Synchronization of Dynamical Networks With Nonidentical Nodes: Criteria and Control

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***Abstract*—This paper presents a framework for global synchronization of dynamical networks with nonidentical nodes. Several criteria for synchronization are given using free matrices for both cases of synchronizing to a common equilibrium solution of all isolated nodes and synchronizing to the average state trajectory. These criteria can be viewed as generalizations of the master stability function method for local synchronization of networks with identical nodes to the case of nonidentical nodes. The controlled synchronization problem is also studied. The control action, which is subject to certain constraints, is viewed as reorganization of the connection topology of the network. Synchronizability conditions via control are put forward. The synchronizing controllers can be obtained by solving an optimization problem.**

***Index Terms*—Dynamical networks, master stability function, nonidentical nodes, synchronization.**

I. Introduction

DYNAMICAL networks have attracted extensive attention recently (see[2] and the references therein). Synchronization, as an emergent phenomenon of dynamical networks, is one of the key issues that have been addressed. Some recent overviews can be found in[21] and [33]. The main concerns of the investigations are to try to understand the synchronization phenomenon in many disciplines including physics, biology, and social interactions, to study the underlying mechanisms, and to establish synchronization criteria mathematically.

A dynamical network can, of course, be regarded as a dynamical system with a special structure. Likewise, synchronization problems can be treated as some types of stability issues. However, applying standard stability analysis methods to a network as a dynamical system usually produces full-dimensional conditions that are often very hard to check or compute. This is mainly because the number of nodes is often very large, and thus, the dimension is huge. Therefore, how to have checkable and computable synchronization criteria, in particular, lower dimensional conditions, is one of the key points in the study of network synchronization. Taking the isolated node dynamics and the network topology into account is an effective way to establish such synchronization criteria.

In almost all the existing results, a common assumption is that all nodes of a network are identical. Indeed, this assumption makes it much easier to analyze the network, particularly for the synchronizability problems. In particular, under this assumption, a constant, symmetric, and irreducible coupling configuration matrix can always give rise to local synchronization criteria that only require the checking of simultaneous stability of several lower dimensional dynamical systems [2], [5], [17], [32]–[34]. Some relevant extensions and results using other methods can be found, for example, in [4], [7], [11], [20], [24], [28], and [38]. Controlled synchronization has been addressed by several research works—see [14], [16], [17], [22], [25], and [36] for results relevant here.

However, most dynamical networks in engineering have different nodes. Taking a power system as an example [12], the generators (power sources) and loads (power sinks) are connected to buses which are interconnected by transmission lines in a network structure. Therefore, the power system can be viewed as a dynamical network where the nodes consist of generators and (dynamical) loads. Due to different physical parameters of individual generators, the generator models have different dynamics, and the power system is obviously a dynamical network with nonidentical nodes.

The behavior of dynamical networks with nonidentical nodes is much more complicated than that of the identical-node case. In terms of the synchronization issue, unlike the identical-node case, decompositions into a number of lower dimensional systems are, in general, no longer possible, even for the local synchronization problem. Thus, the study of synchronization of dynamical networks with nonidentical nodes is very hard, and very few results have been reported by now. A simple case where all nonidentical nodes have the same equilibrium was considered in [35], where a synchronization criterion using V-stability Lyapunov functions was given by constructing a common Lyapunov function for all the nodes. Controlled synchronization was considered for the case where each node has a normal form with a linear main part [23], and distributed controllers were designed to achieve synchronization. Several collective properties for coupled nonidentical chaotic systems were respectively discussed in [8], [15], [9], [10], [30], and [31]. As asymptotic synchronization of a network with nonidentical nodes is very hard to achieve, most researchers focus on partial synchronization, particularly for nonidentical Kuramoto oscillators [3], or output synchronization [6]. Bounded synchronization is another type

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of weaker form of synchronization when asymptotic synchronization is impossible [13], [27].

This paper addresses the issue of asymptotic synchronization for complex dynamical networks with nonidentical nodes. Free matrices are introduced in the analysis, and global synchronization criteria are given based on solving a number of lower dimensional matrix inequalities and scalar inequalities, which generalize the criteria using the method of the master stability function for networks with identical nodes. Controlled synchronizability conditions are put forward. The synchronizing controllers can be designed by solving an optimization problem.

II. Preliminaries

We study a dynamical network modeled as

$$\dot{x}_i = f_i(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i = 1, \ldots, N \tag{1}$$

where $x_i = (x_{i1}, \ldots, x_{in})^T \in R^n$ is the state of the ith node. Assume that the matrix $A = (a_{ij})_{N\times N}$, which represents the outer coupling configuration of the network, is symmetric and that $\sum_{j=1}^{N} a_{ij} = 0$, where $i = 1, \ldots, N$, and f_i are continuously differentiable with Jacobian Df_i. The dynamics of the isolated nodes are $\dot{x}_i = f_i(x_i)$, where $i = 1, \ldots, N$.

The network (1) is said to synchronize if $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$, where $1 \le i$ and $j \le N$. An equivalent definition of synchronization is that there exists a function $s(t)$ such that $\lim_{t\to\infty}(x_i(t) - s(t)) = 0$, where $1 \le i \le N$. Obviously, $s(t)$ is not unique. In fact, any function $\bar{s}(t)$ satisfying $\lim_{t\to\infty}(\bar{s}(t) - s(t)) = 0$ serves the same purpose.

Let $s(t)$ be a function to which all $x_i(t)$'s are expected to synchronize. Then, the deviations $e_i = x_i - s(t)$ satisfy

$$\begin{aligned}\dot{e}_i &= f_i(s+e_i) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j - \dot{s}\\ &= f_i(s) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + \int_0^1 Df_i(s+\tau e_i)e_{id}\tau - \dot{s}.\end{aligned} \tag{2}$$

Let $e = (e_1^T, \ldots, e_N^T)^T$. Then, (2) becomes

$$\begin{aligned}\dot{e} &= (cA\otimes\Gamma)e\\ &\quad + \mathrm{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau, \ldots, \int_0^1 Df_N(s+\tau e_N)d\tau\right\}e\\ &\quad + \left(f_1^T(s), \ldots, f_N^T(s)\right)^T - \left(\dot{s}^T, \ldots, \dot{s}^T\right)^T.\end{aligned} \tag{3}$$

The linearized error dynamics of the network (1) are given by

$$\begin{aligned}\dot{e} &= (cA\otimes\Gamma)e + \mathrm{diag}\{Df_1(s), \ldots, Df_N(s)\}e\\ &\quad + \left(f_1^T(s), \ldots, f_N^T(s)\right)^T - (\dot{s}^T, \ldots, \dot{s}^T)^T\end{aligned} \tag{4}$$

which is the basis to study local synchronization.

