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NOWHERE-ZERO 3-FLOWS IN SIGNED GRAPHS*

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Abstract. Tutte observed that every nowhere-zero k-flow on a plane graph gives rise to a k-vertex-coloring of its dual, and vice versa. Thus nowhere-zero integer flow and graph coloring can be viewed as dual concepts. Jaeger further shows that if a graph G has a face-k-colorable 2-cell embedding in some *orientable* surface, then it has a nowhere-zero k-flow. However, if the surface is *nonorientable*, then a face-k-coloring corresponds to a nowhere-zero k-flow in a *signed graph* arising from G. Graphs embedded in orientable surfaces are therefore a special case that the corresponding signs are all positive. In this paper, we prove that if an 8-edge-connected signed graph admits a nowhere-zero integer flow, then it has a nowhere-zero 3-flow. Our result extends Thomassen's 3-flow theorem on 8-edge-connected graphs to the family of all 8-edge-connected signed graphs. And it also improves Zhu's 3-flow theorem on 11-edge-connected signed graphs.

Key words. integer flow, signed graph, modulo orientation

AMS subject classifications. 05C21, 05C22, 05C20

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1. Introduction.

Graphs considered in this paper may have multiple edges and loops unless otherwise stated. Let $G = (V, E)$ be a graph and let k be a positive integer. An ordered pair (D, f) is called a *k-flow* of G if $D = (V, A)$ is an orientation of G and $f : A \mapsto \{0, \pm 1, \dots, \pm(k-1)\}$ is an assignment of flows, such that, for each $v \in V$,

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e),$$

where $E^+(v)$ is the set of all arcs leaving vertex v in D and $E^-(v)$ is the set of all arcs entering vertex v. We say that the k-flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for any $e \in A$. The concept of nowhere-zero integer flow was introduced by Tutte in 1954, and the theory of integer flows provides an interesting way to extend theorems about region-coloring planar graphs to general graphs [12, 13] (see also [15]). Tutte observed that every nowhere-zero k-flow on a plane graph gives rise to a k-vertex-coloring of its dual, and vice versa. Thus nowhere-zero integer flow and graph coloring can be

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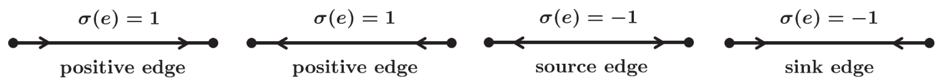

FIG. 1. *Orientations of positive and negative edges.*

viewed as dual concepts, and the above Tutte's observation is often referred to as the *duality theorem*. One of the major open problems in this research area is Tutte's 3-flow conjecture, which is exactly the dual version of Grötzsch's 3-color theorem on planar graphs [3, 4].

CONJECTURE 1.1 (Tutte [12]). *Every* 4*-edge-connected graph has a nowhere-zero* 3*-flow.*

Thomassen [11] made a breakthrough in this conjecture by establishing the following weaker version.

THEOREM 1.1 (Thomassen [11]). *Every* 8*-edge-connected graph has a nowhere-zero* 3*-flow.*

This 3-flow theorem has recently been strengthened by Lovász et al. [8] as follows.

THEOREM 1.2 (Lovász et al. [8]). *Every* 6*-edge-connected graph has a nowhere-zero* 3*-flow.*

As proved by Kochol [7], a minimum counterexample to the 3-flow conjecture is 5-edge-connected. Therefore, the above theorem is actually just one step away from the resolution.

The aforementioned duality theorem cannot be extended directly to embedded graphs. (See DeVos et al. [2] for an asymptotic version.) Nevertheless, Jaeger [5] showed that if a graph G has a face-k-colorable 2-cell embedding in some orientable surface, then it has a nowhere-zero k-flow. Interestingly, if the surface is nonorientable, then this coloring corresponds to a nowhere-zero k-flow in a signed graph arising from G. It is due to their great theoretical interest that integer flows in sign graphs have also been subjects of extensive research.

Let us define a few terms before proceeding. A *signed graph* is a pair (G, σ), where G is a graph and $\sigma : E(G) \to \{1, -1\}$ is a *signature* of G. An edge e is called *positive* if $\sigma(e) = 1$ and *negative* otherwise. Each edge $e = xy$ of a signed graph, (G, σ) is composed of two half-edges h_x and h_y, where h_x is incident with x and h_y is incident with y. An *orientation* D of (G, σ) assigns every half-edge a direction in the following way: if $e = xy$ is positive, then h_x and h_y are directed both from x to y, or both from y to x (see Figure 1); if $e = xy$ is negative, then the directions of h_x and h_y are opposite. (There are two possibilities: (1) h_x is directed to x h_y is directed to y; (2) h_x is directed from x and h_y is directed from y. See Figure 1.)

A negative edge $e = xy$ is called a *source edge* if e is directed toward both x and y, and it is called a *sink edge* otherwise. In the literature, an oriented signed graph is also called a *bidirected graph*. If all edges of (G, σ) are positive, then a signed graph is equivalent to a graph. So we can view signed graphs as generalizations of graphs.

