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Title: A new schedule-based transit assignment model with travel strategies and supply uncertainties

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Statement of contributions

A transit assignment model is useful in estimating or predicting how passengers utilize a given transit system. In the literature of transit assignment studies, these models used either frequency-based (static) or schedule-based (dynamic) approach to model transit route choice. The optimal strategy approach is one of the commonly adopted formulations in these approaches. However, most of existing related studies did not consider the effect of uncertainty in transit networks on route choice.

In fact, due to supply side uncertainty, in-vehicle travel times and waiting times, especially for buses and mini-buses, are highly uncertain. Studies such as Jackson and Jucker (1982) and Szeto et al. (2011b) found that travel time uncertainty does affect the route choice of passengers. It is essential to capture this realistic travel behavior into the transit modeling framework. Therefore, transit assignment models have recently emphasized the influence of uncertainties in frequency-based frameworks and their transit network design applications (Yang and Lam, 2006; Li et al. 2008, 2009; Sumalee et al., 2011, Szeto et al., 2011a, b). These transit assignment models can be used to study aggregated stochastic effects of a specific line from a static perspective. However, uncertainties exist in both vehicle running and dwelling process in line operation and the schedule-based models provide a means to investigate uncertainties within the vehicle process (Zhang et al., 2010). Hence, Zhang et al. (2010) developed a schedule-based transit assignment model to capture the uncertainties. Nevertheless, they proposed a path-based model and hence path enumeration or column generation is needed to obtain solutions. Optimal strategies and hence the concept of set of attractive lines are also not explicitly considered in their model.

The objective of the paper is to extend the schedule-based transit assignment model in Hamdouch and Lawphongpanich (2008) to consider supply uncertainties in the transit network and optimal strategies. This extension is not straightforward, as the resultant problem is a stochastic and dynamic optimization problem. We propose an analytical model that captures the stochastic nature of the transit schedules and in-vehicle travel times due to road conditions, incidents or adverse weather. We adopt a mean variance approach that can consider the covariance of travel time between links in a space time graph but still lead to a robust transit network loading procedure when optimal strategies are adopted. The method of successive averages (MSA) is adopted to solve the model. Numerical studies are performed to illustrate the properties of the model and the effectiveness of the algorithm. This paper differs from Zhang et al. (2010) in threefold. First, this paper adopts a mean-variance approach to consider strategies while they adopt effective travel cost as the factor affecting passengers' line choice. Second, their model is path-based and requires path enumeration and column generation, but ours is strategy-based and relies on Bellman's recursion principle to deal with network loading. Third, we consider hard capacity constraints but they consider a chance constraint for dealing with the capacity.

The contributions of this paper include the following:

1. This paper proposes a schedule-based transit assignment model with the consideration of both supply uncertainties and optimal strategies.
2. The proposed solution method does not rely on path enumeration or column generation technique. The transit network loading procedure relies on the usage of Bellman’s recursion principle, and is quite robust.
3. The model and the solution method allow us to evaluate the performance of transit systems under supply uncertainties, assess the effectiveness of operational strategies, and develop a larger model to plan transit schedules.

A new schedule-based transit assignment model with travel strategies and supply uncertainties

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Abstract

This paper proposes a new scheduled-based transit assignment model. Unlike other schedule-based models in the literature, we consider supply uncertainties and assume that users adopt strategies to travel from their origins to their destinations. We present an analytical formulation to ensure that on-board passengers continuing to the next stop have priority and waiting passengers are loaded on a first-come-first-serve basis. We propose an analytical model that captures the stochastic nature of the transit schedules and in-vehicle travel times due to road conditions, incidents, or adverse weather. We adopt a mean variance approach that can consider the covariance of travel time between links in a space-time graph but still lead to a robust transit network loading procedure when optimal strategies are adopted. The proposed model is formulated as a user equilibrium problem and solved by an MSA-type algorithm. Numerical results are reported to show the effects of supply uncertainties on the travel strategies and departure times of passengers.

Keywords: User equilibrium; Schedule-based transit assignment; Strategy; Supply uncertainty

1 Introduction

A transit assignment model is useful in estimating or predicting how passengers utilize a given transit system. In the literature of transit assignment studies, these models

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*Corresponding author, Email: ceszeto@hku.hk used either the frequency-based (static) or the schedule-based (dynamic) approach to model transit route choice. Similar to the traditional static user equilibrium assignment models, frequency-based transit assignment models (Spiess and Florian, 1989; De Cea and Fernandez, 1993; Cantarella, 1997; Lam et al., 1999, 2002; Kurauchi et al., 2003; Cepeda et al., 2006; Schmöcker et al., 2009; Sumalee et al., 2009; Schmöcker et al., 2011; Cortés et al., 2013; Trozzi et al., 2013; Szeto and Jiang, 2014) often assume that passengers select transit routes to minimize their perceived expected travel cost, and departure time is not the concern. These static transit assignment models are commonly adopted for the strategic and long-term planning/evaluation of transit networks.

Schedule-based transit assignment models (Wilson and Nuzzolo, 2004; Poon et al., 2004; Hamdouch and Lawphongpanich, 2008; Hamdouch et al., 2011; Zhang et al., 2010; Nuzzolo et al., 2012) are typically dynamic and are better suited to short-term transit operations and service planning such as transit timetabling and vehicle scheduling. In a schedule-based model, the temporal dimension is the most important part as it is assumed that transit passengers choose not only their transit routes, but also their departure times for minimizing their individual generalized cost. Researchers incorporate this time dependent choice in different ways which is classified by Poon et al. (2004) as (a) diachronic graph representation (Nuzzolo et al., 2001); (b) dual graph representation (Moller-Pedersen, 1999); (c) forward star network formulation (Tong and Wong, 1998), and; (d) space-time formulation (Nguyen et al., 2001; Hamdouch and Lawphongpanich, 2008; Hamdouch et al., 2011). In the last representation, the schedule-based transit network is represented by a time-expanded graph. This graph has an explicit representation of single runs and allows a more straightforward treatment of congestion when capacity constraints are considered. Moreover, it can explicitly represent passenger movements through the in-vehicle and waiting links in the space-time network. This representation and the first one both consider space-time nodes and links. However, a time-expanded network is built on a two dimension graph with one time axis and one space axis. A diachronic network is built in a three dimension graph with two space axes and one time axis.

To model the route choice, one commonly approach is to adopt the concept of optimal strategy. In the frequency-based approach, the core idea for an optimal strategy is that a traveler selects, at each node of the network, a set of attractive lines that allows him/her to reach his/her destination at a minimum expected cost (Spiess and Florian, 1989; Wu et al., 1994; Cepeda et al., 2006; Schmöcker et al., 2009). Different from the previous static models, Hamdouch and Lawphongpanich (2008) developed a dynamic schedule-based transit assignment where the choice of strategy is an integral part of user behavior.

In that study, passengers specified their individual travel strategy by providing, at each transit station and each point in time, an ordered list of transit lines they preferred to use to continue their own journey. For a given passenger, the user-preference set at each time-expanded (TE) node collectively yielded a set of potential paths that departed from the passenger's origin at the same time and generally arrived at the destination at different times. Also, when loading a transit vehicle at a station, on-board passengers continuing to the next station remained on the vehicle and waiting passengers were loaded in a first-come-first-serve (FCFS) basis. To explicitly consider vehicle capacities, the model assigned the fail-to-board passengers to the wait arc to wait for their next preferred transit services with residual capacities. Hamdouch et al. (2011) extended the model in Hamdouch and Lawphongpanich (2008) to differentiate the discomfort level experienced by the sitting and standing passengers. Each class of passengers, grouped by their remaining journey lengths and times already spent on-board, was assigned success-to-sit, success-to-stand, and failure-to-board probabilities. These probabilities were computed by performing a dynamic network loading. The stimulus of a standing passenger to sit increased with his/her remaining journey length and time already spent on-board. When a vehicle was full, passengers unable to board must wait for the next vehicle to arrive.

The above studies do not consider the effect of the uncertainties of transit networks on route choice. In fact, due to supply side uncertainties, in-vehicle travel times and waiting times, especially for buses and mini-buses, are highly uncertain. Studies such as Jackson and Jucker (1982) and Szeto et al. (2011b) found that travel time uncertainty does affect the route choice of passengers. It is essential to capture this realistic travel behaviour into the transit modelling framework. Therefore, transit assignment models have recently emphasized the influence of uncertainties in the frequency-based framework and their transit network design applications (Yang and Lam, 2006; Li et al., 2008, 2009; Sumalee et al., 2011; Szeto et al., 2011b, 2013) as in traffic assignment (Shao et al., 2006; Szeto et al. 2011a). These transit assignment models can be used to study the aggregated stochastic effects of transit lines from a static perspective. However, uncertainties exist in both the vehicle running and dwelling processes in line operation and the schedule-based models provide means to investigate uncertainties within the vehicle processes (Zhang et al., 2010). Hence, Zhang et al. (2010) developed a schedule-based transit assignment model to capture the uncertainties, wherein they adopted the effective travel cost as the factor affecting the route choice of passengers and considered chance constraint for dealing with the capacity. Nevertheless, they proposed a path-based model and hence path enumeration or column generation is needed to obtain solutions. Optimal strategies and hence the concept of the set of attractive lines are also not explicitly considered in their model.

The objective of the paper is to extend the schedule-based transit assignment model proposed by Hamdouch and Lawphongpanich (2008) to consider supply uncertainties in the transit network, optimal strategies, and hard capacity constraints. This extension is not straightforward, as the resultant problem is a stochastic and dynamic optimization problem. We propose an analytical model that captures the stochastic nature of the transit schedules and in-vehicle travel times due to road conditions, incidents, or adverse weather. We adopt a mean variance approach that can consider the covariance of travel time between links in a space-time graph but still lead to a robust transit network loading procedure when optimal strategies are adopted. We formulate the problem as a user equilibrium problem. We adopt a user equilibrium (UE) framework instead of a stochastic user equilibrium (SUE) framework because of the following:

i) It is easier to illustrate the concept of travel strategy and the model formulation clearly and analyze the model properties without being smeared by other factors such as the perception error of passengers on travel costs.

ii) SUE transit assignment models require a probabilistic choice model to depict the travel choice behavior of passengers. However, a realistic choice model always has some limitations. For example, the Probit model used in SUE transit assignment (e.g., Nielsen, 2000 and Nielsen and Frederiksen, 2006) relies on simulation that suffers from computational burden. The Logit model used in transit assignment models (e.g., Lam et al., 1999; 2002) suffer from the path overlapping issue. Solving C-Logit (Cassetta et al., 1996) and other path-based choice models often requires a path set generation or path enumeration algorithm, and an efficient link based algorithm that obviates the path set generation or enumeration procedure has not yet been developed to solve these models.

iii) A UE framework has a good mathematical property that allows the dynamic programming technique to be used during the solution process. The technique does not rely on path set generation or path enumeration during that process.