Since A is symmetric and irreducible, there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \ldots, \Phi_N)$ such that

$$\Phi^T A\Phi = \Lambda = \mathrm{diag}\{\lambda_1(A), \lambda_2(A), \ldots, \lambda_N(A)\} \tag{5}$$

where Φ_i is the ith column of Φ with $\Phi_1 = ((1/\sqrt{N}), \ldots, (1/\sqrt{N}))^T$ and $\lambda_1(A), \lambda_2(A), \ldots, \lambda_N(A)$ are the eigenvalues of A satisfying $\lambda_1 = \lambda_1(A) = 0$.

Let $\mathcal{PC}_{n\times n}$ ($\mathcal{PC}^1_{n\times n}$) be the linear space of the uniformly bounded continuous (continuously differentiable) real matrix-valued functions defined on $[0, \infty)$. For any $P \in \mathcal{PC}_{n\times n}$, the norm of P is defined by $\|P\| = \sup_{0\le t<\infty}\{\|P(t)\|\}$.

A time-varying matrix $P(t)$ is said to be uniformly positive definite, denoted by $P \succ 0$, if $x^T P(t)x \ge \alpha\|x\|^2$ holds for all $x \in R^n$, $t \in R$, and some $\alpha > 0$. $P(t)$ is uniformly negative definite, denoted by $P \prec 0$, if $-P(t)$ is uniformly positive definite.

III. Synchronization Criteria

In this section, we will present the criteria for synchronizing to an equilibrium solution and to the average of all node states, respectively.

A. Synchronization to an Equilibrium Solution

Suppose that $s(t)$ is a known equilibrium solution to all isolated nodes, i.e., $s(t)$ satisfies

$$\dot{s}(t) = f_i(s(t)), \qquad i = 1, \ldots, N. \tag{6}$$

Substituting (6) into (3) gives

$$\begin{aligned}\dot{e} &= (cA\otimes\Gamma)e\\ &\quad + \mathrm{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau, \ldots, \int_0^1 Df_N(s+\tau e_N)d\tau\right\}e.\end{aligned} \tag{7}$$

Note that $e = 0$ is an equilibrium point of (7), so the asymptotic stability of (7) implies the synchronization.

Theorem 3.1: Suppose that there exist $n\times n$ matrix Π, which may be time varying, and uniformly positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$ with $\|P_i\| \le 1$ and constant $\alpha \ge 0$, such that

$$\dot{P}_i(t) + P_i(t)(\Pi + c\lambda_i\Gamma) + (\Pi + c\lambda_i\Gamma)^T P_i(t) + \alpha I \prec 0, \quad i = 1, \ldots, N \tag{8}$$

$$\left\|\int_0^1 Df_i(s+\tau e_i)d\tau - \Pi\right\| \le \frac{1}{2}\alpha, \qquad i = 1, \ldots, N. \tag{9}$$

Then, the network (1) globally synchronizes.

Proof: Let $\omega = (\Phi^T \otimes I_n)e$. Then, it follows from (7) that

$$\begin{aligned}\dot{\omega} &= (\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)\omega\\ &\quad + (\Phi^T\otimes I_n)\end{aligned}$$

$$\begin{aligned}&\times \operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau,\ldots,\int_0^1 Df_N(s+\tau e_N)d\tau\right\}\\&\times(\Phi\otimes I_n)\omega\\
=&\operatorname{diag}\{\Pi+c\lambda_1\Gamma,\ldots,\Pi+c\lambda_N\Gamma\}\,\omega\\
&+(\Phi^T\otimes I_n)\times\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau-\Pi,\ldots,\right.\\
&\left.\int_0^1 Df_N(s+\tau e_N)d\tau-\Pi\right\}\\
&\times(\Phi\otimes I_n)\omega. \qquad (10)\end{aligned}$$

Equation (8) implies that there exists a sufficiently small constant $\epsilon > 0$ such that

$$\begin{aligned}&\omega_i^T(\dot{P}_i(t)+P_i(t)(\Pi+c\lambda_i\Gamma)+(\Pi+c\lambda_i\Gamma)^TP_i(t))\omega_i\\&\quad\le-(\alpha+\epsilon)\|\omega_i\|^2. \qquad (11)\end{aligned}$$

Choose $V(\omega)=\omega^TP\omega$ with $P=\operatorname{diag}\{P_1,\ldots,P_N\}$. Differentiating $V(\omega)$ along the trajectory of (10) and in view of (9), we have

$$\begin{aligned}\dot{V}=&\,\omega^T\{\dot{P}+\operatorname{diag}\{\Pi+c\lambda_1\Gamma,\ldots,\Pi+c\lambda_N\Gamma\}^TP\\
&\quad+P\operatorname{diag}\{\Pi+c\lambda_1\Gamma,\ldots,\Pi+c\lambda_N\Gamma\}\}\omega\\
&+2\omega^T(\Phi^T\otimes I_n)\\
&\quad\times\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau-\Pi,\ldots,\right.\\
&\quad\left.\int_0^1 Df_N(s+\tau e_N)d\tau-\Pi\right\}^T(\Phi\otimes I_n)P\omega\\
\le&-(\alpha+\epsilon)\|\omega\|^2\\
&+2\left\|\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau-\Pi,\ldots,\right.\right.\\
&\quad\left.\left.\int_0^1 Df_N(s+\tau e_N)d\tau-\Pi\right\}\right\|\|\omega\|^2\\
\le&\left(2\max_i\left\|\int_0^1 Df_i(s+\tau e_i)d\tau-\Pi\right\|-\alpha-\epsilon\right)\|\omega\|^2\\
\le&-\epsilon\|\omega\|^2. \qquad (12)\end{aligned}$$

This completes the proof.

It is crucial to find a matrix Π to satisfy the conditions of Theorem 3.1. This can be done by solving the following optimization problem.

Corollary 3.2: If the optimization problem

$$\begin{aligned}&\max_{\alpha,P_i,\Pi}\left(\alpha-2\max_j\left\|\int_0^1 Df_j(s+\tau e_j)d\tau-\Pi\right\|\right)\\
&\text{s.t.}\quad \dot{P}_i(t)+P_i(t)(\Pi+c\lambda_i\Gamma)+(\Pi+c\lambda_i\Gamma)^TP_i(t)+\alpha I\prec 0;\\
&\qquad P_i\succ 0;\qquad \|P_i\|\le 1;\\
&\qquad \Pi(t)\in R^{n\times n} \qquad (13)\end{aligned}$$

has a positive maximum, then the network (1) globally synchronizes.

When such a matrix Π does not exist or is hard to find, we may apply the following criterion to test the synchronizability.

