Fourier-cosine Method for Gerber-Shiu Functions

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Abstract

In this article, we provide a systematic study on effectively approximating the Gerber-Shiu functions, which is a hardly touched topic in the current literature, by incorporating the recently popular Fourier-cosine method. Fourier-cosine method has been a prevailing numerical method in option pricing theory since the work of Fang and Oosterlee (2009). Our approximant of Gerber-Shiu functions under Lévy subordinator model has $O(n)$ computational complexity in comparison with that of $O(n \log n)$ via the fast Fourier transform algorithm. Also, for Gerber-Shiu functions within our proposed refined Sobolev space, we introduce an explicit error bound, which seems to be absent from the literature. In contrast with our previous work (Chau et al., 2015), this error bound is more conservative without making heavy assumptions on the Fourier transform of the Gerber-Shiu function. The effectiveness of our result will be further demonstrated in the numerical studies.

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1. Introduction

Ever since its introduction in Gerber and Shiu (1998), Gerber-Shiu theory has motivated numerous research with diverse applications. The time of ruin, the surplus before ruin, and the deficit at ruin are the three quantities that have received enormous attention in actuarial science. Incorporating these three values into a single function, Gerber-Shiu function is used to assess the effect and the impact to an insurance company at the time of bankruptcy. Over the decades, the importance of Gerber-Shiu risk theory has advanced rapidly, attracting a massive number of research. Most of the existing literature has focused on exploring the explicit solutions of Gerber-Shiu functions under different models, see, for example, Gerber and Shiu (1998); Lin and Willmot (1999, 2000) and bi-annual international workshops on the latest development of the Gerber-Shiu theory.

However, there are some limits in the existing literature. Due to the inherent complexity of the Gerber-Shiu functions, explicit expression can only been found in a limited number of classical models with exponential or Erlangs–distributed claims. Finding an expression for the Gerber-Shiu functions often involves solving the defective renewal equation, see Landriault and Willmot (2008) and the references therein, or transforming the problem into the boundary value problem, see Albrecher et al. (2010) and the references therein. However, the calculation involved is often intractable. Another popular approach of evaluating the Gerber-Shiu functions is done via their Laplace transforms. The popularity of the Laplace-transformed approach stems from the fact that the Laplace transforms of the Gerber-Shiu functions admit much simpler forms, see, for example, Garrido and Morales (2006). Nevertheless, calculating the Gerber-Shiu functions via their Laplace transforms still involves some fundamental difficulties. The possibility of explicitly inverting the Laplace transform is normally rare for practical models, if not impossible. Moreover, even if explicit solution could be obtained, such expression is often tedious with possibly infinitely many terms that practically prohibits accurate computation (Lin and Willmot, 2000).

Therefore, finding an effective and accurate method to numerically approximate the Gerber-Shiu functions under general model is of utmost importance in the field of actuarial science.

Despite the importance of this topic, research on the efficient estimation of the Gerber-Shiu functions remains rare in the existing literature. One exception is the recent work of Pitts and Politis (2007) with the tools in functional analysis. They approximated the Gerber-Shiu functions for the general claim sizes by the non-linear transformations of the linear combinations of exponentially or Erlang-distributed claim size functions with their explicit Gerber-Shiu functions. Indeed, they regarded the Gerber-Shiu functions as functional of the claim size density functions in the classical model, and constructed a corresponding normed space U. They claimed that when the norm is small between two claim size functions, the functional operations of the Gerber-Shiu functions will be closed with respect to the functional norm in U. Therefore, this allowed them to construct the approximations for the general claim sizes based on the claim size distributions that is close in U and admit explicit Gerber-Shiu functions. However, the calculations in their method are still challenging and the rationale on choosing an appropriate class for approximation remains ad hoc.

In this paper, we take a drastically different approach on estimating the Gerber-Shiu functions. We shall further extend our previous work (Chau et al., 2015) and adopt the recently developed Fourier-cosine method to the Gerber-Shiu functions. Fourier-cosine method, first proposed by Fang and Oosterlee (2009), is a linear estimation method with an explicit error bound. It is proven to be effective and easy to implement. We propose two slightly different error bounds for the Fourier-cosine method by adopting two different derivation processes for the method. While this change does not affect the approximant, it does affect the two parts of the total error (ϵ_1 and ϵ_2 in (30) and (31) respectively) as shown in Section 4. The derivation in this paper is mostly adopted from the Fang and Oosterlee (2009), while the alternative has been covered extensively in our previous paper. Therefore, we will only list the key result for the alternative error bound in Section 4. Readers are refereed to Chau et al. (2015)

for further explanations and proofs. The error bound in our former work depends on the algebraic index of convergence of F_k, which is tighter but requires extra assumption on the Fourier transform of V, which will be defined in Section 2. The error bound in the present work depends on the Fourier transform of V satisfying certain integral condition. Although it provides with a weaker result but it is far more robust.

This paper is organized as follows. We first introduce the model and review the definition of the Gerber-Shiu functions in Section 2. We then explain the Fourier-cosine method in Section 3. In Section 4, we derive both the explicit error bound under the Fourier-cosine method and the mild technical condition under which our method is applicable. In the course of arguments, we shall apply some results from the modified version of the classical Sobolev space theory. We present some numerical examples in Section 5 that demonstrate the effectiveness and the convergence of error in our method. Finally, we summarize our findings in Section 6.

2. Review on the Gerber-Shiu Functions

2.1. Model Setting

Throughout the paper, let R_t be the surplus process of an insurance company.

$$R_t := u + ct - L_t, \tag{1}$$

where $u \geq 0$ is the initial reserve of the company. Premium rate charged by the company is denoted by $c > 0$, and L_t is a Lévy subordinator used to model the accumulation of claims. Define $L_0 = 0$ and its characteristic function is given by

$$\phi_{L_t}(\omega) = \mathbb{E}[\exp(i\omega L_t)] = \exp(ib\omega t + t\int_{(0,\infty)}(e^{i\omega x}-1)\nu(dx)) =: \exp(t\Lambda(\omega)). \tag{2}$$

Here, b is a fixed real number, ν is the Lévy measure on $(0,\infty)$, which means that it is a positive Borel measure with $\int_0^\infty (|x|^2 \wedge 1)\nu(dx) < \infty$ and $\Lambda(\omega)$ represents the characteristic component of L_t. For further details, one may check the relevant textbook such as Applebaum (2009).

Without any loss of generality, we assume that $b = 0$ throughout this article, which means that L_t is now a pure jump Lévy process with only positive jumps. The reason behind this assumption is that whenever we have a model with non-zero b, we can consider a new model with $c' = c - b$ and L'_t be the pure jump part of L_t and the two models will agree with each other. Here ν is also assumed to satisfy $\mu_1 := \int_{(0,\infty)} x\nu(dx) < \infty$. Moreover, the safety loading condition $c > \mu_1$ is assumed for avoiding almost sure ruin.

2.2. Gerber-Shiu Functions

Expected discounted penalty functions, otherwise known as the Gerber-Shiu functions, are used to study the distribution of surplus at time of ruin, surplus prior to ruin, as well as the time of ruin at a time. Gerber-Shiu function, denoted by φ, is defined as:

$$\varphi(u) := \mathbb{E}[e^{-\delta\tau}\kappa(R_{\tau-}, |R_\tau|)\mathbb{1}_{[0,\infty)}(\tau)|R_0 = u], \tag{3}$$

where $\tau := \inf\{t > 0 | R_t < 0\}$ is the time of bankruptcy, $R_{\tau-}$ is the surplus right before τ, $|R_\tau|$ is the deficit at the time of ruin and $\kappa(x, y)$ represents a non-negative penalty for the company that has bankrupted. Here $\mathbb{1}$ denotes an indicator function and δ is a given positive constant standing for interest rate.

When Gerber and Shiu (1998) introduced the Gerber-Shiu functions, they also showed that $\varphi(u)$ can be written as an infinite sum of convolutions under the classical ruin theory. Their result has been generalized to other risk processes, with the following infinite series representation as shown in Garrido and Morales (2006):

$$\varphi(u) = \sum_{k=0}^{\infty} h_1 * h_2^{*k}(u), \tag{4}$$

for some functions h_1 and h_2 depend on the surplus process and v^{*k} denotes the k–th order convolutions for a function v, i.e.

$$v^{*j}(x) = \int_0^x v^{*(j-1)}(x - y)v(y)dy, \text{ for } j \geq 1, \tag{5}$$

with $f * v^{*0} = f$ by convention. Under the model in (1), h_1 and h_2 are given by

$$h_1(x) = \frac{1}{c}\int_x^\infty \int_0^\infty e^{-\rho(z-x)}\kappa(z,y)\zeta(z+y)dydz, \quad x \geq 0, \tag{6}$$

$$h_2(x) = \frac{1}{c}\int_x^\infty e^{-\rho(y-x)}\zeta(y)dy, \quad x \geq 0, \tag{7}$$

where ζ is the density for the Lévy measure $\nu(dy) = \zeta(y)dy$, and the constant ρ is the non-negative solution of the equation in λ,

$$\delta - c\lambda + \Lambda(i\lambda) = 0. \tag{8}$$

For detailed derivation of this formula, readers are referred to Garrido and Morales (2006) and Gerber and Shiu (1998).