The proposed model is formulated as a variational inequality (VI) model, unlike the nonlinear complementarity problem (NCP) model (e.g., Lo et al. 2003) and the fixed point (FP) model (Cantarella, 1997) in the transit assignment literature. Nevertheless, according to Nagurney (1993), our proposed VI model can be reformulated into an NCP model and a FP model so that other solution techniques developed for solving NCP and FP models can be used. In this paper, the method of successive averages, which is often used to solve FP models, is adopted to solve our model. Numerical studies are given to illustrate the effects of supply uncertainties, vehicle capacity, and early/late arrival penalty parameters on travel strategies and/or departure times of passengers. The effects of the value of travel time variability (which was termed by Jenelius (2012) and Brjesson et al. (2012)) or equivalently the degree of risk aversion (termed by Jackson and Jucker (1982)) are also investigated.

The contributions of this paper include the following:

i) This paper proposes a schedule-based transit assignment model with the consideration of both supply uncertainties and optimal strategies.

ii) The solution method developed does not rely on any path enumeration or column generation technique. The transit network loading procedure relies on the usage of Bellman's recursion principle, and is quite robust.

iii) The model and the solution method allow us to evaluate the performance of transit systems under supply uncertainties, assess the effectiveness of operational strategies under these uncertainties, and develop a larger model to plan transit schedules to cope with these uncertainties.

For the remainder, Section 2 presents the network representation, notations, and assumptions of the proposed model. Section 3 depicts how to determine the mean and variance travel times and arrival probabilities. Travel strategies and the computation of the effective strategy costs are described in Section 4. Section 5 formulates the transit assignment problem as a variational inequality and proposes an MSA-based solution algorithm. Section 6 presents numerical results and Section 7 discusses the applicability of our model in real-life applications. Finally, Section 8 concludes the paper.

2 Network representation, notations, and assumptions

2.1 Network representation

Consider a transit network that consists of nodes and arcs. Nodes include origins, destinations, and station nodes where a transit vehicle stops to load and unload passengers. Arcs are used to connect nodes. They consist of walk arcs and in-vehicle arcs. An example is given in Figure 1 that displays a transit system with two origin nodes q and o, two destination nodes r and y, and three transit lines l_1, l_2, and l_3. Nodes labeled a, b, c, and d are station nodes. In this example, there are four walk arcs: two access arcs (q, a) and (o, b), and two egress arcs (d, r) and (c, y). The remaining arcs correspond to route segments of the three transit lines. As an example, Line 1 or l_1 begins its route at node a, travels to node b, then to node c, and finally terminates at node d. Thus, $\{a, b, c, d\}$ is the route sequence associated with line l_1.

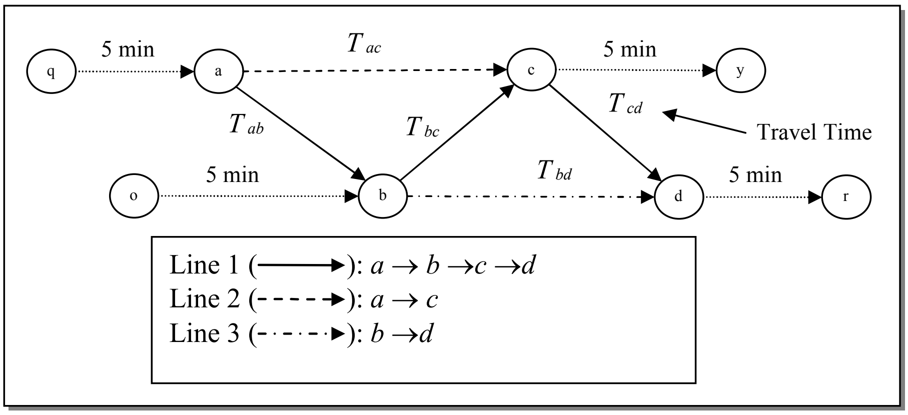

Figure 1: A small network with three transit lines

In the transit network, the number next to each arc (j, k) is the "travel time" T_{jk}. For walk arcs, T_{jk} is assumed to be constant ($T_{jk} = t_{jk}$) and represents the time to walk from j to k. When (j, k) corresponds to a transit-line segment, $\{T_{jk}\}$ is assumed to follow a discrete distribution with the probabilities $P_{jk}(t)$, a mean $E(T_{jk}) = \mu_{jk}$ and a variance $Var(T_{jk}) = \sigma^2_{jk}$.

As in Hamdouch and Lawphongpanich (2008) and Hamdouch et al. (2011), we use a time-expanded (TE) approach to model transit supply in a schedule-based setting. The time horizon is represented as a set of discrete points of the form $\Gamma = \{t_0, t_0 + \delta, t_0 + 2\delta, \cdots, t_0 + n\delta\}$, where δ is the duration of each time interval and $\Omega = \{0, 1, 2, 3, \cdots, n\}$ is the set of time intervals. All time related variables in the model are then specified as a multiple of δ. In general, each node j in the transit network is expanded into multiple nodes j_τ, where $\tau \in \Omega$, in the TE network. Similarly, an in-vehicle arc (j, k) in the transit network is expanded into multiple in-vehicle arcs $(j_\tau, k_{\tau''})$ where τ'' denotes the time interval to reach node k. Similarly, arcs (q, k) and (j, r) are expanded into multiple access arcs $(q_\tau, k_{(\tau+T_{qk})})$, and egress arcs $(j_\tau, r_{(\tau+T_{jr})})$, respectively. These two types of arcs represent walking from an origin to a station and from another station to a destination, respectively. In addition, there are arcs of the form $(j_\tau, j_{\tau+1})$ that represents passengers having to wait at station j from time τ to $(\tau + 1)$.

2.2 Notations

Sets

N	set of nodes (with $i, j, k \in N$)
A	set of arcs (with $a \in A$)
Ω	set of time intervals (with τ, τ', $\tau'' \in \Omega$, where τ stands for the time interval considered; τ' and τ'', respectively, represent the arrival time interval not later and earlier than the current time interval, i.e., $\tau' \leq \tau$,$\tau'' \geq \tau$
L	set of transit lines (with $l \in L$)
L_j	set of transit lines that traverse node j
L_{jk}	set of lines traversing on arc (j, k) with N_{jk} its cardinality
$I^+(j_\tau)$	set of successor nodes for the time-expanded node j_τ
$I^-(j_\tau)$	set of predecessor nodes for the time-expanded node j_τ
$S_{(q,r)}$	set of strategies for OD pair (q, r) (with $s \in S_{(q,r)}$)
$E_j^{s,\tau}$	user-preference set for strategy s, node j, and time τ
$W_j^{\tau,\tau'}$	set of passengers who have reached node j at time $\tau' \leq \tau$
$W_j^{\tau,0,1}$	set of passengers who have continuance priority at node j at time τ and travel on the run with the highest probability to reach node j at time τ
$\{j_1(l), j_2(l), \ldots, j_{N_l}(l)\}$	set of route sequence nodes associated with line l
$\{DT_{1,j_n(l)}, DT_{2,j_n(l)}, \ldots, DT_{M_l,j_n(l)}\}$	set of the departure/arrival times at transit node $j_n(l)$ with the first subscript is for run
$M_{j_\tau,l}$	set of runs of line l that have positive probabilities to reach node j at time τ

Parameters

δ	duration of a time interval
$d^g_{(q,r)}$	travel demand for OD pair (q,r) and group g
$[t^-_{(q,r)}(g), t^+_{(q,r)}(g)]$	desired arrival time interval for OD pair (q,r) and group g
t_{jk}	travel time for access/egress arc $a=(j,k)\in A$
T_{jk}	random travel time for in-vehicle arc $a=(j,k)\in A$
$Y^{l_m}_{jk}$	random travel time for arc (j,k) and transit line l_m under no effects from the previous arc
μ_{jk}	mean travel time for in-vehicle arc $a=(j,k)\in A$
σ^2_{jk}	variance of the travel time for in-vehicle arc $a=(j,k)\in A$
ϕ, ϕ^l_0, ϕ^l_m	coefficients used in the autoregressive model
c_{jk}, c^l_{jk}	constants used in the autoregressive model
$P_{jk}(t)$	probability that the travel time T_{jk} is equal to t
M_l	number of runs for transit line l (with $1\le m\le M_l$)
N_l	number of transit nodes for line l (with $1\le n\le N_l$)
$P_{m,j_n(l)}(\tau)$	probability that the departure/arrival time for the m^{th} transit vehicle at node $j_n(l)$ is equal to τ
u^τ_{jk}	transit capacity for arc (j,k) at time τ
u^τ_{jkml}	transit capacity for the m^{th} run of line l serving arc (j,k) at time τ
v^τ_{jk}	transit fare on arc (j,k) at time τ
η^g_1	early arrival penalty (in monetary units) for group g
η^g_2	late arrival penalty (in monetary units) for group g
η^g_3	value of travel time variability for group g
η_4	crowding penalty (in monetary units)
γ_{travel}	value of time for travelling
γ_{wait}	value of time for waiting
$e^{\tau,g}_{jr}$	late penalty cost for egress arc $(j_\tau, r_{(\tau+T_{jr})})$ and group g
$\widehat{e}^\tau_{jk}$	crowding cost function on arc (j,k) at time τ

Decision variables

$x^{s,\tau^s}_{(q,r,g)}$	number of passengers for OD pair (q,r) and group g assigned to strategy s and who leaves q at time τ^s (starting time of strategy s)
X	strategy assignment (SA) vector (with its components $x^{s,\tau^s}_{(q,r,g)}$)

Functions of decision variables

$f_{jkml}^{s,\tau,\tau''}$	number of passengers using strategy s and traveling on arc $(j_\tau, k_{\tau''})$ and run m of line l
$f_{jk}^{s,\tau,\tau''}$	number of passengers using strategy s and traveling on arc $(j_\tau, k_{\tau''})$ $(f_{jk}^{s,\tau,\tau''} = \sum\limits_{l \in L_j} \sum\limits_{m \in M_l} f_{jkml}^{s,\tau,\tau''})$
$f_{jk}^{s,\tau}$	number of passengers using strategy s and traveling on arc (j, k) at time τ $(f_{jk}^{s,\tau} = \sum\limits_{\tau'' \geq \tau} f_{jk}^{s,\tau,\tau''})$
f_{jk}^{τ}	number of passengers traveling on arc (j, k) at time τ
$\pi_{jk}^{s,\tau,\tau''}$	probability that a passenger using strategy s travels on arc $(j_\tau, k_{\tau''})$
$\pi_{jk}^{s,\tau}$	probability that a passenger using strategy s accesses arc (j, k) at time τ $(\pi_{jk}^{s,\tau} = \sum\limits_{\tau'' \geq \tau} \pi_{jk}^{s,\tau,\tau''})$
$\pi_{j}^{s,\tau}$	probability that a passenger using strategy s waits at node j from time τ to time $\tau + 1$
$z_{jml}^{s,\tau,\tau'}$	number of passengers using strategy s, travelling on run m of line l, and having reached node j at time $\tau' \leq \tau$; $\tau' = 0$ represents the case that these passengers have continuance priority
$z_{jml}^{s,\tau}$	number of passengers using strategy s and travelling on run m of line l who reach node j at time τ
$z_{j}^{s,\tau,\tau'}$	number of passengers using strategy s and having reached node j at time $\tau' \leq \tau$; $\tau' = 0$ represents the case that these passengers have continuance priority
$z_{j}^{s,\tau}$	number of passengers using strategy s who reach node j at time τ $(z_{j}^{s,\tau} = \sum\limits_{l \in L_j} \sum\limits_{m \in M_l} z_{jml}^{s,\tau})$
$Y_{j}^{s,\tau}$	random variable representing the node selected from the preference set $E_{j}^{s,\tau}$
C_{jk}^{g}	random cost associated with link (j, k) for passenger group g
$C_{(q,r,g)}^{s,\tau^s}$	cost for passenger group g reaching destination r from origin q using strategy s at time τ^s
C	vector of strategy costs (with its components $C_{(q,r,g)}^{s,\tau^s}$)
$EC_{(q,r,g)}^{s,\tau^s}$	effective cost for passenger group g reaching destination r from origin q using strategy s at time τ^s
EC	vector of effective strategy costs (with its components $EC_{(q,r,g)}^{s,\tau^s}$).