The concept of nowhere-zero integer flow in graphs carries over naturally to signed graphs, and the following is a well-known conjecture on integer flows in signed graphs.

CONJECTURE 1.2 (Bouchet [1]). *Every signed graph admitting a nowhere-zero integer flow has a nowhere-zero* 6*-flow.*

Despite tremendous research effort, this conjecture remains open; Xu and Zhang [14] confirmed it for 6-edge-connected signed graphs. In [10], Raspaud and Zhu established that every 4-edge-connected signed graph has a nowhere-zero 4-flow provided it admits a nowhere-zero integer flow. Based on Theorem 1.2, Zhu [16] obtained the following result recently.

THEOREM 1.3 (Zhu [16]). *Every* 11*-edge-connected signed graph admitting a nowhere-zero integer flow has a nowhere-zero* 3*-flow.*

What is the least edge-connectivity that can guarantee the existence of nowhere-zero 3-flows in signed graphs? Zhu posed this as an open question in [16]. With the motivation to improve the bound in Theorem 1.3 and extend the setting of Theorem 1.1, we establish the following main result in this paper.

THEOREM 1.4. *Every* 8*-edge-connected signed graph admitting a nowhere-zero integer flow has a nowhere-zero* 3*-flow.*

It is worthwhile pointing out that the assertion no longer holds if 8 is replaced by 4: Let (G, σ) be the signed graph with three vertices in which each pair of vertices is connected by precisely one positive edge and precisely one negative edge. Clearly, G is 4-edge-connected and has a nowhere-zero 4-flow. Nevertheless, it is routine to check that G admits no nowhere-zero 3-flow.

In response to Zhu's open question [16], we offer the following conjecture whose validity would imply Tutte's 3-flow conjecture (see Kochol [7]).

CONJECTURE 1.3. *Every* 5*-edge-connected signed graph admitting a nowhere-zero integer flow has a nowhere-zero* 3*-flow.*

2. Operations.

In this section we introduce some operations on signed graphs which will be employed in subsequent proofs.

Flipping. Let (G, σ) be a signed graph and let A be a subset of $V(G)$. Define $\sigma' : E(G) \to \{1, -1\}$ as

$$\sigma'(e) = \begin{cases} -\sigma(e) & \text{if } e \in [A, \bar{A}], \\ \sigma(e) & \text{otherwise,} \end{cases}$$

where $\bar{A} = V(G) \setminus A$ and $[A, \bar{A}]$ is the cut in G consisting of all edges between A and $\bar{A}$. We say that the signed graph (G, σ') is obtained from (G, σ) by *flipping* all edges in $[A, \bar{A}]$.

Two signed graphs (G, σ) and (G, σ') are called *equivalent* if one can be obtained from the other by flipping all edges in a cut. The following two lemmas are well-known facts (see [10] and [16]) in graph theory, that is, that this flipping operation does not affect the existence of a nowhere-zero integer flow in a signed graph.

LEMMA 2.1. *Let (G, σ) and (G, σ') be two equivalent signed graph and let k be a positive integer. Then (G, σ) has a nowhere-zero k-flow if and only if so does (G, σ').*

Throughout we use $n(G, \sigma)$ to denote the minimum number of negative edges contained in a signed graph equivalent to (G, σ).

LEMMA 2.2. *If a signed graph (G, σ) admits a nowhere-zero integer flow, then $n(G, \sigma) \neq 1$.*

Contraction. Let (G, σ) be a signed graph and let A be a subset of $V(G)$. The signed graph obtained from (G, σ) by *contracting* A, denoted by $(G/A, \sigma)$, is the graph arising from (G, σ) by identifying all vertices in A to a single vertex, in which each edge of G with both ends in A becomes a loop, and each edge has the same sign as in (G, σ).

Since the sign of a loop is not effected by a flipping operation, the following statement holds.

LEMMA 2.3. *Let (G, σ) be a signed graph with precisely $n(G, \sigma)$ negative edges. Then $n(G/A, \sigma) = n(G, \sigma)$ for any proper subset A of $V(G)$.* □

Lifting. Let (G, σ) be a signed graph, let xy, xz be two edges of G, and let G' be obtained from G by deleting xy, xz and adding a new edge e_0 between y and z.

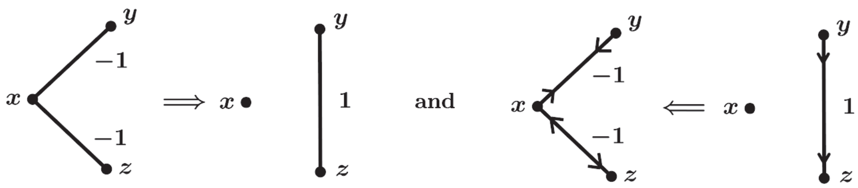

FIG. 2. *A lifting of xy and xz and an orientation extension.*

Define $\sigma' : E(G') \to \{1, -1\}$ as

$$\sigma'(e) = \begin{cases} \sigma(xy)\sigma(xz) & \text{if } e = e_0, \\ \sigma(e) & \text{otherwise.} \end{cases}$$

We say that the signed graph (G', σ') is obtained from (G, σ) by *lifting* xy and xz; see Figure 2 for an illustration. Note that x, y, z are not necessary distinct in this definition.

