Lévy insurance risk process with Poissonian taxation

Zhimin Zhang[*], Eric C.K. Cheung[†], Hailiang Yang[†]

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Abstract

The idea of taxation in risk process was first introduced by Albrecher and Hipp (2007), who suggested that a certain proportion of the insurer's income is paid immediately as tax whenever the surplus process is at its running maximum. In this paper, a spectrally negative Lévy insurance risk model under taxation is studied. Motivated by the concept of randomized observations proposed by Albrecher et al. (2011b), we assume that the insurer's surplus level is only observed at a sequence of Poisson arrival times, at which the event of ruin is checked and tax may be collected from the tax authority. In particular, if the observed (pre-tax) level exceeds the maximum of the previously observed (post-tax) values, then a fraction of the excess will be paid as tax. Analytic expressions for the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) and the expected discounted tax payments until ruin are derived. The Cramér-Lundberg asymptotic formula is shown to hold true for the Gerber-Shiu function, and it differs from the case without tax by a multiplicative constant. Delayed start of tax payments will be discussed as well. We also take a look at the case where solvency is monitored continuously (while tax is still paid at Poissonian time points), as many of the above results can be derived in a similar manner. Some numerical examples will be given at the end.

Keywords: Lévy insurance risk model; Randomized observation periods; Poissonian observer; Gerber-Shiu expected discounted penalty function; Discounted tax payments.

1 Introduction

In this paper, the baseline surplus process (before taxation) of an insurance company is assumed to follow a spectrally negative Lévy process $X = \{X_t\}_{t\geq 0}$ (see Kyprianou (2014)). For $x \in \mathbb{R}$, let P_x and E_x be the probability law and the expectation operator respectively when X starts with an initial level $X_0 = x \geq 0$. For notational convenience, we denote $P = P_0$ and $E = E_0$. The Lévy process X can be characterized by the Laplace exponent

$$\psi(s) = \frac{1}{t}\ln E[e^{sX_t}] = cs + \frac{\sigma^2}{2}s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx\mathbf{1}_{(x<1)})\nu(dx), \tag{1.1}$$

where $\sigma \geq 0$, and $\nu(\cdot)$ is a non-negative measure on $(0,\infty)$ that satisfies the usual condition $\int_{(0,\infty)}(1 \wedge x^2)\nu(dx) < \infty$. We additionally assume $\int_{(1,\infty)} x\nu(dx) < \infty$ so that X has finite expectation. Moreover,

[*]College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P.R. China.

[†]Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong.

the constant c in (1.1) is such that the safety loading condition $\psi'(0+) = E[X_1] > 0$ is satisfied, and we exclude the case where X has monotone sample paths.

The class of Lévy insurance risk processes has gained popularity among researchers in the past few years. Formulas of the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) in Lévy risk models can be found in e.g. Asmussen and Albrecher (2010, Chapters XI and XII) and Kyprianou (2013), and related extensions to the cases of path dependent penalties have been proposed by Biffis and Morales (2010), Biffis and Kyprianou (2010), and Feng and Shimizu (2013). Other recent contributions in Lévy risk models with dividends and/or Parisian implementation delays include Kyprianou and Palmowski (2007), Renaud and Zhou (2007), Loeffen (2008), Kyprianou and Loeffen (2010), Loeffen and Renaud (2010), Czarna and Palmowski (2011, 2014), Loeffen et al. (2013), and Landriault et al. (2014), among others. Apart from the classical compound Poisson risk model, the class of Lévy risk processes contains a number of well known risk models as special cases, such as the perturbed compound Poisson model (e.g. Gerber and Landry (1998), and Tsai and Willmot (2002)), Gamma risk process (e.g. Dufresne et al. (1991)), α-stable risk model (e.g. Furrer (1998)), and perturbed risk model driven by subordinator (e.g. Garrido and Morales (2006), and Morales (2007)). Therefore, with powerful tools such as scale functions, resolvent measures and level-crossing arguments available for Lévy processes, the analysis of Lévy risk models can serve to unify the study of the afore-mentioned risk models.

The concept of taxation under a loss-carry-forward system was first applied to insurance risk models by Albrecher and Hipp (2007), who suggested that a certain proportion (known as the tax rate) of the insurer's income is paid immediately as tax whenever the surplus process is at its running maximum. In the context of the classical compound Poisson risk process, they derived expressions for the survival probability and the expected discounted tax payments until ruin. In particular, they showed that the survival probability with tax simply equals a power of the survival probability without tax, and such a result is now commonly referred to as the tax identity. Since then, risk models with taxation have been studied by a number of researchers. A simple proof of a generalized version of the tax identity, which assumes that the tax rate can possibly depend on the insurer's surplus level, was presented in Albrecher et al. (2009). These results were extended to the Lévy risk model by Albrecher et al. (2008b), Kyprianou and Zhou (2009) and Renaud (2009), where the higher moments of discounted tax payments until ruin were also analyzed. Moreover, related tax identity was established by Li et al. (2013) for a time-homogeneous diffusion process. Compound Poisson risk model under taxation was also considered with credit or debit interest by Wei (2009), Ming et al. (2010) and Wang et al. (2010), where the former two papers contain some asymptotic results for the ruin probability. A further generalization in Cheung and Landriault (2012), who studied the Gerber-Shiu function further incorporating the maximum surplus before ruin, allows both the premium rate and the tax rate to be surplus-dependent. This encompasses models with credit interest, threshold or multi-threshold dividend strategy. Another development was made by Wei et al. (2010) and Albrecher et al. (2014), who looked at the effect of taxation in risk models with Markovian claim arrivals. In addition, Albrecher et al. (2011a) discussed the idea of ruin excursions and obtained asymptotic survival probability for a renewal risk model with tax; whereas Albrecher and Ivanovs (2014) derived power identities for Lévy risk models under taxation and capital injections.

In all the afore-mentioned works concerning taxation, it is implicitly assumed that the insurer's surplus is observed continuously as tax payments are made immediately once the surplus process is at its running maximum. As commented in Hao and Tang (2009), tax is usually collected periodically (e.g. monthly, quarterly or annually) by the tax authority, leading them to study a model in which a fixed portion of the net income (if positive) of each period (of length 1) is paid as tax. Note that their taxation rule differs from the loss-carry-forward system, since a period of positive net income does not necessarily result in a new running maximum. In this paper, we shall apply periodic taxation to the Lévy model (1.1) under a loss-carry-forward system. Specifically, it is assumed that $\{Z_j\}_{j=1}^{\infty}$ is the sequence of time points where the insurance company reports its financial status, i.e. the insurer's surplus level is observed. At these time points, the event of ruin is checked and tax (if any) is paid. For convenience, we denote $Z_0 = 0$. At the jth observation time Z_j ($j = 1, 2, \ldots$), if the surplus level (before tax) exceeds the running maximum of previous surplus levels (after tax) observed at the time points $Z_0, Z_1, \ldots, Z_{j-1}$, then a fraction θ ($0 \leq \theta < 1$) of the excess will be paid as tax. On the other hand, if the observed surplus level is below zero, then ruin is declared. The risk model after the above modifications will be denoted by $X^{\theta} = \{X_t^{\theta}\}_{t\geq 0}$, which can be mathematically described as follows. Let $\{C_{Z_j}^{\theta}\}_{j=1}^{\infty}$ be the sequence of surplus levels observed at $\{Z_j\}_{j=1}^{\infty}$ just before any tax payment, and $\{M_{Z_j}^{\theta}\}_{j=0}^{\infty}$ be the sequence of running maximums observed at $\{Z_j\}_{j=1}^{\infty}$ immediately after tax. The processes X^{θ}, $\{C_{Z_j}^{\theta}\}_{j=1}^{\infty}$ and $\{M_{Z_j}^{\theta}\}_{j=0}^{\infty}$ are jointly described by

$$X_t^{\theta} = \begin{cases} X_t, & 0 \leq t < Z_1, \\ C_{Z_j}^{\theta} - \theta(C_{Z_j}^{\theta} - M_{Z_{j-1}}^{\theta})_+ + X_t - X_{Z_j}, & Z_j \leq t < Z_{j+1}; j = 1, 2, \ldots, \end{cases}$$

$$C_{Z_j}^{\theta} = X_{Z_{j-1}}^{\theta} + X_{Z_j} - X_{Z_{j-1}}, \qquad j = 1, 2, \ldots,$$

and

$$M_{Z_j}^{\theta} = \sup_{0\leq i\leq j} X_{Z_i}^{\theta}, \qquad j = 0, 1, \ldots.$$

Note that the running maximum at time 0 is simply the initial surplus level X_0^{θ}. Obviously, X^{θ} reduces to X when $\theta = 0$.

So far we have not made any specific assumptions on the sequence of observation times $\{Z_j\}_{j=1}^{\infty}$. We first recall the idea of randomized observation periods proposed in Albrecher et al. (2011b, 2013), who assumed Erlang(n) inter-observation times in the compound Poisson risk process with and without dividends respectively. Such a randomized approach often leads to tractable expressions of ruin-related quantities as opposed to deterministic intervals. In particular, the case of exponential inter-observation times (i.e. $n = 1$) is known to result in nice explicit formulas (see Albrecher et al. (2011b, Sections 2 and 4.1; 2013, Section 2)). Since then, ruin theory under a Poissonian observer has been further developed by Albrecher and Ivanovs (2013) and Albrecher et al. (2015), who looked at a Markov additive risk process and a Lévy risk process respectively. Indeed, exponential inter-observation times are also related to the case of constant bankruptcy rate in the (Gamma-)Omega risk model. See Albrecher et al. (2011c, Section 9), Gerber et al. (2012a, Section 3), and Albrecher and Lautscham (2013, Section 2.1.1). In the spirit of the above contributions, in this entire paper we shall assume that $\{Z_j\}_{j=1}^{\infty}$ are the arrival epochs of a Poisson process with rate $\gamma > 0$, which is independent of the attributes of X. Therefore, the inter-observation times $\{Z_j - Z_{j-1}\}_{j=1}^{\infty}$ are independent and identically distributed (i.i.d.), each following an exponential distribution with mean $1/\gamma$.

Under discrete monitoring of solvency, the time to ruin of the process X^{θ} is defined by $\tau_{\theta} = Z_{J_{\theta}^*}$, where $J_{\theta}^* = \inf\{j \geq 1 : X_{Z_j}^{\theta} < 0\}$ is the number of observations before ruin. Let $w(\cdot)$ be the penalty function defined on $[0, \infty)$, which is assumed to be non-negative and bounded by some constant A (i.e. $0 \leq w(\cdot) \leq A$). A quantity of our interest is the Gerber-Shiu expected discounted penalty function (see Gerber and Shiu (1998))

$$m_{\theta,\delta}(u) = E_u[e^{-\delta\tau_{\theta}} w(|X_{\tau_{\theta}}^{\theta}|)\mathbf{1}_{(\tau_{\theta}<\infty)}], \qquad u \geq 0, \tag{1.2}$$

where $\delta \geq 0$ can be regarded as the force of interest or a Laplace transform argument. A notable special case of $m_{\theta,\delta}(u)$ is the ruin probability

$$\Psi_\theta(u) = P_u(\tau_\theta < \infty), \qquad u \geq 0,$$

which can be retrieved by letting $\delta = 0$ and $w(\cdot) \equiv 1$. For a positive force of interest $\delta > 0$, we are also interested in the expected discounted tax payments before ruin defined by

$$V_{\theta,\delta}(u) = E_u\left[\sum_{j=1}^{J_\theta^*} e^{-\delta Z_j}\theta(C_{Z_j}^\theta - M_{Z_{j-1}}^\theta)_+\right], \qquad u \geq 0. \tag{1.3}$$

The remainder of this paper is organized as follows. In Section 2, we apply the discounted density pertaining to the increment of the embedded random walk $\{X_{Z_j}\}_{j=0}^\infty$ to study the time until the first positive tax payment and the resulting amount. It will be shown that the amount of the first tax payment (given that there is such a payment) is exponentially distributed, and the Laplace transform of the time to the first tax payment is identified. Equipped with the above results, the Gerber-Shiu function $m_{\theta,\delta}(u)$ and the expected discounted tax payments before ruin $V_{\theta,\delta}(u)$ are analyzed in Section 3, where both analytic expressions and asymptotic formulas are derived. In particular, the Cramér-Lundberg asymptotic formula for $m_{\theta,\delta}(u)$ only differs from the case without tax by a multiplicative constant. A taxation system with delayed tax payment is also studied. In Section 4, we consider the situation where solvency is monitored continuously but tax is still payable at Poissonian time points (see Avanzi et al. (2013, 2014), Zhang (2014) and Zhang and Cheung (2014a,b) for similar assumptions in dividend problems), and results analogous to those in Section 3 can readily be obtained. Section 5 ends the paper with some numerical illustrations.

2 Discounted density and the first tax payment

2.1 Discounted increment of X observed at Poissonian times

We begin by looking at the discounted density of the increment of the embedded random walk $\{X_{Z_j}\}_{j=0}^\infty$. Note that the pairs $\{(Z_j - Z_{j-1}, X_{Z_{j-1}} - X_{Z_j})\}_{j=1}^\infty$ form an i.i.d. sequence of bivariate random vector with common joint Laplace transform $E[e^{-\delta Z_1 + sX_{Z_1}}]$. As in Albrecher et al. (2011b, 2013), we introduce the discounted density of $X_0 - X_{Z_1}$, namely $g_\delta(\cdot)$, which satisfies

$$E[e^{-\delta Z_1 + sX_{Z_1}}] = \int_{-\infty}^\infty e^{-sx} g_\delta(x)dx.$$

Since $g_\delta(\cdot)$ is a two-sided density, it admits the decomposition

$$g_\delta(x) = g_{\delta,-}(-x)\mathbf{1}_{(x<0)} + g_{\delta,+}(x)\mathbf{1}_{(x>0)},$$

where the densities $g_{\delta,-}(\cdot)$ and $g_{\delta,+}(\cdot)$ respectively represent the cases of net gain and net loss of the process X during the period $(0, Z_1]$. Then

$$E[e^{-\delta Z_1 + sX_{Z_1}}] = \int_0^\infty e^{sx} g_{\delta,-}(x)dx + \int_0^\infty e^{-sx} g_{\delta,+}(x)dx. \tag{2.1}$$

To identify $g_{\delta,-}(\cdot)$ and $g_{\delta,+}(\cdot)$, it will be helpful to write the above joint Laplace transform as

$$E[e^{-\delta Z_1 + sX_{Z_1}}] = E[e^{-\delta Z_1} E[e^{sX_{Z_1}} | Z_1]] = \int_0^\infty e^{-\delta t} E[e^{sX_t}]\gamma e^{-\gamma t}dt = \gamma \int_0^\infty e^{-(\gamma+\delta)t} \int_{-\infty}^\infty e^{sx} P(X_t \in dx)dt$$

$$= \gamma \int_{x\in(-\infty,\infty)} e^{sx} \int_{t\in[0,\infty)} e^{-(\gamma+\delta)t} P(X_t \in dx)dt = \gamma \int_{-\infty}^{\infty} e^{sx}\Theta^{(\gamma+\delta)}(0, dx), \tag{2.2}$$

where $\Theta^{(q)}(0, dx)$ is the q-potential density of X (starting at the origin) defined by

$$\Theta^{(q)}(0, dx) = \int_{t\in[0,\infty)} e^{-qt} P(X_t \in dx)dt, \qquad -\infty < x < \infty.$$

From Kyprianou (2014, Corollary 8.9), one has that

$$\Theta^{(q)}(0, dx) = \left(\frac{1}{\psi'(\phi(q))} e^{-\phi(q)x} - W^{(q)}(-x) \right) dx. \tag{2.3}$$

In the above expressions, $\phi(\cdot)$ is the right inverse of the Laplace exponent given by

$$\phi(q) = \sup\{s \geq 0 : \psi(s) = q\}, \qquad q \geq 0.$$

Due to the loading assumption $\psi'(0+) > 0$, it is known that $\phi(q)$ is the unique solution to $\psi(s) = q$ in $[0, \infty)$, and $\phi(\gamma + \delta) > 0$ for all $\delta \geq 0$ (see Kyprianou (2014, p.85)). Moreover, $W^{(q)}(\cdot)$ appearing in (2.3) is the q-scale function (see Kyprianou (2014, Theorem 8.1(i))). In particular, $W^{(q)}(x) = 0$ for $x < 0$, while for $x \geq 0$ it is characterized by the Laplace transform

$$\widehat{W}^{(q)}(s) = \int_0^{\infty} e^{-sx} W^{(q)}(x)dx = \frac{1}{\psi(s) - q}, \qquad s > \phi(q). \tag{2.4}$$

Upon substitution of (2.3) into (2.2) followed by comparison with (2.1), one concludes that

$$g_{\delta,-}(x) = a_{\gamma,\delta} e^{-\phi(\gamma+\delta)x}, \qquad x > 0, \tag{2.5}$$

and

$$g_{\delta,+}(x) = a_{\gamma,\delta} e^{\phi(\gamma+\delta)x} - \gamma W^{(\gamma+\delta)}(x), \qquad x > 0, \tag{2.6}$$

where $a_{\gamma,\delta} = \gamma/\psi'(\phi(\gamma + \delta))$. The exponential form of the discounted density $g_{\delta,-}(\cdot)$ in (2.5) plays a crucial role in the derivations in Section 2.2.