2.3 Model assumptions

Seven main assumptions are made within the model as in the literature and are presented below.

i) The demand for each OD pair and group g is assumed to be fixed. However, network uncertainties are incorporated in the model through the stochastic nature of the transit schedules and in-vehicle travel times due to road conditions, incidents or adverse weather.

ii) The dwelling time (time for passengers to board and alight) is negligible and the mean travel time, μ_{jk}, denotes the difference between the scheduled departure times (arrival times) at stations j and k.

iii) When loading a vehicle, on-board passengers continuing to the next station remain on the transit vehicle and waiting passengers are loaded on a First-Come-First-Serve (FCFS) basis.

iv) Transit fares are collected based on arcs. This assumption is reasonable for the cases of additive or distance-based fare structures. (i.e., the fares are directly proportional to the travel distance or time.) However, if the fares are not directly proportional to the travel distance or the fares are non-additive over arcs (such as the zone-based fare), one can construct a direct in-vehicle arc between each pair of connected nodes in the TE network. The drawback is an increase in the number of arcs in the TE network (see, e.g., Lo et al. (2003) for more details).

v) All wait arcs have zero fares, zero penalties, and infinite capacities.

vi) All access and egress arcs have zero fares and infinite capacities. However, there are penalties associated with egress arcs to account for lost opportunities associated with arrivals outside the desired interval. Typically, these penalties are different for various groups because of their different values of time or trip purposes.

For egress arcs $(j_\tau, r_{(\tau+T_{jr})})$, one form of such penalty is as follows:

$$e_{jr}^{\tau,g} = \eta_1^g \max\{0, t_{(q,r)}^-(g) - (\tau + T_{jr})\} + \eta_2^g \max\{0, (\tau + T_{jr}) - t_{(q,r)}^+(g)\}. \quad (1)$$

vii) All in-vehicle arcs have transit fares and transit capacities. In addition, there is a discomfort penalty for having too many passengers on board. For example, such a discomfort function can be defined as follows:

$$\widehat{e}_{jk}^\tau(f_{jk}^\tau) \; = \; \eta_4 \left(\frac{f_{jk}^\tau}{u_{jk}^\tau} \right)^2, \quad (2)$$

where u^{τ}_{jk} is the capacity of arc (j,k) at time τ, and f^{τ}_{jk} is the total number of passengers on arc (j,k) at time τ.

3 Travel time and arrival probabilities

Let $L_j \subset L$ be the set of transit lines that traverse node j. For each line $l \in L_j$, node j can be viewed as $j = j_n(l)$, $(1 \leq n \leq N_l)$. To consider correlation between arcs belonging to the same transit line l, we adopt a first-order discrete autoregressive (DAR(1)) model (see Brockwell and Davis, 1991; Biswas and Song (2009)) that accounts for the travel time' effects of an arc on its subsequent one within transit line l. For each $2 \leq n \leq N_l - 1$, given $\{T_{j_{n-1}(l)j_n(l)}\} = \{T_{ij}\}$ with the probabilities $P_{ij}(t)$, a mean μ_{ij} and a variance σ^2_{ij}, $\{T_{j_n(l)j_{n+1}(l)}\} = \{T_{jk}\}$ can be determined as as a mixture distribution of $\{T_{ij}\}$ and $\{Y_{jk}\}$:

$$T_{jk} \quad = \quad (T_{ij}, \phi) * (Y_{jk}, 1-\phi) + c_{jk}, \tag{3}$$

with $c_{jk} = \phi(E(Y_{jk}) - \mu_{ij})$ and the marginal probability function given by:

$$P(T_{jk} + c_{jk} = t) = \phi P(T_{ij} = t) + (1-\phi)P(Y_{jk} = t), \tag{4}$$

where $\{Y_{jk}\}$ are i.i.d with given probabilities, a mean $E(Y_{jk})$, and a variance $Var(Y_{jk})$. Y_{jk} represents the travel time for arc (j,k) under no effects from the previous arc (i,j) and c_{jk} is a constant added in the model to ensure that the mean travel time μ_{jk} is not affected by the mean travel time of arc (i,j) and the correlation between arcs (i,j) and (j,k) is measured by the variance travel time. $\phi\,(0 \leq \phi < 1)$ is the coefficient in the autoregressive model that measures the effects of the previous arc (i,j) on the travel time T_{jk}. If ϕ is close to 0, then the travel time T_{jk} is not affected by the previous arc (i,j) but as ϕ approaches 1, the travel time T_{jk} gets a larger contribution from the previous arc (i,j).

Using (4), we have:

$$\begin{aligned}
\mu_{jk} &= \sum_t tP(T_{jk} = t) + c_{jk} \qquad (5)\\
&= \sum_t t(\phi P(T_{ij} = t) + (1-\phi)P(Y_{jk} = t)) + c_{jk}\\
&= \phi\mu_{ij} + (1-\phi)E(Y_{jk}) + \phi(E(Y_{jk}) - \mu_{ij})\\
&= E(Y_{jk}).
\end{aligned}$$

Also, we can compute the variance and covariance terms (see Appendix A):

$$\begin{aligned}
\sigma^2_{jk} &= \phi\sigma^2_{ij} + (1-\phi)Var(Y_{jk}) + \phi(1-\phi)\big(\mu_{ij} - E(Y_{jk})\big)^2 \quad (6)\\
Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+n'}(l)i_{n+n'+1}(l)}) &= \phi^{n'}\sigma^2_{i_n(l)i_{n+1}(l)},\ 1 \leq n \leq N_l - 2,\ 1 \leq n' \leq N_l - 1 - n.
\end{aligned}$$

In the case we have overlapping lines from node j to k, the travel time T_{jk} depends on not only the travel time of the previous arc (i,j) but also the travel times of all transit lines serving arc (j,k). If L_{jk} denotes the set of all transit lines l_m traversing arc (j,k) with N_{jk} its cardinality and T^l_{jk} is the travel time associated with arc (j,k) and transit line $l \in L_{jk}$, $\{T^l_{jk}\}$ can be determined as a mixture distribution of $\{T^l_{ij}\}$, $\{Y^{l_1}_{jk}\}$, $\{Y^{l_2}_{jk}\}, \cdots \{Y^{l_{N_{jk}}}_{jk}\}$:

$$T^l_{jk} \;=\; (T^l_{ij}, \phi^l_0) * (Y^{l_1}_{jk}, \phi^l_1)*, (Y^{l_2}_{jk}, \phi^l_2) * \cdots * (Y^{l_{N_{jk}}}_{jk}, \phi^l_{l_{N_{jk}}}) + c^l_{jk},$$

with $c^l_{jk} = E(Y^l_{jk}) - \phi^l_0 E(T^l_{ij}) - \sum\limits_{m=1}^{N_{jk}} \phi^l_m E(Y^{l_m}_{jk})$ and the marginal probability function given by

$$P(T^l_{jk} + c^l_{jk} = t) = \phi^l_0 P(T^l_{ij} = t) + \sum_{m=1}^{N_{jk}} \phi^l_m P(Y^{l_m}_{jk} = t),$$

where $\{Y^{l_m}_{jk}\}$ are i.i.d with given probabilities, a mean $E(Y^{l_m}_{jk})$, and a variance $Var(Y^{l_m}_{jk})$. For each $1 \le m \le N_{jk}$, $Y^{l_m}_{jk}$ represents the travel time for arc (j,k) and transit line l_m under no effects from the previous arc (i,j), and ϕ^l_m $(0 \le m < N_{jk})$ are the coefficients in the autoregressive model with $\sum\limits_{m=0}^{N_{jk}} \phi^l_m = 1$. Following the proofs of (5) and (6), we can show that

$$\begin{aligned} E(T^l_{jk}) &= E(Y^l_{jk}), \\ Var(T^l_{jk}) &= \phi^l_0 Var(T^l_{ij}) + \sum_{m=1}^{N_{jk}} \phi^l_m Var(Y^{l_m}_{jk}) + \sum_{m=1}^{N_{jk}} \phi^l_0 \phi^l_m \big(E(T^l_{ij}) - E(Y^{l_m}_{jk})\big)^2 \\ &\quad + \sum_{m=1}^{N_{jk}} \sum_{m'=m+1}^{N_{jk}} \phi^l_m \phi^l_{m'} \big(E(T^{l_m}_{jk}) - E(Y^{l_{m'}}_{jk})\big)^2, \text{ and} \end{aligned}$$

$$Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+n'}(l)i_{n+n'+1}(l)}) = (\phi^l_0)^{n'} \sigma^2_{i_n(l)i_{n+1}(l)},\; 1 \le n \le N_l - 2,\; 1 \le n' \le N_l - 1 - n.$$

To illustrate the discrete autoregressive model, Table 1 displays the input data of all in-vehicle arcs in Figure 1. Using equations (5) and (6) and setting $\phi = 0.3$, we can compute all probability distributions and all mean and variance/covariance terms (see Tables 2 and 3).