An orientation of (G', σ') can be extended naturally to an orientation of (G, σ) by orienting the two half-edges incident with x as follows: one enters x and the other leaves x; see Figure 2 for the case when $\sigma(xy) = \sigma(xz) = -1$.

LEMMA 2.4. *Let (G, σ) be a signed graph and let xy, xz be two edges of G. If (G', σ') is the signed graph obtained from (G, σ) by lifting xy and xz, then*

$$n(G', \sigma') \geq n(G, \sigma) - 2.$$

Proof. For each $U \subseteq V(G)$, let $[U, \bar{U}]_{G'}$ (resp., $[U, \bar{U}]_G$) denote the cut consisting of all edges between U and $\bar{U}$ in G' (resp., in G). Suppose the signed graph (G', σ'') obtained from (G', σ') by flipping all edges in a cut $[A, \bar{A}]_{G'}$ has precisely $n(G', \sigma')$ negative edges. Consider the signed graph $(G, \bar{\sigma})$ obtained from (G, σ) by flipping all edges in $[A, \bar{A}]_G$. It is easy to see that the number of negative edges in $(G, \bar{\sigma})$ is at most two plus the number of negative edges in (G', σ''). Hence, $n(G, \sigma) \leq n(G', \sigma') + 2$, as desired. ◻

Let G be a graph and let x, y be two distinct vertices of G. The *local edge-connectivity* of G between x and y, denoted by $\lambda_G(x, y)$, is the maximum number of edge-disjoint paths connecting x and y in G. The following Mader's theorem [9] asserts that the local edge-connectivity is preserved under some lifting operation.

THEOREM 2.5 (Mader [9]). *Let G be a connected loopless graph and let v_0 be a vertex of degree at least 4 such that no edge incident with v_0 is a cut-edge of G. Then G contains two edges v_0v_1 and v_0v_2 such that $\lambda_H(x, y) = \lambda_G(x, y)$ for any two vertices x, y different from v_0, where H is the graph obtained from G by lifting v_0v_1 and v_0v_2.*

3. Orientations: Modulo and beyond. Let (G, σ) be a signed graph. For each $A \subseteq V(G)$, the *degree* of A, denoted by $d(A)$, is the number of edges between A and $\bar{A}$; we write $d(A) = d(a)$ if $A = \{a\}$. (Notice that the contribution to $d(a)$ made by any loop incident with a, if any, is zero.) For each orientation D of (G, σ), let $d^+_D(v)$ (resp., $d^-_D(v)$) denote the number of half-arcs leaving (resp., entering) a vertex v; we may drop the subscript D if there is no danger of confusion. Note that, by definition, each loop incident with v (if any) contributes two to $d^+_D(v) + d^-_D(v)$, so $d(v) < d^+_D(v) + d^-_D(v)$ if such a loop exists.

An orientation D of (G, σ) is called a *modulo* 3*-orientation* if $d^+_D(v) \equiv d^-_D(v) \pmod 3$ for all $v \in V(G)$. As shown by Tutte [12], a graph G admits a modulo 3-orientation if and only if it has a nowhere-zero 3-flow; this equivalence relation can be further extended to signed graphs.

LEMMA 3.1 (Xu and Zhang [14]). *Let (G, σ) be a* 2*-edge-connected singed graph. Then (G, σ) admits a modulo* 3*-orientation if and only if it has a nowhere-zero* 3*-flow.*

To prove Theorem 1.4, we shall actually establish the following assertion.

THEOREM 3.2. *Let (G, σ) be a signed graph with $n(G, \sigma) \geq 2$, and let $V_0 = \emptyset$ or $V_0 = \{v_0\}$, where v_0 is a vertex of G such that no loop is incident with v_0 and that $d(v_0) \leq 6$ and is even. If $|V(G)\backslash V_0| \geq 2$ and $\lambda_G(x, y) \geq 8$ for any distinct vertices x, y in $V(G)\backslash V_0$, then (G, σ) admits a modulo* 3*-orientation.* To see the implication, let (G, σ) be an 8-edge-connected singed graph with a nowhere-zero integer flow. By Lemma 2.2, we have $n(G, \sigma) \neq 1$. From Theorem 1.1 and Lemma 2.1 (if $n(G, \sigma) = 0$) and from Theorem 3.2 with $V_0 = \emptyset$ and Lemma 3.1 (if $n(G, \sigma) \geq 2$), we can thus deduce that (G, σ) admits a nowhere-zero 3-flow.