Remark 1 Note that the joint Laplace transform $E[e^{-\delta Z_1 + sX_{Z_1}}]$ can also be represented as

$$E[e^{-\delta Z_1 + sX_{Z_1}}] = E[e^{-\delta Z_1} E[e^{sX_{Z_1}} | Z_1]] = E[e^{(\psi(s)-\delta)Z_1}] = \frac{\gamma}{\gamma + \delta - \psi(s)}, \tag{2.7}$$

which can be resolved into partial fractions if $\psi(s)$ is a rational function in s. This leads to an alternative way to identify $g_{\delta,-}(\cdot)$ and $g_{\delta,+}(\cdot)$ instead of using (2.5) and (2.6). See Examples 1 and 2. □

2.2 Discounted amount of the first tax payment

Let $\epsilon_0 = 0$, and for $n = 1, 2, \ldots$ define $\epsilon_n = \inf_{k\in\mathbb{N}}\{k > \epsilon_{n-1} : X_{Z_k} - X_{Z_{\epsilon_{n-1}}} > 0\}$ to be the number of observations up to and including the nth record high in the sequence of observed surplus levels $\{X_{Z_j}\}_{j=0}^{\infty}$. Note that Z_{ϵ_n} $(n = 1, 2, \ldots)$ represents the time of the nth (positive) tax payment, if ruin has not been observed in the interim for the process X^{θ}. Define

$$\overline{\zeta}_{\delta}(u, x) = E_u\Big[e^{-\delta Z_{\epsilon_1}}; X_{Z_{\epsilon_1}} - X_{Z_0} > x, \inf_{0\leq k\leq \epsilon_1 - 1} X_{Z_k} \geq 0\Big], \qquad u, x \geq 0, \tag{2.8}$$

to be the discounted survival function of the amount of the first observed overshoot avoiding ruin enroute. Then, the Laplace transform of the first passage time Z_{ϵ_1} avoiding ruin enroute is its special case, as

$$\zeta_\delta(u) = E_u\Big[e^{-\delta Z_{\epsilon_1}}; \inf_{0\le k\le \epsilon_1-1} X_{Z_k} \ge 0\Big] = \overline{\zeta}_\delta(u,0), \qquad u \ge 0. \tag{2.9}$$

Owing to the stationary and independent increments of X, one has that

$$\begin{aligned}\overline{\zeta}_\delta(u,x) &= E_u[e^{-\delta Z_1}; X_{Z_1} - X_{Z_0} > x] \\ &\quad + \sum_{k=1}^{\infty}\int_0^u E_y[e^{-\delta Z_1}; X_{Z_1} - X_{Z_0} > u + x - y] Q_k(u,dy),\end{aligned} \tag{2.10}$$

where

$$Q_k(u,dy) = E_u\Big[e^{-\delta Z_k}; 0 \le \inf_{0\le j\le k-1} X_{Z_j} \le \sup_{0\le j\le k-1} X_{Z_j} \le u, X_{Z_k} \in dy\Big].$$

Because X is spatially homogeneous, the initial levels in the expectations E_u and E_y in (2.10) are irrelevant. Thus, the discounted density $g_{\delta,-}(\cdot)$ in (2.5) is applicable, leading to

$$\begin{aligned}\overline{\zeta}_\delta(u,x) &= \int_x^\infty g_{\delta,-}(z)dz + \sum_{k=1}^{\infty}\int_0^u \left(\int_{u+x-y}^\infty g_{\delta,-}(z)dz\right) Q_k(u,dy) \\ &= \frac{a_{\gamma,\delta} e^{-\phi(\gamma+\delta)x}}{\phi(\gamma+\delta)}\left(1 + \sum_{k=1}^{\infty}\int_0^u e^{-\phi(\gamma+\delta)(u-y)} Q_k(u,dy)\right).\end{aligned} \tag{2.11}$$

Noting that the dependence of $\overline{\zeta}_\delta(u,x)$ on x only appears via the exponential term $e^{-\phi(\gamma+\delta)x}$, one can use (2.9) to represent $\overline{\zeta}_\delta(u,x)$ as

$$\overline{\zeta}_\delta(u,x) = \zeta_\delta(u) e^{-\phi(\gamma+\delta)x}. \tag{2.12}$$

The density of the overshoot corresponding to the survival function $\overline{\zeta}_\delta(u,x)$ is then

$$\zeta_\delta(u,x) = -\frac{d}{dx}\overline{\zeta}_\delta(u,x) = \zeta_\delta(u)\phi(\gamma+\delta)e^{-\phi(\gamma+\delta)x}, \qquad u,x \ge 0. \tag{2.13}$$

The implication of the above result is that the amount of the first observed overshoot over the initial surplus level, given that such an overshoot occurs without ruin being observed in the interim, is exponentially distributed with mean $1/\phi(\gamma+\delta)$. Since a fraction θ of this overshoot is paid as tax, one concludes that the amount of the first tax payment (conditional on it being paid) is exponential with mean $\theta/\phi(\gamma+\delta)$.

With the representation (2.13) derived, it remains to analyze, $\zeta_\delta(u)$, namely the Laplace transform of the time of the first tax payment. Although $\zeta_\delta(u)$ can in principle be obtained by putting $x = 0$ in (2.11), the expression involves the quantity $Q_k(u,dy)$ which is unknown. Let $J^*(b) = \inf\{j \ge 1 : X_{Z_j} > b\}$ be the number of observations until the first overshoot of the sequence $\{X_{Z_j}\}_{j=0}^\infty$ over a given level $b \ge X_0$. We shall study the more general quantity defined by

$$\chi_\delta(u,b) = E_u[e^{-\delta Z_{J^*(b)}}; Z_{J^*(b)} < \tau_0], \qquad 0 \le u \le b, \tag{2.14}$$

where $\tau_0 = \inf\{Z_k : X_{Z_k} < 0\}$ is the time of ruin of the tax-free process X observed at Poissonian times. Then we have $\zeta_\delta(u) = \chi_\delta(u,u)$. The function $\chi_\delta(u,b)$ will be useful in Section 3.4 concerning delayed start of tax payments. By conditioning on the first increment of $\{X_{Z_j}\}_{j=0}^\infty$ and using (2.5), we arrive at

$$\chi_\delta(u,b) = \int_{b-u}^\infty g_{\delta,-}(x)dx + \int_0^{b-u} \chi_\delta(u+x,b)g_{\delta,-}(x)dx + \int_0^u \chi_\delta(u-x,b)g_{\delta,+}(x)dx$$

$$= \frac{a_{\gamma,\delta}}{\phi(\gamma+\delta)} e^{-\phi(\gamma+\delta)(b-u)} + a_{\gamma,\delta} \int_u^b \chi_\delta(x,b) e^{-\phi(\gamma+\delta)(x-u)} dx + \int_0^u \chi_\delta(u-x,b) g_{\delta,+}(x) dx. \quad (2.15)$$

Application of the operator $d/du - \phi(\gamma+\delta)$ to the above equation yields

$$\left(\frac{d}{du} - \phi(\gamma+\delta) + a_{\gamma,\delta} \right) \chi_\delta(u,b) = \left(\frac{d}{du} - \phi(\gamma+\delta) \right) \int_0^u \chi_\delta(u-x,b) g_{\delta,+}(x) dx, \qquad 0 \le u \le b, \quad (2.16)$$

which is a homogeneous integro-differential equation in u satisfied by $\chi_\delta(u,b)$. Let $v_\delta(\cdot)$ be the solution of the homogeneous integro-differential equation

$$\left(\frac{d}{du} - \phi(\gamma+\delta) + a_{\gamma,\delta} \right) v_\delta(u) = \left(\frac{d}{du} - \phi(\gamma+\delta) \right) \int_0^u v_\delta(u-x) g_{\delta,+}(x) dx, \qquad u \ge 0, \quad (2.17)$$

with boundary condition $v_\delta(0) = 1$. From the theory of integro-differential equations (e.g. Lakshmikantham and Rao (1995)), the solution $v_\delta(\cdot)$ is unique (and it is given in (2.24) below). Then, the general solution of (2.16) is

$$\chi_\delta(u,b) = A_\delta(b) v_\delta(u), \quad (2.18)$$

for some constant $A_\delta(b)$ independent of u but dependent on b. Substituting (2.18) back into (2.15) with $u = b$ followed by rearrangements gives rise to

$$A_\delta(b) = \frac{a_{\gamma,\delta}/\phi(\gamma+\delta)}{v_\delta(b) - \int_0^b v_\delta(b-x) g_{\delta,+}(x) dx}.$$

Hence,

$$\chi_\delta(u,b) = \frac{[a_{\gamma,\delta}/\phi(\gamma+\delta)] v_\delta(u)}{v_\delta(b) - \int_0^b v_\delta(b-x) g_{\delta,+}(x) dx}, \qquad 0 \le u \le b. \quad (2.19)$$

In particular, setting $b = u$ results in

$$\zeta_\delta(u) = \frac{[a_{\gamma,\delta}/\phi(\gamma+\delta)] v_\delta(u)}{v_\delta(u) - \int_0^u v_\delta(u-x) g_{\delta,+}(x) dx}, \qquad u \ge 0. \quad (2.20)$$

To determine $v_\delta(\cdot)$ which appears in both (2.19) and (2.20), we define $\widehat{v}_\delta(s) = \int_0^\infty e^{-sx} v_\delta(x) dx$ and take Laplace transforms on both sides of (2.17) to obtain

$$[s - \phi(\gamma+\delta) + a_{\gamma,\delta}] \widehat{v}_\delta(s) - 1 = [s - \phi(\gamma+\delta)] \widehat{v}_\delta(s) \widehat{g}_{\delta,+}(s), \quad (2.21)$$

where $\widehat{g}_{\delta,+}(s) = \int_0^\infty e^{-sx} g_{\delta,+}(x) dx$ is given by

$$\widehat{g}_{\delta,+}(s) = \frac{\gamma}{\gamma+\delta-\psi(s)} + \frac{a_{\gamma,\delta}}{s - \phi(\gamma+\delta)} \quad (2.22)$$

according to (2.4) and (2.6). Upon rearrangements of (2.21) along with the use of (2.22), one has that

$$\widehat{v}_\delta(s) = \frac{1}{s - \phi(\gamma+\delta) + a_{\gamma,\delta} - [s - \phi(\gamma+\delta)] \widehat{g}_{\delta,+}(s)} = \frac{\frac{1}{s-\phi(\gamma+\delta)}}{1 - \frac{a_{\gamma,\delta}}{\phi(\gamma+\delta)-s} - \widehat{g}_{\delta,+}(s)} = \frac{1}{\delta - \psi(s)} \frac{\gamma+\delta-\psi(s)}{s - \phi(\gamma+\delta)}. \quad (2.23)$$

Hence, by inverting the Laplace transforms in (2.23) we get

$$v_\delta(u) = e^{\phi(\gamma+\delta)u} - \gamma \int_0^u e^{\phi(\gamma+\delta)(u-x)} W^{(\delta)}(x) dx, \qquad u \ge 0. \quad (2.24)$$

In what follows, we present two examples where explicit results for $g_{\delta,+}(\cdot)$ and $v_\delta(\cdot)$ are obtainable. Note that these two quantities are required in (2.20) as far as the computation of $\zeta_\delta(\cdot)$ is concerned.

Example 1 (Brownian motion risk model) Assuming $\psi(s) = cs + (\sigma^2/2)s^2$ where $c, \sigma^2 > 0$, the process X corresponds to the Brownian motion risk model. The equation $\psi(s) = q$ is simply a quadratic equation with roots

$$\tilde{\phi}(q) = -\frac{c}{\sigma^2} - \sqrt{\frac{c^2}{\sigma^4} + \frac{2q}{\sigma^2}} \quad \text{and} \quad \phi(q) = -\frac{c}{\sigma^2} + \sqrt{\frac{c^2}{\sigma^4} + \frac{2q}{\sigma^2}}.$$

First, upon substitution of $\psi(s)$ along with the use of partial fractions, (2.7) can be represented as

$$\begin{aligned} E[e^{-\delta Z_1 + sX_{Z_1}}] &= \frac{\gamma}{-\frac{\sigma^2}{2}[s-\phi(\gamma+\delta)][s-\tilde{\phi}(\gamma+\delta)]} \\ &= \frac{2\gamma/\sigma^2}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} \frac{1}{\phi(\gamma+\delta)-s} + \frac{2\gamma/\sigma^2}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} \frac{1}{s-\tilde{\phi}(\gamma+\delta)}. \end{aligned}$$

Therefore, comparison with (2.1) and (2.5) yields

$$a_{\gamma,\delta} = \frac{2\gamma/\sigma^2}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} \quad \text{and} \quad g_{\delta,+}(x) = a_{\gamma,\delta} e^{\tilde{\phi}(\gamma+\delta)x}$$

by the uniqueness of Laplace transforms. The above results can also be found in Gerber et al. (2012a, Equation (68); 2012b, Remark 3.2). Next, (2.23) implies

$$\begin{aligned} \widehat{v}_\delta(s) &= \frac{1}{-\frac{\sigma^2}{2}[s-\phi(\delta)][s-\tilde{\phi}(\delta)]} \frac{-\frac{\sigma^2}{2}[s-\phi(\gamma+\delta)][s-\tilde{\phi}(\gamma+\delta)]}{s-\phi(\gamma+\delta)} = \frac{s-\tilde{\phi}(\gamma+\delta)}{[s-\phi(\delta)][s-\tilde{\phi}(\delta)]} \\ &= \frac{\phi(\delta)-\tilde{\phi}(\gamma+\delta)}{\phi(\delta)-\tilde{\phi}(\delta)} \frac{1}{s-\phi(\delta)} - \frac{\tilde{\phi}(\delta)-\tilde{\phi}(\gamma+\delta)}{\phi(\delta)-\tilde{\phi}(\delta)} \frac{1}{s-\tilde{\phi}(\delta)}. \end{aligned}$$

By Laplace transforms inversion, we obtain

$$v_\delta(u) = \frac{\phi(\delta)-\tilde{\phi}(\gamma+\delta)}{\phi(\delta)-\tilde{\phi}(\delta)} e^{\phi(\delta)u} - \frac{\tilde{\phi}(\delta)-\tilde{\phi}(\gamma+\delta)}{\phi(\delta)-\tilde{\phi}(\delta)} e^{\tilde{\phi}(\delta)u}, \qquad u \geq 0. \tag{2.25}$$

□

Example 2 (Compound Poisson risk model with exponential claims) In this example, it is assumed that $\psi(s) = \beta s - \lambda s/(s+\mu)$ where $\lambda, \mu > 0$ and $\beta > \lambda/\mu$. Then X is a classical compound Poisson model under Poisson claim arrival rate λ and exponential claims each with mean $1/\mu$. The parameter β can be interpreted as incoming premium rate. With the equation $\psi(s) = q$ having solutions

$$\tilde{\phi}(q) = \frac{-(\beta\mu-\lambda-q) - \sqrt{(\beta\mu-\lambda-q)^2 + 4\beta q\mu}}{2\beta} \quad \text{and} \quad \phi(q) = \frac{-(\beta\mu-\lambda-q) + \sqrt{(\beta\mu-\lambda-q)^2 + 4\beta q\mu}}{2\beta}, \tag{2.26}$$

(2.7) can be represented as

$$\begin{aligned} E[e^{-\delta Z_1 + sX_{Z_1}}] &= \frac{-\gamma(s+\mu)/\beta}{[s-\phi(\gamma+\delta)][s-\tilde{\phi}(\gamma+\delta)]} \\ &= \frac{\gamma[\phi(\gamma+\delta)+\mu]/\beta}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} \frac{1}{\phi(\gamma+\delta)-s} + \frac{\gamma[\tilde{\phi}(\gamma+\delta)+\mu]/\beta}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} \frac{1}{s-\tilde{\phi}(\gamma+\delta)}. \end{aligned}$$

Hence, we have

$$a_{\gamma,\delta} = \frac{\gamma[\phi(\gamma+\delta)+\mu]/\beta}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} \quad \text{and} \quad g_{\delta,+}(x) = \frac{\gamma[\tilde{\phi}(\gamma+\delta)+\mu]/\beta}{\phi(\gamma+\delta)-\tilde{\phi}(\gamma+\delta)} e^{\tilde{\phi}(\gamma+\delta)x},$$

which are in agreement with Albrecher et al. (2011b, Equation (27); 2013, Example 4.1). Next, it is easily seen from (2.23) that $v_\delta(\cdot)$ still takes the form (2.25), but with $\phi(\cdot)$ and $\tilde{\phi}(\cdot)$ defined in (2.26). □

3 Analysis of $m_{\theta,\delta}(u)$ and $V_{\theta,\delta}(u)$

3.1 Analytic expressions for $m_{\theta,\delta}(u)$

First, we assert that the regular condition

$$\lim_{u\to\infty} m_{\theta,\delta}(u) = 0 \tag{3.1}$$

holds for the Gerber-Shiu function (1.2) under the net profit condition $E[X_1] > 0$. Since the penalty function $w(\cdot)$ is assumed to be such that $0 \le w(\cdot) \le A$, it is immediate that $m_{\theta,\delta}(u) \le A\Psi_\theta(u)$ and hence it suffices to argue that $\lim_{u\to\infty} \Psi_\theta(u) = 0$. Under the same tax rate θ, the ruin probability $\Psi_\theta(u)$ in the present model with periodic taxation (and periodic monitoring of ruin) must be no larger than the counterpart in a model where tax is paid immediately when the surplus process is at its running maximum (which corresponds to $\gamma \to \infty$). In the latter model with continuous observation, a sufficient condition for $\lim_{u\to\infty} \Psi_\theta(u) = 0$ is the loading condition $E[X_1] > 0$ according to Albrecher et al. (2008b, Section 3). Therefore, one concludes that (3.1) holds true in our model.

