lambda + q_i + q_j - 2q_k) = 0, \text{ if } w_{-1} = -\lambda, \end{cases} \tag{22}$$

where (ijk) is any permutation of (123).

Now we compare the ODE (17) with the following second order ODE

$$[D - f_2(w)][D - f_1(w)](w - \alpha) = 0, \tag{23}$$

where $D = \dfrac{d}{dz}, \alpha \in \mathbb{C}$ and $f_i(w) = A_i w + B_i, A_i, B_i \in \mathbb{C}, i = 1, 2$. Expanding (23) gives

$$w'' - (f_1 + f_2 + \frac{df_1}{dw} w - \alpha\frac{df_1}{dw})w' + f_1 f_2 (w - \alpha) = 0. \tag{24}$$

Identifying the equations (17) and (24) leads to the conditions

$$\begin{cases} f_1 + f_2 + \dfrac{df_1}{dw} w - \alpha\dfrac{df_1}{dw} + c = 0, \\ f_1 f_2 (w - \alpha) = -\frac{2}{\lambda^2}(w - q_1)(w - q_2)(w - q_3). \end{cases} \tag{25}$$

One can check that the compatibility conditions (22) hold if and only if the conditions (25) are satisfied or $c = 0$.

If $c = 0$, then the ODE (17) reduces to a first order Briot-Bouquet differential equation through multiplying it by w' and integration. Therefore all its meromorphic solutions belong to class W and they are given in the Appendix A.

For $c \neq 0$ and assuming (22) from now on, due to the symmetry in (22) and the fact that w has at least one pole on $\mathbb{C}$, it suffices to consider the case $c = (-q_1 + 2q_2 - q_3)/\lambda \neq 0$ and one choice for $A_i, B_i, i = 1, 2$ and α is

$$A_1 = -\frac{1}{\lambda}, A_2 = \frac{2}{\lambda}, B_1 = \frac{q_3}{\lambda}, B_2 = -\frac{2q_2}{\lambda}, \alpha = q_1. \tag{26}$$

As a consequence, if $c = (-q_1 + 2q_2 - q_3)/\lambda \neq 0$, then (17) can be written as

$$[D - \frac{2}{\lambda}w - B_2][D + \frac{w}{\lambda} - B_1](w - \alpha) = 0. \tag{27}$$

Let $G(z) = [D + \frac{w}{\lambda} - B_1](w - \alpha)$, then we have $[D - \frac{2}{\lambda}w - B_2]G(z) = 0$ from which one can solve for $G(z) = \beta e^{\int \frac{2}{\lambda} w dz} e^{B_2 z}, \beta \in \mathbb{C}$. If $\beta = 0$, from $G(z) = [D + \frac{w}{\lambda} - B_1](w - \alpha) = 0$, we are able to obtain the first family of meromorphic solutions of the ODE (17)

$$w(z) = \frac{q_1 e^{\frac{q_1(z-z_0)}{\pm\lambda}} - q_3 e^{\frac{q_3(z-z_0)}{\pm\lambda}}}{e^{\frac{q_1(z-z_0)}{\pm\lambda}} - e^{\frac{q_3(z-z_0)}{\pm\lambda}}},\ z_0 \in \mathbb{C}. \tag{28}$$

For $\beta \neq 0$, we let $H(z) = e^{\int \frac{2}{\lambda} u dz}$ which satisfies $H'(z) = 2w(z)H(z)/\lambda$ and

$$[D + \frac{u}{\lambda} - B_1](w - \alpha) = \beta e^{B_2 z} H(z), \tag{29}$$

hence, w is meromorphic if and only if H is meromorphic. By the substitution of $w = \frac{\lambda}{2}\frac{H'}{H}$ into (29), we have

$$-2B_1\lambda HH' + 4\alpha B_1 H^2 - 4\beta e^{B_2 z} H^3 + 2\lambda HH'' - 2\alpha HH' - \lambda H'^2 = 0. \tag{30}$$

If we let $H(z) = e^{-B_2 z} h(z)$, then the ODE (30) reduces to

$$(2B_1 + B_2)(2\alpha + B_2\lambda) h^2 - 2h\left((\alpha + (B_1 + B_2)\lambda) h' - \lambda h''\right) - \lambda h'^2 - 4\beta h^3 = 0. \tag{31}$$