Theorem 3.3: Suppose that there exist $n\times n$ matrices Π_{ij} with $\Pi_{ij}=\Pi_{ji}$, which may be time varying and depend on the variable e, constants $\delta_{ij}>0$ with $\delta_{ij}=\delta_{ji}$, where $1\le i$, $j\le N$, and $i\ne j$, constants $\epsilon_{ij}>0$, where $1\le i<j\le N$, uniformly positive definite matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$ with $\|P_i\|\le 1$, and constants $\alpha_i\ge 0$ such that

$$\begin{aligned}&\dot{P}_i(t)+P_i(t)\left(\sum_{j=1}^N\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau+c\lambda_i\Gamma\right)\\
&\quad+\left(\sum_{j=1}^N\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau+c\lambda_i\Gamma\right)^TP_i(t)\\
&\quad+\alpha_iI\prec 0,\qquad i=1,\ldots,N \qquad (14)\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^N|\phi_{jk}\phi_{ji}|\left\|\left(\int_0^1 Df_j(s+\tau e_j)d\tau-\Pi_{ik}\right)\right\|\le\delta_{ik},\\
&\qquad\qquad 1\le i;\quad k\le N;\quad i\ne k \qquad (15)\end{aligned}$$

$$\begin{aligned}&2\sum_{k=2}^N\delta_{1k}\epsilon_{1k}^{-1}\le\alpha_1\\
&2\sum_{k=i+1}^N\delta_{ik}\epsilon_{ik}^{-1}+2\sum_{l=1}^{i-1}\delta_{li}\epsilon_{li}\le\alpha_i,\qquad 2\le i\le N-1;\\
&\qquad\qquad\text{if } N\ge 3\\
&2\sum_{k=1}^{N-1}\delta_{kN}\epsilon_{kN}\le\alpha_N. \qquad (16)\end{aligned}$$

Then, the network (1) globally synchronizes.

Proof: Choose the new variables $\omega=(\Phi^T\otimes I_n)e$. Then, it follows from (7) that

$$\begin{aligned}\dot{\omega}=&(\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)\omega\\
&+(\Phi^T\otimes I_n)\\
&\quad\times\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau,\ldots,\int_0^1 Df_N(s+\tau e_N)d\tau\right\}\\
&\quad\times(\Phi\otimes I_n)\omega\\
=&\operatorname{diag}\{c\lambda_1\Gamma,\ldots,c\lambda_N\Gamma\}\omega\\
&+(\Phi^T\otimes I_n)\\
&\quad\times\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau,\ldots,\int_0^1 Df_N(s+\tau e_N)d\tau\right\}\\
&\quad\times(\Phi\otimes I_n)\omega. \qquad (17)\end{aligned}$$

Equivalently, we have

$$\begin{aligned}\dot{\omega}_i=c\lambda_i\Gamma\omega_i+&\left(\phi_{1i}\int_0^1 Df_1(s+\tau e_1)d\tau,\ldots,\right.\\
&\left.\phi_{Ni}\int_0^1 Df_N(s+\tau e_N)d\tau\right)(\Phi\otimes I_n)\,\omega\end{aligned}$$

$$
\begin{aligned}
&= c\lambda_i\Gamma\omega_i + \left(\sum_{j=1}^{N}\phi_{j1}\phi_{ji}\int_0^1 Df_j(s+\tau e_j)d\tau,\ldots,\right.\\
&\qquad\left.\sum_{j=1}^{N}\phi_{jN}\phi_{ji}\int_0^1 Df_j(s+\tau e_j)d\tau\right)\omega\\
&= \left(\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right)\omega_i\\
&\quad+\sum_{k=1,k\neq i}^{N}\left(\sum_{j=1}^{N}\phi_{jk}\phi_{ji}\int_0^1 Df_j(s+\tau e_j)d\tau\right)\omega_k.
\end{aligned}
\tag{18}
$$

Since Φ is a unitary matrix, we have

$$
\sum_{j=1}^{N}\phi_{jk}\phi_{ji} = \begin{cases}1, & k=i\\ 0, & k\neq i.\end{cases}
$$

Thus, (18) is equivalent to

$$
\begin{aligned}
\dot{\omega}_i &= \left(\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right)\omega_i\\
&\quad+\sum_{k=1,k\neq i}^{N}\left(\sum_{j=1}^{N}\phi_{jk}\phi_{ji}\left(\int_0^1 Df_j(s+\tau e_j)d\tau - \Pi_{ik}\right)\right)\omega_k.
\end{aligned}
\tag{19}
$$

Choose $V_i(\omega_i) = \omega_i^T P_i \omega_i$ and $V(\omega) = \sum_{i=1}^{N} V_i(\omega_i)$. Differentiating $V_i(\omega_i)$ along the trajectory of (18), we have

$$
\begin{aligned}
\dot{V}_i &= \omega_i^T\\
&\quad\times\left(\dot{P}_i(t) + P_i(t)\left(\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right)\right.\\
&\qquad\left.+\left(\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right)^T P_i(t)\right)\omega_i\\
&\quad+2\omega_i^T P_i\sum_{k=1,k\neq i}^{N}\left(\sum_{j=1}^{N}\phi_{jk}\phi_{ji}\left(\int_0^1 Df_j(s+\tau e_j)d\tau - \Pi_{ik}\right)\right)\omega_k.
\end{aligned}
\tag{20}
$$

Applying (14) and (15) gives

$$
\begin{aligned}
\dot{V}_i &\leq -(\alpha_i+\epsilon)\|\omega_i\|^2\\
&\quad+2\|\omega_i\|\|P_i\|\\
&\quad\times\sum_{k=1,k\neq i}^{N}\left(\sum_{j=1}^{N}|\phi_{jk}\phi_{ji}|\left\|\left(\int_0^1 Df_j(s+\tau e_j)d\tau - \Pi_{ik}\right)\right\|\right)\\
&\quad\times\|\omega_k\|\\
&\leq -(\alpha_i+\epsilon)\|\omega_i\|^2 + 2\sum_{k=1,k\neq i}^{N}\delta_{ik}\|\omega_i\|\|\omega_k\|
\end{aligned}
\tag{21}
$$

with a sufficiently small constant $\epsilon > 0$.