When analyzing the Gerber-Shiu function $m_{\theta,\delta}(u)$, we need to distinguish between two situations based on whether ruin is observed before the time Z_{ϵ_1} of the first overshoot. Applying the overshoot density (2.13), we obtain the integral equation

$$\begin{aligned} m_{\theta,\delta}(u) &= \alpha_\delta(u) + \int_0^\infty m_{\theta,\delta}(u + (1-\theta)x)\zeta_\delta(u,x)dx \\ &= \alpha_\delta(u) + \zeta_\delta(u)\int_0^\infty m_{\theta,\delta}(u+(1-\theta)x)\phi(\gamma+\delta)e^{-\phi(\gamma+\delta)x}dx \\ &= \alpha_\delta(u) + \frac{\phi(\gamma+\delta)}{1-\theta}\zeta_\delta(u)\int_u^\infty m_{\theta,\delta}(x)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}dx, \qquad u \ge 0, \end{aligned} \tag{3.2}$$

where

$$\alpha_\delta(u) = E_u[e^{-\delta\tau_\theta}w(|X^\theta_{\tau_\theta}|)\mathbf{1}_{(\tau_\theta < Z_{\epsilon_1})}]. \tag{3.3}$$

Since no tax is paid before Z_{ϵ_1}, on the set $\{\tau_\theta < Z_{\epsilon_1}\}$, we have $\tau_\theta = \tau_0$ and $X^\theta_{\tau_\theta} = X_{\tau_0}$. Thus, $\alpha_\delta(u)$ does not depend on θ. In particular, setting $\theta = 0$ in (3.2) leads to

$$\alpha_\delta(u) = m_{0,\delta}(u) - \phi(\gamma+\delta)\zeta_\delta(u)\int_u^\infty m_{0,\delta}(x)e^{-\phi(\gamma+\delta)(x-u)}dx, \qquad u \ge 0, \tag{3.4}$$

where $m_{0,\delta}(\cdot)$ is the Gerber-Shiu function pertaining to the tax-free risk model X based on Poissonian monitoring of solvency. Expressions for $m_{0,\delta}(\cdot)$ are available in Gerber et al. (2012a, Section 3) and Albrecher et al. (2013) for the Brownian motion risk model and the compound Poisson model respectively. The Laplace transform of $m_{0,\delta}(\cdot)$ will be given in (3.15) for the general Lévy insurance risk process. This together with $\zeta_\delta(\cdot)$ derived in (2.20) characterizes $\alpha_\delta(\cdot)$ in (3.4).

To solve for $m_{\theta,\delta}(u)$, it is instructive to note that the integral equation (3.2) is structurally identical to Equation (22) in Albrecher et al. (2008a), who studied taxation problems in a dual risk model with exponential gains. Such a similarity is not surprising, since the amount of the first overshoot (conditional on ruin not occurring in the interim) in their model is also exponentially distributed. Hence, we can directly apply their Equation (26) if $\lim_{u\to\infty} m_{\theta,\delta}(u) = \lim_{u\to\infty}\alpha_\delta(u) = 0$ and $\zeta_\delta(u) > 0$ (as these conditions are used in their derivation). The first condition $\lim_{u\to\infty} m_{\theta,\delta}(u) = 0$ is simply (3.1), which has already been proved. From (3.4), it is clear that $\alpha_\delta(u) \le m_{0,\delta}(u)$. Meanwhile, we also have

$m_{0,\delta}(u) \le A\Psi_0(u)$, where $\Psi_0(u)$ is the ruin probability of the tax-free process X with solvency monitored at Poisson arrival times. Because $\Psi_0(u)$ is the ruin probability of the embedded random walk $\{X_{Z_j}\}_{j=0}^{\infty}$ which has positive drift thanks to the loading assumption $E[X_1] > 0$, by the theory of random walk one has that $\lim_{u\to\infty}\Psi_0(u) = 0$. This in turn implies that the second condition $\lim_{u\to\infty}\alpha_\delta(u) = 0$ holds. Lastly, $\zeta_\delta(u)$ defined by (2.9) is positive again because of the loading condition. Therefore, Albrecher et al. (2008a, Equation (26)) gives the solution

$$m_{\theta,\delta}(u) = \alpha_\delta(u) + \frac{\phi(\gamma+\delta)}{1-\theta}\zeta_\delta(u)\int_u^\infty \alpha_\delta(x)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_u^x[1-\zeta_\delta(y)]dy}dx. \tag{3.5}$$

In particular, if $\delta = 0$ and $w(\cdot) \equiv 1$, then $\alpha_0(u) = P_u(\tau_\theta < Z_{\epsilon_1}) = 1 - \zeta_0(u)$ according to (2.9) and (3.3). In this case, (3.5) becomes the ruin probability which can be represented as

$$\begin{aligned}\Psi_\theta(u) &= 1-\zeta_0(u) + \frac{\phi(\gamma)}{1-\theta}\zeta_0(u)\int_u^\infty[1-\zeta_0(x)]e^{-\frac{\phi(\gamma)}{1-\theta}\int_u^x[1-\zeta_0(y)]dy}dx \\ &= 1-\zeta_0(u) - \zeta_0(u)\int_{x\in[u,\infty)} de^{-\frac{\phi(\gamma)}{1-\theta}\int_u^x[1-\zeta_0(y)]dy} \\ &= 1-\zeta_0(u)e^{-\frac{\phi(\gamma)}{1-\theta}\int_u^\infty[1-\zeta_0(y)]dy}.\end{aligned}$$

When $\theta = 0$, this further reduces to the ruin probability without tax (but solvency is still monitored at Poissonian times), leading to

$$\Psi_0(u) = 1-\zeta_0(u)e^{-\phi(\gamma)\int_u^\infty[1-\zeta_0(y)]dy}.$$

Combining the above two equations, we obtain the power relationship

$$1-\Psi_\theta(u) = [\zeta_0(u)]^{1-\frac{1}{1-\theta}}[1-\Psi_0(u)]^{\frac{1}{1-\theta}}, \tag{3.6}$$

which can be regarded as the tax identity in the present model.

Remark 2 As $\gamma \to \infty$ (i.e. the insurer's surplus is observed continuously for both taxation and solvency), overshoot of X over the initial level occurs immediately because of the diffusion component (if any) and/or the loading condition. Consequently, for any $u \ge 0$, one has $\zeta_0(u) \to 1$ as $\gamma \to \infty$. Therefore, the tax identity in Albrecher and Hipp (2007, Theorem 1) and Albrecher et al. (2008b, Corollary 3.1) is recovered from (3.6). □

Apart from the exact expression (3.5) for the Gerber-Shiu function $m_{\theta,\delta}(u)$, we can also obtain an alternative solution as follows. Defining

$$\zeta_{\delta,1}(u,x) = \frac{1}{1-\theta}\zeta_\delta\left(u, \frac{x-u}{1-\theta}\right) = \zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}, \qquad x > u \ge 0, \tag{3.7}$$

(3.2) can be rewritten as

$$m_{\theta,\delta}(u) = \alpha_\delta(u) + \int_u^\infty \zeta_{\delta,1}(u,x)m_{\theta,\delta}(x)dx,$$

which is a linear Volterra integral equation of the second kind (see e.g. Polyanin and Manzhirov (2008, Chapter 11.9)). It follows from successive substitution that its solution admits the representation

$$m_{\theta,\delta}(u) = \alpha_\delta(u) + \sum_{n=1}^\infty\int_u^\infty \zeta_{\delta,n}(u,x)\alpha_\delta(x)dx, \tag{3.8}$$

where $\zeta_{\delta,n}(u,x)$ is given recursively via, for $n=2,3,\ldots$,

$$\zeta_{\delta,n}(u,x)=\int_u^x \zeta_{\delta,n-1}(u,y)\zeta_{\delta,1}(y,x)dy=\int_u^x \zeta_{\delta,1}(u,y)\zeta_{\delta,n-1}(y,x)dy, \qquad x>u\geq 0, \tag{3.9}$$

with the starting point (3.7). The solution (3.8) allows for probabilistic interpretation. Indeed, from (3.7) and (3.9) (as well as the Markov property), it is clear that $\zeta_{\delta,n}(u,x)$ represents the discounted density of the process X^θ being at level x immediately after the nth tax payments (without ruin being observed in the interim). More specifically, the quantity $\zeta_{\delta,n}(u,x)$ is such that, for $n=1,2,\ldots$,

$$\zeta_{\delta,n}(u,x)dx=E_u\Big[e^{-\delta Z_{\epsilon_n}};X^\theta_{Z_{\epsilon_n}}\in dx,\ \inf_{0\leq k\leq \epsilon_n-1}X^\theta_{Z_k}\geq 0\Big], \qquad x>u\geq 0. \tag{3.10}$$

Together with (3.3), we easily see that

$$\int_u^\infty \zeta_{\delta,n}(u,x)\alpha_\delta(x)dx=E_u[e^{-\delta\tau_\theta}w(|X^\theta_{\tau_\theta}|)\mathbf{1}_{(Z_{\epsilon_n}<\tau_\theta<Z_{\epsilon_{n+1}})}].$$

Hence, (3.8) is actually a decomposition of the Gerber-Shiu function according to the number of tax payments before ruin. See also Remark 6 for further application of (3.8) in relation to asymptotics of $m_{\theta,\delta}(u)$.

3.2 Asymptotic formula for $m_{\theta,\delta}(u)$

Although two formulas for the Gerber-Shiu function $m_{\theta,\delta}(u)$ have been derived in Section 3.1, both (3.5) and (3.8) are expressed in terms of $\zeta_\delta(\cdot)$ and $\alpha_\delta(\cdot)$. Examination of (2.6), (2.20), (2.24) and (3.4) reveals that the quantities $m_{0,\delta}(\cdot)$ and $W^{(\gamma+\delta)}(\cdot)$ are required. While explicit expressions for $m_{0,\delta}(\cdot)$ are available in the literature only for certain models, $W^{(\gamma+\delta)}(\cdot)$ is generally characterized by its Laplace transform in (2.4). In this subsection, we aim at obtaining some asymptotic results for $m_{\theta,\delta}(u)$ as $u\to\infty$.

First, we consider the Gerber-Shiu function $m_{0,\delta}(u)$ in the tax-free case. By conditioning on the first increment of $\{X_{Z_j}\}_{j=0}^\infty$ and taking into account appropriate discounting (see Albrecher et al. (2013, Equation (3.20))), one has that

$$\begin{aligned}m_{0,\delta}(u)&=\int_0^\infty m_{0,\delta}(u+x)g_{\delta,-}(x)dx+\int_0^u m_{0,\delta}(u-x)g_{\delta,+}(x)dx+\int_u^\infty w(x-u)g_{\delta,+}(x)dx\\&=a_{\gamma,\delta}\mathcal{T}_{\phi(\gamma+\delta)}m_{0,\delta}(u)+\int_0^u m_{0,\delta}(u-x)g_{\delta,+}(x)dx+\omega(u), \qquad u\geq 0,\end{aligned} \tag{3.11}$$

where (2.5) has been used in the last equality, and

$$\omega(u)=\int_u^\infty w(x-u)g_{\delta,+}(x)dx. \tag{3.12}$$

The Dickson-Hipp operator $\mathcal{T}_s$ appearing in (3.11) is defined as (see Dickson and Hipp (2001))

$$\mathcal{T}_sf(y)=\int_y^\infty e^{-s(x-y)}f(x)dx=\int_0^\infty e^{-sx}f(x+y)dx, \qquad y\geq 0,$$

for some function $f(\cdot)$ on $(0,\infty)$. The above definition is valid as long as the integral exists, which must be the case if $f(\cdot)$ is an integrable function and s is a complex number such that $\Re(s)\geq 0$. Define the Laplace transforms $\widehat{m}_{0,\delta}(s)=\int_0^\infty e^{-su}m_{0,\delta}(u)du$ and $\widehat{\omega}(s)=\int_0^\infty e^{-su}\omega(u)du$. Taking Laplace transforms on both sides of (3.11) followed by application of Properties 1 and 2 in Li and Garrido (2004, Section 3) leads to

$$\widehat{m}_{0,\delta}(s) = a_{\gamma,\delta}\frac{\widehat{m}_{0,\delta}(s) - \widehat{m}_{0,\delta}(\phi(\gamma+\delta))}{\phi(\gamma+\delta) - s} + \widehat{m}_{0,\delta}(s)\widehat{g}_{\delta,+}(s) + \widehat{\omega}(s).$$

Rearrangements result in

$$\begin{aligned}\widehat{m}_{0,\delta}(s) &= \frac{\widehat{\omega}(s) - \frac{a_{\gamma,\delta}}{\phi(\gamma+\delta)-s}\widehat{m}_{0,\delta}(\phi(\gamma+\delta))}{1 - \frac{a_{\gamma,\delta}}{\phi(\gamma+\delta)-s} - \widehat{g}_{\delta,+}(s)} = \frac{\widehat{\omega}(s) - \frac{a_{\gamma,\delta}}{\phi(\gamma+\delta)-s}\widehat{m}_{0,\delta}(\phi(\gamma+\delta))}{1 - \frac{\gamma}{\gamma+\delta-\psi(s)}} \\ &= \frac{\gamma+\delta-\psi(s)}{\delta-\psi(s)}\left(\widehat{\omega}(s) - \frac{a_{\gamma,\delta}}{\phi(\gamma+\delta)-s}\widehat{m}_{0,\delta}(\phi(\gamma+\delta))\right), \end{aligned} \tag{3.13}$$

where (2.22) is utilized in the second equality. Since $\widehat{m}_{0,\delta}(s)$ is analytic for $\Re(s) \geq 0$ and $\psi(\phi(\delta)) = \delta$, the term inside the big brackets in the above expression must be zero when $s = \phi(\delta)$, i.e.

$$\widehat{m}_{0,\delta}(\phi(\gamma+\delta)) = \frac{\phi(\gamma+\delta) - \phi(\delta)}{a_{\gamma,\delta}}\widehat{\omega}(\phi(\delta)).$$