Now, consider

$$\begin{aligned} h_1'(x) &= -\frac{1}{c}\int_0^\infty \kappa(x,y)\zeta(x+y)dy + \frac{1}{c}\int_x^\infty \int_0^\infty \rho e^{-\rho(z-x)}\kappa(z,y)\zeta(z+y)dydz \\ &= \rho h_1(x) - h_3(x), \end{aligned} \tag{9}$$

where $h_3(x) := \frac{1}{c}\int_0^\infty \kappa(x,y)\zeta(x+y)dy$. Therefore, by using (9),

$$\begin{aligned} &\varphi(u) \\ = &\sum_{k=0}^\infty h_1 * h_2^{*k}(u) = h_1(u) + \sum_{k=1}^\infty \int_0^u h_1(u-y)h_2^{*k}(y)dy \\ = &h_1(0) + \int_0^u h_1'(x)dx + \sum_{k=1}^\infty \int_0^u \left(h_1(0)h_2^{*k}(x) + \int_0^x h_1'(x-y)h_2^{*k}(y)dy\right)dx \\ = &h_1(0) + \int_0^u (\rho h_1(x) - h_3(x))dx \\ &+ \sum_{k=1}^\infty \int_0^u \left(h_1(0)h_2^{*k}(x) + \rho\int_0^x h_1(x-y)h_2^{*k}(y)dy - \int_0^x h_3(x-y)h_2^{*k}(y)dy\right)dx \\ = &h_1(0) + \int_0^u (\rho h_1(x) - h_3(x))dx \\ &+ \int_0^u \sum_{k=1}^\infty \left(h_1(0)h_2^{*k}(x) + \rho\int_0^x h_1(x-y)h_2^{*k}(y)dy - \int_0^x h_3(x-y)h_2^{*k}(y)dy\right)dx \\ = &h_1(0) + \int_0^u V(x)dx, \end{aligned} \tag{10}$$

where $V(x) := h_1(0)\sum_{k=1}^\infty h_2^{*k}(x) + \rho\sum_{k=0}^\infty h_1 * h_2^{*k}(x) - \sum_{k=0}^\infty h_3 * h_2^{*k}(x)$. The interchange of summation and integration in the second last equality can be justified by Fubini's theorem because $\sum h_1(0)h_2^{*k}(x)$, $\sum \rho \int_0^x h_1(x-y)h_2^{*k}(y)dy$ and $\sum \int_0^x h_3(x-y)h_2^{*k}(y)dy$ are all monotone series. We shall assume that $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ in this paper.

3. Fourier-cosine Expansion Method

In this section, we derive an approximation for the Gerber-Shiu functions based on the Fourier-cosine series, which was first proposed in Fang and Oosterlee (2009). We shall consider the integral in (10) by replacing V with its Fourier-cosine series. We first provide a self-contained introduction of the Fourier-cosine method.

For any function g defined on $[0, \pi]$, there is a natural extension for transforming this function into an even function on $[-\pi, \pi]$. Define $\breve{g}$ as

$$\breve{g}(x) = \begin{cases} g(x), & x \geq 0 \\ g(-x), & x < 0 \end{cases}. \tag{11}$$

Every even function can be expressed as a Fourier-cosine series (Boyd, 2001); indeed

$$\breve{g}(x) = {\sum_{k=0}^{\infty}}' A_k \cos(kx), \tag{12}$$

where

$$A_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \breve{g}(x) \cos(kx) dx = \frac{2}{\pi} \int_0^{\pi} g(x) \cos(kx) dx. \tag{13}$$

The notation $\sum'$ denotes a summation with its first terms weighted by half. Since g is a part of $\breve{g}$, the expansion is also valid for g itself. Fourier-cosine series expansion for the function supported on $[0, a]$ can be obtained through a simple change of variable $y = \frac{x}{a}\pi$.

Next, we formulate the Gerber-Shiu functions in terms of the Fourier-cosine representation. The first step is to rewrite (10) into the following form:

$$\varphi(u) = h_1(0) + \int_0^a \mathbb{1}_{\{x \leq u\}} V(x) dx, \text{ for } a \geq u. \tag{14}$$

While V is defined on $[0, \infty)$ in Section 2, hereafter we restrict V as a function defined on $[0, a]$, where a is some fixed number greater than the initial capital u.

The method of determining an appropriate a will be deferred to Section 4. The reason for such restriction is that Fourier series expansion can only be applied to the functions over the finite domain. Yet, this restriction would not pose any error as the original integration is only up to $u < a$. Moreover, it will be shown later that the total estimation error of the Fourier-cosine method can be controlled by a judicious, if not immediate, choice of a.

Applying Fourier-cosine expansion on function $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$V(x) = \sum_{k=0}^{\infty}{}' A_k \cos(k\pi \frac{x}{a}), \tag{15}$$

where

$$A_k = \frac{2}{a} \int_0^a V(x) \cos(k\pi \frac{x}{a}) dx, \tag{16}$$

and substitute V in (14) by its Fourier-cosine expansion (15),

$$\varphi(u) = h_1(0) + \int_0^a \mathbb{1}_{\{x \le u\}} \sum_{k=0}^{\infty}{}' A_k \cos(k\pi \frac{s}{a}) ds. \tag{17}$$

Interchanging the order of integration and summation in (17), we get

$$\varphi(u) = h_1(0) + \sum_{k=0}^{\infty}{}' A_k \int_0^u \cos(k\pi \frac{x}{a}) dx = h_1(0) + \sum_{k=0}^{\infty}{}' A_k \chi_k(0, u), \tag{18}$$

where

$$\chi_k(c, d) := \begin{cases} [\sin(k\pi \frac{d}{a}) - \sin(k\pi \frac{c}{a})] \frac{a}{k\pi}, & k \neq 0 \\ d - c, & k = 0 \end{cases}, \tag{19}$$

which follows from basic calculus. Next, we truncate the series by only including the first N terms, and it arrives with

$$\varphi(u) \approx h_1(0) + \sum_{k=0}^{N-1}{}' A_k \chi_k(0, u). \tag{20}$$

Finally, we should note that each A_k can be rewritten as:

$$A_k = \frac{2}{a} \Re \left\{ \int_0^a V(x) e^{i \frac{k\pi x}{a}} dx \right\}. \tag{21}$$

We can compare the integral in (21) with the characteristic function of V.

$$\int_0^a V(x) e^{i \frac{k\pi x}{a}} dx \approx \int_0^{\infty} V(x) e^{i \frac{k\pi x}{a}} dx = \phi_V(\frac{k\pi}{a}), \tag{22}$$

where ϕ_V is the Fourier transform of V. Due to (22), one can adopt ϕ_V in place of the original integral in (21). Defining

$$F_k := \frac{2}{a}\Re\{\phi_V(\frac{k\pi}{a})\}, \tag{23}$$

and replacing every A_k by F_k, one can get the approximation:

$$\varphi(u) \approx h_1(0) + \sum_{k=0}^{N-1}{}' F_k \chi_k(0,u). \tag{24}$$

There are two key advantages with this Fourier-cosine approach. The first one is that instead of computing convolution directly, which is usually computationally complex, we here only need to have the Fourier transform of V to apply the Fourier-cosine method. Indeed, we shall examine the Fourier transform of V and demonstrate that it is easy to calculate.

We first consider $\phi_{h_2}(\omega)$,

$$\begin{aligned}
\phi_{h_2}(\omega) &= \frac{1}{c}\int_0^\infty e^{(i\omega+\rho)x}\int_x^\infty e^{-\rho y}\zeta(y)dydx \\
&= \frac{-1}{c(i\omega+\rho)}\int_0^\infty e^{-\rho y}\zeta(y)dy + \frac{1}{c(i\omega+\rho)}\int_0^\infty e^{(i\omega+\rho)x}e^{-\rho x}\zeta(x)dx \\
&= \frac{1}{c(i\omega+\rho)}\int_0^\infty (e^{i\omega x} - e^{-\rho x})\zeta(x)dx.
\end{aligned} \tag{25}$$

Remark 3.1. It is clear that $|e^{i\omega x} - e^{-\rho x}| \le |i\omega+\rho| x$ for $x \in [0,\infty)$,

$$|\phi_{h_2}(\omega)| \le \frac{1}{c}\int_0^\infty \left|\frac{e^{i\omega x} - e^{-\rho x}}{i\omega+\rho}\right|\zeta(x)dx \le \frac{1}{c}\int_0^\infty x\zeta(x)dx \le 1.$$

As a consequence, the following three formulae are well-defined. Using $\mathcal{F}$ to denote the Fourier transform,

$$\mathcal{F}\left(\sum_{k=1}^\infty h_2^{*k}\right)(\omega) = \sum_{k=1}^\infty \phi_{h_2}^k(\omega) = \frac{\phi_{h_2}(\omega)}{1-\phi_{h_2}(\omega)}, \tag{26}$$

$$\mathcal{F}\left(\sum_{k=0}^\infty h_1 * h_2^{*k}\right)(\omega) = \sum_{k=0}^\infty \phi_{h_1}(\omega)\phi_{h_2}^k(\omega) = \frac{\phi_{h_1}(\omega)}{1-\phi_{h_2}(\omega)}, \tag{27}$$

and $$\mathcal{F}\left(\sum_{k=0}^\infty h_3 * h_2^{*k}\right)(\omega) = \sum_{k=0}^\infty \phi_{h_3}(\omega)\phi_{h_2}^k(\omega) = \frac{\phi_{h_3}(\omega)}{1-\phi_{h_2}(\omega)}. \tag{28}$$

The Fourier transform of V follows as a linear combination of (26)-(28).

$$\phi_V(\omega) = \frac{h_1(0)\phi_{h_2}(\omega) + \rho\phi_{h_1}(\omega) - \phi_{h_3}(\omega)}{1 - \phi_{h_2}(\omega)}. \tag{29}$$

The second advantage of the Fourier-cosine method is that despite the fact that our derivation of the approximation formula is done via Fourier transform, our estimator only involves basic arithmetic operation, without any need of taking inverse Fourier transform, which is normally computationally demanding. Therefore, the numerical calculation is much faster, with the computational order of $O(N)$.