The remainder of this paper is devoted to a proof of Theorem 3.2. The proof proceeds by induction on $|V(G)| + |E(G)|$; to make the induction work, we need a generalized concept of graph orientation and a set function from [8], which is a variant of the one introduced by Thomassen in [11].

Let G be a loopless graph. A mapping $\beta : V(G) \mapsto \mathbb{Z}_3 = \{0, 1, 2\}$ is called a $\mathbb{Z}_3$*-boundary* of G if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3$ [6]. Given a $\mathbb{Z}_3$-boundary β of G, an orientation D of G is called a β*-orientation* if $d^+_D(v) - d^-_D(v) \equiv \beta(v) \pmod 3$ for all $v \in V(G)$. The set function is a mapping $\tau : V(G) \mapsto \{0, \pm 1, \pm 2, \pm 3\}$ such that

$$\tau(v) \equiv \begin{cases} \beta(v) \pmod 3, \\ d(v) \pmod 2 \end{cases}$$

for all $v \in V(G)$. This mapping τ can be further extended to any nonempty $A \subseteq V(G)$ as follows:

$$\tau(A) \equiv \begin{cases} \beta(A) \pmod 3, \\ d(A) \pmod 2, \end{cases}$$

where $\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod 3$. Since $d(A)$ and $\tau(A)$ have the same parity, the following inequality holds.

LEMMA 3.3 (Lovász et al. [8]). *If $d(A) \geq 6$, then $d(A) \geq 4 + |\tau(A)|$.* □

Theorem 1.2 is an immediate corollary of the following result, which was derived by refining Thomassen's technique [11] and will be used in our proof.

THEOREM 3.4 (Lovász et al. [8]). *Let G be a loopless graph, let β be a $\mathbb{Z}_3$-boundary of G, let $z_0 \in V(G)$, and let $D(z_0)$ be a preorientation of the set $E(z_0)$ of all edges incident with z_0. Assume that*

(i) $|V(G)| \geq 3$*;*

(ii) $d(z_0) \leq 4 + |\tau(z_0)|$ *and* $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod 3$*;*

(iii) $d(A) \geq 4 + |\tau(A)|$ *for each nonempty* $A \subseteq V(G) \setminus \{z_0\}$ *with* $|V(G) \setminus A| \geq 2$*.*

Then $D(z_0)$ can be extended to a β-orientation D of the entire graph G.

When restricted to the disjoint union of an isolated vertex z_0 and a 6-edge-connected loopless graph, the preceding theorem yields the following statement.

THEOREM 3.5 (Lovász et al. [8]). *Let G be a loopless graph and let β be a $\mathbb{Z}_3$-boundary of G. If G is* 6*-edge-connected, then G has a β-orientation.*

We now proceed to prove two technical lemmas, which will play important roles in our proof of Theorem 3.2.

LEMMA 3.6. *Let (G,σ) be a 6-edge-connected signed graph with only 2 or 3 negative edges. Then (G,σ) admits a modulo 3-orientation.*

Proof. Let m be the number of negative edges of (G,σ). Set $r=1$ if $m=2$ and $r=0$ if $m=3$. Let H be the graph obtained from G by first orienting r negative edges as sink edges and the remaining $m-r$ negative edges as source edges, then inserting a new vertex to each negative edge, and finally identifying all these newly inserted vertices to a single vertex z_0. Let $G'=H$ if $m=2$ and let G' be obtained from H by replacing one arc leaving z_0 with two parallel arcs entering z_0 if $m=3$. For each $A\subseteq V(G')$, we use $d'(A)$ and $\tau'(A)$ to denote the degree of A in G' and the value of the set function at A, respectively. If $m=2$, then $d'(z_0)=4\le 4+|\tau'(z_0)|$. If $m=3$, then $d'(z_0)=7$. So $|\tau'(z_0)|=3$ by definition and thus $d'(z_0)=4+|\tau'(z_0)|$. Hence the inequality $d'(z_0)\le 4+|\tau'(z_0)|$ holds in either case. By Lemma 3.3, we have $d'(A)\ge 6\ge 4+|\tau'(A)|$ for each nonempty $A\subseteq V(G')\setminus\{z_0\}$ with $|V(G')\setminus A|\ge 2$. Therefore, by Theorem 3.4, the preorientation of the arcs incident with z_0 can be extended to a modulo 3-orientation of the entire graph G', which clearly yields a modulo 3-orientation of (G,σ). ∎

LEMMA 3.7. *Let G be a loopless graph, let β be a $\mathbb{Z}_3$-boundary of G, let $z_0\in V(G)$, let $D(z_0)$ be a preorientation of the set $E(z_0)$ of all edges incident with z_0, and let $S=\{v\in V(G)\setminus\{z_0\}\mid d(v)=5$ and $\beta(v)=0\}$. Assume that*

(i) $|V(G)|\ge 3$;

(ii) $d(z_0)\le 5$ *and* $d^+(z_0)-d^-(z_0)\equiv\beta(z_0)\pmod 3$;

(iii) $d(v)\ge 4+|\tau(v)|$ *for each* $v\in V(G)\setminus(S\cup\{z_0\})$; *and*

(iv) $d(A)\ge 6$ *for each* $A\subseteq V(G)\setminus\{z_0\}$ *with* $\min\{|A|,|V(G)\setminus A|\}\ge 2$.