Back substitution into (3.13) yields

$$\widehat{m}_{0,\delta}(s) = \frac{\gamma+\delta-\psi(s)}{\delta-\psi(s)}\left(\widehat{\omega}(s) - \frac{\phi(\gamma+\delta)-\phi(\delta)}{\phi(\gamma+\delta)-s}\widehat{\omega}(\phi(\delta))\right) \tag{3.14}$$

$$= \frac{\gamma+\delta-\psi(s)}{\delta-\psi(s)}\frac{1}{\phi(\gamma+\delta)-s}\{\phi(\gamma+\delta)[\widehat{\omega}(s) - \widehat{\omega}(\phi(\delta))] - [s\widehat{\omega}(s) - \phi(\delta)\widehat{\omega}(\phi(\delta))]\}$$

$$= \frac{\gamma+\delta-\psi(s)}{\delta-\psi(s)}\frac{\phi(\delta)-s}{\phi(\gamma+\delta)-s}\widehat{\xi}_{\delta}(s), \tag{3.15}$$

where

$$\begin{aligned}\widehat{\xi}_{\delta}(s) &= \frac{\phi(\gamma+\delta)[\widehat{\omega}(s) - \widehat{\omega}(\phi(\delta))] - [s\widehat{\omega}(s) - \phi(\delta)\widehat{\omega}(\phi(\delta))]}{\phi(\delta)-s} \\ &= \phi(\gamma+\delta)\frac{\widehat{\omega}(s) - \widehat{\omega}(\phi(\delta))}{\phi(\delta)-s} - \frac{[s-\phi(\delta)]\widehat{\omega}(s) + \phi(\delta)[\widehat{\omega}(s) - \widehat{\omega}(\phi(\delta))]}{\phi(\delta)-s} \\ &= [\phi(\gamma+\delta) - \phi(\delta)]\mathcal{T}_s\mathcal{T}_{\phi(\delta)}\omega(0) + \widehat{\omega}(s)\end{aligned}$$

is the Laplace transform of

$$\xi_{\delta}(u) = [\phi(\gamma+\delta) - \phi(\delta)]\mathcal{T}_{\phi(\delta)}\omega(u) + \omega(u), \qquad u \geq 0, \tag{3.16}$$

i.e. $\widehat{\xi}_{\delta}(s) = \int_0^\infty e^{-su}\xi_{\delta}(u)du$. By applying the initial value theorem for Laplace transforms to (3.14) (see Remark 3) and noting that $\psi(\infty) = \infty$ (see Kyprianou (2014, p.85)), we obtain

$$\begin{aligned} m_{0,\delta}(0) &= \lim_{s\to\infty} s\widehat{m}_{0,\delta}(s) = \lim_{s\to\infty} s\widehat{\omega}(s) + [\phi(\gamma+\delta) - \phi(\delta)]\widehat{\omega}(\phi(\delta)) \\ &= \omega(0) + [\phi(\gamma+\delta) - \phi(\delta)]\int_0^\infty e^{-\phi(\delta)x}\omega(x)dx \\ &= \int_0^\infty w(y)g_{\delta,+}(y)dy + [\phi(\gamma+\delta) - \phi(\delta)]\int_0^\infty e^{-\phi(\delta)x}\int_x^\infty w(y-x)g_{\delta,+}(y)dydx \\ &= \int_0^\infty w(y)h_{\delta}^*(y|0)dy, \end{aligned} \tag{3.17}$$

where

$$h_{\delta}^*(y|0) = g_{\delta,+}(y) + [\phi(\gamma+\delta) - \phi(\delta)]\mathcal{T}_{\phi(\delta)}g_{\delta,+}(y), \qquad y > 0. \tag{3.18}$$

Remark 3 Application of the initial value theorem in obtaining (3.17) requires that the functions $m_{0,\delta}(\cdot)$ and $\omega(\cdot)$ are differentiable on $[0,\infty)$ (and therefore also continuous at zero). To check this, we first rewrite (3.11) as

$$m_{0,\delta}(u) = a_{\gamma,\delta} e^{\phi(\gamma+\delta)u} \int_u^\infty m_{0,\delta}(x) e^{-\phi(\gamma+\delta)x} dx + \int_0^u m_{0,\delta}(x) g_{\delta,+}(u-x) dx + \omega(u), \qquad u \geq 0, \quad (3.19)$$

where $\omega(u) = \int_0^\infty w(x) g_{\delta,+}(u+x) dx$ according to (3.12). Obviously, the first term on the right-hand side of (3.19) is continuous in u. The second and third terms are continuous as well because $g_{\delta,+}(\cdot)$ given in (2.6) is continuous thanks to the continuity of $W^{(\gamma+\delta)}(\cdot)$ (see Kyprianou (2014, Theorem 8.1(i))). Hence, $m_{0,\delta}(\cdot)$ is continuous on $[0,\infty)$. Having established the continuity of $m_{0,\delta}(\cdot)$, the first term in (3.19) is clearly differentiable. For the second and third terms to be differentiable, a sufficient condition is that $W^{(\gamma+\delta)}(\cdot)$ is differentiable (so that $g_{\delta,+}(\cdot)$ is differentiable). From Lemma 8.2 of Kyprianou (2014) (and the discussions following it) along with Exercise 8.4 therein, it is known that $W^{(\gamma+\delta)}(\cdot)$ is differentiable if (i) X has unbounded variation; or (ii) X has bounded variation and the Lévy measure $\nu(\cdot)$ has no atoms. Therefore, the use of the initial value theorem is justified for a wide range of Lévy processes. □

From the representation (3.17) of the Gerber-Shiu function $m_{0,\delta}(0)$, it is clear that $h^*_\delta(\cdot|0)$ in (3.18) is the discounted density of the deficit observed at ruin when the risk process X starts with zero initial surplus. Owing to the spatial homogeneity of X, the quantity $h^*_\delta(\cdot|0)$ is also the discounted density of the amount of the first observed drop of X below its initial level $X_0 = u$. Therefore, analogous to Albrecher et al. (2013, Equation (3.1)), we have

$$m_{0,\delta}(u) = \int_0^u m_{0,\delta}(u-y) h^*_\delta(y|0) dy + \int_u^\infty w(y-u) h^*_\delta(y|0) dy, \qquad u \geq 0. \quad (3.20)$$

Because of the loading condition $E[X_1] > 0$, we have that $\int_0^\infty h^*_\delta(y|0)dy = E[e^{-\delta\tau_0}\mathbf{1}_{(\tau_0<\infty)}] < 1$ and therefore one asserts that (3.20) is a defective renewal equation satisfied by $m_{0,\delta}(\cdot)$. With (3.12) and (3.18), it is straightforward to verify that the non-homogeneous term of the above defective renewal equation can be written as

$$\int_u^\infty w(y-u) h^*_\delta(y|0) dy = \xi_\delta(u), \quad (3.21)$$

where $\xi_\delta(u)$ is given in (3.16). In the rest of the paper, we shall use the notation $f_1(u) \sim f_2(u)$ to denote $\lim_{u\to\infty}[f_1(u)/f_2(u)] = 1$. It follows from e.g. Willmot and Lin (2001, Theorem 9.1.3) that if $e^{\mathcal{K}_\delta x}\xi_\delta(x)$ is directly Riemann integrable on $x \in (0,\infty)$ (see Remark 4) and $\mathcal{K}_\delta > 0$ is the adjustment coefficient satisfying

$$\int_0^\infty e^{\mathcal{K}_\delta y} h^*_\delta(y|0) dy = 1, \quad (3.22)$$

then the Cramér-Lundberg asymptotic formula

$$m_{0,\delta}(u) \sim \mathcal{C}_\delta e^{-\mathcal{K}_\delta u} \quad (3.23)$$

holds true, where

$$\mathcal{C}_\delta = \frac{\int_0^\infty e^{\mathcal{K}_\delta y}\xi_\delta(y)dy}{\int_0^\infty y e^{\mathcal{K}_\delta y} h^*_\delta(y|0)dy} = \frac{\widehat{\xi}_\delta(-\mathcal{K}_\delta)}{\int_0^\infty y e^{\mathcal{K}_\delta y} h^*_\delta(y|0)dy}. \quad (3.24)$$

Although the constants $\mathcal{K}_\delta$ and $\mathcal{C}_\delta$ are in principle obtainable from (3.22) and (3.24), in what follows we shall provide an alternative (and more direct) way to compute them. Taking Laplace transforms on both sides of (3.20) along with the use of (3.21) results in

$$\widehat{m}_{0,\delta}(s) = \frac{1}{1-\widehat{h}^*_\delta(s|0)}\widehat{\xi}_\delta(s),$$

where $\widehat{h}^*_\delta(s|0) = \int_0^\infty e^{-sy} h^*_\delta(y|0)dy$. Comparison with (3.15) yields the identity

$$\frac{1}{1-\widehat{h}^*_\delta(s|0)} = \frac{\gamma+\delta-\psi(s)}{\delta-\psi(s)} \frac{\phi(\delta)-s}{\phi(\gamma+\delta)-s}. \tag{3.25}$$

(This can also be proved using (3.18) and (2.22) together with Properties 1 and 2 in Li and Garrido (2004, Section 3) about Dickson-Hipp operators.) Note that (3.22) is equivalent to $1-\widehat{h}^*_\delta(-\mathcal{K}_\delta|0) = 0$. Assuming that the Laplace exponent $\psi(\cdot)$ exists in a neighbourhood of the origin (as far as asymptotics of Gerber-Shiu functions are concerned), it is clear from the above identity that $-\mathcal{K}_\delta < 0$ must be the root of the equation $\psi(s) = \delta$ which has the largest real part in the left half of the complex plane. As for $\mathcal{C}_\delta$, the asymptotic formula (3.23) implies

$$\mathcal{C}_\delta = \lim_{u\to\infty} e^{\mathcal{K}_\delta u} m_{0,\delta}(u). \tag{3.26}$$

With $\widehat{m}_{0,\delta}(s)$ being analytic for $\Re(s) > -\mathcal{K}_\delta$, from the damping property of Laplace transforms we know that $\widehat{m}_{0,\delta}(s-\mathcal{K}_\delta) = \int_0^\infty e^{-(s-\mathcal{K}_\delta)u} m_{0,\delta}(u)du$ represents the Laplace transform of $e^{\mathcal{K}_\delta u} m_{0,\delta}(u)$. Application of the final value theorem for Laplace transforms to (3.26) followed by the use of (3.15) and the L'Hospital rule gives

$$\begin{aligned}\mathcal{C}_\delta &= \lim_{s\to 0} s\widehat{m}_{0,\delta}(s-\mathcal{K}_\delta) = \lim_{s\to 0}\left(s\frac{\gamma+\delta-\psi(s-\mathcal{K}_\delta)}{\delta-\psi(s-\mathcal{K}_\delta)} \frac{\phi(\delta)-s+\mathcal{K}_\delta}{\phi(\gamma+\delta)-s+\mathcal{K}_\delta}\widehat{\xi}_\delta(s-\mathcal{K}_\delta)\right)\\ &= \gamma\frac{\phi(\delta)+\mathcal{K}_\delta}{\phi(\gamma+\delta)+\mathcal{K}_\delta}\widehat{\xi}_\delta(-\mathcal{K}_\delta)\lim_{s\to 0}\frac{s}{\delta-\psi(s-\mathcal{K}_\delta)} = \frac{\phi(\delta)+\mathcal{K}_\delta}{\phi(\gamma+\delta)+\mathcal{K}_\delta}\frac{\gamma}{[-\psi'(-\mathcal{K}_\delta)]}\widehat{\xi}_\delta(-\mathcal{K}_\delta).\end{aligned} \tag{3.27}$$

See Asmussen and Albrecher (2010, Chapter IV, Remark 5.6) for comments on the above procedure. (One may also use (3.25) and the fact that $\int_0^\infty y e^{\mathcal{K}_\delta y} h^*_\delta(y|0)dy = -(d/ds)\widehat{h}^*_\delta(s|0)|_{s=-\mathcal{K}_\delta}$ to prove that (3.24) and (3.27) are identical.)

Remark 4 Recall that $-\mathcal{K}_\delta$ is a root of $\psi(s) = \delta$. We shall show that a sufficient condition for $e^{\mathcal{K}_\delta x}\xi_\delta(x)$ to be directly Riemann integrable on $x \in (0,\infty)$ is that $\psi(\cdot)$ is continuous in the neighborhood of $-\mathcal{K}_\delta$. For example, it can be checked that such a condition is satisfied by a (perturbed) compound Poisson risk model where each claim size is distributed as a combination of exponentials (and this includes Examples 1 and 2 as special cases). Since the penalty $w(\cdot)$ is bounded by a constant A, use of (3.18) and (3.21) leads to

$$e^{\mathcal{K}_\delta x}\xi_\delta(x) \le A e^{\mathcal{K}_\delta x}\int_x^\infty h^*_\delta(y|0)dy = A e^{\mathcal{K}_\delta x}\mathcal{T}_0 g_{\delta,+}(x) + A[\phi(\gamma+\delta)-\phi(\delta)]e^{\mathcal{K}_\delta x}\mathcal{T}_0\mathcal{T}_{\phi(\delta)} g_{\delta,+}(x). \tag{3.28}$$

From the definition (2.1), the first Dickson-Hipp operator above can be represented as

$$\mathcal{T}_0 g_{\delta,+}(x) = \int_x^\infty g_{\delta,+}(y)dy = E[e^{-\delta Z_1}\mathbf{1}_{(-X_{Z_1}\ge x)}] = \gamma\int_0^\infty e^{-(\gamma+\delta)t}P(-X_t \ge x)dt.$$

Applying the Markov's inequality to $P(-X_t \ge x) = P(e^{-rX_t} \ge e^{rx})$ for any $r > 0$ such that $E[e^{-rX_t}] < \infty$, one has that

$$\mathcal{T}_0 g_{\delta,+}(x) \le \gamma\int_0^\infty e^{-(\gamma+\delta)t}\frac{E[e^{-rX_t}]}{e^{rx}}dt = \gamma e^{-rx}\int_0^\infty e^{-(\gamma+\delta)t+\psi(-r)t}dt = \frac{\gamma}{\gamma+\delta-\psi(-r)}e^{-rx}. \tag{3.29}$$

Under the assumption that $\psi(\cdot)$ is continuous in the neighborhood of $-\mathcal{K}_\delta$, we can find a small constant $\varepsilon > 0$ such that $|\psi(-(\mathcal{K}_\delta+\varepsilon)) - \psi(-\mathcal{K}_\delta)| \le \gamma/2$, or equivalently, $\gamma/2 \le \gamma+\delta-\psi(-(\mathcal{K}_\delta+\varepsilon)) \le 3\gamma/2$ since $\psi(-\mathcal{K}_\delta) = \delta$. Utilizing the lower bound, (3.29) with $r = \mathcal{K}_\delta + \varepsilon$ gives

$$\mathcal{T}_0 g_{\delta,+}(x) \le 2e^{-(\mathcal{K}_\delta+\varepsilon)x}. \tag{3.30}$$

Because the operators $\mathcal{T}_0$ and $\mathcal{T}_{\phi(\delta)}$ commute according to Property 2 of Li and Garrido (2004, Section 3), the above result implies

$$\mathcal{T}_0\mathcal{T}_{\phi(\delta)}g_{\delta,+}(x) = \int_0^\infty e^{-\phi(\delta)y}\mathcal{T}_0 g_{\delta,+}(x+y)dy \leq 2\int_0^\infty e^{-\phi(\delta)y}e^{-(\mathcal{K}_\delta+\varepsilon)(x+y)}dy = \frac{2e^{-(\mathcal{K}_\delta+\varepsilon)x}}{\phi(\delta)+\mathcal{K}_\delta+\varepsilon}. \tag{3.31}$$

Plugging (3.30) and (3.31) into (3.28) yields

$$e^{\mathcal{K}_\delta x}\xi_\delta(x) \leq 2A\frac{\phi(\gamma+\delta)+\mathcal{K}_\delta+\varepsilon}{\phi(\delta)+\mathcal{K}_\delta+\varepsilon}e^{-\varepsilon x}. \tag{3.32}$$

Since the exponential function on the right-hand side of (3.32) is non-negative, decreasing and Riemann integrable on $x \in (0,\infty)$, it is directly Riemann integrable (see e.g. Willmot and Lin (2001, p.157, condition (a))). As $e^{\mathcal{K}_\delta x}\xi_\delta(x)$ is bounded by a directly Riemann integrable function, it is directly Riemann integrable according to e.g. Willmot and Lin (2001, p.157, condition (c)). $\square$