Remark 3.2. Comparing the derivations and arguments used here with that in Section 3 of our previous paper (Chau et al., 2015), there is a substantial difference. In this paper, we first truncate the infinite sum in (20), and then replace A_k by F_k. The order of the procedure is reversed in our former work. The differences between the two derivations will be further illustrated in Section 4.

4. Error Estimate

After identifying the form of the proposed approximation formula, we now establish that there is a reasonable bound for the error incurred in the Fourier-cosine approximation. Following from our derivation, the total error of the Fourier-cosine estimation consists of two parts:

1. The series truncation error for including only the first N terms:

$$\epsilon_1 = \left| \sum_{k=N}^{\infty} A_k \chi_k(0,u) \right| = \left| \sum_{k=N}^{\infty} \frac{aA_k}{k\pi} \sin(k\pi \frac{u}{a}) \right|, \tag{30}$$

2. The error in connection with replacing A_k by F_k in (24):

$$\epsilon_2 = \left| \frac{2}{a} \sum_{k=0}^{N-1}{}' \mathrm{Re} \left\{ \int_a^{\infty} e^{ik\pi\frac{x}{a}} V(x) dx \right\} \chi_k(0,u) \right|. \tag{31}$$

The total error ϵ is bounded by these two parts, i.e. $\epsilon \leq \epsilon_1 + \epsilon_2$. In this section, we shall consider the error bound for these two parts separately. For the error ϵ_1, we shall show that if the even extended function $\breve{V}$ belongs to a revised Sobolev space, ϵ_1 will converge with N. Also, we shall apply result from the previous paper (Chau et al., 2015) so that ϵ_2 is bounded by the upper tail integral of $|V|$. While some steps or techniques used is similar to Fang and Oosterlee (2009), the whole establishment of the error bound has been significantly modified in comparison with that in Section 4 in Fang and Oosterlee (2009) in order to cater for our current consideration.

4.1. Refined Sobolev Space

Define $\mathcal{H}^k$ as a subspace of L^2 space, so that for any $f \in \mathcal{H}^k$, the norm

$$||f||^2_{\mathcal{H}^k} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_f(\omega)|^2 (1 + |\omega|^2)^k d\omega < \infty.$$

Also define an inner product $< f, g >_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_f(\omega)\overline{\phi_g(\omega)}(1 + |\omega|^2)^k d\omega$ on $\mathcal{H}^k$. In fact, this norm $|| \cdot ||_{\mathcal{H}^k}$ has been used to define an extended Sobolev space, and the Hilbert space $\mathcal{H}^k$ boils down to the classical Sobolev space when k is a positive integer; see Rosenberg (1997) for more details.

We shall establish that when $\breve{V} \in \mathcal{H}^k$, ϵ_1 can be shown to be bounded by a suitable function of N. We shall apply the celebrated Sobolev Embedding Theorem to derive an error bound for ϵ_1; however, the derivation of this embedding theorem needs to be modified as the arguments we needed seem to be far from immediate in the existing literature. In this subsection, we provide some basic properties for $\mathcal{H}^k$. To begin with, we establish some results on the differentiability of functions in $\mathcal{H}^k$. Define $C_c^{\infty}(\mathbb{R})$ as the set of all smooth test functions defined on compact domain in $\mathbb{R}$.

Lemma 4.1. *Let $g \in \mathcal{H}^1$, there exists a weak derivative $g'(x)$ such that $\int_{-\infty}^{\infty} g(x)\varphi'(x)dx = -\int_{-\infty}^{\infty} g'(x)\varphi(x)dx$ for any $\varphi(x) \in C_c^{\infty}(\mathbb{R})$.*

Proof. By the Plancherel theorem , there exists $\Phi_M(x) := \frac{1}{2\pi} \int_{-M}^{M} \phi_g(\omega)e^{-i\omega x}d\omega$

such that $||\Phi_M - g||_2 \to 0$ as $M \to \infty$. Next,

$$\frac{\Phi_M(x+h) - \Phi_M(x)}{h} = \frac{1}{2\pi}\int_{-M}^{M} \phi_g(\omega)e^{-ix\omega}\frac{e^{-ih\omega}-1}{h}d\omega. \tag{32}$$

Since $\lim_{h\to 0} \phi_g(\omega)e^{-ix\omega}\frac{e^{ih\omega}-1}{h} = -i\omega\phi_g(\omega)e^{-ix\omega}$ and $\left|\phi_g(\omega)e^{-ix\omega}\frac{e^{-ih\omega}-1}{h}\right| \leq$ $|\omega\phi_g(\omega)|$ for all ω and h in $\mathbb{R}$, for any fixed $M > 0$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-M}^{M}|\omega\phi_g(\omega)|d\omega &\leq \sqrt{\frac{M}{\pi}}\left(\int_{-M}^{M}|\omega\phi_g(\omega)|^2 d\omega\right)^{\frac{1}{2}} \\ &< \sqrt{\frac{M}{\pi}}\left(\int_{-\infty}^{\infty}(1+|\omega|^2)|\phi_g(\omega)|^2 d\omega\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

The first inequality follows from Hölder's inequality. Then by Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{h\to 0}\frac{\Phi_M(x+h)-\Phi_M(x)}{h} &= \lim_{h\to 0}\frac{1}{2\pi}\int_{-M}^{M}\phi_g(\omega)e^{-ix\omega}\frac{e^{-ih\omega}-1}{h}d\omega \\ &= -\frac{i}{2\pi}\int_{-M}^{M}\omega\phi_g(\omega)e^{-ix\omega}d\omega.\end{aligned}$$

Define $\Phi'_M(x) := -\frac{i}{2\pi}\int_{-M}^{M}\omega\phi_g(\omega)e^{-ix\omega}d\omega$. As $\Phi'_M(x)$ is the classical derivative of $\Phi_M(x)$,

$$\int_{-\infty}^{\infty}\Phi'_M(x)\varphi(x)dx = -\int_{-\infty}^{\infty}\Phi_M(x)\varphi'(x)dx, \tag{33}$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. Since $g \in \mathcal{H}^1$ and $-i\omega\phi_g(\omega) \in L^2$,

$$\tilde{g}^{(1)}(x) := -\frac{i}{2\pi}\int_{-\infty}^{\infty}\omega\phi_g(\omega)e^{ix\omega}dx$$

is well defined by the Plancherel theorem, and Φ'_M converges to $\tilde{g}^{(1)}$ in L^2 sense when M goes to infinity. Also note that $\tilde{g}^{(1)} \in L^2(\mathbb{R})$. What remains is to show that $\tilde{g}^{(1)}$ is the weak derivative of g. Consider the following,

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\tilde{g}^{(1)}(x)\varphi(x)dx - \int_{-\infty}^{\infty}\Phi'_M(x)\varphi(x)dx\right| &= \left|\int_{-\infty}^{\infty}(\tilde{g}^{(1)}(x)-\Phi'_M(x))\varphi(x)dx\right| \\ &\leq \int_{-\infty}^{\infty}|\tilde{g}^{(1)}(x)-\Phi'_M(x)||\varphi(x)|dx \\ &\leq ||\tilde{g}^{(1)}-\Phi'_M||_2\left(\int_{-\infty}^{\infty}|\varphi(x)|^2dx\right)^{\frac{1}{2}}\end{aligned}$$

from the Cauchy-Schwarz inequality. Note that the L^2 norm of any test function must be finite and bounded by some constant C. So

$$\left|\int_{-\infty}^{\infty} \tilde{g}^{(1)}(x)\varphi(x)dx - \int_{-\infty}^{\infty} \Phi'_M(x)\varphi(x)dx\right| \leq C||\tilde{g}^{(1)} - \Phi'_M||_2 \to 0, \text{ as } M \to \infty.$$

Therefore, $\lim_{M\to\infty} \int_{-\infty}^{\infty} \Phi'_M(x)\varphi(x)dx = \int_{-\infty}^{\infty} \tilde{g}^{(1)}(x)\varphi(x)dx$. Similarly, we have

$$\left|\int_{-\infty}^{\infty} g(x)\varphi'(x)dx - \int_{-\infty}^{\infty} \Phi_M\varphi'(x)dx\right| \to 0 \text{ as } M \to \infty.$$

Finally,

$$\begin{aligned}
\int_{-\infty}^{\infty} g'(x)\varphi(x)dx &= \lim_{M\to\infty} \int_{-\infty}^{\infty} \Phi'_M(x)\varphi(x)dx \\
&= -\lim_{M\to\infty} \int_{-\infty}^{\infty} \Phi_M\varphi'(x)dx = -\int_{-\infty}^{\infty} g(x)\varphi'(x)dx.
\end{aligned}$$

This deduces our claim and $g' = \tilde{g}^{(1)}$. □

Given that $g \in \mathcal{H}^1$, we have

$$\begin{aligned}
\phi_{g'}(\omega) &= \int_{-\infty}^{\infty} g'(x)e^{i\omega x}dx \\
&= \lim_{M\to\infty} \int_{-\infty}^{\infty} \Phi'_M(x)e^{i\omega x}dx \\
&= \lim_{M\to\infty} \left(\Phi_M(x)e^{i\omega x}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Phi_M(x)i\omega e^{i\omega x}dx \right) \\
&= -\lim_{M\to\infty} \int_{-\infty}^{\infty} \Phi_M(x)i\omega e^{i\omega x}dx \\
&= -\int_{-\infty}^{\infty} g(x)i\omega e^{i\omega x}dx \\
&= -i\omega\phi_g(\omega). \qquad (34)
\end{aligned}$$

The limit convergences in the second and fourth equalities are justified by the Plancherel theorem and the first term in the third equality vanishes because of Riemann-Lebesgue Lemma. Moreover, since $|\omega|^{2(s-1)}|(1+|\omega|^2) \leq (1+|\omega|^2)^s \leq (1+|\omega|^2)^k$ for any integer s and real number $k \geq s$, the following corollary can be proven by applying an induction on s.