Using the probabilities $P_{jk}(t)$, we can calculate the arrival probabilities $P_{m,j_n(l)}(\tau)$ associated with the m^{th} transit vehicle at node $j_n(l)$. We first set all arrival probabilities

Line 1						Line 2		Line 3	
T_{ab}		Y_{bc}		Y_{cd}		T_{ac}		T_{bd}	
Time	Prob	Time	Prob	Time	Prob	Time	Prob	Time	Prob
4	0.25	3	0.25	3	0.1	9	0.25	8	0.1
5	0.5	5	0.5	4	0.15	10	0.5	9	0.15
6	0.25	7	0.25	5	0.4	11	0.25	10	0.4
				6	0.35			11	0.35

Table 1: Input data for in-vehicle arcs in Figure 1

Line 1						Line 2		Line 3	
T_{ab}		T_{bc}		T_{cd}		T_{ac}		T_{bd}	
Time	Prob	Time	Prob	Time	Prob	Time	Prob	Time	Prob
4	0.25	3	0.175	3	0.1225	9	0.25	8	0.1
5	0.5	4	0.075	4	0.1275	10	0.5	9	0.15
6	0.25	5	0.5	5	0.43	11	0.25	10	0.4
		6	0.075	6	0.2675			11	0.35
		7	0.175	7	0.0525				

Table 2: Probability distributions for in-vehicle arcs in Figure 1

$P_{m,j_n(l)}(\tau)$ to 0 and then update them recursively as follows:

$$P_{m,j_n(l)}(\tau) = \begin{cases} 1 & \text{if } n = 1 \text{ and } \tau \text{ is the starting time} \\ & \quad \text{of the } m^{th} \text{ run of line } l; \\ \sum\limits_{\tau' < \tau} P_{m,j_{n-1}(l)}(\tau') P_{j_{n-1}(l) j_n(l)}(\tau - \tau') & \text{otherwise.} \end{cases} \tag{7}$$

Using equation (7), we can obtain the probability distributions of all transit lines in Figure 1 (as shown in Table 4).

In our example, the time horizon is $[7h00, 8h00]$, $\Omega = \{0, 1, 2, \cdots, 60\}$, $\delta = 1$ min. We assume line l_1 has 4 runs ($M_{l_1} = 4$) and lines l_2 and l_3 have 3 runs ($M_{l_2} = M_{l_3} = 3$). Associated with each line l, there are fixed departure times, $DT_{m,j_1(l)}$, at which each m^{th} transit vehicle must leave its starting station $j_1(l)$. At node a, there are four departure times ($DT_{1,a(l_1)} = 5$, $DT_{2,a(l_1)} = 15$, $DT_{3,a(l_1)} = 25$ and $DT_{4,a(l_1)} = 35$) corresponding to transit line l_1 and three departure times ($DT_{1,a(l_2)} = 5$, $DT_{2,a(l_2)} = 20$ and $DT_{3,a(l_2)} = 35$) corresponding to transit line l_2. At node b, there are three departure times ($DT_{1,b(l_3)} = 10$,

	Line 1			Line 2	Line 3
	T_{ab}	T_{bc}	T_{cd}	T_{ac}	T_{bd}
Mean	5	5	5	10	10
Var/Cov					
T_{ab}	0.5	0.15	0.045		
T_{bc}	0.15	1.55	0.465		
T_{cd}	0.045	0.465	1.095		
T_{ac}				0.5	
T_{bd}					0.9

Table 3: Mean and variance/covariance terms for in-vehicle arcs in Figure 1

$DT_{2,b(l_3)} = 20$ and $DT_{3,b(l_3)} = 30$) associated with transit line l_3.

4 Travel strategies and effective strategy costs

In this section, we show how the concept of travel strategies is adopted in the TE networks with supply uncertainties and illustrate how to compute the effective cost of a strategy.

4.1 Travel strategies

As in previous studies (Hamdouch and Lawphongpanich, 2008 and Hamdouch et al., 2011), we assume that passengers use strategies when travelling. To specify a strategy (denoted as s), passengers must provide, at each node j_τ, a preference set $E_j^{s,\tau}$ of subsequent nodes at which they want to reach via a transit line, walking, or waiting at a station. The order in which nodes are listed in $E_j^{s,\tau}$ gives the passengers' preference, i.e., the first node in the set is the most preferred and the last is the least. To each node k in the preference set that can be reached via a walking or a wait arc, we associate a time interval index representing the actual time interval to reach node k. To each node k that can be reached via an in-vehicle arc, we associate an index representing the corresponding transit line. It is important to note that this strategy definition is different from the one used in previous studies with fixed timetables. Indeed, while we can identify the actual time passengers reach node k via a walking or a wait arc, the time to reach node k via an in-vehicle arc is random and passengers can only include transit line indices in their preference set. For example, Table 5 displays one valid strategy s^1 for OD pair (q, r).

	Line 1								Line 2				Line 3			
	a		*b*		*c*		*d*		*a*		*c*		*b*		*d*	
	Time	Pr	Time	Pr	Time	Pr	Time	Pr	Time	Pr	Time	Pr	Time	Pr	Time	Pr
Run 1	5	1	9	0.25	12	0.044	15	0.005	5	1	14	0.25	10	1	18	0.1
			10	0.5	13	0.106	16	0.019			15	0.5			19	0.15
			11	0.25	14	0.206	17	0.058			16	0.25			20	0.4
					15	0.288	18	0.119							21	0.35
					16	0.206	19	0.181								
					17	0.106	20	0.224								
					18	0.044	21	0.195								
							22	0.122								
							23	0.058								
							24	0.017								
							25	0.002								
Run 2	15	1	19	0.25	22	0.044	25	0.005	20	1	29	0.25	20	1	28	0.1
			20	0.5	23	0.106	26	0.019			30	0.5			29	0.15
			21	0.25	24	0.206	27	0.058			31	0.25			30	0.4
					25	0.288	28	0.119							31	0.35
					26	0.206	29	0.181								
					27	0.106	30	0.224								
					28	0.044	31	0.195								
							32	0.122								
							33	0.058								
							34	0.017								
							35	0.002								
Run 3	25	1	29	0.25	32	0.044	35	0.005	35	1	44	0.25	30	1	38	0.1
			30	0.5	33	0.106	36	0.019			45	0.5			39	0.15
			31	0.25	34	0.206	37	0.058			46	0.25			40	0.4
					35	0.288	38	0.119							41	0.35
					36	0.206	39	0.181								
					37	0.106	40	0.224								
					38	0.044	41	0.195								
							42	0.122								
							43	0.058								
							44	0.017								
							45	0.002								
Run 4	35	1	39	0.25	42	0.044	45	0.005								
			40	0.5	43	0.106	46	0.019								
			41	0.25	44	0.206	47	0.058								
					45	0.288	48	0.119								
					46	0.206	49	0.181								
					47	0.106	50	0.224								
					48	0.044	51	0.195								
							52	0.122								
							53	0.058								
							54	0.017								
							55	0.002								

Table 4: Probability distributions for transit lines in Figure 1

For a passenger using s^1, the order of nodes in the user-preference set at node a_5, i.e., $[b_{l_1}, c_{l_2}, a_6]$, indicates that the passenger prefers Line 1 over Line 2 and Line 2 over waiting. Using this strategy, there are several directed paths emanating from q_0 and reaching node r at different times. The arrival time at the destination depends on the probabilities to access various lines at nodes a, b, c, and d as well as the probabilities associated with the random travel times T_{ab}, T_{ac}, T_{bc}, T_{bd}, and T_{cd}.

The effective cost of a strategy s depends directly on the arc probabilities $\pi_{jk}^{s,\tau}$ and $\pi_j^{s,\tau}$ associated with in-vehicle and wait arcs at time τ. The procedure for computing this strategy cost comprises two main steps. In the first step, a *stochastic loading* of the TE network is performed according to a given strategy assignment vector X and

Strategy: s^1									
Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.
q_0	$[a_5]$	a_5	$[b_{l_1}, c_{l_2}, a_6]$	b_9	$[c_{l_1}, b_{10}]$	c_{12}	$[d_{l_1}, c_{13}]$	d_{15}	$[r_{20}]$
		a_{15}	$[b_{l_1}, a_{16}]$	b_{10}	$[c_{l_1}, d_{l_3}, b_{11}]$	c_{13}	$[d_{l_1}, c_{14}]$	d_{16}	$[r_{21}]$
		a_{20}	$[c_{l_2}, a_{21}]$	b_{11}	$[c_{l_1}, b_{12}]$	c_{14}	$[d_{l_1}, c_{15}]$	d_{17}	$[r_{22}]$
		a_{25}	$[b_{l_1}, a_{26}]$	b_{19}	$[c_{l_1}, b_{20}]$	c_{15}	$[d_{l_1}, c_{16}]$	d_{18}	$[r_{23}]$
		a_{35}	$[b_{l_1}, c_{l_2}, a_{36}]$	b_{20}	$[c_{l_1}, d_{l_3}, b_{21}]$	c_{16}	$[d_{l_1}, c_{17}]$	d_{19}	$[r_{24}]$
				b_{21}	$[c_{l_1}, b_{22}]$	c_{17}	$[d_{l_1}, c_{18}]$	d_{20}	$[r_{25}]$
				b_{29}	$[c_{l_1}, b_{30}]$	c_{18}	$[d_{l_1}, c_{19}]$	d_{21}	$[r_{26}]$
				b_{30}	$[c_{l_1}, d_{l_3}, b_{31}]$	c_{22}	$[d_{l_1}, c_{23}]$	d_{22}	$[r_{27}]$
				b_{31}	$[c_{l_1}, b_{32}]$	c_{23}	$[d_{l_1}, c_{24}]$	d_{23}	$[r_{28}]$
				b_{39}	$[c_{l_1}, b_{40}]$	c_{24}	$[d_{l_1}, c_{25}]$	d_{24}	$[r_{29}]$
				b_{40}	$[c_{l_1}, b_{41}]$	c_{25}	$[d_{l_1}, c_{26}]$	d_{25}	$[r_{30}]$
				b_{41}	$[c_{l_1}, b_{42}]$	c_{26}	$[d_{l_1}, c_{27}]$	d_{26}	$[r_{31}]$
						c_{27}	$[d_{l_1}, c_{28}]$	d_{27}	$[r_{32}]$
						c_{28}	$[d_{l_1}, c_{29}]$	d_{28}	$[r_{33}]$
						c_{32}	$[d_{l_1}, c_{33}]$	d_{29}	$[r_{34}]$
						c_{33}	$[d_{l_1}, c_{34}]$	d_{30}	$[r_{35}]$
						c_{34}	$[d_{l_1}, c_{35}]$	d_{31}	$[r_{36}]$
						c_{35}	$[d_{l_1}, c_{36}]$	d_{32}	$[r_{37}]$
						c_{36}	$[d_{l_1}, c_{37}]$	d_{33}	$[r_{38}]$
						c_{37}	$[d_{l_1}, c_{38}]$	d_{34}	$[r_{39}]$
						c_{38}	$[d_{l_1}, c_{39}]$	d_{35}	$[r_{40}]$
						c_{42}	$[d_{l_1}, c_{43}]$	d_{36}	$[r_{41}]$
						c_{43}	$[d_{l_1}, c_{44}]$	d_{37}	$[r_{42}]$
						c_{44}	$[d_{l_1}, c_{45}]$	d_{38}	$[r_{43}]$
						c_{45}	$[d_{l_1}, c_{46}]$	d_{39}	$[r_{44}]$
						c_{46}	$[d_{l_1}, c_{47}]$	d_{40}	$[r_{45}]$
						c_{47}	$[d_{l_1}, c_{48}]$	d_{41}	$[r_{46}]$
						c_{48}	$[d_{l_1}, c_{49}]$	d_{42}	$[r_{47}]$
								d_{43}	$[r_{48}]$
								d_{44}	$[r_{49}]$
								d_{45}	$[r_{50}]$
								d_{46}	$[r_{51}]$
								d_{47}	$[r_{52}]$
								d_{48}	$[r_{53}]$
								d_{49}	$[r_{54}]$
								d_{50}	$[r_{55}]$
								d_{51}	$[r_{56}]$
								d_{52}	$[r_{57}]$
								d_{53}	$[r_{58}]$
								d_{54}	$[r_{59}]$
								d_{55}	$[r_{60}]$

Table 5: One travel strategy for OD pair (q, r)

is an extension to the one proposed by Hamdouch and Lawphongpanich (2008). The stochastic loading process computes the arc flows, $f_{jk}^{s,\tau}$, and the arc probabilities, $\pi_{jk}^{s,\tau}(X)$ and $\pi_{j}^{s,\tau}(X)$, by processing TE nodes one at a time and in topological and chronological (T&C) order, i.e., a node with no predecessor and the smallest time interval index is processed first. Given all the arc flows and probabilities, the second step computes the effective strategy cost using a mean variance approach. This step involves scanning TE nodes in reverse T&C order and applying Bellman's generalized recursion. Note that this procedure is different from the one adopted in previous studies with fixed timetables. Using a mean variance approach, Bellman's recursion is essential to account for both expected and variance cost terms in calculating the effective cost of a strategy.