Suppose h is a meromorphic solution of (31). W.L.O.G, we assume that it has a pole at $z = 0$ and $h(z) = \sum_{j=p}^{+\infty} h_j z^j, -p \in \mathbb{N}, h_p \neq 0$ then one can check that $p = -2$ and the Fuchs indices of the ODE (31) are $-1, 4$ with the compatibility condition

$$(\alpha + (B_1 + B_2)\lambda)^2 \left(2\alpha\lambda(10B_1 + B_2) - 8\alpha^2 + \left(-8B_1^2 + 2B_2B_1 + B_2^2\right)\lambda^2\right) = 0, \tag{32}$$

which by the substitution of (26) reduces to

$$(q_1 + q_2 - 2q_3)(2q_1 - q_2 - q_3)(q_1 - 2q_2 + q_3) = 0, \tag{33}$$

which implies $q_3 = (q_1 + q_2)/2$ or $q_1 = (q_2 + q_3)/2$ since $c = (2q_2 - q_1 - q_3)/\lambda \neq 0$.

Then by the substitution of (26), the ODE (31) reduces to

$$\begin{cases} -\lambda^2 h'(z)^2 + \lambda h(z)\left(2\lambda h''(z) + 3\left(q_2 - q_1\right) h'(z)\right) + 2\left(q_2 - q_1\right)^2 h(z)^2 - 4\beta\lambda h(z)^3 = 0, \\ q_3 = \frac{1}{2}\left(q_1 + q_2\right). \end{cases}$$
$$\begin{cases} -\lambda^2 h'(z)^2 + \lambda h(z)\left(2\lambda h''(z) + 3\left(q_2 - q_3\right) h'(z)\right) + 2\left(q_2 - q_3\right)^2 h(z)^2 - 4\beta\lambda h(z)^3 = 0, \\ q_1 = \frac{1}{2}\left(q_2 + q_3\right). \end{cases}$$

Next, it suffices to consider the case $q_3 = (q_1 + q_2)/2$ due to the symmetry in the above two equations. By the translation against the dependent variable u, we may further assume $q_3 = 0$ which implies $q_1 + q_2 = 0$. Let us come back to equation (30), which by the substitution of (26) with $q_1 = -q_2 \neq 0, q_3 = 0$ reduces to

$$-\lambda H'(z)^2 + 2H(z)\left(\lambda H''(z) + q_2 H'(z)\right) - 4\beta H(z)^3 e^{-\frac{2q_2 z}{\lambda}} = 0. \tag{34}$$

Performing the transformation $H(z) = v(\zeta), \zeta = e^{-\frac{q_2}{\lambda}z}$ gives

$$v'^2 - \frac{2\beta\lambda}{q_2^2} v^3 + Cv = 0, \tag{35}$$

whose general solution is meromorphic and can be found in the Appendix A.

Finally, for $c = (-q_1 + 2q_2 - q_3)/\lambda \neq 0$ and $q_3 = (q_1 + q_2)/2$, which implies $c = -(2q_1 - q_2 - q_3)/\lambda$, we obtain the meromorphic solutions of the ODE (17) (which meanwhile is the general solution)

$$w(z) = -\frac{q_2 - q_3}{2} e^{-\frac{q_2 - q_3}{\lambda}z} \frac{\wp'(e^{-\frac{q_2 - q_3}{\lambda}z} - \zeta_0; g_2, 0)}{\wp(e^{-\frac{q_2 - q_3}{\lambda}z} - \zeta_0; g_2, 0)} + q_3,\ \zeta_0, g_2 \in \mathbb{C}. \tag{36}$$

Lemma 4.7 *([6, p. 5]) Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set F with finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) < h(\alpha r)$ holds for all $r \geq r_0$.*

Proof of Theorem 3.4. From the expression of solutions in Lemmas 4.4 and 4.5, it suffices to focus on $w_2(z)$ and $w_6(z)$ because other solutions belong to the class W which only consists of meromorphic functions of finite order.

We claim that for every $\alpha\,(\neq 0) \in \mathbb{C}$, there exists $A, B \in \mathbb{R}^+$ such that

$$T(r, \wp(e^{\alpha z}; \omega_1, \omega_2)) < A\exp(Br), 0 \leq r < \infty,$$

where $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}\,(\omega_1/\omega_2 \notin \mathbb{R})$ are the periods of $\wp(z)$. Since $\wp(z; \omega_1, \omega_2) = \wp(z/\omega_1; 1, \tau)/\omega_1^2$, where $\tau = \omega_2/\omega_1$, we only need to prove the claim for $\wp(e^{\alpha z}; 1, \tau)$, where $\tau \in \mathbb{H}$. For brevity, we denote $\wp(e^{\alpha z}; 1, \tau)$ by $\wp(e^{\alpha z})$.