If $|S|\le 2$, then $D(z_0)$ can be extended to a β-orientation D of the entire graph G.

Proof. By definition, $d(z_0)$ and $\tau(z_0)$ have the same parity, so $|\tau(z_0)|\ge 1$ if $d(z_0)=5$. Hence, $d(z_0)\le 4+|\tau(z_0)|$. If $S=\emptyset$, then the statement follows instantly from Theorem 3.4. Thus we may assume $S\ne\emptyset$.

Let p be the integer in $\mathbb{Z}_3$ with $\beta(z_0)-d(z_0)+1\equiv 2p\pmod 3$ and let $q=7-d(z_0)-p$. Then $q\ge 0$ and $p+q\ge 2$ as $d(z_0)\le 5$. Let G' be obtained from G by adding a set P of p arcs from S to z_0 and adding a set Q of q arcs from z_0 to S such that each vertex in S has degree at least six in G'. (This G' is available because $|S|\le 2$.) Let $\beta'(z_0)$ be the integer in $\mathbb{Z}_3$ with $\beta'(z_0)\equiv\beta(z_0)+q-p\pmod 3$. By the definitions of p and q, we obtain $\beta'(z_0)\equiv(d(z_0)-1+2p)+(7-d(z_0)-p)-p\equiv 0\pmod 3$. So $\beta'(z_0)=0$. For each vertex $v\ne z_0$, let $P(v)$ (resp., $Q(v)$) be the set of all arcs in P (resp., Q) incident with v, and let $\beta'(v)$ be the integer in $\mathbb{Z}_3$ with $\beta'(v)\equiv\beta(v)+|P(v)|-|Q(v)|\pmod 3$. Then $\sum_{v\in V(G')}\beta'(v)=\sum_{v\in V(G)}\beta(v)+(q-p)+\sum_{v\ne z_0}(|P(v)|-|Q(v)|)=\sum_{v\in V(G)}\beta(v)\equiv 0\pmod 3$. Hence, β' is a $\mathbb{Z}_3$-boundary of G'.

Let $d'(A)$ and $\tau'(A)$ denote the degree of A in G' and the value of the set function at A, respectively. Since $d'(z_0)=7$ and $\beta'(z_0)=0$, we have $|\tau'(z_0)|=3$. So $d'(z_0)=4+|\tau'(z_0)|$. Since $d'(v)\ge 6$ for each $v\in S$, from Lemma 3.3 it follows that $d'(v)\ge 4+|\tau'(v)|$. Therefore, by Theorem 3.4, the preorientation of the arcs incident with z_0 can be extended to a β'-orientation of the entire graph G', which clearly yields a β-orientation of (G,σ). ∎

4. Proof of Theorem 3.2.

The proof proceeds by induction on $|V(G)|+|E(G)|$. Assume on the contrary that (G,σ) is a smallest counterexample and, subject to this, the number of negative edges in (G,σ) is minimum.

For each nonempty proper subset $A\subseteq V(G)$, we use $g(A,\sigma)$ (resp., $h(A,\sigma)$) to denote the number of positive (resp., negative) edges of (G,σ) contained in the cut $[A,\bar{A}]$ of G, and set $g(A,\sigma)=g(a,\sigma)$ (resp., $h(A,\sigma)=h(a,\sigma)$) if $A=\{a\}$.

Claim 1. For each nonempty proper subset $A \subseteq G$, we have $g(A, \sigma) \geq h(A, \sigma)$. Hence, (G, σ) contains exactly $n(G, \sigma)$ negative edges.

Otherwise, $g(A, \sigma) < h(A, \sigma)$. Let (G, σ') be the signed graph obtained from (G, σ) by flipping all edges in the cut $[A, \bar{A}]$. Then the number of negative edges in (G, σ') is less than that in (G, σ). By Lemmas 2.1 and 3.1, (G, σ') admits no modulo 3-orientation. Thus the existence of (G, σ') contradicts the minimality assumption on (G, σ).

From the definition, it follows instantly that (G, σ) contains exactly $n(G, \sigma)$ negative edges. Thus Claim 1 is justified.

Claim 2. $n(G, \sigma) \geq 4$.