Therefore

$$
\begin{aligned}
\dot{V} &\leq -\sum_{i=1}^{N}(\alpha_i+\epsilon)\|\omega_i\|^2 + 2\sum_{i=1}^{N}\sum_{k=1,k\neq i}^{N}\delta_{ik}\|\omega_i\|\|\omega_k\|\\
&= -\sum_{i=1}^{N}(\alpha_i+\epsilon)\|\omega_i\|^2 + 4\sum_{1\leq i<k\leq N}\delta_{ik}\|\omega_i\|\|\omega_k\|.
\end{aligned}
\tag{22}
$$

Applying Young's inequality results in

$$
\begin{aligned}
&4\sum_{1\leq i<k\leq N}\delta_{ik}\|\omega_i\|\|\omega_k\|\\
&\quad\leq 2\sum_{1\leq i<k\leq N}\left(\delta_{ik}\epsilon_{ik}^{-1}\|\omega_i\|^2 + \delta_{ik}\epsilon_{ik}\|\omega_k\|^2\right)\\
&\quad= 2\sum_{k=2}^{N}\delta_{1k}\epsilon_{1k}^{-1}\|\omega_1\|^2\\
&\qquad+\sum_{i=2}^{N-1}\left(2\sum_{k=i+1}^{N}\delta_{ik}\epsilon_{ik}^{-1} + 2\sum_{l=1}^{i-1}\delta_{li}\epsilon_{li}\right)\|\omega_i\|^2\\
&\qquad+2\sum_{k=1}^{N-1}\delta_{kN}\epsilon_{kN}\|\omega_N\|^2.
\end{aligned}
\tag{23}
$$

Combining (16), (22), and (23) yields

$$
\dot{V} \leq -\epsilon\|\omega\|^2 \tag{24}
$$

which completes the proof.

Remark 3.4: In Theorem 3.1, Corollary 3.2, and Theorem 3.3, the condition $\|P_i\| \leq 1$ is only for simplicity since any nonzero matrix can be normalized by being divided by its norm.

Remark 3.5: If we replace $\int_0^1 Df_j(s+\tau e_j)d\tau$ with $Df_j(s)$ in (9) and (13)–(15), we will have local synchronization criteria. In this case, more particularly, if all nodes are identical, then (9), (15), and (16) are automatically satisfied with $\Pi_{ik} = \Pi = Df_j = Df$ and $\delta_{ik} = 0$. In this case, both Theorems 3.1 and 3.3 degenerate into the well-known master Lyapunov function condition [2].

Remark 3.6: It is worth mentioning that the inequalities (14) are not pure linear matrix inequalities due to the presence of the matrices $\int_0^1 Df_j(s+\tau e_j)d\tau$, so we have no general methods to solve these inequalities. Fortunately, in some special cases, it is still possible to find solutions. We only mention the following two cases.

1) $\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau$ are bounded in the sense of the following decompositions:

$$
\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau = W_i(t) + E_i(t)F_i(t,e)D_i(t)
$$

where $F_i^T(t,e)F_i(t,e) \leq I$ and $W_i(t)$, $E_i(t)$, and $D_i(t)$ are time-varying matrices, or each $\int_0^1 Df_j(s+\tau e_j)d\tau$ has the decomposition

$$
\int_0^1 Df_j(s+\tau e_j)d\tau = W_j(t) + E_j(t)F_j(t,e)D_j(t).
$$

In this case, a linear matrix inequality-type condition can be easily established to make (14) hold.

2) Let $\mu_i(t,e)$ denote the largest eigenvalue of the matrix

$$\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma + \left(\sum_{j=1}^{N}\phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right)^T.$$

If $\mu_i(t,e) \leq \mu_i < 0$ for some constants μ_i, then (14) automatically holds with $P_i = I$ and $\alpha_i = \mu_i$.

Remark 3.7: Similar to Corollary 3.2, the constants δ_{ik} satisfying (15) can be obtained by solving the following optimization problems:

$$\begin{aligned}\min_{x_j,\Pi}\quad & \sum_{j=1}^{N}|\phi_{jk}\phi_{ji}|x_j\\ \text{s.t.}\quad & \left\|\int_0^1 Df_j(s+\tau e_j)d\tau - \Pi\right\| \leq x_j\\ & x_j \geq 0\\ & \Pi(t,e) \in R^{n\times n}.\end{aligned} \tag{25}$$

B. Synchronization to the Average Trajectory

If not all the nodes share a common equilibrium solution, the synchronization analysis becomes more complicated. First of all, we have to choose a proper $s(t)$ to which all the nodes are expected to synchronize so that the analysis can be carried out. Here, we choose the average of all node states, i.e.,

$$s(t) = \frac{1}{N}\sum_{k=1}^{N}x_k(t).$$

The average dynamics of all node dynamics are defined by the vector field

$$\bar{f}(x) = \frac{1}{N}\sum_{k=1}^{N}f_k(x).$$

Again, let $e_i = x_i - s$. Obviously, $\sum_{i=1}^{N}e_i = 0$. A straightforward calculation gives

$$\begin{aligned}\dot{s} &= \frac{1}{N}\sum_{k=1}^{N}\left(f_k(x_k) + c\sum_{j=1}^{N}a_{kj}\Gamma x_j\right)\\ &= \frac{1}{N}\sum_{k=1}^{N}f_k(s+e_k)\\ &= \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + \bar{f}(s).\end{aligned} \tag{26}$$

Substituting (26) into (2), we have

$$\begin{aligned}\dot{e}_i &= c\sum_{j=1}^{N}a_{ij}\Gamma e_j + f_i(s+e_i) - \frac{1}{N}\sum_{k=1}^{N}f_k(s+e_k)\\ &= c\sum_{j=1}^{N}a_{ij}\Gamma e_j + \int_0^1 Df_i(s+\tau e_i)e_i d\tau\\ &\quad - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s).\end{aligned} \tag{27}$$

Thus

$$\begin{aligned}\dot{e} &= (cA\otimes\Gamma)e\\ &\quad + \mathrm{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau \cdots \int_0^1 Df_N(s+\tau e_N)d\tau\right\}e\\ &\quad - \frac{1}{N}\begin{pmatrix}\int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau\\ \vdots & \ddots & \vdots\\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau\end{pmatrix}e\\ &\quad + \begin{pmatrix}f_1(s)-\bar{f}(s)\\ \vdots\\ f_N(s)-\bar{f}(s)\end{pmatrix}.\end{aligned} \tag{28}$$

It is worth pointing out that $e = 0$ is no longer an equilibrium point of (28). Therefore, the synchronization problem cannot be solved as the asymptotic stability problem of (28). Instead, the synchronizability can be checked by the attractiveness of (28) to the manifold defined by $e \equiv 0$.