Corollary 4.2. *If $g \in \mathcal{H}^k$, then g is weakly differentiable up to order $s \leq k$.*

As a result, $\breve{V} \in \mathcal{H}^k$ implies that $\breve{V}$ is weakly differentiable up to order $\lfloor k \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number. This result is consistence with identifying $\mathcal{H}^k$ as an extended Sobolev space. Next, we find a dense subset of $\mathcal{H}^k$. In particular, we consider the set of all Schwartz functions.

Definition 4.3. A Schwartz function is a smooth function $f \in C^\infty(\mathbb{R})$ so that

$$\sup_{x\in\mathbb{R}} |x^\beta f^{(\alpha)}(x)| < \infty \tag{35}$$

for all integer α and β. Intuitively, it means that f and all its derivatives decrease rapidly when x goes to infinity. Define the linear space of all Schwartz functions as $S(\mathbb{R})$.

It is well-known that $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, and Fourier transform is an automorphism for $S(\mathbb{R})$. We shall show that it is also dense in $\mathcal{H}^k$, see Stein and Shakarchi (2011) for more details.

Lemma 4.4. *$S(\mathbb{R})$ is dense in $\mathcal{H}^k$.*

Proof. It is clear that $S(\mathbb{R})$ is a subset of $\mathcal{H}^k$; indeed, let N be an integer strictly greater than $k + \frac{1}{2}$,

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\infty}^{\infty} |\phi_g(\omega)|^2(1+|\omega|^2)^k d\omega \\ \leq\ & \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\sup_{\omega\in\mathbb{R}}(1+|\omega|)^N|\phi_g(\omega)|\right)^2 \frac{(1+|\omega|^2)^k}{(1+|\omega|)^{2N}} d\omega \\ =\ & \left(\sup_{\omega\in\mathbb{R}}(1+|\omega|)^N|\phi_g(\omega)|\right)^2 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1+|\omega|^2)^k}{(1+|\omega|)^{2N}} d\omega < \infty. \end{aligned} \tag{36}$$

The first term in the last equality is bounded because of the definition of Schwartz functions and ϕ_g is a Schwartz function. The integration term is bounded because $2k - 2N < -1$. Therefore, $S(\mathbb{R}) \subset \mathcal{H}^k$.

The density result can be derived through the inner product of $\mathcal{H}^k$. We assume that $u \in \mathcal{H}^k$ and $< u, g >_k = 0$ for all $g \in S(\mathbb{R})$. It means that

$$\int_{-\infty}^{\infty} \phi_u(\omega)\overline{\phi_g(\omega)}(1+|\omega|^2)^k d\omega = 0,$$

for all $g \in S(\mathbb{R})$ and this implies that $\phi_u(\omega)(1+|\omega|^2)^k$, and hence $\phi_u(\omega)$ both vanish almost everywhere on $\mathbb{R}$. Since Fourier transform is an isometry of $L^2(\mathbb{R})$, we have $||u||_2 = ||\phi_u||_2 = 0$. This shows that $u = 0$ almost everywhere, and therefore the orthogonal complement of $S(\mathbb{R})$ in $\mathcal{H}^k$ is just the trivial subspace. As a result, $S(\mathbb{R})$ is dense in $\mathcal{H}^k$.

4.2. Series Truncation Error

Following the idea developed in Fang and Oosterlee (2009), we shall derive the error bound for ϵ_1 by taking into account of the algebraic index of convergence of A_k. The algebraic index of convergence is defined by:

Definition 4.5. (Boyd (2001) Definition 2 in Section 2.3) A_k has an algebraic index of convergence of s if s is the greatest number such that

$$\limsup_{k\to\infty} |A_k| k^s < \infty. \tag{37}$$

This also implies that $|A_k| \sim O\left(\frac{1}{k^s}\right)$.

We aim at establishing that for any function $\breve{V} \in \mathcal{H}^k$, $\breve{V}$ has some algebraic index of convergence $\beta \in \mathbb{R}$. As it turns out, the algebraic index of convergence of a function is closely related to its smoothness. In this subsection, we first show some smoothness properties of the functions in the space $\mathcal{H}^k$. The first half of the proof in Theorem 4.6 is adopted from Theorem 1.20 of Rosenberg (1997), so that it is regarded as an extension of the latter theorem. Theorem 4.6 is also an alternative version of Sobolev Embedding Theorem.

Theorem 4.6. *$g \in \mathcal{H}^k \Rightarrow g \in C^{s,\gamma}(\mathbb{R})$ for*

$$\begin{cases} s = \lfloor k - \frac{1}{2} \rfloor & \gamma < k - s - \frac{1}{2} \quad \textit{when } k - \frac{1}{2} > 0 \textit{ is not an integer,} \\ s = k - \frac{1}{2} - 1 & 0 < \gamma < 1 \quad \textit{when } k - \frac{1}{2} > 0 \textit{ is an integer.} \end{cases}$$

Proof. For the sake of convenience, C will denote a general constant in the rest of this proof and its exact value may vary from line to line. Firstly, we show that $g \in C^0(\mathbb{R})$ when $g \in \mathcal{H}^k$ and $k > \frac{1}{2}$.

$$\begin{aligned}
|g(x)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\omega} \phi_g(\omega) d\omega \right| \\
&= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} e^{ix\omega} (1+|\omega|^2)^{-\frac{k}{2}} (1+|\omega|^2)^{\frac{k}{2}} \phi_g(\omega) d\omega \right| \\
&\leq \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} (1+|\omega|^2)^{-k} d\omega \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\phi_g(\omega)|^2 (1+|\omega|^2)^k d\omega \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{-\infty}^{\infty} |\phi_g(\omega)|^2 (1+|\omega|^2)^k d\omega \right)^{\frac{1}{2}}, \qquad (38)
\end{aligned}$$

implying that $||g||_\infty \leq C||g||_{\mathcal{H}^k}$. Since any $g \in \mathcal{H}^k$ can be written as a $\mathcal{H}^k$ limit of Schwartz functions due to Lemma 4.4, g is the uniform limit of continuous functions and is therefore continuous.

Next, for any $r \leq s$, let $D^r g$ denote the rth weak derivative of g,

$$\begin{aligned}
\|D^r g\|^2_{\mathcal{H}^{k-r}} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_{D^r g}|^2 (1+|\omega|)^{k-r} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (|\omega|^r |\phi_g(\omega)|)^2 (1+|\omega|^2)^{k-r} d\omega \\
&\leq C\|g\|^2_{\mathcal{H}^k} < \infty,
\end{aligned}$$

since $|\omega|^{2r}(1+|\omega|^2)^{k-r} \leq C(1+|\omega|^2)^k$. This implies that $D^r : \mathcal{H}^k \to \mathcal{H}^{k-r}$ is continuous. By the first part of the proof, we conclude that $D^r g \in C^0$ for $r \leq s$ and $g \in C^s$. Also, we have $\|D^s g\|^2_{\mathcal{H}^{k-s}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_{D^s g}|^2 (1+|\omega|^2)^{k-s} d\omega < \infty$. Then, for $x \neq y$,

$$\begin{aligned}
\frac{|D^s g(x) - D^s g(y)|}{|x-y|^\gamma} &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \phi_{D^s g}(\omega) \frac{e^{-ix\omega} - e^{-iy\omega}}{|x-y|^\gamma} d\omega \right| \\
&= \frac{1}{\pi} \left| \int_{-\infty}^{\infty} \phi_{D^s g}(\omega) \xi \frac{\sin(\frac{x-y}{2}\omega)}{|x-y|^\gamma} d\omega \right|,
\end{aligned}$$

where ξ is a unit complex number. The second equality follows from the sum to product identity. There are two possible cases.

Case 1: For $r = |x-y| \leq 1$,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} \phi_{D^s g}(\omega)\xi \frac{\sin(\frac{x-y}{2}\omega)}{|x-y|^\gamma} d\omega\right| \\
\leq\ & \int_{-\infty}^{\infty} |\phi_{D^s g}(\omega)| \frac{|\sin(\frac{x-y}{2}\omega)|}{|x-y|^\gamma} d\omega \\
\leq\ & \int_{-\infty}^{\infty} |\phi_{D^s g}(\omega)| \frac{1}{2^\beta} \frac{(|x-y||\omega|)^\beta}{|x-y|^\gamma} d\omega \quad \text{for } \gamma \leq \beta < 1 \\
\leq\ & \int_{-\infty}^{\infty} |\phi_{D^s g}(\omega)|(1+|\omega|^2)^{\frac{\beta}{2}} d\omega \\
=\ & \int_{-\infty}^{\infty} |\phi_{D^s g}(\omega)|(1+|\omega|^2)^{\frac{\beta-(k-s)}{2}}(1+|\omega|^2)^{\frac{k-s}{2}} d\omega \\
\leq\ & \left(\int_{-\infty}^{\infty} |\phi_{D^s g}|^2(1+|\omega|^2)^{(k-s)} d\omega\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{d\omega}{(1+|\omega|^2)^{(k-s)-\beta}}\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

for $\gamma \leq \beta < k - s - \frac{1}{2} \leq 1$. The second inequality follows from the fact that for any fixed $\beta < 1$, $|\omega|^\beta - |\sin(\omega)| \geq 0$ for all real number ω; indeed, it is obviously true for $|\omega| \geq 1$ and for $\omega \in (-1, 1)$, $|\omega|^\beta \geq |\omega| \geq |\sin\omega|$. Note that $|\omega|^\beta - |\sin\omega|$ is also an even function. Therefore, for any given x, y, $|\frac{x-y}{2}\omega|^\beta - |\sin(\frac{x-y}{2}\omega)|$ still holds.