Assume the contrary: $n(G, \sigma) = 2$ or 3. By Lemma 3.6, we have $V_0 = \{v_0\}$ and $d(v_0) \leq 4$. In view of Claim 1, $g(v_0, \sigma) \geq h(v_0, \sigma)$. Thus we can partition all the edges incident with v_0 into pairs so that each pair contains at most one negative edge. Let (G', σ') be the signed graph obtained from (G, σ) by lifting each of these edge pairs and deleting the resulting isolated vertex v_0. Then (G', σ') has the same number of negative edges as (G, σ). For each nonempty proper subset $A \subseteq V(G')$, let $d'(A)$ be the degree of A in G' and let $\bar{A} = V(G') \setminus A$. Then $d'(A) + d'(\bar{A}) \geq d(A) + d(\bar{A}) - d(v_0) \geq 8 + 8 - 4 = 12$. Since $d'(A) = d'(\bar{A})$, we have $d'(A) \geq 6$. Thus G' is 6-edge-connected. By Lemma 3.6, (G', σ') admits a modulo 3-orientation, which clearly yields a modulo 3-orientation of (G, σ); this contradiction proves Claim 2.

Claim 3. (G, σ) contains no loops.

Suppose on the contrary that e_1 is a loop incident with a vertex x. Let e_2 be an edge connecting x and one of its neighbors y, and let (G', σ') be the signed graph obtained from (G, σ) by lifting e_1 and e_2. By Claim 2 and Lemma 2.4, we have $n(G', \sigma') \geq n(G, \sigma) - 2 \geq 4 - 2 = 2$. Hence, by induction hypothesis, (G', σ') admits a modulo 3-orientation, which clearly yields a modulo 3-orientation of (G, σ); this contradiction establishes Claim 3.

Claim 4. $|V(G)| \neq 2$.

Otherwise, $|V(G)| = 2$; let $V(G) = \{x, y\}$. By hypothesis, we have $V_0 = \emptyset$. By Claim 3, the edges of (G, σ) are all between x and y. Recall Claim 1; the number of negative edges between x and y is $n(G, \sigma)$, so the number of positive edges between x and y is $|E(G)| - n(G, \sigma) \geq n(G, \sigma) \geq 4$ by Claim 2.

Let p be the integer in $\mathbb{Z}_3$ such that $p \equiv n(G, \sigma) - p \pmod 3$. Orient p negative edges as source edges and the remaining $n(G, \sigma) - p$ negative edges as sink edges.

Let q be the integer in $\mathbb{Z}_3$ such that $q \equiv (E(G) - n(G, \sigma)) - q \pmod 3$. Orient q positive edges from x to y and the remaining $(E(G) - n(G, \sigma)) - q$ positive edges from y to x.

Clearly, the resulting orientation is a modulo 3-orientation of (G, σ); this contradiction implies Claim 4.

Claim 5. $d(v)$ is odd for each $v \in V(G)$. So $V_0 = \emptyset$ and hence $|V(G)| \geq 4$ by Claim 4.

Suppose on the contrary that some vertex of G has even degree; let u be such vertex with the smallest $d(u)$. By Theorem 2.5, G contains two edges uv_1 and uv_2 such that $\lambda_{G'}(x, y) = \lambda_G(x, y)$ for any two distinct vertices x and y different from u, where (G', σ') is the signed graph obtained from (G, σ) by lifting uv_1 and uv_2. Let $d'(v)$ stand for the degree of a vertex v in G'. Then $d'(u) = d(u) - 2$. Depending on the value of $d(u)$, we define V_0' as follows.

CASE 1. $d(u) \leq 6$.

In this case, $V_0 = \{u\}$ because, by hypothesis and Menger's theorem, all vertices except v_0 in V_0 have degree at least eight. If $d'(u) = 0$, with a slight abuse of notation, we still use G' to denote the graph obtained from G' by deleting u, and set $V_0' = \emptyset$. If $d'(u) > 0$, set $V_0' = \{u\}$. Since $V(G')\backslash V_0' = V(G)\backslash V_0$, by hypothesis $|V(G')\backslash V_0'| \geq 2$.

Case 2. $d(u) \geq 8$.

In this case, $V_0 = \emptyset$ by the choice of u. If $d(u) \geq 10$, then $d'(u) \geq 8$; set $V_0' = \emptyset$. If $d(u) = 8$, then $d'(u) = 6$; set $V_0' = \{u\}$. By Claim 4, $|V(G')\backslash V_0'| \geq |V(G)\backslash\{u\}| \geq 2$.

In either case, by Claim 2 and Lemma 2.4, we obtain $n(G', \sigma') \geq n(G, \sigma) - 2 \geq 4 - 2 = 2$. Thus, by induction hypothesis, (G', σ') admits a modulo 3-orientation, which clearly yields a modulo 3-orientation of (G, σ), a contradiction. So Claim 5 is established.

Claim 6. For each $v \in V(G)$, either $g(v, \sigma) \geq 6$ or $g(v, \sigma) = 5$ and $h(v, \sigma) = 4$.

By Claim 1, $g(v, \sigma) \geq h(v, \sigma)$. By Claim 5, $g(v, \sigma) + h(v, \sigma)$ is odd. By hypothesis, $g(v, \sigma) + h(v, \sigma) \geq 8$ and hence is at least 9. So the statement follows.

Claim 7. For some nonempty proper subset $A \subseteq V(G)$, we have $g(A, \sigma) \leq 5$.