Now, we analyze the limit $\lim_{u\to\infty}\zeta_\delta(u) = \zeta_\delta(\infty)$, which is needed to derive asymptotic formulas for $\alpha_\delta(u)$ via (3.4) and then $m_{\theta,\delta}(u)$ via (3.5). Utilizing (2.7), (3.25) can be rewritten as

$$\frac{1}{1-\widehat{h}^*_\delta(s|0)} = \frac{1}{1-E[e^{-\delta Z_1+sX_{Z_1}}]}\left(1-\frac{\phi(\gamma+\delta)-\phi(\delta)}{\phi(\gamma+\delta)-s}\right). \tag{3.33}$$

On the other hand, the Wiener-Hopf factorization of the embedded random walk $\{X_{Z_j}\}_{j=0}^\infty$ (see Tang and Wei (2010, Lemma 4.6)) means that

$$1-E[e^{-\delta Z_1+sX_{Z_1}}] = (1-E[e^{-\delta Z_{\tilde{\epsilon}_1}+sX_{Z_{\tilde{\epsilon}_1}}}\mathbf{1}_{(Z_{\tilde{\epsilon}_1}<\infty)}])(1-E[e^{-\delta Z_{\epsilon_1}+sX_{Z_{\epsilon_1}}}]). \tag{3.34}$$

In the above equation, $\tilde{\epsilon}_1$ is defined as $\tilde{\epsilon}_1 = \inf_{k\in\mathbb{N}}\{k>0: X_{Z_k}-X_{Z_0}\leq 0\}$, so that $Z_{\tilde{\epsilon}_1}$ represents the first (weak) descending ladder epoch of the random walk $\{X_{Z_j}\}_{j=0}^\infty$. Also recall from Section 2.2 that Z_{ϵ_1} is the first (strict) ascending ladder epoch. It is instructive to note that the words 'weak' and 'strict' in the descriptions of ladder epochs are not necessary, since X_{Z_1} is a continuous random variable with discounted distribution characterized by the densities $g_{\delta,-}(\cdot)$ and $g_{\delta,+}(\cdot)$. Moreover, under the initial surplus $X_0 = 0$, the time of ruin τ_0 equals the descending ladder epoch $Z_{\tilde{\epsilon}_1}$ almost surely (if ruin occurs), and the resulting deficit at ruin $|X_{\tau_0}|$ coincides almost surely with the height of the descending ladder $|X_{Z_{\tilde{\epsilon}_1}}|$. Hence, upon substitution of $w(y) = e^{-sy}$ into (3.17), one sees that

$$\widehat{h}^*_\delta(s|0) = E[e^{-\delta\tau_0}e^{-s|X_{\tau_0}|}\mathbf{1}_{(\tau_0<\infty)}] = E[e^{-\delta Z_{\tilde{\epsilon}_1}+sX_{Z_{\tilde{\epsilon}_1}}}\mathbf{1}_{(Z_{\tilde{\epsilon}_1}<\infty)}].$$

Thus, by comparing (3.33) and (3.34), we arrive at

$$E[e^{-\delta Z_{\epsilon_1}+sX_{Z_{\epsilon_1}}}] = \frac{\phi(\gamma+\delta)-\phi(\delta)}{\phi(\gamma+\delta)-s}. \tag{3.35}$$

To see how $\zeta_\delta(\infty)$ can be obtained from the above result, we utilize monotone convergence to the limit of the definition (2.9) as $u\to\infty$, so that

$$\begin{aligned}\zeta_\delta(\infty) &= \lim_{u\to\infty}\zeta_\delta(u) = \lim_{u\to\infty}E_u\Big[e^{-\delta Z_{\epsilon_1}};\ \inf_{0\leq k\leq\epsilon_1-1}X_{Z_k}\geq 0\Big] = \lim_{u\to\infty}E\Big[e^{-\delta Z_{\epsilon_1}};\ u+\inf_{0\leq k\leq\epsilon_1-1}X_{Z_k}\geq 0\Big]\\ &= E[e^{-\delta Z_{\epsilon_1}}],\end{aligned}$$

where the spatial homogeneity of X and the loading condition $E[X_1] > 0$ have also been used. Therefore, setting $s = 0$ in (3.35) leads to

$$\zeta_\delta(\infty) = 1 - \frac{\phi(\delta)}{\phi(\gamma+\delta)}. \tag{3.36}$$

In particular, when $\delta = 0$, we have $\zeta_0(\infty) = 1$ as $\phi(0) = 0$.

Next, application of (3.26) and (3.36) to the limit of (3.4) yields the asymptotic result

$$\begin{aligned}
\lim_{u\to\infty} e^{\mathcal{K}_\delta u}\alpha_\delta(u) &= \lim_{u\to\infty} e^{\mathcal{K}_\delta u}\left(m_{0,\delta}(u) - \phi(\gamma+\delta)\zeta_\delta(u)\int_0^\infty m_{0,\delta}(u+x)e^{-\phi(\gamma+\delta)x}dx\right) \\
&= \lim_{u\to\infty} e^{\mathcal{K}_\delta u} m_{0,\delta}(u) - \phi(\gamma+\delta)\left(\lim_{u\to\infty}\zeta_\delta(u)\right)\left(\lim_{u\to\infty}\int_0^\infty e^{\mathcal{K}_\delta u} m_{0,\delta}(u+x)e^{-\phi(\gamma+\delta)x}dx\right) \\
&= \mathcal{C}_\delta - \phi(\gamma+\delta)\left(1 - \frac{\phi(\delta)}{\phi(\gamma+\delta)}\right)\int_0^\infty \left(\lim_{u\to\infty} e^{\mathcal{K}_\delta(u+x)} m_{0,\delta}(u+x)\right)e^{-[\mathcal{K}_\delta+\phi(\gamma+\delta)]x}dx \\
&= \mathcal{C}_\delta - [\phi(\gamma+\delta) - \phi(\delta)]\frac{\mathcal{C}_\delta}{\mathcal{K}_\delta + \phi(\gamma+\delta)} \\
&= \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\mathcal{C}_\delta,
\end{aligned} \tag{3.37}$$

where dominated convergence has been used in the third equality. Finally, asymptotic formula for $m_{\theta,\delta}(u)$ can be obtained in a similar manner using (3.5), (3.36), (3.37) and dominated convergence, giving

$$\begin{aligned}
&\lim_{u\to\infty} e^{\mathcal{K}_\delta u} m_{\theta,\delta}(u) \\
&= \lim_{u\to\infty} e^{\mathcal{K}_\delta u}\alpha_\delta(u) + \frac{\phi(\gamma+\delta)}{1-\theta}\left(\lim_{u\to\infty}\zeta_\delta(u)\right)\int_0^\infty\left(\lim_{u\to\infty} e^{\mathcal{K}_\delta u}\alpha_\delta(u+x)\right)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_0^x[1-\lim_{u\to\infty}\zeta_\delta(u+y)]dy}dx \\
&= \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\mathcal{C}_\delta + \frac{\phi(\gamma+\delta)}{1-\theta}\left(1 - \frac{\phi(\delta)}{\phi(\gamma+\delta)}\right)\int_0^\infty \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta x}e^{-\frac{\phi(\delta)}{1-\theta}x}dx \\
&= \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\mathcal{C}_\delta\left(1 + \frac{\phi(\gamma+\delta) - \phi(\delta)}{1-\theta}\frac{1}{\mathcal{K}_\delta + \frac{\phi(\delta)}{1-\theta}}\right) \\
&= \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\frac{(1-\theta)\mathcal{K}_\delta + \phi(\gamma+\delta)}{(1-\theta)\mathcal{K}_\delta + \phi(\delta)}\mathcal{C}_\delta.
\end{aligned}$$

Thus, one has the asymptotic formula

$$m_{\theta,\delta}(u) \sim \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\frac{(1-\theta)\mathcal{K}_\delta + \phi(\gamma+\delta)}{(1-\theta)\mathcal{K}_\delta + \phi(\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u} \sim \mathcal{A}_{\theta,\delta} m_{0,\delta}(u), \tag{3.38}$$

where the last relationship is due to (3.23), and

$$\mathcal{A}_{\theta,\delta} = \frac{\mathcal{K}_\delta + \phi(\delta)}{\mathcal{K}_\delta + \phi(\gamma+\delta)}\frac{(1-\theta)\mathcal{K}_\delta + \phi(\gamma+\delta)}{(1-\theta)\mathcal{K}_\delta + \phi(\delta)}. \tag{3.39}$$

In other words, the Cramér-Lundberg asymptotic expression for the Gerber-Shiu function with tax is just a constant multiple of that without tax. By some simple algebra, it can be easily checked that $\mathcal{A}_{\theta,\delta} > 1$ for $0 < \theta < 1$, i.e. the Gerber-Shiu function in the presence of taxation is asymptotically larger than the counterpart without tax. This is not surprising because ruin of X^θ occurs no later than X for the same sample path of X and any realization of $\{Z_j\}_{j=1}^\infty$.

Remark 5 Since $\phi(0) = 0$, if $\delta = 0$ then the asymptotic formula (3.38) simplifies to

$$m_{\theta,0}(u) \sim \frac{(1-\theta)\mathcal{K}_0 + \phi(\gamma)}{(1-\theta)(\mathcal{K}_0 + \phi(\gamma))} \mathcal{C}_0 e^{-\mathcal{K}_0 u} \sim \mathcal{A}_{\theta,0} m_{0,0}(u),$$

which can be rewritten as $m_{\theta,0}(u)/m_{0,0}(u) \sim \mathcal{A}_{\theta,0}$. Since $\mathcal{A}_{\theta,0}$ does not depend on the choice of the penalty function $w(\cdot)$, for the ruin probability (i.e. $w(\cdot) \equiv 1$) we also have $\Psi_\theta(u)/\Psi_0(u) \sim \mathcal{A}_{\theta,0}$. Comparing these asymptotics, one has $m_{\theta,0}(u)/m_{0,0}(u) \sim \Psi_\theta(u)/\Psi_0(u)$. Rearrangements yield $m_{\theta,0}(u)/\Psi_\theta(u) \sim m_{0,0}(u)/\Psi_0(u)$, or equivalently,

$$E_u[w(|X^\theta_{\tau_\theta}|)|\tau_\theta < \infty] \sim E_u[w(|X_{\tau_0}|)|\tau_0 < \infty].$$

In particular, when $w(y) = e^{-sy}$, both sides of the above relationship correspond to the Laplace transform of the deficit at ruin conditional on ruin occurring. Therefore, we conclude that the asymptotic conditional distribution of the deficit with taxation is the same as that without taxation. □

Remark 6 Instead of using the solution (3.5), we can also utilize the series expression (3.8) to obtain the asymptotic formula (3.38) for the Gerber-Shiu function $m_{\theta,\delta}(u)$. First, because X is spatial homogeneous, by sample path arguments one observes from the definition (2.9) that $\zeta_\delta(u)$ is increasing (i.e. non-decreasing) in u. This together with (3.7) gives the inequality

$$\zeta_{\delta,1}(u,x) \le \zeta_\delta(\infty)\frac{\phi(\gamma+\delta)}{1-\theta}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}, \qquad x > u \ge 0. \tag{3.40}$$

Now we want to prove that, for $n = 1, 2, \ldots$,

$$\left(\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^n \frac{(x-u)^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}}{(n-1)!} \le \zeta_{\delta,n}(u,x) \le \left(\zeta_\delta(\infty)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^n \frac{(x-u)^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}}{(n-1)!},$$
$$x > u \ge 0. \tag{3.41}$$

We can verify (3.41) by mathematical induction. Obviously, the starting point $n = 1$ holds true due to (3.7) and (3.40). Suppose that (3.41) holds true for some $n \ge 1$. For the case of $n+1$, by (3.7), (3.9), the first inequality in (3.41) and the fact that $\zeta_\delta(y) \ge \zeta_\delta(u)$ for $y \ge u$, we get

$$\begin{aligned}
\zeta_{\delta,n+1}(u,x) &= \int_u^x \zeta_{\delta,1}(u,y)\zeta_{\delta,n}(y,x)dy \\
&\ge \left(\frac{\phi(\gamma+\delta)}{1-\theta}\right)^{n+1} \int_u^x \zeta_\delta(u)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(y-u)}[\zeta_\delta(y)]^n \frac{(x-y)^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-y)}}{(n-1)!}dy \\
&\ge \left(\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^{n+1} e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)} \int_u^x \frac{(x-y)^{n-1}}{(n-1)!}dy \\
&= \left(\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^{n+1} \frac{(x-u)^n e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}}{n!}.
\end{aligned}$$

Similarly, (3.40) together with the second inequality in (3.41) implies

$$\begin{aligned}
\zeta_{\delta,n+1}(u,x) &\le \left(\zeta_\delta(\infty)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^{n+1} \int_u^x e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(y-u)} \frac{(x-y)^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-y)}}{(n-1)!}dy \\
&= \left(\zeta_\delta(\infty)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^{n+1} \frac{(x-u)^n e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}}{n!}.
\end{aligned}$$

Hence, the inductive step of $n+1$ is completed and one asserts that (3.41) holds true for $n=1,2,\ldots$.

Next, we apply (3.37), the first inequality in (3.41) and dominated convergence to (3.8). This leads to

$$\begin{aligned}
m_{\theta,\delta}(u) &\geq \alpha_\delta(u) + \sum_{n=1}^{\infty}\int_u^\infty \left(\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^n \frac{(x-u)^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}}{(n-1)!}\alpha_\delta(x)dx \\
&= \alpha_\delta(u) + \sum_{n=1}^{\infty}\left(\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^n \int_0^\infty \frac{x^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}x}}{(n-1)!}\alpha_\delta(u+x)dx \\
&\sim \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u} + \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\mathcal{C}_\delta \sum_{n=1}^{\infty}\left(\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^n \int_0^\infty \frac{x^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}x}}{(n-1)!}e^{-\mathcal{K}_\delta(u+x)}dx \\
&= \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u}\left[1+\sum_{n=1}^{\infty}\left(\frac{\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}}{\mathcal{K}_\delta+\frac{\phi(\gamma+\delta)}{1-\theta}}\right)^n\right] \\
&= \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u}\frac{1}{1-\frac{\zeta_\delta(u)\frac{\phi(\gamma+\delta)}{1-\theta}}{\mathcal{K}_\delta+\frac{\phi(\gamma+\delta)}{1-\theta}}} \\
&\sim \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u}\frac{1}{1-\frac{\zeta_\delta(\infty)\frac{\phi(\gamma+\delta)}{1-\theta}}{\mathcal{K}_\delta+\frac{\phi(\gamma+\delta)}{1-\theta}}} \\
&= \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\frac{(1-\theta)\mathcal{K}_\delta+\phi(\gamma+\delta)}{(1-\theta)\mathcal{K}_\delta+\phi(\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u},
\end{aligned} \tag{3.42}$$

where the last line follows from the use of (3.36) along with some straightforward simplifications. Omitting the details, similar procedure results in

$$\begin{aligned}
m_{\theta,\delta}(u) &\leq \alpha_\delta(u) + \sum_{n=1}^{\infty}\int_u^\infty \left(\zeta_\delta(\infty)\frac{\phi(\gamma+\delta)}{1-\theta}\right)^n \frac{(x-u)^{n-1}e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}}{(n-1)!}\alpha_\delta(x)dx \\
&\sim \frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\frac{(1-\theta)\mathcal{K}_\delta+\phi(\gamma+\delta)}{(1-\theta)\mathcal{K}_\delta+\phi(\delta)}\mathcal{C}_\delta e^{-\mathcal{K}_\delta u}.
\end{aligned} \tag{3.43}$$

Finally, the asymptotic formula (3.38) is recovered by combining (3.42) and (3.43). □

3.3 Analysis of $V_{\theta,\delta}(u)$

In this subsection, we turn our attention to the expected discounted tax payments payable until ruin $V_{\theta,\delta}(u)$ defined in (1.3). If ruin has not been observed before the first overshoot of X over the initial level, the tax payments for X^θ consist of a fraction θ of the overshoot plus potential future payments. Hence, the use of the density (2.13) leads to

$$\begin{aligned}
V_{\theta,\delta}(u) &= \theta\int_0^\infty x\zeta_\delta(u,x)dx + \int_0^\infty V_{\theta,\delta}(u+(1-\theta)x)\zeta_\delta(u,x)dx \\
&= \theta\zeta_\delta(u)\int_0^\infty x\phi(\gamma+\delta)e^{-\phi(\gamma+\delta)x}dx + \zeta_\delta(u)\int_0^\infty V_{\theta,\delta}(u+(1-\theta)x)\phi(\gamma+\delta)e^{-\phi(\gamma+\delta)x}dx \\
&= \frac{\theta}{\phi(\gamma+\delta)}\zeta_\delta(u) + \frac{\phi(\gamma+\delta)}{1-\theta}\zeta_\delta(u)\int_u^\infty V_{\theta,\delta}(x)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}dx, \qquad u\geq 0.
\end{aligned} \tag{3.44}$$