From the theory of elliptic functions [21, 22], we know that $\wp(z; 1, \tau)$ satisfies the first order ODE $\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$, where $e_1 = \wp(1/2), e_2 = \wp(\tau/2), e_3 = \wp((1+\tau)/2)$ are distinct. Next we consider $\overline{N}_{\wp(e^{\alpha z})}(r, e_j) := \int_0^r \frac{\overline{n}_{\wp(e^{\alpha z})}(t, e_j)}{t} dt, j =$

$1,2,3$, where $\overline{n}_f(r,a)$ denotes the number of poles of $1/(f-a)$ in $\mathbb{D}(r)=\{z\in\mathbb{C}||z|<r\}$, without counting multiplicity.

Let $T=2\pi i/\alpha=|T|e^{i\beta},\beta\in[0,2\pi)$, R be the region enclosed by the rectangle $\{z\in\mathbb{C}|z=x+iy,0\leq|x|,|y|<r\}$ and $R'=e^{-i\beta}R=\{z'\in\mathbb{C}|z'=e^{-i\beta}z,z\in R\}\supset\mathbb{D}(r)$.

Then we have $\overline{n}_{\wp(e^{\alpha z})}(t,e_j)\leq\overline{n'}_{\wp(e^{\alpha z})}(t,e_j)$, where $\overline{n'}_f(r,a)$ denotes the number of poles of $1/(f-a)$ in R', without counting multiplicity. As $e^{\alpha z}$ has a period T, we have $\overline{n'}_{\wp(e^{\alpha z})}(t,e_j)\leq\dfrac{2([t]+1)}{T}\overline{n''}_{\wp(e^{\alpha z})}(t,e_j)$, where $[r]$ is the integer part of $r\geq 0$ and $\overline{n''}_f(t,a)$ is the number of poles of $1/(f-a)$ in $R_1=\{e^{-i\beta}z|z=x+iy,-t<x<t,0\leq y<T\}$, without counting multiplicity. Since $\wp^{-1}(e_1)=\{\frac{1}{2}+m+n\tau|m,n\in\mathbb{Z}\}$, $\overline{n''}_{\wp(e^{\alpha z})}(t,e_1)\leq(2[e^{|\alpha|t}]+1)\times\dfrac{2e^{|\alpha|t}}{|\tau|}$. Therefore,

$$\begin{aligned}\overline{N}_{\wp(e^{\alpha z})}(r,e_1) &= \int_0^r\frac{\overline{n}_{\wp(e^{\alpha z})}(t,e_1)}{t}dt\\ &= \int_\delta^r\frac{\overline{n}_{\wp(e^{\alpha z})}(t,e_1)}{t}dt\\ &\leq \int_\delta^r\frac{2([t]+1)}{Tt}\overline{n''}_{\wp(e^{\alpha z})}(t,e_1)dt\\ &\leq \frac{4(1+\frac{1}{\delta})}{T|\tau|}\int_\delta^r e^{|\alpha|t}(2e^{|\alpha|t}+1)dt\\ &\leq \frac{4(1+\frac{1}{\delta})}{T|\tau|}\frac{e^{2|\alpha|r}+e^{|\alpha|r}-2}{|\alpha|}\\ &< a_1e^{b_1r},\end{aligned}$$

where $a_1=\dfrac{4(1+\frac{1}{\delta})}{T|\alpha\tau|}>0,b_1=2|\alpha|>0$, and $\delta>0$ is chosen such that $\wp(e^{\alpha z})$ omits e_1 in $\mathbb{D}(\delta)$. Applying the same argument, we can obtain the upper bounds of $\overline{N}_{\wp(e^{\alpha z})}(r,e_j),j=2,3$ which are given by

$$\overline{N}_{\wp(e^{\alpha z})}(r,e_j)<a_je^{b_jr},0<a_j,b_j,j=2,3.$$

According to the Second Main Theorem of Nevanlinna theory, we have

$$T(r,\wp(e^{\alpha z}))\ \leq\ \sum_{j=1}^3\overline{N}_{\wp(e^{\alpha z})}(r,e_j)+S(r,\wp(e^{\alpha z})),$$