4.2 Stochastic loading process

In loading the TE network, we ensure that, at each node j_τ, the summation of the probabilities associated with outgoing arcs in the preference set $E_j^{s,\tau}$ are equal to one:

$$\sum_{k \in E_j^{s,\tau} - \{j_{\tau+1}\}} \pi_{jk}^{s,\tau}(X) + \pi_j^{s,\tau}(X) = 1, \ \forall j_\tau, \ \forall s. \tag{8}$$

Consider processing node j at time τ. For each line $l \in L_j$, node j can be viewed as $j = j_n(l)$, $(1 \leq n \leq N_l)$. Let $M_{j_\tau,l}$ be the set of runs of line l that have positive probabilities to reach node j at time τ (ordered from the highest probability to the smallest):

$$M_{j_\tau,l} = \{m \in M_l | P_{m,j_n(l)}(\tau) > 0\}.$$

Note that due to the variability in travel time, more than one run of the same line l can reach node j at time τ, resulting in bus bunching (Bartholdi and Eisenstein, 2012). This *bunching* issue occurs when at least one of the transit vehicles of line l is unable to keep to its schedule and therefore reaches node j as one or more other vehicles of the same transit line l at the same time τ. For example, Table 4 shows a bunching issue at node d_{25} with $M_{d_{25},l_1} = \{2, 1\}$, $P_{2,d}(25) = 0.005$, and $P_{1,d}(25) = 0.002$.

For each line l such that $1 < n < N_l$ (i.e., $j_n(l)$ is neither the starting nor the ending node of line l) and for each strategy s such that the first choice in the user-preference set $E_j^{s,\tau}(1) = \{j_{n+1}(l)\}$, the passengers using strategy s on arc $(j_{n-1}(l), j)$ have priority to board line l on arc $(j, j_{n+1}(l))$. In case we have more than one run of the same line l that reach node j at time τ, it is intuitively to give priority to the passengers on the run with the highest probability first. This assumption can be relaxed by loading together all the passengers on the runs belonging to $M_{j_\tau,l}$. Therefore, the first priority class, $W_j^{\tau,0,1}$, consists of all the passengers who have continuance priority at node j_τ and travelling on run $M_{j_\tau,l}(1)$ with the highest probability:

$$W_j^{\tau,0,1} = \cup_s \cup_{l\in L_j} \left\{z_{jml}^{s,\tau,0}, E_j^{s,\tau}(1) = \{j_{n+1}(l)\}, m = M_{j_\tau,l}(1)\right\}, \tag{9}$$

where $z_{jml}^{s,\tau,0}$ denotes the number of passengers using strategy s, travelling on run m of line l, and having continuance priority at node j_τ:

$$z_{jml}^{s,\tau,0} = \sum_{\tau_c<\tau} \pi_{ij}^{s,\tau_c,\tau} z_{iml}^{s,\tau_c}, \ i = j_{n-1}(l). \tag{10}$$

For each line $l \in L_j$ and for each strategy s such that $E_j^{s,\tau}(1) = \{j_{n+1}(l)\}$, the passengers using strategy s on arc $(j_{n-1}(l), j)$ and travelling on run m of line l, $\sum\limits_{\tau_c<\tau} f_{j_{n-1}(l)jml}^{s,\tau_c,\tau}$, have priority to board line l on arc $(j, j_{n+1}(l))$ and the flows $f_{jj_{n+1}(l)ml}^{s,\tau,\tau''}$ and $f_{jj_{n+1}(l)ml}^{s,\tau}$ are computed as follows:

$$\begin{aligned} f_{jj_{n+1}(l)ml}^{s,\tau,\tau''} &= P_{jj_{n+1}(l)}(\tau''-\tau) \sum_{\tau_c<\tau} f_{j_{n-1}(l)jml}^{s,\tau_c,\tau} \\ &= P_{jj_{n+1}(l)}(\tau''-\tau) z_{jml}^{s,\tau,0}, \quad \forall \tau'' > \tau \\ f_{jj_{n+1}(l)ml}^{s,\tau} &= \sum_{\tau''>\tau} f_{jj_{n+1}(l)ml}^{s,\tau,\tau''}. \end{aligned} \tag{11}$$

Then, the residual capacities of all arcs $(j, j_{n+1}(l))$, $u_{jj_{n+1}(l)ml}^{\tau}$, are updated ($u_{jj_{n+1}(l)ml}^{\tau} = u_{jj_{n+1}(l)ml}^{\tau} - \sum\limits_{s} f_{jj_{n+1}(l)ml}^{s,\tau}$) and the process ends for class $W_j^{\tau,0,1}$. We repeat the same process for the priority classes $W_j^{\tau,0,m'}$ for $m' = 2, ..., \max_l\{|M_{j_\tau,l}|\}$.

After loading all on-board passengers who want to continue their journey in the same transit vehicle, the process loads passengers who arrive at node j at time τ on various transit lines and want to transfer to other transit lines as well as those who have been waiting at node j_τ. To enforce the FCFS rule, we classify these passengers according to their arrival times at node j. We denote $z_j^{s,\tau,\tau'}$ as the number of passengers using strategy s at node j_τ and having reached node j at time $\tau' \le \tau$ and group all flows into a class $W_j^{\tau,\tau'}$ restricted to passengers having reached node j at time τ':

$$W_j^{\tau,\tau'} = \cup_s \left\{z_j^{s,\tau,\tau'}, E_j^{s,\tau} \neq \emptyset\right\},$$

where $z_j^{s,\tau,\tau'} = \sum\limits_{l\in L_j} \sum\limits_{m\in M_{j_\tau,l}} z_{jml}^{s,\tau,\tau'}$ and $z_{jml}^{s,\tau,\tau'}$ is computed according to the following recursion:

$$z_{jml}^{s,\tau,\tau'} = \begin{cases} \pi_j^{s,\tau-1} z_{jml}^{s,\tau-1,\tau'} & \text{if } \tau' \le \tau - 1 \\ \sum\limits_{\tau_c<\tau} \pi_{ij}^{s,\tau_c,\tau} z_{iml}^{s,\tau_c} & \text{if } \tau' = \tau,\ i = j_{n-1}(l) \text{ and} \\ & \quad (j, E_j^{s,\tau}(1)) \in l' \neq l, \end{cases} \tag{12}$$

In equation (12), the first term denotes the passengers who reach node j before time $\tau - 1$ and the second term denotes those who reach node j at time τ.

As for the priority classes, we load passengers on the runs with the highest probabilities in the sets $M_{j_\tau,l}$ and then we repeat the process for the subsequent runs following the descendent order of probabilities. In loading passengers belonging to the classes $W_j^{\tau,\tau'}(\tau' \leq \tau)$, the process loads, in the FCFS order, the passengers who, according to their strategy s, prefer to access arcs $(j_\tau, k_{\tau''})$ for all $\tau'' > \tau$, i.e., the process loads those passengers who arrive earlier at time $\tau_1 \leq \tau$ (z_j^{s,τ,τ^1}) before those who arrive later at time $\tau_2 > \tau_1$, $\tau_2 \leq \tau$ (z_j^{s,τ,τ^2}) until the remaining capacity of the arc is exhausted $(u_{jk}^{\tau} = 0)$. Those who cannot be loaded must use wait arc $(j_\tau, j_{\tau+1})$.

Once all the arcs emanating from j_τ are loaded, the arc probabilities are computed as follows:

$$\pi_{jk}^{s,\tau,\tau''} = \frac{\sum\limits_{l\in L_j} \sum\limits_{m\in M_{j_\tau,l}} f_{jkml}^{s,\tau,\tau''}}{\sum\limits_{l\in L_j} \sum\limits_{m\in M_{j_\tau,l}} z_{jml}^{s,\tau}} = \frac{f_{jk}^{s,\tau,\tau''}}{z_j^{s,\tau}}, \tag{13}$$

$$\pi_{jk}^{s,\tau} = \sum_{\tau''\geq\tau} \pi_{jk}^{s,\tau,\tau''} = \frac{f_{jk}^{s,\tau}}{z_j^{s,\tau}}, \text{ and} \tag{14}$$

$$\pi_{j}^{s,\tau} = \frac{z_j^{s,\tau} - f_{jk}^{s,\tau}}{z_j^{s,\tau}}. \tag{15}$$

The stochastic loading procedure will be explained in detail using the example in Figure 2 which is built based upon Figure 1. Not all nodes and links are shown for the sake of clarity. We focus on the loading process at nodes q_{10}, o_{15}, a_{15}, b_{20}, and c_{25}. The loading process starts at node q_{10} where 10 passengers using strategy s^1 and 5 passengers using s^3 are loaded onto access arc (q_{10}, a_{15}). Thus, $f_{qa}^{s^1,10} = 10$, $f_{qa}^{s^3,10} = 5$, $\pi_{qa}^{s^1,10} = \pi_{qa}^{s^3,10} = 1$, and $\pi_{q}^{s^1,10} = \pi_{q}^{s^3,10} = 0$. At node o_{15}, 30 passengers using strategy s^2 are loaded onto access arc (o_{15}, b_{20}) and we get $f_{ob}^{s^2,15} = 30$, $\pi_{ob}^{s^1,15} = 1$, and $\pi_{q}^{s^2,15} = 0$. At node a_{15}, the 10 passengers using strategy s^1 and the 5 passengers using s^3 want to board the second run of line 1 and access arc (a, b) at time 15 ($P_{2,a(l_1)}(15) = 1$, $z_{a2l_1}^{s^1,15} = 10$, and $z_{a2l_1}^{s^3,15} = 5$). The time to reach node b depends on the probabilities associated with the random travel time T_{ab}. From Tables 2 and 4, we know that $P_{ab}(4) = 0.25$, $P_{ab}(5) = 0.5$, and $P_{ab}(6) = 0.25$.