Case 2: For $r = |x-y| > 1$,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} \phi_{D^s g}(\omega)\xi \frac{\sin(\frac{x-y}{2}\omega)}{|x-y|^\gamma} d\omega\right| \\
\leq\ & \int_{-\infty}^{\infty} |\phi_{D^s g}| d\omega \\
=\ & \int_{-\infty}^{\infty} |\phi_{D^s g}|(1+|\omega|^2)^{\frac{k-s}{2}}(1+|\omega|^2)^{-\frac{k-s}{2}} d\omega \\
\leq\ & \left(\int_{-\infty}^{\infty} |\phi_{D^s g}|^2(1+|\omega|^2)^{k-s} d\omega\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{d\omega}{(1+|\omega|^2)^{k-s}}\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Note that $k - s > \frac{1}{2}$.

In these two cases, the bounds we derived are independent of x and y. So we can fix β in Case 1 according to our condition and pick the maximum of these two bounds. This finite number will be greater than $\dfrac{|D^s g(x) - D^s g(y)|}{|x-y|^\gamma}$ for all $x \neq y$. It means that $\sup\limits_{x\neq y} \dfrac{|D^s g(x) - D^s g(y)|}{|x-y|^\gamma}$ is finite and our claim follows.

Knowing that $\breve{V} \in \mathcal{H}^k$ implies $\breve{V} \in C^{s,\gamma}(\mathbb{R})$, we shall derive a bound for the error ϵ_1 based on the smoothness of $\breve{V}$. Parts of the calculation below, namely applying integration by parts, is adopted from Boyd (2001). We shall show that A_k converges with respect to k and the algebraic index of convergence is related to s and γ, and we shall also discuss the cases with various values of s step by step.

i.) For $s = 0$,

$$A_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \breve{V}(x) \cos(kx) dx.$$

Since cosine is a periodic function and $\cos(x + \pi) = -\cos(x)$, for any interval $\left[\frac{l}{k}\pi, \frac{l+2}{k}\pi\right] \subset [-\pi, \pi]$, where $l = -k, -k+2, -k+4, \ldots, k-2$, we have

$$\begin{aligned}
& \int_{\frac{l}{k}\pi}^{\frac{l+2}{k}\pi} \breve{V}(x) \cos(kx) dx \\
= & \int_{\frac{l+1}{k}\pi}^{\frac{l+2}{k}\pi} \breve{V}(x) \cos(kx) dx + \int_{\frac{l}{k}\pi}^{\frac{l+1}{k}\pi} \breve{V}(x) \cos(kx) dx \\
= & \int_{\frac{l}{k}\pi}^{\frac{l+1}{k}\pi} \breve{V}\left(x + \frac{\pi}{k}\right) \cos(kx + \pi) dx + \int_{\frac{l}{k}\pi}^{\frac{l+1}{k}\pi} \breve{V}(x) \cos(kx) dx \\
= & \int_{\frac{l}{k}\pi}^{\frac{l+1}{k}\pi} \left(\breve{V}(x) - \breve{V}\left(x + \frac{\pi}{k}\right)\right) \cos(kx) dx.
\end{aligned}$$

Now consider the integration over the whole interval $[-\pi, \pi]$,

$$\begin{aligned}
|A_k| &= \frac{1}{\pi} \left| \int_{-\pi}^{\pi} \breve{V}(x) \cos(kx) dx \right| \\
&= \frac{1}{\pi} \left| \int_I \left(\breve{V}(x) - \breve{V}^{(s)}\left(x + \frac{\pi}{k}\right)\right) \cos(kx) dx \right| \\
&\leq \frac{1}{\pi} \int_I \left| \breve{V}(x) - \breve{V}\left(x + \frac{\pi}{k}\right) \right| dx \\
&\leq C \int_I \left(\frac{\pi}{k}\right)^{\gamma} dx \\
&\leq \frac{C}{k^{\gamma}} \sim O\left(\frac{1}{k^{\gamma}}\right),
\end{aligned}$$

where $I = \bigcup_l \left[\frac{l}{k}\pi, \frac{l+1}{k}\pi\right]$ with the union taking over $l = -k, -k + 2, -k+4, \ldots, k-2$.

ii.) When $s = 1$, we apply integration by parts and derive

$$\begin{aligned} A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \breve{V}(x)\cos(kx)dx \\ &= \left.\frac{\breve{V}(x)\sin(kx)}{k\pi}\right|_{-\pi}^{\pi} - \frac{1}{k\pi}\int_{-\pi}^{\pi} \breve{V}'(x)\sin(kx)dx \\ &= -\frac{1}{k\pi}\int_{-\pi}^{\pi} \breve{V}'(x)\sin(kx)dx \sim O\left(\frac{1}{k^{1+\gamma}}\right). \end{aligned}$$

The expression on the right hand side is of order $O\left(\frac{1}{k^{1+\gamma}}\right)$ following from the similar reasoning as in case i.

iii.) For $s = 2$,

$$\begin{aligned} A_k &= -\frac{1}{k\pi}\int_{-\pi}^{\pi} \breve{V}'(x)\sin(kx)dx \\ &= \left.\frac{\breve{V}'(x)\cos(kx)}{k^2\pi}\right|_{-\pi}^{\pi} - \frac{1}{k^2\pi}\int_{-\pi}^{\pi} \breve{V}''(x)\cos(kx)dx \\ &= \frac{\breve{V}'(\pi)\cos(k\pi)}{k^2\pi} - \frac{\breve{V}'(-\pi)\cos(-k\pi)}{k^2\pi} - \frac{1}{k^2\pi}\int_{-\pi}^{\pi} \breve{V}''(x)\cos(kx)dx. \end{aligned}$$

Since $\breve{V}$ is even, $\breve{V}'$ is odd and $\cos(k\pi) = \cos(-k\pi)$, we have

$$A_k = 2\frac{\breve{V}'(\pi)\cos(k\pi)}{k^2\pi} + O(\frac{1}{k^{2+\gamma}}).$$

iv.) For $s \geq 3$,

$$\begin{aligned} A_k &= 2\frac{\breve{V}'(\pi)\cos(k\pi)}{k^2\pi} - \frac{1}{k^2\pi}\int_{-\pi}^{\pi} \breve{V}''(x)\cos(kx)dx \\ &= 2\frac{\breve{V}'(\pi)\cos(k\pi)}{k^2\pi} - \left.\frac{\breve{V}''(x)\sin(kx)}{k^3\pi}\right|_{-\pi}^{\pi} + \frac{1}{k^3\pi}\int_{-\pi}^{\pi} \breve{V}^{(3)}(x)\sin(kx)dx \\ &= 2\frac{\breve{V}'(\pi)\cos(k\pi)}{k^2\pi} + O\left(\frac{1}{k^3}\right). \end{aligned}$$

We next use the convergence nature of A_k to derive a bound for the error ϵ_1. Again, we proceed on case by case. For the fixed number $a \in \mathbb{R}_+$, we have the following.

a.) For $s = 0, 1$,

$$\frac{aA_k}{k\pi}\sin\left(k\pi\frac{u}{a}\right) \sim O\left(\frac{1}{k^{s+\gamma+1}}\right).$$

As

$$\left|\sum_{k=N}^{\infty}\frac{aA_k}{k\pi}\sin\left(k\pi\frac{u}{a}\right)\right| \le \sum_{k=N}^{\infty}\left|\frac{aA_k}{k\pi}\sin\left(k\pi\frac{u}{a}\right)\right|,$$

we have

$$\epsilon_1 \le \sum_{k=N}^{\infty}\left|\frac{aA_k}{k\pi}\sin\left(k\pi\frac{u}{a}\right)\right| \le \sum_{k=N}^{\infty}\frac{P_a}{k^{s+\gamma+1}} \le \frac{\bar{P}_a}{(N-1)^{s+\gamma}}.$$

The last inequality comes from approximating the summation as for usual Maclaurin-Cauchy test. Here $\bar{P}_a$ is a constant related to a and increases with a.

b.) For $s \ge 2$, we have

$$\begin{aligned}\left|\sum_{k=N}^{\infty}\frac{aA_k}{k\pi}\sin(k\pi\frac{u}{a})\right| &\le \left|\sum_{k=N}^{\infty}P_a(1)\frac{\cos(k\pi)\sin\left(k\pi\frac{u}{a}\right)}{k^3}+\sum_{k=N}^{\infty}O\left(\frac{1}{k^{r+1}}\right)\right| \\ &\le P_a(1)\left\{\left|\sum_{k=N}^{\infty}\frac{\sin\left[k\pi\left(\frac{u}{a}+1\right)\right]}{k^3}\right|\right. \\ &\qquad \left.+\left|\sum_{k=N}^{\infty}\frac{\sin\left[k\pi\left(\frac{u}{a}-1\right)\right]}{k^3}\right|\right\}+\frac{P_a(2)}{(N-1)^r},\end{aligned}$$

where $P_a(1)$ and $P_a(2)$ are constant independent of N but may change value from line to line and $r = \min\{s+r, 3\}$. Note that for $u = a$, $\cos(k\pi)\sin(k\pi) = 0$, for all $k \in \mathbb{N}\cup\{0\}$. In this case, the summation is bounded above by $\frac{P_a(2)}{(N-1)^r}$. Otherwise, for $x \in (-\pi, 2\pi)\backslash\{0\}$, we have

$$
\begin{aligned}
& \sum_{k=N}^{\infty} \frac{\sin(kx)}{k^3} \\
= \;& \sum_{k=N}^{\infty} \left\{ \left[\sum_{l=k}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \right] \sin(kx) \right\} \\
= \;& \sum_{l=N}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \left(\sum_{k=N}^{l} \sin(kx) \right) \\
= \;& \frac{1}{2\sin\frac{x}{2}} \sum_{l=N}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \left(\cos\left[(N - \frac{1}{2})x\right] - \cos\left[(l + \frac{1}{2})x\right] \right) \\
= \;& \frac{\cos\left[(N-\frac{1}{2})x\right]}{2\sin\frac{x}{2}} \left[\sum_{l=N}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \right] \\
& - \frac{1}{2\sin\frac{x}{2}} \sum_{l=N}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \cos\left[(l + \frac{1}{2})x\right] \\
= \;& \frac{\cos\left[(N-\frac{1}{2})x\right]}{2N^3\sin\frac{x}{2}} - \frac{1}{2\sin\frac{x}{2}} \sum_{l=N}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \cos\left[(l + \frac{1}{2})x\right]. \qquad (39)
\end{aligned}
$$