The above equation is the same as Equation (35) in Albrecher et al. (2008a) with $n = 1$. Thus, the solution given in their Equation (41) is applicable (see Remark 7), yielding

$$V_{\theta,\delta}(u) = \frac{\theta}{1-\theta}\zeta_\delta(u)\int_0^\infty e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_0^x [1-\zeta_\delta(u+y)]dy}dx. \tag{3.45}$$

On the other hand, by rewriting (3.44) as the Volterra integral equation

$$V_{\theta,\delta}(u) = \frac{\theta}{\phi(\gamma+\delta)}\zeta_\delta(u) + \int_u^\infty \zeta_{\delta,1}(u,x)V_{\theta,\delta}(x)dx,$$

successive substitution results in the alternative expression

$$V_{\theta,\delta}(u) = \frac{\theta}{\phi(\gamma+\delta)}\zeta_\delta(u) + \frac{\theta}{\phi(\gamma+\delta)}\sum_{n=1}^\infty \int_u^\infty \zeta_{\delta,n}(u,x)\zeta_\delta(x)dx, \tag{3.46}$$

where $\zeta_{\delta,n}(u,x)$ is defined via (3.7) and (3.9). Because each tax payment (if paid) is exponential with mean $\theta/\phi(\gamma+\delta))$, one can follow (3.10) and the subsequent discussion to observe that (3.46) is indeed equivalent to the decomposition

$$V_{\theta,\delta}(u) = \sum_{n=1}^\infty E_u[e^{-\delta Z_{\epsilon_n}}\theta(C^\theta_{Z_{\epsilon_n}} - M^\theta_{Z_{\epsilon_{n-1}}})\mathbf{1}_{(Z_{\epsilon_n}<\tau_\theta)}].$$

Remark 7 To apply Equation (41) of Albrecher et al. (2008a), we need to check two conditions that are used in their derivation. First, because $\zeta_\delta(y)$ is increasing in y, we can utilize (3.36) to assert that $1-\zeta_\delta(y) \geq 1-\zeta_\delta(\infty) = \phi(\delta)/\phi(\gamma+\delta) > 0$. (The last inequality is due to $\phi(\delta) > 0$ as we are only concerned with the case $\delta > 0$ for discounted tax payments.) Hence, one has the condition

$$\lim_{x\to\infty} e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_u^x [1-\zeta_\delta(y)]dy} = 0.$$

The next required condition is $\lim_{u\to\infty} V_{\theta,\delta}(u) < \infty$, which can be proved using arguments analogous to those in Remark 6. In fact, it is easy to see that

$$V_{\theta,\delta}(u) \sim \frac{\theta}{\phi(\gamma+\delta)}\zeta_\delta(\infty)\left(1+\sum_{n=1}^\infty[\zeta_\delta(\infty)]^n\right) = \theta\left(\frac{1}{\phi(\delta)} - \frac{1}{\phi(\gamma+\delta)}\right) < \infty, \tag{3.47}$$

where (3.36) is used to simplify the expression in the last equality. □

3.4 Tax payments with delayed start

In this subsection, we analyze an extension of the tax system, in which the first tax payment starts only when the surplus process overshoots a fixed threshold level b (that is no less than the initial surplus u) for the first time without ruin being observed in the interim. If the first tax payment occurs, the amount will be a fraction θ of the overshoot. The resulting Gerber-Shiu function and expected discounted tax payments until ruin are denoted by $m_{\theta,\delta}(u,b)$ and $V_{\theta,\delta}(u,b)$ respectively. Clearly, $m_{\theta,\delta}(u,u) = m_{\theta,\delta}(u)$ and $V_{\theta,\delta}(u,u) = V_{\theta,\delta}(u)$. Following almost identical proof as in Section 2.2, we can show that the discounted amount of the overshoot over the threshold b is again exponentially distributed with mean $1/\phi(\gamma+\delta)$, given that the overshoot occurs avoiding ruin enroute. Hence, similar to (3.2), we can apply $\chi_\delta(u,b)$ defined in (2.14) to obtain

$$m_{\theta,\delta}(u,b) = \alpha_\delta(u,b) + \frac{\phi(\gamma+\delta)}{1-\theta}\chi_\delta(u,b)\int_b^\infty m_{\theta,\delta}(x)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-b)}dx, \qquad 0 \le u \le b, \tag{3.48}$$

where

$$\alpha_\delta(u,b) = E_u[e^{-\delta\tau_\theta}w(|X^\theta_{\tau_\theta}|)\mathbf{1}_{(\tau_\theta < Z_{J^*(b)})}]$$

is the Gerber-Shiu function for ruin occurring before the surplus is observed to overshoot level b. Letting $\theta = 0$ in (3.48) gives

$$\begin{aligned}\alpha_\delta(u,b) &= m_{0,\delta}(u) - \phi(\gamma+\delta)\chi_\delta(u,b)\int_b^\infty m_{0,\delta}(x)e^{-\phi(\gamma+\delta)(x-b)}dx \\ &= m_{0,\delta}(u) - \chi_\delta(u,b)\frac{m_{0,\delta}(b)-\alpha_\delta(b)}{\zeta_\delta(b)}, \qquad 0 \le u \le b,\end{aligned} \tag{3.49}$$

where (3.4) at $u = b$ has been utilized in the last line. Note that $\alpha_\delta(u,b)$ can be regarded as a generalization of the quantity $\alpha_\delta(u)$ defined in (3.3) with solution (3.4), since $\alpha_\delta(u,u) = \alpha_\delta(u)$. Moreover, $\alpha_\delta(u,b)$ does not depend on θ. Using (3.2) (at $u = b$) along with (3.49), we can rewrite (3.48) neatly as

$$\begin{aligned}m_{\theta,\delta}(u,b) &= m_{0,\delta}(u) - \chi_\delta(u,b)\frac{m_{0,\delta}(b)-\alpha_\delta(b)}{\zeta_\delta(b)} + \chi_\delta(u,b)\frac{m_{\theta,\delta}(b)-\alpha_\delta(b)}{\zeta_\delta(b)} \\ &= m_{0,\delta}(u) + \chi_\delta(u,b)\frac{m_{\theta,\delta}(b)-m_{0,\delta}(b)}{\zeta_\delta(b)}.\end{aligned} \tag{3.50}$$

For the expected discounted tax payments until ruin $V_{\theta,\delta}(u,b)$ (where $\delta > 0$) under delayed start, (3.44) is extended to

$$\begin{aligned}V_{\theta,\delta}(u,b) &= \frac{\theta}{\phi(\gamma+\delta)}\chi_\delta(u,b) + \frac{\phi(\gamma+\delta)}{1-\theta}\chi_\delta(u,b)\int_b^\infty V_{\theta,\delta}(x)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-b)}dx \\ &= \chi_\delta(u,b)\frac{V_{\theta,\delta}(b)}{\zeta_\delta(b)}, \qquad 0 \le u \le b,\end{aligned} \tag{3.51}$$

where the last step again follows from (3.44) at $u = b$.

Motivated by the optimization problems considered in Dickson and Waters (2004, Section 6.1) and Gerber at al. (2006), we look at the optimal threshold level b^* (if it exists) that maximizes with respect to b the function

$$\eta_{\theta,\delta}(u,b) = V_{\theta,\delta}(u,b) - m_{\theta,\delta}(u,b), \qquad 0 \le u \le b,$$

which represents the tax payments minus a penalty applied at ruin. (We further restrict $\theta > 0$, otherwise no tax will ever be paid regardless of the value of b.) Note that the optimal threshold when the surplus process is monitored continuously for both solvency and taxation was also analyzed by Cheung and Landriault (2012, Section 4.2). Using (3.50) and (3.51), it is immediate that

$$\eta_{\theta,\delta}(u,b) = -m_{0,\delta}(u) + \chi_\delta(u,b)\left(\frac{V_{\theta,\delta}(b)-m_{\theta,\delta}(b)+m_{0,\delta}(b)}{\zeta_\delta(b)}\right). \tag{3.52}$$

Following the same arguments leading to Equation (45) in Albrecher et al. (2008a), an expression for $\chi_\delta(u,b)$ can be obtained as

$$\chi_\delta(u,b) = \zeta_\delta(u)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_u^b[1-\zeta_\delta(y)]dy}, \qquad 0 \le u \le b, \tag{3.53}$$

which is in terms of $\zeta_\delta(\cdot)$. This result can be regarded as an alternative solution to (2.19). Hence, substitution of (3.53) into (3.52) followed by differentiation with respect to b yields

$$\begin{aligned}\frac{d}{db}\eta_{\theta,\delta}(u,b) = &-\frac{\phi(\gamma+\delta)}{1-\theta}[1-\zeta_\delta(b)]\chi_\delta(u,b)\left(\frac{V_{\theta,\delta}(b)-m_{\theta,\delta}(b)+m_{0,\delta}(b)}{\zeta_\delta(b)}\right)\\ &+\chi_\delta(u,b)\frac{d}{db}\left(\frac{V_{\theta,\delta}(b)-m_{\theta,\delta}(b)+m_{0,\delta}(b)}{\zeta_\delta(b)}\right).\end{aligned}$$

Since $\chi_\delta(u,b)$ is always positive, if a positive b^* exists then it is a root of the equation (in b),

$$\frac{d}{db}\left(\frac{V_{\theta,\delta}(b)-m_{\theta,\delta}(b)+m_{0,\delta}(b)}{\zeta_\delta(b)}\right) = \frac{\phi(\gamma+\delta)}{1-\theta}[1-\zeta_\delta(b)]\left(\frac{V_{\theta,\delta}(b)-m_{\theta,\delta}(b)+m_{0,\delta}(b)}{\zeta_\delta(b)}\right),$$

which is independent of the initial surplus u. In other words, if a positive b^* satisfying the above equation exists, then it is the same b^* that maximizes $\eta_{\theta,\delta}(u,b)$ with respect to b for all u such that $0 \le u \le b^*$.

4 Continuous monitoring of solvency

In this section, we analyze the situation in which the event of ruin is monitored continuously but tax is only paid at Poissonian time points, i.e. the model dynamics of the taxed surplus process still follows X^θ with the time of ruin now defined by $\tau_\theta^c = \inf\{t>0 : X_t^\theta < 0\}$. Such a model assumption can be viewed as a complement to risk processes with periodic dividend barrier strategy and continuous monitoring of solvency, which were studied by Avanzi et al. (2013, 2014), Zhang (2014) and Zhang and Cheung (2014a,b). In addition, one can also treat the present model as the reverse case of Albrecher and Ivanovs (2014, Section 6), which is concerned with Poissonian monitoring of ruin but continuous checking for tax payments. Similar to (1.2) and (1.3), we are interested in the Gerber-Shiu function

$$m_{\theta,\delta}^c(u) = E_u[e^{-\delta\tau_\theta^c}w(|X_{\tau_\theta^c}^\theta|)\mathbf{1}_{(\tau_\theta^c<\infty)}], \qquad u>0, \tag{4.1}$$

under $\delta \ge 0$, as well as the expected discounted tax payments until ruin

$$V_{\theta,\delta}^c(u) = E_u\left[\sum_{Z_j<\tau_\theta^c} e^{-\delta Z_j}\theta(C_{Z_j}^\theta - M_{Z_{j-1}}^\theta)_+\right], \qquad u\ge 0,$$

under $\delta > 0$. Clearly, $m_{\theta,\delta}^c(u)$ contains the ruin probability $\Psi_\theta^c(u) = P_u(\tau_\theta^c < \infty)$ as a special case. Because the surplus process possibly drops below zero level between the time points $\{Z_j\}_{j=0}^\infty$, we have $\tau_\theta^c \le \tau_\theta$ and hence $\Psi_\theta(u) \le \Psi_\theta^c(u)$.

Remark 8 Although the domain of the initial surplus is written as $u > 0$ instead of $u \ge 0$ in the definition (4.1), our upcoming results indeed also hold true for $u = 0$ when (i) X has bounded variation; or (ii) X has unbounded variation due to Brownian motion but not jumps. If X has unbounded variation and $\sigma = 0$, some formulas (e.g. Equation (4.6)) are only valid for $u > 0$, but the case $u = 0$ is not interesting anyway since ruin occurs immediately (see Kyprianou (2014, Lemma 8.6)). □

Since many of the subsequent analyses under the modified assumption still resemble those in earlier sections, we shall only highlight the key steps and omit the straightforward details wherever appropriate. Because the Gerber-Shiu function $m_{\theta,\delta}^c(u)$ will be analyzed by considering the discounted amount of the first tax payment, we define the analogues of (2.8) and (2.9) in the present model as

$$\overline{\zeta}_\delta^c(u,x) = E_u[e^{-\delta Z_{\epsilon_1}}; X_{Z_{\epsilon_1}} - X_{Z_0} > x,\ \tau_0^c > Z_{\epsilon_1}], \qquad u,x\ge 0,$$

and

$$\zeta_\delta^c(u) = E_u[e^{-\delta Z_{\epsilon_1}};\ \tau_0^c > Z_{\epsilon_1}] = \overline{\zeta}_\delta^c(u,0), \qquad u \geq 0,$$

respectively, where $\tau_0^c = \inf\{t > 0 : X_t < 0\}$ is the traditional ruin time for the Lévy insurance risk process X without tax. From Zhang and Cheung (2014b, Proposition 6), one has that

$$\overline{\zeta}_\delta^c(u,x) = \frac{\gamma}{\phi(\gamma+\delta)} e^{-\phi(\gamma+\delta)(u+x)} \left(W^{(\gamma+\delta)}(u) + \sum_{k=1}^{\infty} \int_0^u W^{(\gamma+\delta)}(y) Q_k^c(u,dy) \right), \tag{4.2}$$

where

$$Q_k^c(u,dy) = E_u\Big[e^{-\delta Z_k}; \tau_0^c > Z_k, \sup_{0\leq j\leq k-1} X_{Z_j} \leq u, X_{Z_k} \in dy\Big].$$

Equation (4.2) implies $\overline{\zeta}_\delta^c(u,x)/\zeta_\delta^c(u) = e^{-\phi(\gamma+\delta)x}$ (which is the analogue of (2.12)). It follows that the amount of the first observed overshoot $X_{Z_{\epsilon_1}} - X_{Z_0}$ is again exponentially distributed with mean $1/\phi(\gamma+\delta)$, even it is now conditional on that X has never dropped below zero in the interim. Thus, similar to (3.2), we arrive at

$$m_{\theta,\delta}^c(u) = \alpha_\delta^c(u) + \frac{\phi(\gamma+\delta)}{1-\theta} \zeta_\delta^c(u) \int_u^\infty m_{\theta,\delta}^c(x) e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)} dx, \qquad u > 0, \tag{4.3}$$

where

$$\alpha_\delta^c(u) = E_u[e^{-\delta\tau_\theta^c} w(|X_{\tau_\theta^c}^\theta|)\mathbf{1}_{(\tau_\theta^c < Z_{\epsilon_1})}]$$

has solution

$$\alpha_\delta^c(u) = m_{0,\delta}^c(u) - \phi(\gamma+\delta)\zeta_\delta^c(u) \int_u^\infty m_{0,\delta}^c(x) e^{-\phi(\gamma+\delta)(x-u)} dx, \qquad u > 0. \tag{4.4}$$

In the above expression, $m_{0,\delta}^c(\cdot)$ is the Gerber-Shiu function for the tax-free Lévy risk model X under continuous monitoring of solvency. Defining the q-potential measure for X killed on exiting $[0,\infty)$ as

$$\mathcal{R}^{(q)}(u,dx) = \int_0^\infty e^{-qt} P_u(X_t \in dx, \tau_0^c > t) dt, \qquad u,x \geq 0,$$

it is known that $\mathcal{R}^{(q)}(u,dx) = r^{(q)}(u,x)dx$, where

$$r^{(q)}(u,x) = e^{-\phi(q)x} W^{(q)}(u) - W^{(q)}(u-x). \tag{4.5}$$

See the proof of Theorem 8.7 in Kyprianou (2014). According to e.g. Asmussen and Albrecher (2010, Chapter XII.4), $m_{0,\delta}^c(\cdot)$ can be represented as

$$m_{0,\theta}^c(u) = \frac{\sigma^2}{2}[W^{(\delta)\prime}(u) - \phi(\delta)W^{(\delta)}(u)]w(0) + \int_0^\infty \int_0^\infty w(y) r^{(\delta)}(u,x)\nu(x+dy)dx, \qquad u > 0. \tag{4.6}$$

With the regular condition $\lim_{u\to\infty} m_{\theta,\delta}^c(u) = 0$ (which follows from $\lim_{u\to\infty} \Psi_\theta^c(u) = 0$ thanks to Albrecher et al. (2008b, Section 3) again) and $\alpha_\delta^c(u)$ given in (4.4), we can solve (4.3) to obtain the solution

$$m_{\theta,\delta}^c(u) = \alpha_\delta^c(u) + \frac{\phi(\gamma+\delta)}{1-\theta} \zeta_\delta^c(u) \int_u^\infty \alpha_\delta^c(x) e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_u^x [1-\zeta_\delta^c(y)]dy} dx.$$

In particular, for the ruin probability one easily observes power relationship in the form of (3.6) also holds true with $\Psi_\theta(\cdot)$, $\Psi_0(\cdot)$ and $\zeta_0(\cdot)$ replaced by $\Psi_\theta^c(\cdot)$, $\Psi_0^c(\cdot)$ and $\zeta_0^c(\cdot)$ respectively. Note that the traditional ruin probability for the Lévy risk model X based on continuous monitoring of solvency admits the simple representation (see Kyprianou (2014, Theorem 8.1(ii)))

$$\Psi_0^c(u) = P_u(\tau_0^c < \infty) = 1 - \psi'(0+)W^{(0)}(u), \qquad u \geq 0.$$

It remains to fully identify $\zeta_\delta^c(\cdot)$.