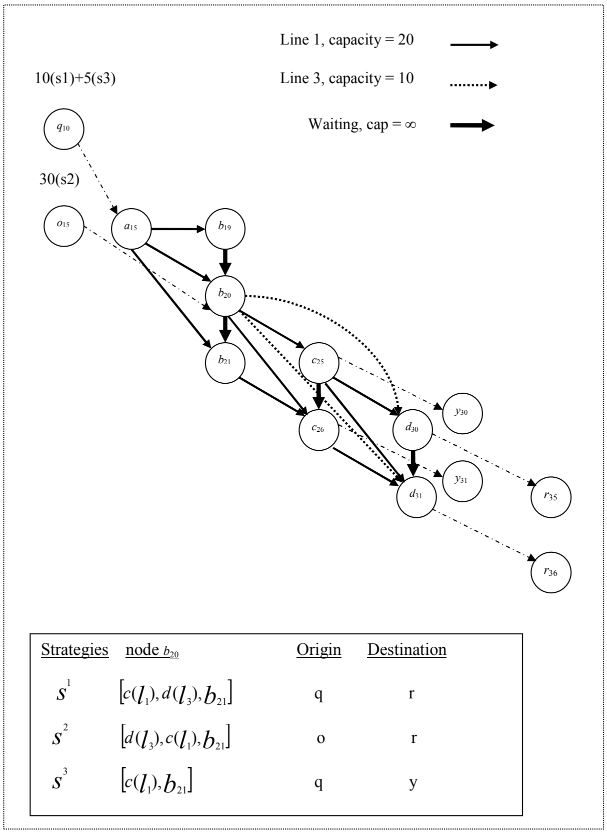

Figure 2: An example of stochastic loading

Therefore, from equation (10), we obtain the following:

$$\begin{aligned}
f_{ab2l_1}^{s^1,15,19} &= P_{ab}(4) z_{a2l_1}^{s^1,15} = 0.25(10) = 2.5, \\
f_{ab2l_1}^{s^3,15,19} &= P_{ab}(4) z_{a2l_1}^{s^3,15} = 0.25(5) = 1.25, \\
f_{ab2l_1}^{s^1,15,20} &= P_{ab}(5) z_{a2l_1}^{s^1,15} = 0.5(10) = 5, \\
f_{ab2l_1}^{s^3,15,20} &= P_{ab}(5) z_{a2l_1}^{s^3,15} = 0.5(5) = 2.5, \\
f_{ab2l_1}^{s^1,15,21} &= P_{ab}(6) z_{a2l_1}^{s^1,15} = 0.25(10) = 2.5, \\
f_{ab2l_1}^{s^3,15,21} &= P_{ab}(6) z_{a2l_1}^{s^3,15} = 0.25(5) = 1.25, \\
f_{ab}^{s^1,15} &= f_{ab2l_1}^{s^1,15,19} + f_{ab2l_1}^{s^1,15,20} + f_{ab2l_1}^{s^1,15,21} = 10, \\
f_{ab}^{s^3,15} &= f_{ab2l_1}^{s^3,15,19} + f_{ab2l_1}^{s^3,15,20} + f_{ab2l_1}^{s^3,15,21} = 5, \\
\pi_{ab}^{s^1,15} &= \pi_{ab}^{s^3,15} = 1, \\
\pi_{a}^{s^1,15} &= \pi_{a}^{s^3,15} = 0, \\
\pi_{ab}^{s^1,15,19} &= \pi_{ab}^{s^3,15,19} = 0.25, \\
\pi_{ab}^{s^1,15,20} &= \pi_{ab}^{s^3,15,20} = 0.5, \\
\pi_{ab}^{s^1,15,21} &= \pi_{ab}^{s^3,15,21} = 0.25.
\end{aligned}$$

When the process undergoes node b_{20}, there are two classes of passengers: $W_b^{20,0,1} = \{z_{b2l_1}^{s^1,20} = 5,\ z_{b2l_1}^{s^3,20} = 2.5\}$ and $W_b^{20,20} = \{z_b^{s^2,20} = 30\}$. The process starts by loading the passengers belonging to $W_b^{20,0,1}$, where 5 passengers using strategy s^1 and 2.5 passengers using s^3 board the second run of line 1 and travel on arcs (b_{20}, c_{23}), (b_{20}, c_{24}), (b_{20}, c_{25}), (b_{20}, c_{26}), and (b_{20}, c_{27}) with probabilities 0.175, 0.075, 0.5, 0.075, and 0.175, respectively (see Table 2). Thus, we get $f_{bc2l_1}^{s^1,20,25} = P_{bc}(5) z_{b2l_1}^{s^1,20} = 0.5(5) = 2.5$, $f_{bc2l_1}^{s^3,20,25} = P_{bc}(5) z_{b2l_1}^{s^3,20} = 0.5(2.5) = 1.25$, $f_{bc2l_1}^{s^1,20} = \sum_{\tau''=23}^{26} P_{bc}(\tau'' - 20) f_{bc2l_1}^{s^1,20,\tau''} = 5$, and $f_{bc2l_1}^{s^3,20} = \sum_{\tau''=23}^{26} P_{bc}(\tau'' - 20) f_{bc2l_1}^{s^3,20,\tau''} = 2.5$. Then, the residual capacity, $u_{bc2l_1}^{20}$, is updated ($u_{bc2l_1}^{20} = 20 - 7.5 = 12.5$). The next step is to load passengers belonging to the class $W_b^{20,20}$. Among the 30 passengers using strategy s^2 and belonging to this class, 10 passengers travel on the second run of line 3 and 12.5 passengers travel on the second run of line 1. The times to reach nodes c and d depend on the probabilities associated with the variables T_{bc} and T_{bd}, respectively. The remaining passengers 7.5 use wait arc (b_{20}, b_{21}).

Finally, at node c_{25}, 1.25 passengers using s^3 alight from line 1 to take egress arc (c_{25}, y_{30}). Therefore, only passengers using strategies s^1 and s^2 continue on line 1 and access arcs (c_{25}, d_{28}), (c_{25}, d_{29}), (c_{25}, d_{30}), (c_{25}, d_{31}), and (c_{25}, d_{32}) with probabilities 0.1225, 0.1275, 0.43, 0.2675, and 0.0525, respectively (see Table 2). Relevant arcs flows and probabilities for this stochastic loading example are displayed in Table 6.

	(q, a)	(o, b)	(a, b)	(b, c)	(b, d)	(c, d)
$f_{jk}^{s,\tau}$	$f_{qa}^{s^1,10} = 10$ $f_{qa}^{s^3,10} = 5$	$f_{ob}^{s^2,15} = 30$	$f_{ab}^{s^1,15} = 10$ $f_{ab}^{s^3,15} = 5$	$f_{bc}^{s^1,20} = 5$ $f_{bc}^{s^3,20} = 2.5$ $f_{bc}^{s^2,20} = 12.5$	$f_{bd}^{s^2,20} = 10$	$f_{cd}^{s^1,25} = 2.5$ $f_{cd}^{s^2,25} = 6.25$
$f_{jkml}^{s,\tau,\tau''}$			$f_{ab}^{s^1,15,20} = 5$ $f_{ab}^{s^3,15,20} = 2.5$	$f_{bc}^{s^1,20,25} = 2.5$ $f_{bc}^{s^3,20,25} = 1.25$ $f_{bc}^{s^2,20,25} = 6.25$	$f_{bd}^{s^2,20,30} = 4$	
$\pi_{jk}^{s,\tau}$	$\pi_{qa}^{s^1,10} = 1$ $\pi_{qa}^{s^3,10} = 1$	$\pi_{ob}^{s^2,15} = 1$	$\pi_{ab}^{s^1,15} = 1$ $\pi_{ab}^{s^3,15} = 1$	$\pi_{bc}^{s^1,20} = 1$ $\pi_{bc}^{s^3,20} = 1$ $\pi_{bc}^{s^2,20} = 0.42$	$\pi_{bd}^{s^2,20} = 0.33$	$\pi_{cd}^{s^1,25} = 1$ $\pi_{cd}^{s^2,25} = 1$
$\pi_{jk}^{s,\tau,\tau''}$	$\pi_{qa}^{s^1,10,15} = 1$ $\pi_{qa}^{s^3,10,15} = 1$	$\pi_{ob}^{s^2,15,20} = 1$	$\pi_{ab}^{s^1,15,20} = 0.5$ $\pi_{ab}^{s^3,15,20} = 0.5$	$\pi_{bc}^{s^1,20,25} = 0.5$ $\pi_{bc}^{s^3,20,25} = 0.5$ $\pi_{bc}^{s^2,20,25} = 0.21$	$\pi_{bd}^{s^2,20,30} = 0.13$	
$\pi_j^{s,\tau}$	$\pi_q^{s^1,10} = 0$ $\pi_q^{s^3,10} = 0$	$\pi_o^{s^2,15} = 0$	$\pi_a^{s^1,15} = 0$ $\pi_a^{s^3,15} = 0$	$\pi_b^{s^1,20} = 0$ $\pi_b^{s^3,20} = 0$ $\pi_b^{s^2,20} = 0.25$	$\pi_b^{s^1,20} = 0$ $\pi_b^{s^3,20} = 0$ $\pi_b^{s^2,20} = 0.25$	$\pi_c^{s^1,25} = 0$ $\pi_c^{s^2,25} = 0$

Table 6: Stochastic loading process at nodes q_{10}, o_{15}, a_{15}, b_{20}, and c_{25}

4.3 Effective strategy cost

In our model, passengers are dealing with two types of randomness when deciding on the strategy to travel from their origins to their destinations. The first type of randomness is due to the possibility to fail to board a vehicle as a result of limited transit capacities. At each node j_τ, the node selected from the preference set $E_j^{s,\tau}$ is random and depends on the residual capacities of the transit vehicles passing though j at time τ. The second type of randomness comes from the in-vehicle arc travel times, T_{jk}, that follow a discrete distribution with the probabilities $P_{jk}(t)$, a mean μ_{jk} and a variance σ_{jk}^2. To take into account of these two types of uncertainties, a mean variance cost function is used to model the passengers' averseness to both failure to board a vehicle and link travel time variability.

At each node j_τ, let $Y_j^{s,\tau}$ be the random variable representing the node selected from the preference set $E_j^{s,\tau}$ and $C^g_{jY_j^{s,\tau}}$ the random cost associated with link $(j_\tau, (Y_j^{s,\tau})_{\tau+T_{jk}})$ and group g:

$$C^g_{jY_j^{s,\tau}} = \begin{cases} \gamma_{travel} T_{jk} + v_{jk}^\tau + e_{jk}^{\tau,g} + \widehat{e}_{jk}^\tau(f_{jk}^\tau) & \text{if } Y_j^{s,\tau} = k \in E_j^{s,\tau} - \{j_{\tau+1}\}; \\ \gamma_{wait} & \text{if } Y_j^{s,\tau} = j_{\tau+1}. \end{cases}$$

Using a mean variance approach, the effective cost of a strategy s (according to a strategy assignment vector X) can be determined as

$$EC^{s,\tau^s}_{(q,r,g)}(X) = E(C^{s,\tau^s}_{(q,r,g)}(X)) + \eta_3^g Var(C^{s,\tau^s}_{(q,r,g)}(X)), \tag{16}$$

where τ^s is the starting time of strategy s.