The above derivation is well-defined since $\frac{1}{k^3} \to 0$ as $k \to \infty$, and we can rewrite $\frac{1}{k^3}$ as $\sum_{l=k}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right)$. Taking absolute value on both sides of (39),

$$
\begin{aligned}
& \left| \sum_{k=N}^{\infty} \frac{\sin(kx)}{k^3} \right| \\
\leq \;& \left| \frac{\cos\left[(N-\frac{1}{2})x\right]}{2N^3\sin\frac{x}{2}} \right| + \frac{1}{2|\sin\frac{x}{2}|} \sum_{l=N}^{\infty} \left| \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \cos\left[(l + \frac{1}{2})x\right] \right| \\
\leq \;& \frac{1}{2N^3|\sin\frac{x}{2}|} + \frac{1}{2|\sin\frac{x}{2}|} \sum_{l=N}^{\infty} \left(\frac{1}{l^3} - \frac{1}{(l+1)^3} \right) \\
= \;& \frac{1}{N^3|\sin\frac{x}{2}|}. \qquad (40)
\end{aligned}
$$

Finally,

$$
\left| \sum_{k=N}^{\infty} \frac{aA_k}{k\pi} \sin(k\pi\frac{u}{a}) \right| \leq \frac{P_a(1)}{N^3} + \frac{P_a(2)}{(N-1)^r} \sim O\left(\frac{1}{(N-1)^r} \right). \qquad (41)
$$

We can summarize the result obtained as the next theorem.

Theorem 4.7. *If $\breve{V} \in C^{s,\gamma}(\mathbb{R})$, then $\epsilon_1 \sim O\left(\frac{1}{N^r}\right)$ where $r = \min\{s+\gamma, 3\}$.*

Since the Fourier transform of $\check{V}$ is

$$\begin{aligned}\phi_{\check{V}}(\omega) &= \int_{-\infty}^{0} e^{i\omega x} V(-x)dx + \int_{0}^{\infty} e^{i\omega x} V(x)dx \\ &= \phi_V(-\omega) + \phi_V(\omega) \\ &= 2\Re(\phi_V(\omega)). \end{aligned} \tag{42}$$

We can combine Theorem 4.6 and Theorem 4.7.

Theorem 4.8. *For any $V \in L^2(\mathbb{R})$, and $k > \frac{1}{2}$ such that*

$$\frac{2}{\pi}\int_{-\infty}^{\infty} (\Re(\phi_V(\omega)))^2 (1 + |\omega|^2)^k d\omega < \infty, \tag{43}$$

we have $\epsilon_1 \leq \frac{\bar{P}_a}{(N-1)^r}$ for some constant $\bar{P}_a$, where $r = \min\{s + \gamma, 3\}$. s and γ are defined as in Theorem 4.6.

4.3. Approximating Error for replacing A_k by F_k

Next, we show that ϵ_2 is bounded by an upper tail integration of $|V|$. Since V is a real-valued function in our setting, we have

$$\begin{aligned} & \left| \frac{2}{a} \sum_{k=0}^{N-1}{}' \chi_k(0,u) \int_a^{\infty} \cos(k\pi \frac{x}{a}) V(x) dx \right| \\ = & \left| \frac{2}{a} \int_a^{\infty} \sum_{k=0}^{N-1}{}' \chi_k(0,u) \cos(k\pi \frac{x}{a}) V(x) dx \right| \\ \leq & \int_a^{\infty} \left| \frac{2}{a} \sum_{k=0}^{N-1}{}' \chi_k(0,u) \cos(k\pi \frac{x}{a}) \right| |V(x)|\, dx. \end{aligned}$$

We shall achieve our desired result if $\left| \frac{2}{a} \sum_{k=0}^{N-1}{}' \chi_k(0,u) \cos(k\pi \frac{x}{a}) \right|$ is bounded uniformly for all x and such a bound is also independent of N. This result has been established in our previous paper (Chau et al., 2015).

Proposition 4.9. $\left| \frac{2}{a} \sum_{k=0}^{n}{}' \chi_k(0,u) \cos(k\pi \frac{x}{a}) \right| \leq 1 + \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt$*, which holds independent of x, a and n.*

With the assumption that V is a L^1 function, ϵ_2 will converge to zero for large enough a. As a result, given that V satisfies the condition in Theorem 4.8, the total error of Fourier-cosine approximation is bounded by

$$\epsilon \leq \frac{\bar{P}_a}{(N-1)^r} + \left(1 + \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t} dt\right) \int_a^\infty |V(x)| dx. \tag{44}$$

The second term on the right hand side can be made as small as possible by increasing the value of a and is independent of N; while the first term depends on both a and N. It increases with a but decreases with N. When applying our approximation, one should first pick a large enough value of a to control ϵ_2 and then pick a N to control ϵ_1. In practice, ϵ_2 diminishes fast when a just modestly increases for commonly-used models. As a folklore, one should pick an as small as possible a from the acceptable range of ϵ_2 so that an accurate result can be obtained by a modest size of N.

In our previous work (Chau et al., 2015), we assume extra structure on the Fourier transform of the density V, namely the algebraic index of convergence of the Fourier transform of V. This assumption is stronger in the sense that it requires the Fourier transform of V converges to zero at a certain rate, whereas we only assume its overall integrability in the present article. As a result, the error bound is tighter in our previous paper and can cover models with slower convergent properties, see, for instance, Example 5.4 in the next section.

Theorem 4.10. *For the total error of applying Fourier-cosine method, we have:*

1. *When the real part of the Fourier transform of V has an algebraic index of convergence of $\beta > 0$, the total error for applying Fourier-cosine method in approximating Gerber-Shiu function is*

$$\epsilon \leq \left(1 + \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t} dt\right) \int_a^\infty |V(x)| dx + \frac{\bar{C}}{(N-1)^\beta}, \tag{45}$$

for some constant $\bar{C} > 0$ that depends on a.

2. *For $u \in [\theta, a-\theta]$ and any $\theta > 0$, the real part of the Fourier transform of V, $\Re(\phi_V)$, satisfies:*
   (a) *$\Re(\phi_V)$ has algebraic index of convergence $\beta > 0$, so that $\Re(\phi_V) \to 0$ as $k \to \infty$.*

(b) *There exists a large enough X such that $\Re(\phi_V)(x)$ is monotonously increasing or decreasing for all $x \geq X$.*

Then the total error:

$$\epsilon \leq \left(1 + \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t} dt\right) \int_a^\infty |V(x)| dx + \frac{C_\theta}{N^{\beta+1}}, \tag{46}$$

for some constant C_θ (depending on both θ and a) and $N \geq \frac{aX}{\pi}$.

Proof. It can be analogously shown by following the argument as in Chau et al. (2015).

5. Numerical Studies

We now conduct some numerical studies of using Fourier-cosine method to common examples arising in risk theory. Note that the graph may be of different scale for demonstration purpose and a is set at 200 in all the following examples.

Example 5.1 (Compound Poisson-Exponential Claim Distribution). Let L_t be a compound Poisson process whose arrival intensity is 1.5 with exponentially distributed claim size with mean $\frac{10}{7}$. The Lévy measure for such process is $\nu(dx) = 1.5e^{0.7x}dx$. The premium rate is set as 3. The penalty function $\kappa(x, y) = y^3$ and the discounted factor $\delta = 0.04$. The explicit solution of this type of model is given in the original paper of Gerber and Shiu (1998). The numerical approximation result is shown in Figure 1a.

One can check that $\breve{V} \in H^{\frac{3}{2}-\eta}$ for some small positive constant η, so our error bound can be applied. Theorem 4.8 suggests that ϵ_1 will converge at a rate of $1 - \eta$ with respect to N. However, the numerical result, shown in Figure 1b, suggests that the convergence rate of error with respect to N is approximately 2.9031579842. Since the real part of the Fourier transform of V also has an algebraic index of convergence of 2 and is monotone for large enough x, Theorem 4.10 suggests that the error will converge with order $O(N^3)$, This result corresponds well with our numerical study.