Proposition 3.8: Suppose that $x_i(t)$'s are uniformly continuous with respect to t and $f_i(x)$'s are uniformly continuous with respect to x. If the network (1) synchronizes, then it holds that

$$\lim_{t\to\infty}(f_i(s(t)) - f_j(s(t))) = 0, \qquad 1 \leq i; \quad j \leq N. \tag{29}$$

Proof: From (27), we have

$$\begin{aligned}\dot{e}_i &= c\sum_{j=1}^{N}a_{ij}\Gamma e_j + f_i(s+e_i) - f_i(s)\\ &\quad + \frac{1}{N}\sum_{k=1}^{N}(f_k(s) - f_k(s+e_k)) + f_i(s)\\ &\quad - \frac{1}{N}\sum_{k=1}^{N}f_k(s).\end{aligned} \tag{30}$$

The uniform continuity of $f(\cdot)$ implies

$$c\sum_{j=1}^{N}a_{ij}\Gamma e_j + f_i(s+e_i) - f_i(s) + \frac{1}{N}\sum_{k=1}^{N}(f_k(s) - f_k(s+e_k)) \to 0.$$

Integrating both sides of (30) from 0 to ∞ and using the uniform continuity of $f_i(s) - (1/N)\sum_{k=1}^{N}f_k(s)$ immediately give

$$\lim_{t\to\infty}\left(f_i(s) - \frac{1}{N}\sum_{k=1}^{N}f_k(s)\right) = 0$$

which is equivalent to

$$\lim_{t\to\infty}(f_i(s(t)) - f_j(s(t))) = 0, \qquad 1 \leq i; \quad j \leq N.$$

Obviously, (29) is far from being sufficient. This can be easily seen from the fact that even many networks with identical nodes still do not synchronize.

In order to develop sufficient conditions, we need to rewrite (28) in the new coordinates $\omega = (\Phi^T \otimes I_n)e$

$$\begin{aligned}\dot{\omega} =& (\Phi^T \otimes I_n)\dot{e}\\ =& (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)\omega\\ &+ (\Phi^T \otimes I_n)\\ &\times \operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau, \ldots, \int_0^1 Df_N(s+\tau e_N)d\tau\right\}\\ &\times (\Phi \otimes I_n)\omega\\ &- \frac{1}{N}(\Phi^T \otimes I_n)\\ &\times \begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}\\ &\times (\Phi \otimes I_n)\omega + (\Phi^T \otimes I_n)\begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s) \end{pmatrix}. \end{aligned} \tag{31}$$

Note that

$$\begin{aligned}&\begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}\\ &= \sqrt{N}(\Phi_1, \mathbf{0}, \ldots, \mathbf{0}) \otimes \int_0^1 Df_1(s+\tau e_1)d\tau\\ &\quad + \sqrt{N}(\mathbf{0}, \Phi_1, \ldots, \mathbf{0}) \otimes \int_0^1 Df_2(s+\tau e_2)d\tau\\ &\quad + \cdots + \sqrt{N}(\mathbf{0}, \mathbf{0}, \ldots, \Phi_1) \otimes \int_0^1 Df_N(s+\tau e_N)d\tau \end{aligned} \tag{32}$$

and Φ is a unitary matrix; it turns out that

$$\begin{aligned}&\frac{1}{N}(\Phi^T \otimes I_n)\\ &\times \begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}\\ &\times (\Phi \otimes I_n)\\ &= \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \otimes \int_0^1 Df_1(s+\tau e_1)d\tau(\Phi \otimes I_n)\\ &\quad + \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \otimes \int_0^1 Df_2(s+\tau e_2)d\tau(\Phi \otimes I_n)\\ &\quad + \cdots + \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}\\ &\quad \otimes \int_0^1 Df_N(s+\tau e_N)d\tau(\Phi \otimes I_n). \end{aligned} \tag{33}$$

Thus, a simple calculation gives

$$\begin{aligned}&\frac{1}{N}(\Phi^T \otimes I_n)\\ &\times \begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}\\ &\times (\Phi \otimes I_n)\\ &= \frac{1}{\sqrt{N}}\begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1N} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}\\ &\quad \otimes \int_0^1 Df_1(s+\tau e_1)d\tau\\ &\quad + \frac{1}{\sqrt{N}}\begin{pmatrix} \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2N} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}\\ &\quad \otimes \int_0^1 Df_2(s+\tau e_2)d\tau\\ &\quad + \cdots + \frac{1}{\sqrt{N}}\begin{pmatrix} \varphi_{N1} & \varphi_{N2} & \cdots & \varphi_{NN} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}\\ &\quad \otimes \int_0^1 Df_N(s+\tau e_N)d\tau. \end{aligned} \tag{34}$$

Therefore, combining (31) and (34) yields

$$\begin{aligned}\dot{\omega} =& (c\Lambda \otimes \Gamma)\omega\\ &+ (\Phi^T \otimes I_n)\end{aligned}$$

$$\times \operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau,\ldots,\int_0^1 Df_N(s+\tau e_N)d\tau\right\}$$
$$\times (\Phi\otimes I_n)\omega - \begin{pmatrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\omega$$
$$+ (\Phi^T\otimes I_n)\begin{pmatrix} f_1(s)-\bar{f}(s) \\ \vdots \\ f_N(s)-\bar{f}(s)\end{pmatrix}. \tag{35}$$

Since $\omega_1 \equiv 0$, we only need to consider $\omega_2,\ldots,\omega_N$. Let $\bar{\omega} = (\omega_2,\ldots,\omega_N)^T$. Then, (35) becomes

$$\begin{aligned}\dot{\bar{\omega}} =& \operatorname{diag}\{c\lambda_2\Gamma,\ldots,c\lambda_N\Gamma\}\bar{\omega} \\ &+ \begin{pmatrix}\Phi_2^T \\ \vdots \\ \Phi_N^T\end{pmatrix}\otimes I_n \\ &\times\left(\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau,\ldots,\int_0^1 Df_N(s+\tau e_N)d\tau\right\}\right) \\ &\times((\Phi_2,\ldots,\Phi_N)\otimes I_n)\,\bar{\omega} \\ &+ \begin{pmatrix}\Phi_2^T \\ \vdots \\ \Phi_N^T\end{pmatrix}\otimes I_n \begin{pmatrix} f_1(s)-\bar{f}(s) \\ \vdots \\ f_N(s)-\bar{f}(s)\end{pmatrix}.\end{aligned} \tag{36}$$

Theorem 3.9: Suppose the following.

1)
$$\lim_{t\to\infty}(f_i(s(t)) - f_j(s(t))) = 0, \qquad 1\le i;\quad j\le N.$$

2) There exist $n\times n$ matrix Π, which may be time varying, uniformly positive definite matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$ with $\|P_i\|\le 1$, and constant $\alpha>0$ such that

$$\dot{P}_i(t) + P_i(t)(\Pi + c\lambda_i\Gamma) + (\Pi + c\lambda_i\Gamma)^T P_i(t) + \alpha I \prec 0, \qquad i=2,\ldots,N. \tag{37}$$

3)
$$\left\|\int_0^1 Df_i(s+\tau e_i)d\tau - \Pi\right\| \le \frac{1}{2}\alpha, \qquad i=1,\ldots,N. \tag{38}$$

Then, the network (1) globally synchronizes.