For a given triplet (j, r, g), let

$C^{s,\tau}_{(j,r,g)}(X)$ be the cost for reaching node r from node j_τ using strategy s.

$EC^{s,\tau}_{(j,r,g)}(X)$ be the effective cost for reaching node r from node j_τ using strategy s.

$$EC^{s,\tau}_{(j,r,g)}(X) = E(C^{s,\tau}_{(j,r,g)}(X)) + \eta_3^g Var(C^{s,\tau}_{(j,r,g)}(X)).$$

The effective costs $EC^{s,\tau}_{(j,r,g)}(X)$ are computed by scanning TE nodes in reverse T&C order starting from destination r and applying Bellman's equation.

Ending Conditions at node r:

i) Set $E(C^{s,\tau}_{(r,r,g)}) = 0, \forall \tau \in \Omega$.

ii) Set $Var(C^{s,\tau}_{(r,r,g)}) = 0, \forall \tau \in \Omega$.

Recursions at node $j_\tau (j = j_n(l),\, j \neq q)$:

Bellman's equation for the random cost $C^{s,\tau}_{(j,r,g)}(X)$ at node j_τ is given as follows:

$$C^{s,\tau}_{(j,r,g)}(X) = \begin{cases} \begin{array}{c} \gamma_{travel} T_{jk} + v^\tau_{jk} \\ +e^{\tau,g}_{jk} + \widehat{e}^\tau_{jk}(f^\tau_{jk}) + C^{s,\tau+T_{jk}}_{(k,r,g)}(X) \end{array} & \text{if } Y^{s,\tau}_j = k \in E^{s,\tau}_j - \{j_{\tau+1}\}; \\ \gamma_{wait} + C^{s,\tau+1}_{(j,r,g)}(X) & \text{if } Y^{s,\tau}_j = j_{\tau+1}. \end{cases}$$

In the above expression, the first case represents the cost associated with on-board passengers that consists of the travel cost of link $(j_\tau, k_{\tau+T_{jk}})$ plus the cost for reaching node r from node $k_{\tau+T_{jk}}$, $C^{s,\tau+T_{jk}}_{(k,r,g)}(X)$. The travel cost of link $(j_\tau, k_{\tau+T_{jk}})$ includes the travel time, transit fare, the penalty $e^{\tau,g}_{jk}$ as well as the penalty, $\widehat{e}^\tau_{jk}(f^\tau_{jk})$, for being in a crowded vehicle. The second case represents the cost associated with waiting that comprises the travel cost of link $(j_\tau, j_{\tau+1})$, γ_{wait}, and the cost for reaching node r from node $j_{\tau+1}$, $C^{s,\tau+1}_{(j,r,g)}(X)$.

From the above formulation, the expected cost $E(C^{s,\tau}_{(j,r,g)}(X))$ can be calculated as

$$\begin{aligned} E(C^{s,\tau}_{(j,r,g)}(X)) = & \sum_{k \in E^{s,\tau}_j - \{j_{\tau+1}\}} \pi^{s,\tau}_{jk}(X) \sum_{\tau'' > \tau} P_{jk}(\tau'' - \tau) \Big[\gamma_{travel}(\tau'' - \tau) + v^\tau_{jk} + e^{\tau,g}_{jk} \\ & + \widehat{e}^\tau_{jk}(f^\tau_{jk}) + E(C^{s,\tau''}_{(k,r,g)}(X)) \Big] \\ & + \pi^{s,\tau+1}_j(X) \Big[\gamma_{wait} + E(C^{s,\tau+1}_{(j,r,g)}(X)) \Big]. \end{aligned}$$

Using $\mu_{jk} = \sum_{\tau'' > \tau} (\tau'' - \tau) P_{jk}(\tau'' - \tau)$ and setting

$$\begin{aligned} \varphi^{s,\tau,g}_k &= \gamma_{travel}\mu_{jk} + v^\tau_{jk} + e^{\tau,g}_{jk} + \widehat{e}^\tau_{jk}(f^\tau_{jk}) + \sum_{\tau'' > \tau} P_{jk}(\tau'' - \tau) E(C^{s,\tau''}_{(k,r,g)}(X)), \\ \varphi^{s,\tau,g}_j &= \gamma_{wait} + E(C^{s,\tau+1}_{(j,r,g)}(X)), \end{aligned}$$

the expected cost $E(C^{s,\tau}_{(j,r,g)}(X))$ can be expressed as

$$E(C^{s,\tau}_{(j,r,g)}(X)) = \sum_{k \in E^{s,\tau}_j - \{j_{\tau+1}\}} \pi^{s,\tau}_{jk}(X) \varphi^{s,\tau,g}_k + \pi^{s,\tau+1}_j(X) \varphi^{s,\tau,g}_j.$$

For the variance strategy cost, we use the formula for variance decomposition, $Var(X_1) = E(Var(X_1|X_2)) + Var(E(X_1|X_2))$, where $X_1 = C^{s,\tau}_{(j,r,g)}(X)$ and $X_2 = Y^{s,\tau}_j$. Therefore, we obtain

$$\begin{aligned} Var(C^{s,\tau}_{(j,r,g)}(X)) = & \sum_{k\in E^{s,\tau}_j-\{j_{\tau+1}\}} \pi^{s,\tau}_{jk}(X)\psi^{s,\tau,g}_k + \pi^{s,\tau+1}_j(X)\psi^{s,\tau,g}_j \\ & + \sum_{k\in E^{s,\tau}_j-\{j_{\tau+1}\}} \pi^{s,\tau}_{jk}(X)(\varphi^{s,\tau,g}_k)^2 + \pi^{s,\tau+1}_j(X)(\varphi^{s,\tau,g}_j)^2 \\ & -\Big(\sum_{k\in E^{s,\tau}_j-\{j_{\tau+1}\}} \pi^{s,\tau}_{jk}(X)\varphi^{s,\tau,g}_k + \pi^{s,\tau+1}_j(X)\varphi^{s,\tau,g}_j\Big)^2, \end{aligned}$$

where $\psi^{s,\tau,g}_j = Var(\gamma_{wait} + C^{s,\tau+1}_{(j,r,g)}(X)) = Var(C^{s,\tau+1}_{(j,r,g)}(X))$ and

$$\begin{aligned} \psi^{s,\tau,g}_k &= Var(\gamma_{travel}T_{jk} + v^{\tau}_{jk} + e^{\tau,g}_{jk} + \widehat{e}^{\tau}_{jk}(f^{\tau}_{jk}) + C^{s,\tau+Tjk}_{(k,r,g)}(X)) \\ &= \gamma^2_{travel}Var(T_{jk}) + Var(C^{s,\tau+Tjk}_{(k,r,g)}(X)) + 2\gamma_{travel}Cov(T_{jk}, C^{s,\tau''}_{(k,r,g)}(X)) \\ &= \gamma^2_{travel}\sigma^2_{jk} + \sum_{\tau''>\tau} P_{jk}(\tau''-\tau)\big(Var(C^{s,\tau''}_{(k,r,g)}(X)) + 2\gamma_{travel}Cov(T_{jk}, C^{s,\tau''}_{(k,r,g)}(X))\big). \end{aligned}$$

$Cov(T_{jk}, C^{s,\tau''}_{(k,r,g)}(X)) = \sum\limits_{n'\in N'_s} Cov(T_{j_n(l),j_{n+1}(l)}, T_{j_{n'}(l),j_{n'+1}(l)})$,
where N^s is the set of nodes included in the user-preference sets of strategy s and $N'_s = \{n' : n+1 \le n' \le N_l - 1,\ j_{n'+1}(l) \in N^s\}$.

Determining the effective cost of strategy s:

$$\begin{aligned} EC^{s,\tau^s}_{(q,r,g)}(X) &= E(C^{s,\tau^s}_{(q,r,g)}(X)) + \eta^g_3 Var(C^{s,\tau^s}_{(q,r,g)}(X)), \\ &= \gamma_{travel}t_{qj} + E(C^{s,\tau}_{(j,r,g)}(X)) + \eta^g_3 Var(C^{s,\tau}_{(j,r,g)}(X)), \end{aligned}$$

where $E^{s,\tau^s}_q = \{j_\tau\}$, t_{qj} is the walking time of access arc (q_{τ^s}, j_τ) and $E(C^{s,\tau}_{(j,r,g)}(X))$ and $Var(C^{s,\tau}_{(j,r,g)}(X))$ are available from previous recursions.

5 User equilibrium

A strategic assignment vector X^* is in a user equilibrium if no passenger has any incentive to change his or her strategy based on effective strategy costs. X^* is in a user equilibrium if and only if X^* solves the following variational inequality (denoted as VI$[EC(X), \mathcal{X}]$):

$$EC(X^*)^T(X - X^*) \ge 0, \quad \forall X \in \mathcal{X}, \tag{17}$$

where $EC(X)$ is a vector of the effective strategy costs associated with X and $\mathcal{X}$ is the set of all feasible SA vectors:

$$\mathcal{X} = \{X : \textstyle\sum_{s\in S_{(q,r)}} x^{s,\tau^s}_{(q,r,g)} = d^g_{(q,r)}, \forall (q,r,g)\}. \tag{18}$$

This is an extension of the strategy-based equilibrium conditions to stochastic networks, where we replace an expected strategy cost function by an effective strategy cost function obtained through a mean-variance approach.

5.1 Computation of an optimal strategy

In finding a strategic equilibrium solution, we need to compute, for each triplet (q,r,g), an optimal strategy $s^*_{(q,r,g)}$ with the least effective cost given (or in response to) the current strategy assignment X:

$$\begin{aligned} EC^{s^*,\tau^{s^*}}_{(q,r,g)}(X) &= \min_{s\in S_{(q,r)},\tau^s} EC^{s,\tau^s}_{(q,r,g)}(X) \\ &= \min_{s\in S_{(q,r)},\tau^s} E(C^{s,\tau^s}_{(q,r,g)}(X)) + \eta^g_3 Var(C^{s,\tau^s}_{(q,r,g)}(X)). \end{aligned}$$

As in previous work with fixed timetables, the construction of the optimal strategy s^* is based on dynamic programming and uses the information (strategic flows in the classes $W^{\tau,0}_j$ and $W^{\tau,\tau'}_j (\tau' \leq \tau)$) generated by the stochastic loading process.