As in Section 2.2, similar to (2.14) we shall study the more general quantity

$$\chi_\delta^c(u,b) = E_u[e^{-\delta Z_{J^*(b)}}; Z_{J^*(b)} < \tau_0^c], \qquad 0 \leq u \leq b,$$

so that $\zeta_\delta^c(u) = \chi_\delta^c(u,u)$. Letting $\tau_b^+ = \inf\{t > 0 : X_t = b\}$ be the first upcrossing time at level b, it is well known that (see Kyprianou (2014, Theorem 8.1(iii)))

$$E_u[e^{-\delta\tau_b^+}; \tau_b^+ < \tau_0^c] = \frac{W^{(\delta)}(u)}{W^{(\delta)}(b)}, \qquad 0 \leq u \leq b.$$

Hence, use of the strong Markov property of X and the memoryless property of exponential inter-observation times yields

$$\chi_\delta^c(u,b) = \frac{W^{(\delta)}(u)}{W^{(\delta)}(b)}\chi_\delta^c(b,b). \tag{4.7}$$

On the other hand, conditioning on the surplus level at time Z_1 gives

$$\begin{aligned}
\chi_\delta^c(u,b) &= \int_0^\infty \gamma e^{-(\gamma+\delta)t}\int_b^\infty P_u(X_t \in dx; \tau_0^c > t)dt + \int_0^\infty \gamma e^{-(\gamma+\delta)t}\int_0^b \chi_\delta^c(x,b)P_u(X_t \in dx; \tau_0^c > t)dt \\
&= \gamma\int_b^\infty r^{(\gamma+\delta)}(u,x)dx + \gamma\int_0^b \chi_\delta^c(x,b)r^{(\gamma+\delta)}(u,x)dx \\
&= \frac{\gamma}{\phi(\gamma+\delta)}e^{-\phi(\gamma+\delta)b}W^{(\gamma+\delta)}(u) + \gamma\chi_\delta^c(b,b)\int_0^b \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)}[e^{-\phi(\gamma+\delta)x}W^{(\gamma+\delta)}(u) - W^{(\gamma+\delta)}(u-x)]dx,
\end{aligned}$$

where the last equality follows from the application of (4.5) and (4.7). Plugging in $b = u$ in above equation and solving for $\zeta_\delta^c(u) = \chi_\delta^c(u,u)$ leads to

$$\zeta_\delta^c(u) = \frac{\frac{\gamma}{\phi(\gamma+\delta)}e^{-\phi(\gamma+\delta)u}W^{(\gamma+\delta)}(u)}{1 - \gamma\int_0^u \frac{W^{(\delta)}(x)}{W^{(\delta)}(u)}[e^{-\phi(\gamma+\delta)x}W^{(\gamma+\delta)}(u) - W^{(\gamma+\delta)}(u-x)]dx}, \qquad u \geq 0.$$

Concerning asymptotic formula for $m_{\theta,\delta}^c(u)$ as $u \to \infty$, we need to first derive an asymptotic result for $m_{0,\delta}^c(u)$ in the absence of taxation. By substituting (4.5) into (4.6), taking Laplace transforms and applying (2.4), we arrive at

$$\begin{aligned}
\widehat{m}_{0,\delta}^c(s) &= \frac{\sigma^2}{2}[s\widehat{W}^{(\delta)}(s) - W^{(\delta)}(0) - \phi(\delta)\widehat{W}^{(\delta)}(s)]w(0) + \int_0^\infty\int_0^\infty w(y)e^{-\phi(q)x}\widehat{W}^{(\delta)}(s)\nu(x+dy)dx \\
&\quad - \int_0^\infty\int_0^\infty w(y)e^{-sx}\widehat{W}^{(\delta)}(s)\nu(x+dy)dx \\
&= \frac{\frac{\sigma^2}{2}[s-\phi(\delta)]w(0) + \int_0^\infty(e^{-\phi(\delta)x} - e^{-sx})\int_0^\infty w(y)\nu(x+dy)dx}{\psi(s)-\delta}.
\end{aligned} \tag{4.8}$$

Note that we have used the fact that $\sigma^2 W^{(\delta)}(0) = 0$ in the last equality, since $\sigma^2 > 0$ implies $W^{(\delta)}(0) = 0$ (see Kyprianou (2014, Lemma 8.6)). As in Section 3.2, it is assumed that the Laplace exponent $\psi(\cdot)$

is well defined in a neighborhood of the origin, and we follow the same definition of the adjustment coefficient $\mathcal{K}_\delta$ therein. We further assume

$$\int_0^\infty e^{\mathcal{K}_\delta x}\int_0^\infty w(y)\nu(x+dy)dx<\infty.$$

Suppose that the Cramér-Lundberg asymptotic formula

$$m_{0,\delta}^c(u)\sim\mathcal{C}_\delta^c e^{-\mathcal{K}_\delta u} \tag{4.9}$$

holds true (see Remark 9). Use of the final value theorem for Laplace transforms and (4.8) yields

$$\begin{aligned}\mathcal{C}_\delta^c &= \lim_{u\to\infty} e^{\mathcal{K}_\delta u} m_{0,\delta}^c(u)=\lim_{s\to 0} s\widehat{m}_{0,\delta}^c(s-\mathcal{K}_\delta)\\ &=\frac{\frac{\sigma^2}{2}[-\mathcal{K}_\delta-\phi(\delta)]w(0)+\int_0^\infty(e^{-\phi(\delta)x}-e^{\mathcal{K}_\delta x})\int_0^\infty w(y)\nu(x+dy)dx}{\psi'(-\mathcal{K}_\delta)}.\end{aligned}$$

Remark 9 From Biffis and Morales (2010, Theorem 4.1) and Feng and Shimizu (2013, Section 3.3 and Proposition 5.1), it is known that $m_{0,\delta}^c(\cdot)$ satisfies a defective renewal equation if the process X belongs to the class (i) or (ii) in Remark 8. In this case, it follows from Feng and Shimizu (2013, Theorem 5.1 and Proposition 5.1) that the asymptotic expression (4.9) is valid. □

Next, following the arguments leading to (3.36) and (3.37), it is easy to see that

$$\zeta_\delta^c(\infty)=1-\frac{\phi(\delta)}{\phi(\gamma+\delta)}$$

and

$$\lim_{u\to\infty} e^{\mathcal{K}_\delta u}\alpha_\delta^c(u)=\frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\mathcal{C}_\delta^c.$$

Hence, analogous to (3.38), we arrive at

$$m_{\theta,\delta}^c(u)\sim\frac{\mathcal{K}_\delta+\phi(\delta)}{\mathcal{K}_\delta+\phi(\gamma+\delta)}\frac{(1-\theta)\mathcal{K}_\delta+\phi(\gamma+\delta)}{(1-\theta)\mathcal{K}_\delta+\phi(\delta)}\mathcal{C}_\delta^c e^{-\mathcal{K}_\delta u}\sim\mathcal{A}_{\theta,\delta}m_{0,\delta}^c(u), \tag{4.10}$$

where $\mathcal{A}_{\theta,\delta}$ is given in (3.39). Similar comments as in Remark 5 concerning the asymptotic conditional distribution of the deficit are also applicable.

For the expected discounted tax payments before ruin $V_{\theta,\delta}^c(u)$, in parallel to (3.44), (3.45) and (3.47), we observe that $V_{\theta,\delta}^c(\cdot)$ satisfies the integral equation

$$V_{\theta,\delta}^c(u)=\frac{\theta}{\phi(\gamma+\delta)}\zeta_\delta^c(u)+\frac{\phi(\gamma+\delta)}{1-\theta}\zeta_\delta^c(u)\int_u^\infty V_{\theta,\delta}^c(x)e^{-\frac{\phi(\gamma+\delta)}{1-\theta}(x-u)}dx,\qquad u\geq 0,$$

which has analytic solution

$$V_{\theta,\delta}^c(u)=\frac{\theta}{1-\theta}\zeta_\delta^c(u)\int_0^\infty e^{-\frac{\phi(\gamma+\delta)}{1-\theta}\int_0^x[1-\zeta_\delta^c(u+y)]dy}dx,$$

and asymptotic expression

$$V_{\theta,\delta}^c(u)\sim\theta\left(\frac{1}{\phi(\delta)}-\frac{1}{\phi(\gamma+\delta)}\right). \tag{4.11}$$

It is instructive to note that $V_{\theta,\delta}(u)$ and $V_{\theta,\delta}^c(u)$ possess the same asymptotic formula. This is not surprising because ruin is unlikely to happen for sufficiently large initial surplus regardless of whether ruin is monitored continuously or at Poissonian time points only.

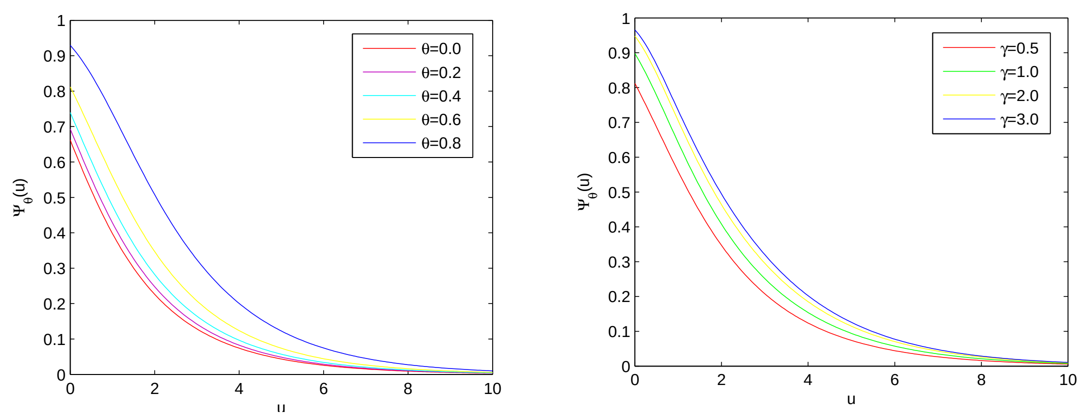
Finally, we omit the case of delayed tax payments under continuous monitoring of solvency, as the analysis is essentially identical to that in Section 3.4.

5 Numerical illustrations

In this section, we present some numerical examples concerning the ruin probability under taxation and the expected discounted tax payments until ruin. Two classes of Lévy insurance risk processes will be considered, namely,

- a Brownian motion risk model with Laplace exponent $\psi(s) = 0.5s + s^2$; and
- compound Poisson models with Laplace exponents (i) $\psi(s) = 1.5s - 1 + \frac{3}{s+3}\frac{1.5}{s+1.5}$; (ii) $\psi(s) = 1.5s - 1 + \frac{1}{s+1}$; and (iii) $\psi(s) = 1.5s - 1 + \frac{1}{3}\frac{0.5}{s+0.5} + \frac{2}{3}\frac{2}{s+2}$.

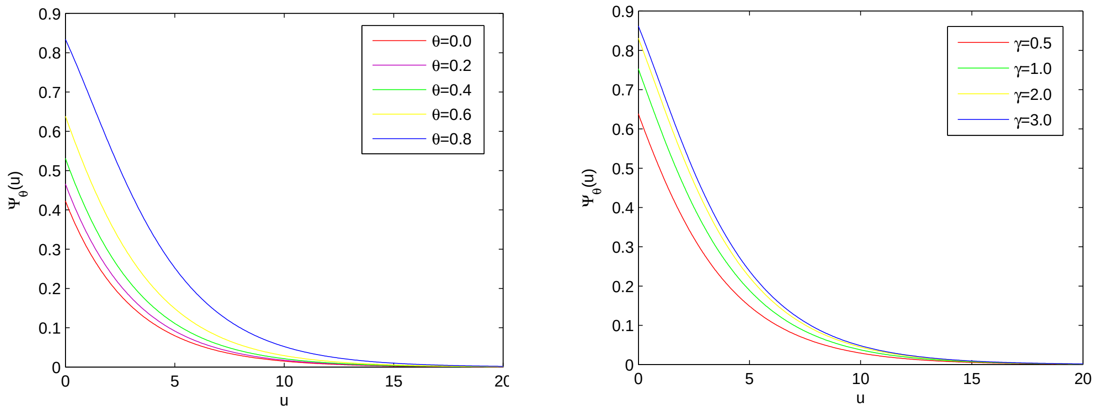
In the above compound Poisson processes, the Poisson claim arrival intensity and the premium rate are always $\lambda = 1$ and $\beta = 1.5$ respectively (see Example 2) whereas the generic claim random variable follows (i) a sum of two independent exponentials; (ii) an exponential distribution; and (iii) a mixture of two exponentials. These three claim distributions were also used in the numerical illustrations in Albrecher et al. (2011b, 2013). They have the same mean of 1 but possess different variances of 0.56, 1 and 2 respectively. Moreover, it is easy to check that the above Brownian motion risk model is the diffusion approximation of the compound Poisson model (ii), i.e. the first two moments $E[X_t]$ and $E[X_t^2]$ in both models match for all $t \geq 0$ (see e.g. Klugman et al. (2013, Chapter 11.6)).