Example 5.2 (Compound Poisson-Mixtures of Erlangs). Let L_t be a compound Poisson process whose arrival intensity is 1.1 with the claim size distribution density function being a mixture of Erlangs:

$$f(x) = \sum_{k=1}^{\tau} q_k \frac{\theta(\theta x)^{k-1} e^{-\theta x}}{(k-1)!}, \forall x \geq 0, \tag{47}$$

where $\{q_1, \ldots, q_\tau\}$ is a probability distribution and $\theta > 0$. The premium rate is set at 8. The penalty function $\kappa(x, y) = y$ and the discounted factor $\delta = 0.01$. For illustration, we take $\tau = 2$ and also set $\{q_1, q_2\} = \{0.05, 0.95\}$ and $\theta = 0.5$. Lin and Willmot (2000) have established an explicit expression for the Gerber-Shiu function in this case. Figure 2a displays the numerical result of Fourier-cosine approximation.

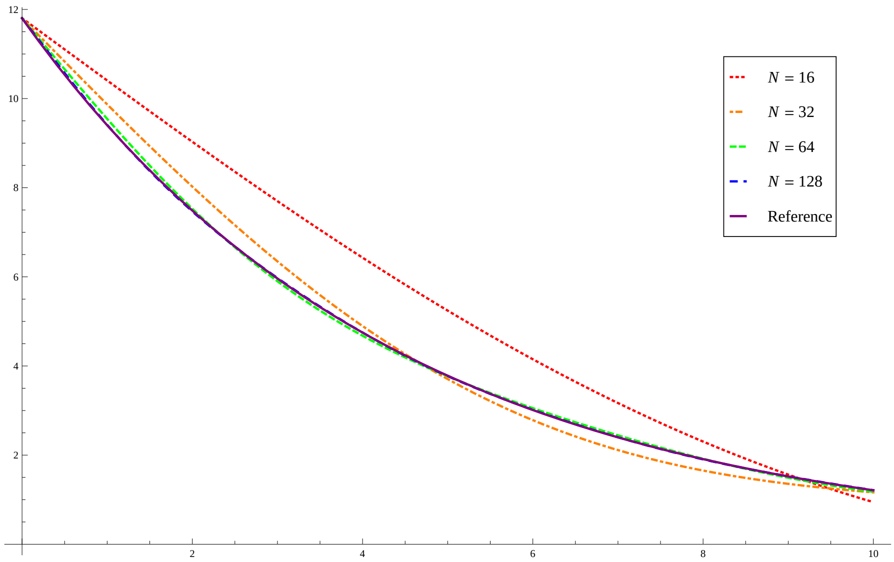

Figure 1a: Comparison of Fourier-cosine approximation with reference curve when a compound Poisson process with exponential claim size is the underlying model. (Example 5.1) Truncation range N is set to be $16, 32, 64$ and 128

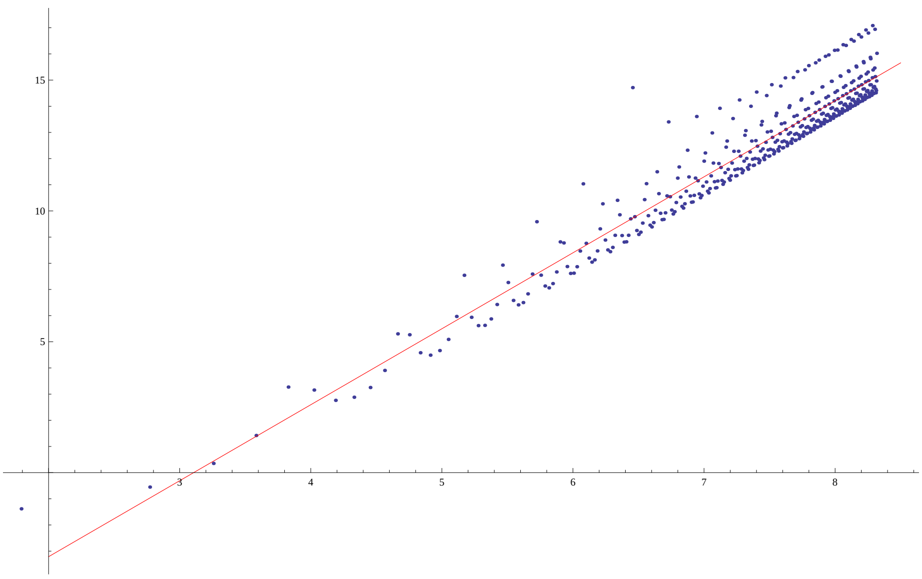

Figure 1b: The graph of $-\log(|\varphi_e(3, N) - \varphi(3)|)$ against $\log N$ in Example 5.1

Again, the $\check{V}$ of this example is within our refined Sobolev space of order $\frac{3}{2} - \eta$ for some small positive constant η. However the actual algebraic index of convergence with respect to N is approximately 2.5681198702 in Figure 2b. The order of convergence is higher than what Theorem 4.8 suggests, that is $1 - \eta$. However, the real part of the Fourier transform of V also has an algebraic index of convergence of 2 and is monotone for great enough x, the numerical result corresponds well with Theorem 4.10.

Example 5.3 (Compound Poisson-Gamma). Pitts and Politis (2007) have illustrated numerically a Gerber-Shiu function in the case that the claim process L_t is a compound Poisson model with Gamma $(\frac{3}{2}, \frac{3}{2})$ when the density function is

$$f(x) = \frac{3\sqrt{6x}e^{-\frac{3x}{2}}}{2\sqrt{\pi}}, \tag{48}$$

and penalty function is 1. The parameters used in their work are $\delta = 1$, $\lambda = 1$ and $c = 2$. Here we use our method to estimate the same function. The numerical result can be seen from Figure 3a.

The algebraic index of convergence of ϵ with respect to N is approximately 2.4815744159 in Figure 3b. Comparing with the fact that the $\check{V}$ of this Gerber-Shiu function is within the refined Sobolev space of order $\frac{3}{2} - \eta$ for some positive η, the actual result is higher than the theoretical result $(1 - \eta)$ in our proposed error bound in Theorem 4.8. Nevertheless, $\Re(\phi_V)$ has an algebraic index of convergence of 2 and is monotone when x is large. Theorem 4.10 suggests that the error will converge with order $O(N^3)$, which is in agreement with our numerical study.

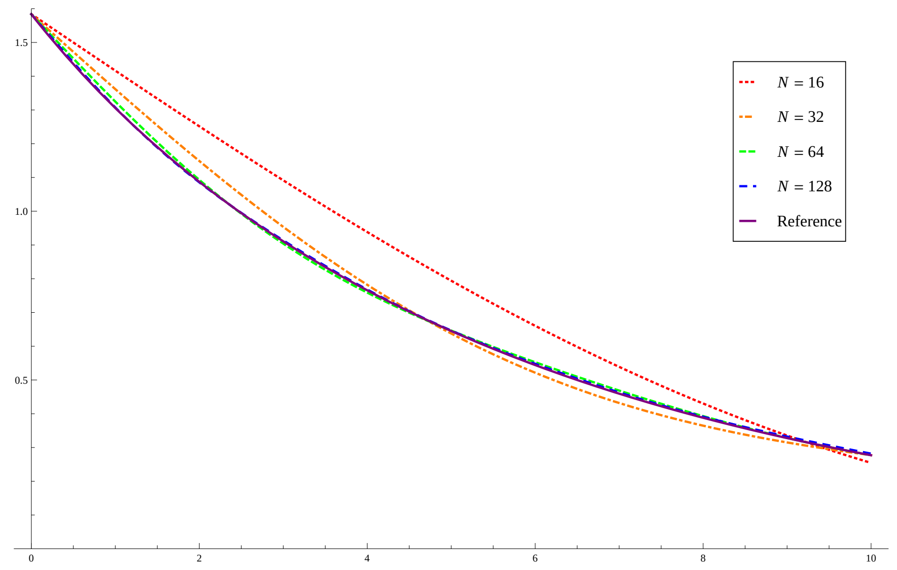

Figure 2a: Comparison of Fourier-cosine approximation with reference curve when a compound Poisson process with mixtures of Erlangs claim size is the underlying model. (Example 5.2) Truncation range N is set to be $16, 32, 64$ and 128

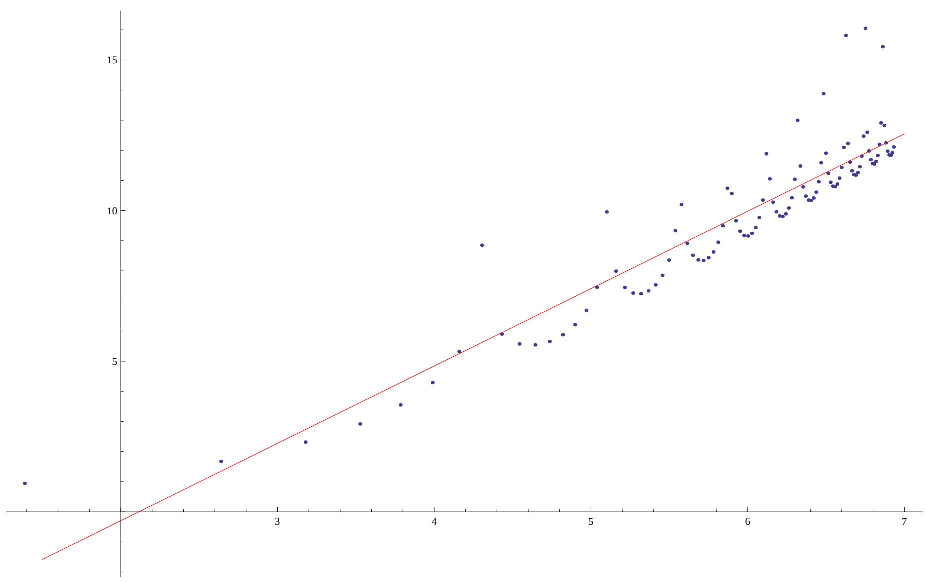

Figure 2b: The graph of $-\log(|\varphi_e(2, N) - \varphi(2)|)$ against $\log N$ in Example 5.2