Proof: From (36), we know

$$\begin{aligned}\dot{\bar{\omega}} =& \operatorname{diag}\{\Pi + c\lambda_2\Gamma,\ldots,\Pi + c\lambda_N\Gamma\}\,\bar{\omega} \\ &+ \begin{pmatrix}\Phi_2^T \\ \vdots \\ \Phi_N^T\end{pmatrix}\otimes I_n \\ &\times\left(\operatorname{diag}\left\{\int_0^1 Df_1(s+\tau e_1)d\tau - \Pi,\ldots,\right.\right. \\ &\left.\left.\int_0^1 Df_N(s+\tau e_N)d\tau - \Pi\right\}\right) \\ &\times((\Phi_2,\ldots,\Phi_N)\otimes I_n)\bar{\omega} \\ &+ \begin{pmatrix}\Phi_2^T \\ \vdots \\ \Phi_N^T\end{pmatrix}\otimes I_n \begin{pmatrix} f_1(s)-\bar{f}(s) \\ \vdots \\ f_N(s)-\bar{f}(s)\end{pmatrix}.\end{aligned} \tag{39}$$

Choosing $V(\bar{\omega}) = \bar{\omega}^T P\bar{\omega}$ with $P = \operatorname{diag}\{P_2,\ldots,P_N\}$ and in view of $\lim_{t\to\infty}(f_i(s(t)) - \bar{f}(s(t))) = 0$, where $1\le i\le N$, similar to the proof of Theorem 3.1, we can complete the proof.

Theorem 3.10: Suppose that there exist $n\times n$ matrices Π_{ij} with $\Pi_{ij} = \Pi_{ji}$, which may be time varying and even depend on variable e, constants $\delta_{ij}>0$ with $\delta_{ij} = \delta_{ji}$, where $2\le i$, $j\le N$, and $i\ne j$, constants $\epsilon_{ij}>0$, where $2\le i<j\le N$, uniformly positive definite matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$ with $\|P_i\|\le 1$, and constants $\alpha_i\ge 0$, where $2\le i\le N$, such that

$$\begin{aligned}&\dot{P}_i(t) + P_i(t)\left(\sum_{j=1}^N \phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right) \\ &+\left(\sum_{j=1}^N \phi_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau + c\lambda_i\Gamma\right)^T P_i(t) \\ &+\alpha_i I\prec 0, \qquad i=2,\ldots,N\end{aligned} \tag{40}$$

$$\sum_{j=1}^N|\phi_{jk}\phi_{ji}|\left\|\left(\int_0^1 Df_j(s+\tau e_j)d\tau - \Pi_{ik}\right)\right\| \le \delta_{ik}, \qquad 2\le i;\quad k\le N;\quad i\ne k \tag{41}$$

and if $N\ge 3$

$$\begin{aligned}&2\sum_{k=3}^N \delta_{2k}\epsilon_{2k}^{-1}\le\alpha_2 \\ &2\sum_{k=i+1}^N\delta_{ik}\epsilon_{ik}^{-1} + 2\sum_{l=2}^{i-1}\delta_{li}\epsilon_{li}\le\alpha_i, \qquad 3\le i\le N-1, \quad \text{if } N\ge 4 \\ &2\sum_{k=2}^{N-1}\delta_{kN}\epsilon_{kN}\le\alpha_N.\end{aligned} \tag{42}$$

If in addition

$$\lim_{t\to\infty}(f_i(s(t)) - f_j(s(t))) = 0, \qquad 1\le i;\quad j\le N$$

then the network (1) globally synchronizes.

Proof: Using a similar method as in the proof of Theorem 3.3 completes the proof.

IV. Synchronization via Control

In this section, we study how to achieve synchronization via design of controllers.

Consider the controlled network

$$\dot{x}_i = f_i(x_i) + c\sum_{j=1}^N a_{ij}\Gamma x_j + u_i, \qquad i=1,\ldots,N \tag{43}$$

where u_i is the control for the ith node.

Unlike general nonlinear control systems for which nonlinear controllers of any form can be designed, network control must take network features into account. The form of feedback information gathered from individual nodes is assumed to be consistent with the interconnection among nodes which are characterized by the outer coupling configuration matrix. Thus, feedback control for networks differs from the control for general control systems and also increases the difficulty of control design for networks. On the other hand, if general nonlinear controllers were allowed for networks, the nonlinear isolated dynamics could be completely canceled, and the synchronization problem would be trivial.

Now, taking signal transmission in a network into account, we consider the control action as a reset of the outer coupling configuration according to certain constraints. Thus, the controllers take the form

$$u_i = c\sum_{j=1}^{N} b_{ij}\Gamma x_j, \qquad i=1,\ldots,N \tag{44}$$

where $B=(b_{ij})_{n\times n}\in\mathcal{B}\subset R^{n\times n}$ and $\mathcal{B}$ is a given control constraint set. Again, the set $\mathcal{B}$ has the property that the matrix $A+B$ for any matrix $B\in\mathcal{B}$ is again symmetric and has zero row-sum property.

Some typical forms of $\mathcal{B}$ are listed as follows.

1) Any $B\in\mathcal{B}$ is formed by adding or removing a certain number of links based on the existing links. The number can be pregiven.
2) b_{ij}'s are obtained by adjusting the values of the corresponding a_{ij}'s.
3) Some boundedness on the entries of B, for example, $\sum_{j=1}^{N}|b_{ij}|\le M_i$ for some pregiven constants $M_i>0$.
4) A combination of all the above.

For simplicity, we only consider the case where all the isolated nodes have the same equilibrium solution $s(t)$, i.e., (6) holds. Similar discussions can be given for the case of the average trajectory.

The constants $\mu_1,\mu_2,\ldots,\mu_n$ are called the eigenvalues of a matrix pair $(W_{n\times n},R_{n\times m})$ if they are the eigenvalues of the matrix $W+RK$ for some matrix $K_{m\times n}$.

We introduce the notion of constrained eigenvalue assignment.

Definition 4.1: Let $(W_{n\times n},R_{n\times m})$ be a matrix pair and $S\subseteq C^n$ and $\mathcal{K}\subseteq R^{m\times n}$ be the given sets. We say that the eigenvalues of the pair (W,R) can be assigned to the set S under the constraint set $\mathcal{K}$ if there exists $K\in\mathcal{K}$ such that the vector of the eigenvalues of $W+RK$ belongs to S.

This notion is a generalization of pole assignment for linear systems when feedback is limited to an admissible set. A similar concept using a special form of $\mathcal{K}$ was adopted in the study of large-scale systems [29].

Let Q be the set of all the vectors $q=(q_1,q_2,\ldots,q_N)^T$ with $q_1=0$ and the following property.