(a) **(b)**

Figure 1: $\Psi_\theta(u)$ in Brownian motion model. (a) $\gamma = 0.5$ (b) $\theta = 0.6$

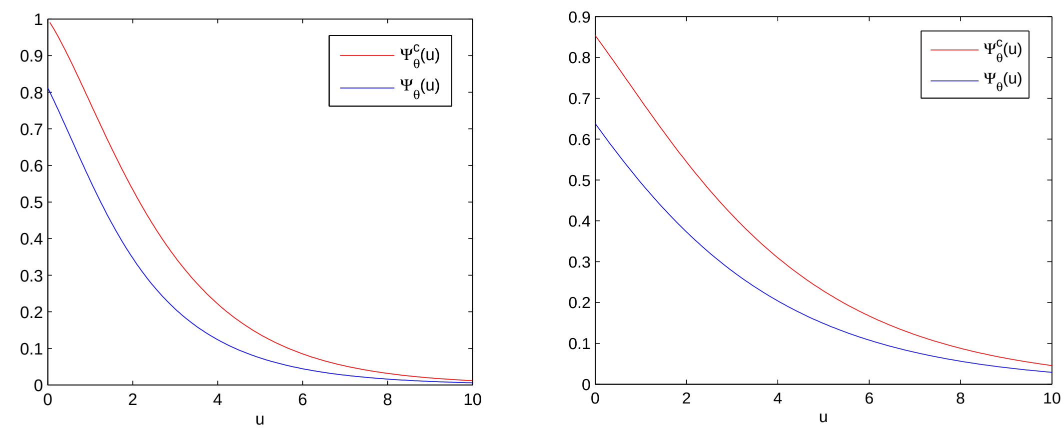
In Figure 1, we plot the ruin probability $\Psi_\theta(u)$ against the initial surplus u in the Brownian motion risk model with a Poissonian observer (for both ruin monitoring and taxation). Specifically, Figure 1(a) plots $\Psi_\theta(u)$ for the tax rates $\theta = 0, 0.2, 0.4, 0.6, 0.8$ when the Poissonian observation rate is fixed at $\gamma = 0.5$; whereas Figure 1(b) shows the behaviour of $\Psi_\theta(u)$ for $\gamma = 0.5, 1, 2, 3$ and fixed $\theta = 0.6$. As expected, $\Psi_\theta(u)$ is decreasing in u. Furthermore, for each fixed u the ruin probability $\Psi_\theta(u)$ becomes larger when θ or γ increases. These can be interpreted as follows. First, all other things being equal, for two processes X^{θ_1} and X^{θ_2} with different tax rates such that $0 \leq \theta_1 \leq \theta_2 \leq 1$, it is clear that $X_t^{\theta_2} \leq X_t^{\theta_1}$ for all $t \geq 0$ and therefore $\Psi_{\theta_2}(u) \geq \Psi_{\theta_1}(u)$. (Note that X^{θ_1} and X^{θ_2} achieve observed maximum at the same time points.) Second, a larger γ not only leads to more frequent checking of ruin but also more frequent tax payments out of the surplus process, thereby increasing the ruin probability $\Psi_\theta(u)$. Figure 2 also depicts $\Psi_\theta(u)$ but for the compound Poisson model (ii) with exponential jumps, and the same behaviour as in the Brownian motion model is observed. To compare the ruin probabilities $\Psi_\theta(u)$ and $\Psi_\theta^c(u)$ under discrete and continuous monitoring of solvency, Figure 3 shows the plots for the Brownian



(a) **(b)**

Figure 2: $\Psi_\theta(u)$ in compound Poisson model with exponential jumps. (a) $\gamma = 0.5$ (b) $\theta = 0.6$

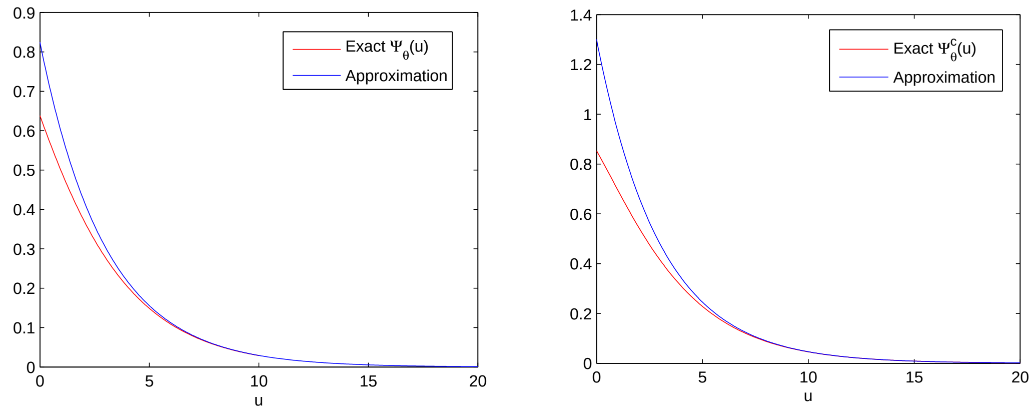
motion model and the compound Poisson model with exponential claims when $\gamma = 0.5$ and $\theta = 0.6$. Not surprisingly, it is found that $\Psi^c_\theta(u) \geq \Psi_\theta(u)$ in both models, which is due to the fact that the first passage time of X^θ below zero may not be observed under discrete checking of ruin. Next, for both discrete and continuous monitoring of ruin, the Cramér-Lundberg approximations (3.38) and (4.10) are compared to the exact ruin probabilities $\Psi_\theta(u)$ and $\Psi^c_\theta(u)$ respectively in Figure 4 under the compound Poisson model with exponential jumps. Note that the approximations perform well for $u \geq 5$ when the ruin probabilities are still significantly above zero (under the setting $\gamma = 0.5$ and $\theta = 0.6$).



(a) **(b)**

Figure 3: Comparison of $\Psi^c_\theta(u)$ and $\Psi_\theta(u)$ when $\gamma = 0.5$ and $\theta = 0.6$. (a) Brownian motion model (b) Compound Poisson model with exponential jumps

For Brownian motion model and the compound Poisson model under three different claim distributions, some exact values of the ruin probability $\Psi_\theta(u)$ are given in Tables 1-4. Comparison of the values across Tables 2-4 reveals that $\Psi_\theta(u)$ is higher when the variance of the claim distribution in the compound Poisson model becomes larger. This intuitively makes sense because a claim distribution with a larger variance represents higher risk for the insurer (with the mean being fixed). In addition, by



(a) (b)

Figure 4: Comparison of the exact ruin probability (red curve) and the Cramér-Lundberg approximation (blue curve) when $\gamma = 0.5$ and $\theta = 0.6$. (a) Discrete monitoring of ruin (b) Continuous monitoring of ruin

examining Tables 1 and 3, it can be seen that the Brownian motion model does not appear to be a very good diffusion approximation of the compound Poisson model with exponential claims as far as the ruin probability $\Psi_\theta(u)$ is concerned.

Table 1: Exact ruin probabilities when $\psi(s) = 0.5s + s^2$ and $\gamma = 0.5$.

θ	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$
0.2	0.6934763	0.4283647	0.2477129	0.1422615	0.0826375	0.0486397	0.0289278	0.0173289	0.0104295
0.4	0.7396074	0.4757191	0.2823157	0.1646833	0.0965598	0.0571542	0.0341071	0.0204735	0.0123373
0.6	0.8120865	0.5589842	0.3468196	0.2077839	0.1237736	0.0739553	0.0443828	0.0267324	0.0161419
0.8	0.9293771	0.7375029	0.5075839	0.3241953	0.2005945	0.1225835	0.0745589	0.0452701	0.0274682

Table 2: Exact ruin probabilities when $\psi(s) = 1.5s - 1 + \frac{3}{s+3}\frac{1.5}{s+1.5}$ and $\gamma = 0.5$.

θ	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$
0.2	0.4337051	0.2855819	0.1854250	0.1198858	0.0773307	0.0498097	0.0320543	0.0206164	0.0132550
0.4	0.4921251	0.3317359	0.2186703	0.1427232	0.0926121	0.0598788	0.0386273	0.0248823	0.0160136
0.6	0.5915068	0.4152914	0.2811458	0.1866355	0.1224198	0.0796982	0.0516396	0.0333584	0.0215075
0.8	0.7874459	0.6083361	0.4401645	0.3053315	0.2060954	0.1366844	0.0896295	0.0583474	0.0378061

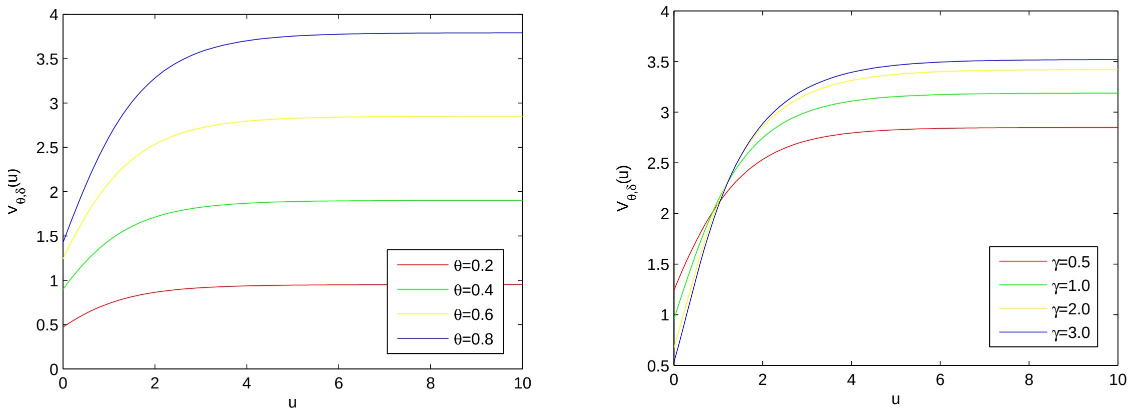
Table 3: Exact ruin probabilities when $\psi(s) = 1.5s - 1 + \frac{1}{s+1}$ and $\gamma = 0.5$.

θ	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$
0.2	0.4659592	0.3390253	0.2454378	0.1771128	0.1275335	0.0916983	0.0658655	0.0472766	0.0339169
0.4	0.5310428	0.3952059	0.2905639	0.2119363	0.1537594	0.1111424	0.0801321	0.0576705	0.0414525
0.6	0.6383815	0.4936476	0.3728812	0.2772246	0.2038698	0.1487907	0.1080151	0.0781194	0.0563476
0.8	0.8341929	0.7028409	0.5668289	0.4423913	0.3370897	0.2524350	0.1866945	0.1368421	0.0996588

Next, we take a look at the expected discounted tax payments before ruin. For simplicity, only the results in the Brownian motion risk model with $\psi(s) = 0.5s + s^2$ will be illustrated, and we set $\delta = 0.1$ throughout. In Figure 5, we observe that $V_{\theta,\delta}(u)$ is an increasing function of u for each fixed pair of γ

Table 4: Exact ruin probabilities when $\psi(s) = 1.5s - 1 + \frac{1}{3}\frac{0.5}{s+0.5} + \frac{2}{3}\frac{2}{s+2}$ and $\gamma = 0.5$.

θ	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=6$	$u=7$	$u=8$
0.2	0.5065377	0.4086923	0.3342859	0.2739977	0.2244695	0.1837271	0.1502537	0.1227941	0.1002973
0.4	0.5808308	0.4789195	0.3974625	0.3293595	0.2721456	0.2242745	0.1844197	0.1513809	0.1240868
0.6	0.6975455	0.5953427	0.5063980	0.4277401	0.3588853	0.2994269	0.2486856	0.2057902	0.1697953
0.8	0.8863755	0.8104888	0.7286341	0.6444473	0.5618635	0.4839525	0.4126641	0.3489676	0.2931008



(a) **(b)**

Figure 5: $V_{\theta,\delta}(u)$ in the Brownian motion model. (a) $\gamma = 0.5$ (b) $\theta = 0.6$

and θ, which is expected. In particular, Figure 5(a) demonstrates that $V_{\theta,\delta}(u)$ appears to increase with θ. Indeed, when θ increases (all else being equal), there are two opposing effects on $V_{\theta,\delta}(u)$. A larger θ means that a larger fraction of an observed overshoot is paid as tax (if ruin has not been observed in the interim). On the other hand, it also implies that the process X^θ is likely to ruin earlier, resulting in potential loss of future tax payments (see also comments in relation to Figure 1(a) concerning the ruin probability $\Psi_\theta(u)$). Figure 5(a) suggests that the former effect dominates in our example. Concerning Figure 5(b), we notice the interesting phenomenon that $V_{\theta,\delta}(u)$ is decreasing in γ for small values of u (around $0 \le u \le 1$) and increasing in γ for larger u. Recall that both checking of ruin and payment of tax occur more frequently when γ is larger. When the initial surplus u is close to zero, a larger γ is likely to result in early observation of ruin. As tax payments cease after ruin is observed, this explains why $V_{\theta,\delta}(u)$ decreases in γ for small u. In contrast, when u gets larger, the process X^θ stays further away from ruin and hence $V_{\theta,\delta}(u)$ moves in the same direction as the frequency of tax payments. Figure 6 compares $V_{\theta,\delta}(u)$ and $V^c_{\theta,\delta}(u)$ under discrete and continuous checking of ruin when $\gamma = 0.5$ and $\theta = 0.6$. Clearly, one has that $V^c_{\theta,\delta}(u) \le V_{\theta,\delta}(u)$ because the tax payments under continuous monitoring of ruin cannot last longer than those in the discrete case. Moreover, as u gets larger, $V^c_{\theta,\delta}(u)$ and $V_{\theta,\delta}(u)$ converge to the same constant, which is in agreement with the asymptotic formulas (3.47) and (4.11) that are identical. Finally, we look at the case of delayed start of taxation as in Section 3.4 when solvency is discretely monitored, and we set $\gamma = 2$ and $\theta = 0.8$. Figure 7(a) depicts $V_{\theta,\delta}(u, b)$ as a function of u for $0 \le u \le b$ when $b = 1, 3, 5, 7$. Of course, $V_{\theta,\delta}(u, b)$ is increasing in u. In Figure 7(b), we plot $V_{\theta,\delta}(u, b)$ against b for $b \ge u$ when $u = 0, 0.5, 1, 1.5, 2$. It is observed that $V_{\theta,\delta}(u, b)$ is first increasing and then decreasing in b for $u = 0, 0.5, 1, 1.5$, and it is a decreasing function of b for $u = 2$. Further numerical investigation reveals that it is the same optimal tax threshold of approximately 1.6 which maximizes $V_{\theta,\delta}(u, b)$ with respect to b for $u = 0, 0.5, 1, 1.5$. When $u = 2$, the optimal threshold level is simply the initial surplus level.

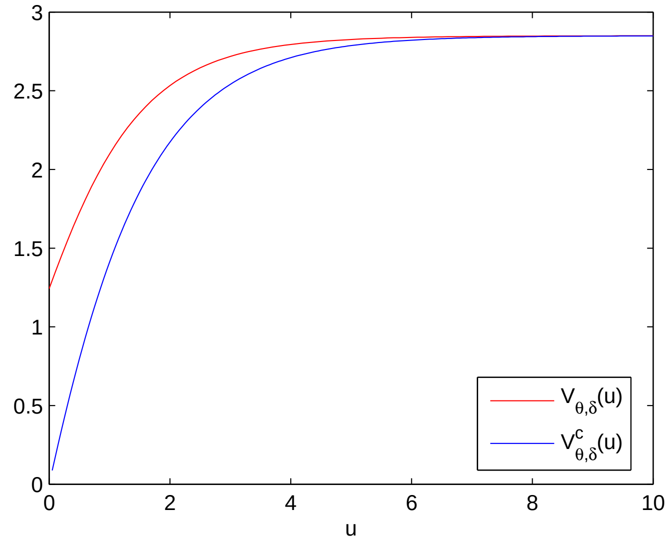


Figure 6: Comparison of $V_{\theta,\delta}(u)$ and $V^c_{\theta,\delta}(u)$ when $\gamma = 0.5$ and $\theta = 0.6$ in the Brownian motion model.

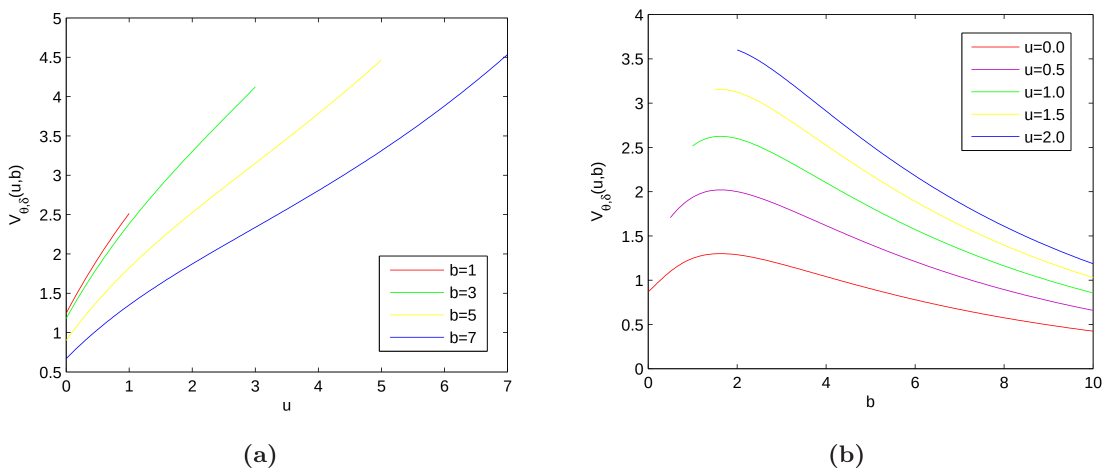


Figure 7: $V_{\theta,\delta}(u, b)$ in the Brownian motion model when $\gamma = 2$ and $\theta = 0.8$. (a) as a function of u (b) as a function of b

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