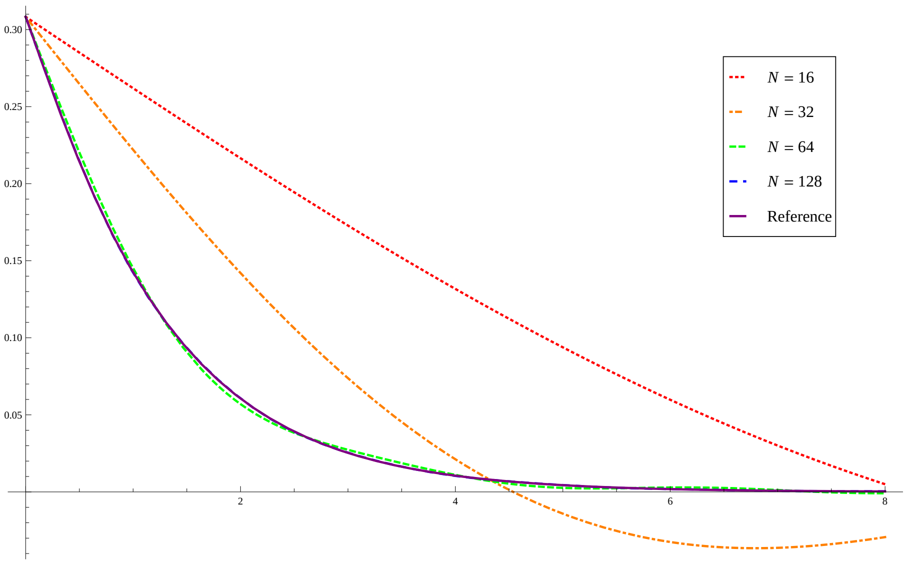

Figure 3a: Comparison of Fourier-cosine approximation with reference curve when a compound Poisson process with Gamma claim size is the underlying model. (Example 5.3) Truncation range N is set to be $16, 32, 64$ and 128

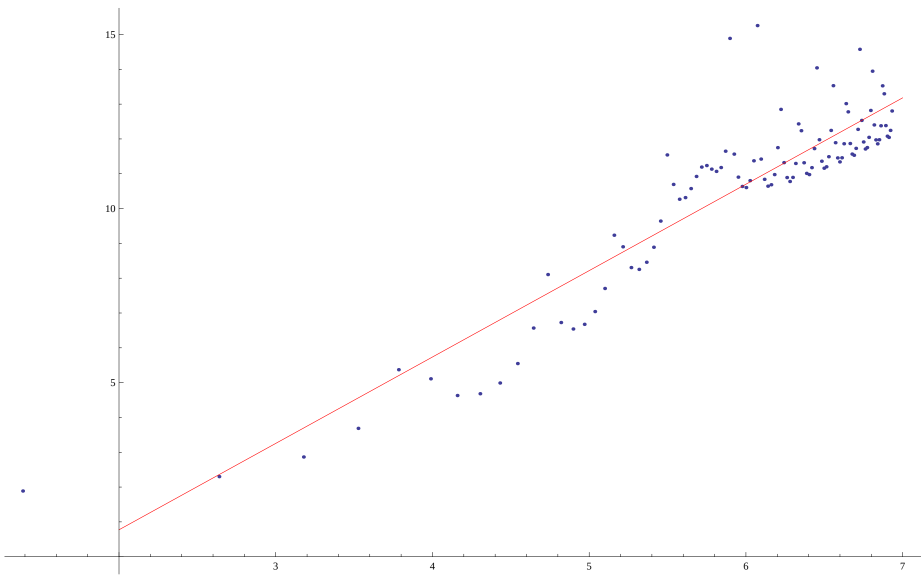

Figure 3b: The graph of $-\log(|\varphi_e(3, N) - \varphi(3)|)$ against $\log N$ in Example 5.3

Example 5.4 (Lévy–Gamma Process). Let L_t be Lévy–Gamma process. The Lévy measure for such a process is $\nu(dx) = 20\frac{e^{-0.5x}}{x}dx$. Note that this model is not covered in the classical compound Poisson setting since $\int_0^\infty \nu(dx) = \infty$. The premium rate is set at 50. The penalty function $\kappa(x, y) = 1$ and the discounted factor $\delta = 1$. This model has been used in Zhang and Yang (2013) as the underlying model for approximating ruin probabilities. We approximate this Gerber-Shiu function with the proposed Fourier-cosine method and the numerical result can be seen in Figure 4a.

In this example, $\check{V}$ is within the Sobolev space of order $\frac{1}{2} - \eta$ for some positive η. Therefore, Theorem 4.8 cannot be applied here. Luckily, the real part of the Fourier transform of V has an algebraic index of convergent of 1 and it satisfies the condition in part two of Theorem 4.10. Therefore, Fourier-cosine can be applied with an error bound. The error converges with N at a rate of approximately 1.7463387854 in Figure 4b.

In all the examples above, since the algebraic indices of convergence of their Fourier transforms are explicitly known, Theorem 4.10 can provide a more accurate error bound for the Fourier-cosine method. However, Theorem 4.8 is required when the converge rate of $\Re(\phi_V)$ is ambiguous or even unknown, for example, when the Fourier transform of V is derived from empirical data.

6. Conclusion

In this paper, we have provided a comprehensive study of using Fourier-cosine approximation for the Gerber-Shiu functions. This method has the advantage of having linear computational complexity and can be easily implemented. Moreover, a sufficient condition for applying the Fourier-cosine method is given by using our refined Sobolev theory as elaborated in detail. We showed that for the functions within a refined Sobolev space at a certain order, the error bound of our approximation will converge with the number N.

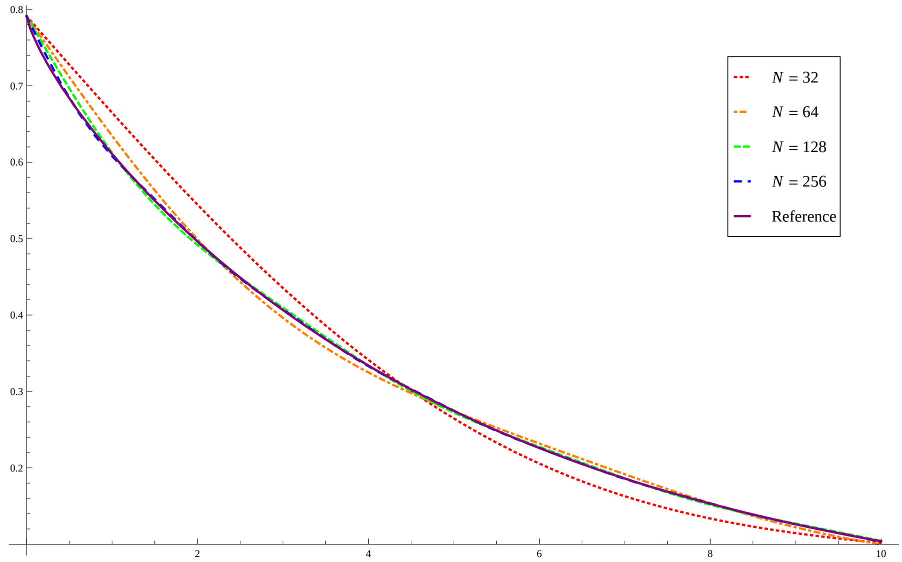

Figure 4a: Comparison of Fourier-cosine approximation with reference curve when a Lévy–Gamma Process is the underlying model. (Example 5.4) Truncation range N is set to be $32, 64, 128$ and 256

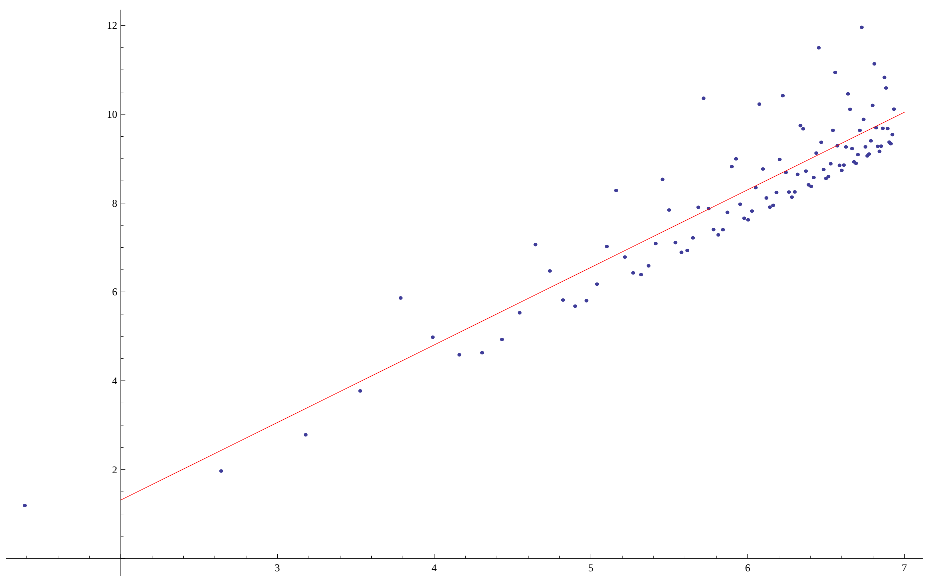

Figure 4b: The graph of $-\log(|\varphi_e(3, N) - \varphi(3)|)$ against $\log N$ in Example 5.4

Further research on enhancing the convergence rate of the Fourier-cosine methods remains open. Our method can also be further enhanced by adopting it to the general risk models or by giving a more accurate error bound.

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