Property 4.2: There exist $n\times n$ matrix Π, which may be time varying, uniformly positive definite matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$ with $\|P_i\|\le 1$, and constant $\alpha\ge 0$; all Π, P_i, and α may be depending on q such that

$$\dot{P}_i(t)+P_i(t)(\Pi+q_i\Gamma)+(\Pi+q_i\Gamma)^TP_i(t)+\alpha I\prec 0, \qquad i=1,\ldots,N \tag{45}$$

$$\left\|\int_0^1 Df_i(s+\tau e_i)d\tau-\Pi\right\|\le\frac{1}{2}\alpha, \qquad i=1,\ldots,N. \tag{46}$$

Theorem 4.3: Suppose that $Q\neq\emptyset$. If the eigenvalues of the matrix pair (A,I) can be assigned to the set $\bar{Q}=\{(1/c)q|q\in Q\}$ under the constraint set $\mathcal{B}$, then there exists $B\in\mathcal{B}$ such that the controllers $u_i=c\sum_{j=1}^{N}b_{ij}\Gamma x_j$ globally synchronize the network (1).

Proof: Applying Theorem 3.1 to the feedback network immediately completes the proof.

The matrix B can be obtained by solving the following optimization problem:

$$\begin{aligned}\min\quad & \sum_{j=1}^{N}\left(\lambda_j(A+B)-\frac{1}{c}q_j\right)^2\\ \text{s.t.}\quad & q\in Q\\ & B\in\mathcal{B}.\end{aligned} \tag{47}$$

Any optimal solution to (47) with minimum zero provides a choice of synchronizing controllers.

Remark 4.4: In order to apply Theorem 4.3 and to solve the optimization problem (47), it is crucial to know the structure of the set Q, which is, in general, very hard. Therefore, it is realistic to replace Q with some proper subset of Q, which makes all results still valid. For the case of identical nodes and local synchronization problems, such a subset is comparatively easier to obtain. In fact, we can choose $\Pi=Df=Df_i$ and $\alpha=0$, which makes (46) automatically satisfied, while q satisfying (47) can be characterized by the master stability function.

Once such a subset is fixed, whether Theorem 4.3 is applicable or the optimization problem (45) has the minimum zero largely depends on "how far" the eigenvalues of the matrix pair (A,I) can be assigned under the constraint set $\mathcal{B}$. If the assignable eigenvalues can reach the subset of $\bar{Q}$, synchronization is reached, and the feedback can be found. According to the linear control theory, (A,I) is completely controllable, and thus, the eigenvalues can be assigned anywhere. However, here, we have to stick to the constraint set $\mathcal{B}$. For a concrete set $\mathcal{B}$, specific methods may be applicable to check the conditions of Theorem 4.3 and to solve the optimization problem (47). For example, when the set $\mathcal{B}$ is characterized by adding a certain number of links, the method in [1] and [26] may be applied.

Next, we study how to apply Theorem 3.3 to design controllers. For convenience, denote $U=\{G_{n\times n}|G^TG=I_n\}$.

Theorem 4.5: Suppose that there exist a vector $q=(q_1,q_2,\ldots,q_N)^T$ with $q_1=0$, a matrix $G=\{g_{ij}\}\in U$, matrices Π_{ij} with $\Pi_{ij}=\Pi_{ji}$, which may be time varying and depend on variable e, constants $\delta_{ij}>0$ with $\delta_{ij}=\delta_{ji}$, where $1\le i,j\le N$, and $i\neq j$, constants $\epsilon_{ij}>0$ with $1\le i<j\le N$, uniformly positive definite matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$ with $\|P_i\|\le 1$, and constants $\alpha_i\ge 0$ such that

$$\begin{aligned}&\dot{P}_i(t)+P_i(t)\left(\sum_{j=1}^{N}g_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau+q_i\Gamma\right)\\&\quad+\left(\sum_{j=1}^{N}g_{ji}^2\int_0^1 Df_j(s+\tau e_j)d\tau+q_i\Gamma\right)^TP_i(t)\\&\quad+\alpha_iI\prec 0, \qquad i=1,\ldots,N\end{aligned} \tag{48}$$

$$\sum_{j=1}^{N}|g_{jk}g_{ji}|\left\|\left(\int_0^1 Df_j(s+\tau e_j)d\tau-\Pi_{ik}\right)\right\|\le\delta_{ik},$$
$$1\le i;\quad k\le N;\quad i\ne k \quad (49)$$

$$2\sum_{k=2}^{N}\delta_{1k}\epsilon_{1k}^{-1}\le\alpha_1$$
$$2\sum_{k=i+1}^{N}\delta_{ik}\epsilon_{ik}^{-1}+2\sum_{l=1}^{i-1}\delta_{li}\epsilon_{li}\le\alpha_i,$$
$$2\le i\le N-1,\quad \text{if } N\ge 3$$
$$2\sum_{k=1}^{N-1}\delta_{kN}\epsilon_{kN}\le\alpha_N. \quad (50)$$

If $(1/c)G\mathrm{diag}\{q_1,\ldots,q_N\}G^T - A \in \mathcal{B}$, then the global synchronization is achieved by the controller (44) with $B=(1/c)G\mathrm{diag}\{q_1,\ldots,q_N\}G^T - A$.

Proof: Applying Theorem 3.3 to the feedback network completes the proof.

The discussion similar to Remark 4.4 can be made for Theorem 4.5.

V. Example

Consider the following dynamical network with five nonidentical nodes:

$$\dot{x}_i=f_i(x_i)+\sum_{j=1}^{5}a_{ij}x_j,\qquad x_i\in R^3;\quad i=1,2,3,4,5 \quad (51)$$

where

$$f_1(x_1)=\begin{pmatrix}-2.5x_{11}+0.3x_{12}+0.9x_{13}+3x_{13}^2\\ 0.6x_{11}-2.6x_{12}+2x_{12}^2+x_{13}\\ -2.8x_{11}+\sin x_{11}\cos x_{12}-2.2x_{13}\end{pmatrix}$$
$$f_2(x_2)=\begin{pmatrix}-2.5x_{21}+x_{22}+x_{21}x_{22}+x_{23}\\ 0.5x_{21}-0.8x_{22}-\sin x_{22}+x_{23}-x_{23}^2\\ -2x_{21}-0.8x_{23}-0.5\sin(2x_{23})\end{pmatrix}$$
$$f_3(x_3)=\begin{pmatrix}-2.5x_{31}+x_{31}\sin x_{31}+x_{32}+1.4x_{33}\\ 0.5x_{31}-2.5x_{32}+x_{33}\\ -2x_{31}+x_{32}x_{33}-0.6x_{33}-x_{33}\cos x_{33}\end{pmatrix}$$
$$f_4(x_4)=\begin{pmatrix}-2.5x_{41}+1.5x_{42}+1.5x_{43}\\ x_{41}-2.5x_{42}+x_{42}^2+1.5x_{43}+x_{43}^2\\ -2x_{41}+x_{41}x_{43}-2.5x_{43}\end{pmatrix}$$
$$f_5(x_5)=\begin{pmatrix}-2.1x_{51}+0.5x_{51}^2+1.4x_{52}+x_{53}\\ 0.9x_{51}-2.1x_{52}+x_{52}x_{53}+x_{53}\\ -2x_{51}+0.4x_{52}-2.1x_{53}\end{pmatrix}$$
$$A=\begin{pmatrix}-1&0&0&1&0\\0&-2&0&1&1\\0&0&-1&1&0\\1&1&1&-3&0\\0&1&0&0&-1\end{pmatrix}.$$