Suppose on the contrary that $g(A, \sigma) \geq 6$ for each nonempty proper subset $A \subseteq V(G)$. Let G' be the graph obtained from G by deleting all negative edges. Then G' is 6-edge-connected. By Claim 3, G' is also loopless.

Let p be the integer in $\mathbb{Z}_3$ such that $p \equiv n(G, \sigma) - p \pmod 3$. We partition the set of all negative edges into two subsets P and Q with $|P| = p$. Then $|Q| = n(G, \sigma) - p$ by Claim 1. Let us orient all negative edges in P (resp., in Q) as source (resp., sink) edges. For each $v \in V(G')$, let $P(v)$ (resp., $Q(v)$) be the set of all arcs in P (resp., Q) incident with v, and let $\beta'(v)$ be the integer in $\mathbb{Z}_3$ with $\beta'(v) \equiv |P(v)| - |Q(v)| \pmod 3$. Clearly, $\sum_{v \in V(G')} \beta'(v) \equiv 0 \pmod 3$. So β' is a $\mathbb{Z}_3$-boundary of G'.

By Theorem 3.5, (G', σ') admits a β-orientation, which clearly yields a modulo 3-orientation of (G, σ); this contradiction justifies Claim 7.

In the remainder of our proof, we reserve the symbol A for a nonempty proper subset of $V(G)$ such that

$$g(A, \sigma) \leq 5; \tag{1}$$

$$|A| \geq 2; \text{ and} \tag{2}$$

$$g(B, \sigma) \geq 6 \text{ for any } B \subseteq A \text{ with } 2 \leq |B| < |A|. \tag{3}$$

Such A is available because $|A| + |\bar{A}| \geq 4$ by Claim 5; we may interchange A and $\bar{A}$ if $|A| = 1$. By hypothesis, $d(A) \geq 8$. So $h(A, \sigma) = d(A) - g(A, \sigma) \geq 8 - g(A, \sigma)$. By (1), we thus have

$$h(A, \sigma) \geq 3. \tag{4}$$

Let $k(A, \sigma)$ be the number of negative edges with both ends in A. By Lemma 2.3 and Lemma 2.4, we obtain $n(G/A, \sigma) = n(G, \sigma) \geq k(A, \sigma) + h(A, \sigma)$. It follows from (4) that

$$n(G/A, \sigma) - k(A, \sigma) \geq 3. \tag{5}$$

Let v_A be the vertex of $(G/A, \sigma)$ resulting from contracting A. By Claim 3, all loops of $(G/A, \sigma)$ are incident with v_A, and precisely $k(A, \sigma)$ of them are negative. By (1) and Claim 1, we have $d(A) \leq 10$. By Claim 5, $V_0 = \emptyset$, so the minimum degree of G is at least eight by Menger's theorem, and hence some edge of G has two ends in A (see (2)). Let (G', σ') be the signed graph obtained from $(G/A, \sigma)$ by replacing all loops incident with v_A by a new loop e, such that

$$\sigma'(e) = 1 \text{ if } k(A, \sigma) \equiv 0 \pmod 3 \text{ and } \sigma'(e) = -1 \text{ otherwise.} \tag{6}$$

Notice that e does not necessarily correspond to an edge of G. Let $d'(U)$ stand for the degree of U in G' for each $U \subseteq V(G')$. Since $d(A) \geq 8$, we have $d'(v_A) \geq 8$. Set

$V_0' = \emptyset$. It is clear that

- $|V(G')\backslash V_0'| = |V(G)\backslash A| + 1 \geq 2$;
- $n(G', \sigma') \geq n(G/A, \sigma) - k(A, \sigma) \geq 3$ by (5); and
- $\lambda_{G'}(x, y) \geq 8$ for any two vertices x and y of G' by Menger's theorem.

Thus, by (2) and induction hypothesis, (G', σ') has a modulo 3-orientation D', which yields a partial orientation of (G, σ). Reversing the directions of all half-arcs in D' if necessary, we may assume that

(7) $$e \text{ is a source edge in } D' \text{ when } \sigma'(e) = -1.$$

Let G'' be the loopless graph (with no signature) obtained from the signed graph $(G/\bar{A}, \sigma)$ by first deleting all negative edges and then deleting all loops incident with z_0, the vertex arising from contracting $\bar{A}$. We orient all edges between A and z_0 in G'' as follows: Suppose edge xz_0 in G'' with $x \in A$ corresponds to edge $v_A y$ in G' with $y \in \bar{A}$. Then the direction of xz_0 in G'' is exactly the same as the direction of $v_A y$ in D'. For convenience, we denote this preorientation of edges incident with z_0 by $D(z_0)$. Let $p(z_0)$ (resp., $q(z_0)$) be the number of all resulting arcs entering (resp., leaving) z_0; we define $\beta''(z_0)$ to be the integer in $\mathbb{Z}_3$ with $\beta''(z_0) \equiv q(z_0) - p(z_0) \pmod 3$.