where $S(r,\wp(e^{\alpha z}))=o(T(r,\wp(e^{\alpha z}))$, for all $r\in[0,+\infty)$ outside an exceptional set $E\subset(0,+\infty)$ with finite linear measure. Hence,

$$T(r,\wp(e^{\alpha z}))<a'e^{b'r}$$

holds for all $r\in[0,+\infty)-E$, where $a'=(1+\varepsilon)(a_1+a_2+a_3),\varepsilon>0,b'=\max\limits_{1\leq j\leq 3}b_j$. According to Lemma 4.7, for $\gamma=3/2$, there exists $r_0>0$ such that $T(r,\wp(e^{\alpha z}))<a'e^{3b'r/2}$ for all $r\geq r_0$. On the other hand, it is obvious that there exist $a'',b''>0$

such that $T(r, \wp(e^{\alpha z})) < a'' e^{b'' r}$ for all $0 \le r < r_0$. As a consequence, we have

$$T(r, \wp(e^{\alpha z})) < Ae^{Br}, 0 \le r < \infty,$$

where $A = \max\{a', a''\}, B = \max\{3b'/2, b''\}$.

From the proof of our claim, it is easy to see that the same conclusion holds for $\wp(k_1 \exp\{\alpha z\} + k_2), \wp'(k_1 \exp\{\alpha z\} + k_2), k_1, k_2, \alpha \in \mathbb{C}$ as well. By making use of the properties $T(r, fg) \le T(r, f) + T(r, g)$ and $T(r, f) = T(r, 1/f) + O(1)$, we conclude that there exists $a, b > 0$ and $c = 1$, such that

$$T(r, w_i(z)) < ae^{br}, 0 \le r < \infty, i = 2, 6.$$

Thus, the proof is complete.

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Appendix A. The second degree Briot and Bouquet equation

We recall here the various expressions for the general solution of the first order second degree binomial equation of Briot and Bouquet

$$\left(\frac{du}{dz}\right)^2 = a_k \prod_{j=1}^{k} (u - e_j), \tag{A1}$$

in which the integer k runs from 0 to 4, a_k is a nonzero complex constant and e_j are complex constants. Let us denote $\wp$ and ζ the functions of Weierstrass,

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,\ \zeta' = -\wp.$$

The general solution is (the arbitrary origin z_0 of z is omitted) [23, Table 1, p. 73],

$$\begin{cases} u = a_4^{-1/2} \left(\zeta(z+a) - \zeta(z-a) - 2\zeta(a) + A \right) = a_4^{-1/2} \left(A - \dfrac{\wp'(a)}{\wp(z) - \wp(a)} \right), \\ \qquad k=4,\ e_j \text{ all distinct}, \\ \dfrac{1}{u - e_1} = A \cosh(Bz) + C,\ k = 4, \quad \text{one double root } e_1, \\ u = \dfrac{e_1 + e_2}{2} + A \coth(Bz),\ k=4, \quad \text{two double roots } e_1, e_2 \\ \dfrac{u - e_4}{u - e_1} = A \left[\dfrac{e_4 - e_1}{2} z \right]^2,\ k=4, \quad \text{one triple root } e_1, \\ u = e_1 \pm \dfrac{a_4^{-1/2}}{z},\ k=4, \quad \text{one quadruple root}, \\ u = \dfrac{e_1 + e_2 + e_3}{3} + \dfrac{4}{a_3} \wp(z, g_2, g_3),\ k=3,\ e_j \text{ all different}, \\ u = A + B \coth^2(Cz),\ k=3,\ e_1 = e_2 \neq e_3, \\ u = e_1 + \dfrac{4}{a_3 z^2},\ k=3,\ e_1 = e_2 = e_3, \\ u = \dfrac{e_1 + e_2}{2} + A \cosh(Bz),\ k=2,\ e_1 \neq e_2, \\ u = e_1 + e^{\pm \sqrt{a_2} z},\ k=2,\ e_1 = e_2, \\ u = e_1 + \dfrac{a_1}{4} z^2,\ k=1, \\ u = \pm \sqrt{a_0} z,\ k=0, \end{cases} \tag{A2}$$

in which $\wp(a), A, B, C$ are algebraic functions of a_k, e_j.