Solving (8) and (9) gives $\alpha=2$ and

$$P_1=\begin{pmatrix}0.5199&0.0481&-0.0229\\0.0481&0.5040&0.0553\\-0.0229&0.0553&0.4854\end{pmatrix}$$

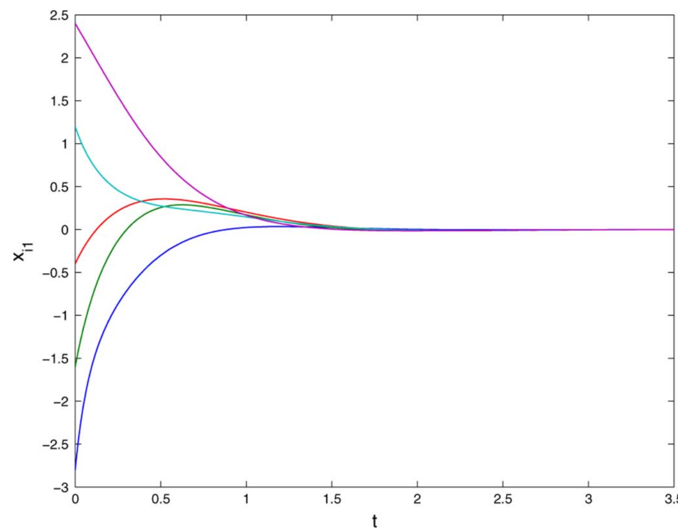

Fig. 1. Synchronization errors of x_{i1}.

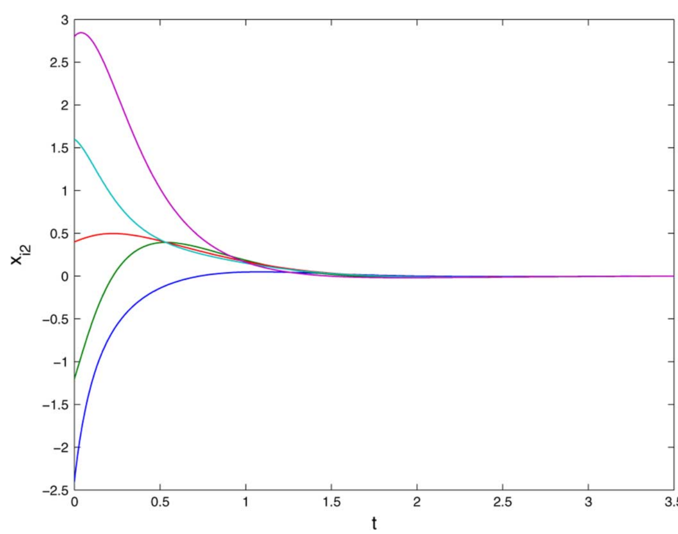

Fig. 2. Synchronization errors of x_{i2}.

$$P_2=\begin{pmatrix}0.4749&0.0340&-0.0173\\0.0340&0.4652&0.0358\\-0.0173&0.0358&0.4537\end{pmatrix}$$
$$P_3=\begin{pmatrix}0.4461&0.0266&-0.0145\\0.0266&0.4391&0.0267\\-0.0145&0.0267&0.4313\end{pmatrix}$$
$$P_4=\begin{pmatrix}0.3962&0.0178&-0.0108\\0.0178&0.3925&0.0162\\-0.0108&0.0162&0.3891\end{pmatrix}$$
$$P_5=\begin{pmatrix}0.3514&0.0137&-0.0088\\0.0137&0.3495&0.0113\\-0.0088&0.0113&0.3479\end{pmatrix}$$
$$\Pi=\begin{pmatrix}-2.5&1&1\\0.5&-2.5&1\\-2&0&-2.5\end{pmatrix}.$$

Applying Theorem 3.1, we know that the network synchronizes. The simulation results are shown in Figs. 1–3.

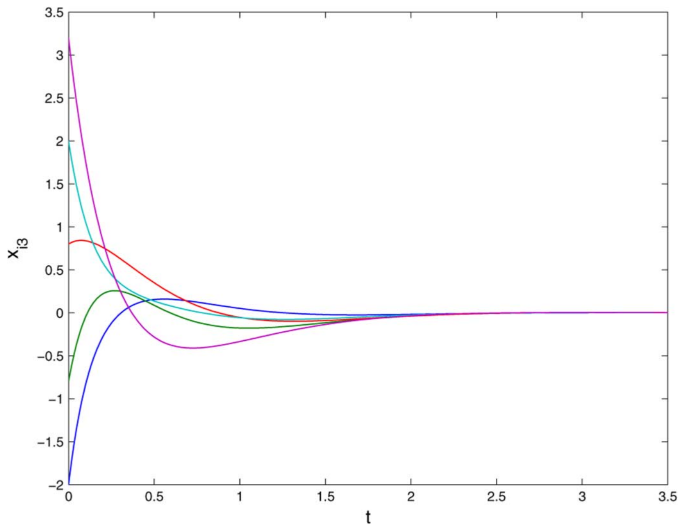

Fig. 3. Synchronization errors of x_{i3}.

VI. Conclusion

We have established a framework for the synchronization of a dynamical network with nonidentical nodes. Several synchronization criteria have been given in terms of matrix inequalities and constraints on a matrix norm. Free matrices Π and Π_{ik} are introduced respectively to establish the criteria, which provide more freedom to check synchronizability. For controlled synchronization, we take the control action as reorganization of the outer coupling topology under admissible structures. This point of view is distinct from the existing results on controlled synchronization where either a general nonlinear controller or error feedback is exploited.

The proposed methods are applicable to practical dynamical networks with nonidentical nodes. For example, the proposed design strategy provides a useful tool to maintain angle and voltage stability of power systems.

For networks with nonidentical nodes, we have little knowledge. In particular, efficient techniques need to be developed to improve the checkability of the conditions of the proposed theorems and to solve the optimization problems.

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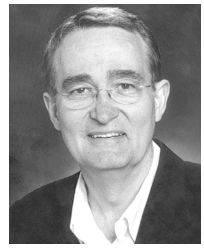

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