Let F_1 be the set of all negative edges of G with both ends in A. Recall that

(8) $$|F_1| = k(A, \sigma).$$

We orient all edges in F_1 as sink edges if $k(A, \sigma) \equiv 2 \pmod 3$, and orient all edges in F_1 as source edges otherwise. Let F_2 be the set of all negative edges between A and $\bar{A}$ in G; for each edge $f \in F_2$, we orient it as in D'. Set $F = F_1 \cup F_2$. For each $v \in A$, let $p(v)$ (resp., $q(v)$) be the number of all half-arcs entering (resp., leaving) v in F; we define $\beta''(v)$ to be the integer in $\mathbb{Z}_3$ with $\beta''(v) \equiv p(v) - q(v) \pmod 3$. We propose to show that

(9) $$\beta'' \text{ is a } \mathbb{Z}_3\text{-boundary of } G''.$$

To justify this, let p_1 (resp., q_1) be the number of positive edges directed from A to $\bar{A}$ (resp., from $\bar{A}$ to A) in D', and let p_2 (resp., q_2) be the number of source (resp., sink) edges between A and $\bar{A}$ in D'. Note that

(10) $$p_1 = p(z_0) \text{ and } q_1 = q(z_0).$$

Since $d^+_{D'}(v_A) \equiv d^-_{D'}(v_A) \pmod 3$, the following equality holds.

(11) $$p_1 + q_2 \equiv q_1 + p_2 \pmod 3 \text{ if } \sigma'(e) = 1$$
$$\text{and } p_1 + q_2 \equiv q_1 + p_2 + 2 \pmod 3 \text{ if } \sigma'(e) = -1.$$

Observe that in F there are precisely p_2 half-arcs entering A and precisely q_2 half-arcs leaving A. By direct computation, we obtain

(12) $$\sum_{v \in A} \beta''(v) = p_2 - q_2 - 2|F_1| \text{ if } k(A, \sigma) \equiv 2 \pmod 3 \text{ and}$$
$$\sum_{v \in A} \beta''(v) = p_2 - q_2 + 2|F_1| \text{ otherwise.}$$

If $k(A, \sigma) \equiv 0 \pmod 3$, then, by (6) and (8), we have $\sigma'(e) = 1$ and $|F_1| \equiv 0 \pmod 3$.

It follows from (12), (10), and (11) that $\sum_{v\in A}\beta''(v)+\beta''(z_0)\equiv p_2-q_2+q_1-p_1\equiv 0 \pmod 3$.

If $k(A,\sigma)\equiv 1 \pmod 3$, then, by (6) and (8), we have $\sigma'(e)=-1$ and $|F_1|\equiv 1 \pmod 3$. It follows from (12), (10), and (11) that $\sum_{v\in A}\beta''(v)+\beta''(z_0)\equiv p_2-q_2+2+q_1-p_1\equiv 0 \pmod 3$.

If $k(A,\sigma)\equiv 2 \pmod 3$, then, by (6) and (8), we have $\sigma'(e)=-1$ and $|F_1|\equiv 2 \pmod 3$. It follows from (12), (10), and (11) that $\sum_{v\in A}\beta''(v)+\beta''(z_0)\equiv p_2-q_2-4+q_1-p_1\equiv 0 \pmod 3$.

Combining the above three cases, we arrive at (9).

Let us now verify that G'' satisfies all the hypotheses of Lemma 3.7. By (2), we have $|V(G'')|\geq |A|+1\geq 3$. From (1) and the construction of G'', we see that $d_{G''}(z_0)=g(A,\sigma)\leq 5$; with respect to $D(z_0)$, the equality $d^+(z_0)-d^-(z_0)\equiv\beta(z_0) \pmod 3$ clearly holds. For each $v\in V(G'')\backslash(S\cup\{z_0\})$, we have $d''(v)\geq 5$ by Claim 6. If $d''(v)\geq 6$, then $d''(v)\geq 4+|\tau''(v)|$ by Lemma 3.3. If $d''(v)=5$, then $|\tau''(v)|=1$ since $\beta''(v)\neq 0$ by definition of S, and hence $d''(v)=4+|\tau''(v)|$. Each $B\subseteq V(G'')\setminus\{z_0\}$ with $\min\{|B|,|V(G'')\setminus B|\}\geq 2$ is a proper subset of A, so $d''(B)\geq 6$ by (3). Moreover, for each $v\in S$, Claim 6 implies $g(v,\sigma)=5$ and $h(v,\sigma)=4$. Since $\beta''(v)=0$ and since negative edges with both ends in A are either all source edges or all sink edges, there are at least two negative edges between v and z_0. Since $h(v,\sigma)\leq 5$ by (1) and Claim 1, we obtain $|S|\leq 2$. Thus, by Lemma 3.7, $D(z_0)$ can be extended to a β''-orientation D'' of the entire graph G. Combining D'' with $D'\setminus\{e\}$, we obtain a modulo 3-orientation of (G,σ); this contradiction completes the proof of our theorem. ∎

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