Since the computation of the effective cost $EC^{s^*,\tau^{s^*}}_{(q,r,g)}(X)$ involves the arc probabilities associated with the optimal (unknown) strategy being constructed, these probabilities have to be computed in reverse T&C order. The resulting procedure resembles the stochastic loading process described in Section 4.2 with the small difference that the flow corresponding to the optimal strategy being computed is set to zero. This micro-loading phase (loading of zero or virtual flow) faces the same challenge occurred in the deterministic case. Indeed, since stochastic loading is performed in reverse T&C order, one might be unaware of the priority status of the virtual flow at loading times. To make up for this, we consider two situations:

i) The virtual (zero) flow arrives at node j_τ with continuance priority and the micro-loading is performed over the set $W^{\tau,0}_j \cup \{s^*\}$ yielding the effective cost $EC^{s^*,\tau,0}_{(j,r,g)}(X) = E(C^{s^*,\tau,0}_{(j,r,g)}(X)) + \eta^g_3 Var(C^{s^*,\tau,0}_{(j,r,g)}(X))$, where $C^{s^*,\tau,0}_{(j,r,g)}(X)$ is the cost for reaching destination r from node j_τ assuming that the passengers using the optimal strategy, s^*, arrive at node j_τ with continuance priority.

ii) The virtual (zero) flow arrives at node j at time $\tau' = 1, 2, \ldots, \tau$ and tries to board transit line l ar node $j = j_n(l)$. The micro-loading is then performed over the sets $W^{\tau,0}_j$, $W^{\tau,1}_j, \cdots W^{\tau,\tau'-1}_j$ and $W^{\tau,\tau'}_j \cup \{s^*\}$ yielding the effective cost $EC^{s^*,\tau,\tau'}_{(j,r,g)}(X) = E(C^{s^*,\tau,\tau'}_{(j,r,g)}(X)) + \eta^g_3 Var(C^{s^*,\tau,\tau'}_{(j,r,g)}(X))$, where $C^{s^*,\tau,\tau'}_{(j,r,g)}(X)$ is the cost for reaching destination r from node j assuming that the passengers using the optimal strategy, s^*, arrive at node j at time τ', where $\tau' = 1, 2, \ldots, \tau$.

The user-preference set $E_j^{s^*,\tau}$ and the effective costs $EC_{(j,r,g)}^{s^*,\tau,\tau'}(\tau' = 0, 1, \ldots, \tau)$ are computed by scanning TE nodes in reverse T&C order and applying Bellman's generalized recursion.

Ending Conditions:

For the current destination r associated with s^*,

i) Set $E_r^{s^*,\tau} = \emptyset, \forall\tau \in \Omega$.

ii) Set $E(C_{(r,r,g)}^{s^*,\tau,\tau'}) = 0, \forall\tau \in \Omega$, and $\tau' = 0, \ldots, \tau$.

iii) Set $Var(C_{(r,r,g)}^{s^*,\tau,\tau'}) = 0, \forall\tau \in \Omega$, and $\tau' = 0, \ldots, \tau$.

For the destination $\widehat{r} \neq r$ not covered by s^*,

i) Set $E_{\widehat{r}}^{s^*,\tau} = \emptyset, \forall\tau \in \Omega$.

ii) Set $E(C_{(\widehat{r},r,g)}^{s^*,\tau,\tau'}) = \infty, \forall\tau \in \Omega$, and $\tau' = 0, \ldots, \tau$.

iii) Set $Var(C_{(\widehat{r},r,g)}^{s^*,\tau,\tau'}) = \infty, \forall\tau \in \Omega$, and $\tau' = 0, \ldots, \tau$.

Recursions at node $j_\tau (j = j_n(l),\ j \neq q)$:

To compute the user-preference set $E_j^{s^*,\tau}$ and the effective costs at node j_τ, we first determine Bellman's equations for the expected and variance costs at node j_τ. Following section 3.3, the equation for the expected cost is:

$$E(C_{(j,r,g)}^{s^*,\tau,\tau'}(X)) = \sum_{k \in E_j^{s^*,\tau} - \{j_{\tau+1}\}} \pi_{jk}^{s^*,\tau}(X)\varphi_k^{s^*,\tau,g} + \pi_j^{s^*,\tau+1}(X)\varphi_j^{s^*,\tau,\tau',g},$$

where $\varphi_j^{s^*,\tau,\tau',g} = \gamma_{wait} + E(C_{(j,r,g)}^{s^*,\tau+1,\tau'}(X))$ and

$$\begin{aligned}\varphi_k^{s^*,\tau,g} &= \gamma_{travel}\mu_{jk} + v_{jk}^{\tau} + e_{jk}^{\tau,g} + \widehat{e}_{jk}(f_{jk}^{\tau}) \\ &\quad + \sum_{\tau'' \geq \tau} P_{jk}(\tau'' - \tau) \begin{cases} E(C_{(k,r,g)}^{s^*,\tau'',0}(X)), & \text{if } (j,k) \text{ and } (k, k_1^{\tau''}) \text{ belong} \\ & \text{to same transit line,} \\ E(C_{(k,r,g)}^{s^*,\tau'',\tau''}(X)), & \text{otherwise,} \end{cases}\end{aligned}$$

$k_1^{\tau''}$ is the first element in the preference set $E_k^{s^*,\tau''}$.

For the variance cost, Bellman's equation can be expressed as:

$$\begin{aligned}Var(C_{(j,r,g)}^{s^*,\tau}(X)) = &\sum_{k \in E_j^{s^*,\tau} - \{j_{\tau+1}\}} \pi_{jk}^{s^*,\tau}(X)\psi_k^{s^*,\tau,g} + \pi_j^{s^*,\tau+1}(X)\psi_j^{s^*,\tau,\tau',g} \\ &+ \sum_{k \in E_j^{s^*,\tau} - \{j_{\tau+1}\}} \pi_{jk}^{s^*,\tau}(X)(\varphi_k^{s^*,\tau,g})^2 + \pi_j^{s^*,\tau+1}(X)(\varphi_j^{s^*,\tau,\tau',g})^2 \\ &-\Big(\sum_{k \in E_j^{s^*,\tau} - \{j_{\tau+1}\}} \pi_{jk}^{s^*,\tau}(X)\varphi_k^{s^*,\tau,g} + \pi_j^{s^*,\tau+1}(X)\varphi_j^{s^*,\tau,\tau',g}\Big)^2,\end{aligned}$$

where $\psi_j^{s^*,\tau,\tau',g} = Var(C_{(j,r,g)}^{s^*,\tau+1,\tau'}(X))$ and

$$\begin{array}{rcl}\psi_k^{s^*,\tau,g} & = & \gamma_{travel}^2 \sigma_{jk}^2 \\ & & + \sum\limits_{\tau'' \geq \tau} P_{jk}(\tau'' - \tau) \left\{ \begin{array}{ll} Var(C_{(k,r,g)}^{s^*,\tau'',0}(X)) & \\ +2\gamma_{travel} C, & \text{if } (j,k) \text{ and } (k, k_1^{\tau''}) \text{ belong} \\ & \text{to same transit line,} \\ Var(C_{(k,r,g)}^{s^*,\tau'',\tau''}(X)), & \text{otherwise,} \end{array} \right.\end{array}$$

where $C = \sum\limits_{n' \in N'_s} Cov(T_{j_n(l),j_{n+1}(l)}, T_{j_{n'}(l),j_{n'+1}(l)})$, N^{s^*} is the set of nodes included in the user-preference sets of strategy s^* and
$N'_{s^*} = \{n' : n+1 \leq n' \leq N_l - 1,\, j_{n'+1}(l) \in N^{s^*}\}$.

After determining Bellman's equations for the expected and variance costs at node j_τ, we must calculate the arc probabilities $\pi_{jk}^{s^*,\tau}(X)$ and $\pi_j^{s^*,\tau}(X)$. As mentioned before, to make up for the unawareness of the priority status at the current time τ, we consider two cases.

i) With continuance priority:

   To consider this case, we should have at least one transit line $l \in L_j$ such that $j = j_n(l)$ and $1 < n < N_l$. In this case, the virtual passenger using strategy s^* is added to the first class $W_j^{\tau,0,1}$ and has continuance priority to access arc (j, j^1) where j^1 is the first element of the set $E_j^{s^*,\tau}$. Node $j^1 = E_j^{s^*,\tau}(1)$ is determined as follows:

$$j^1 = \underset{l \in L_j : 1 < n < N_l}{\arg\min} \{\varphi_{j_{n+1}(l)}^{s^*,\tau,g} + \eta_3^g \psi_{j_{n+1}(l)}^{s^*,\tau,g}\}.$$

   After determining the first element of $E_j^{s^*,\tau}$, the effective cost, $EC_{(j,r,g)}^{s^*,\tau,0}(X)$ is calculated as follows:

$$EC_{(j,r,g)}^{s^*,\tau,0}(X) = \pi_{jj^1}^{s^*,\tau}(X)(\varphi_{j^1}^{s^*,\tau,g} + \eta_3^g \psi_{j^1}^{s^*,\tau,g}) = \varphi_{j^1}^{s^*,\tau,g} + \eta_3^g \psi_{j^1}^{s^*,\tau,g}.$$

ii) Without continuance priority:

   In this case, the virtual passenger using strategy s^* can arrive at node j at time τ', where $\tau' = 1, \ldots, \tau$. For each $\tau' = 1, \ldots, \tau$, we load passengers over the sets $W_j^{\tau,0,1}$, $W_j^{\tau,1}, \cdots W_j^{\tau,\tau'-1}$ and the virtual passenger, $z_j^{s^*,\tau}$, is added to the class $W_j^{\tau,\tau'}$. Then, the effective cost $EC_{(j,r,g)}^{s^*,\tau,\tau'}(X)$ is computed using the recursion:

$$\begin{aligned} EC^{s^*,\tau,\tau'}_{(j,r,g)}(X) &= E(C^{s^*,\tau,\tau'}_{(j,r,g)}(X)) + \eta^g_3 Var(C^{s^*,\tau,\tau'}_{(j,r,g)}(X)) \\ &= \pi^{s^*,\tau}_j(X)\big[\varphi^{s^*,\tau,\tau',g}_j + \eta^g_3\big(\psi^{s^*,\tau,\tau',g}_j + (\varphi^{s^*,\tau,\tau',g}_j)^2\big)\big] \\ &\quad + \sum_{k\in E^{s^*,\tau}_j-\{j_{\tau+1}\}} \pi^{s^*,\tau}_{jk}\big[\varphi^{s^*,\tau,g}_k + \eta^g_3\big(\psi^{s^*,\tau,g}_k + (\varphi^{s^*,\tau,g}_k)^2\big)\big] \\ &\quad -\eta^g_3\big(\pi^{s^*,\tau}_j\varphi^{s^*,\tau,\tau',g}_j + \sum_{k\in E^{s^*,\tau}_j-\{j_{\tau+1}\}} \pi^{s^*,\tau}_{jk}\varphi^{s^*,\tau,g}_k\big)^2, \end{aligned}$$

where the optimal preference set $E^{s^*,\tau}_j$ is the solution of the following combinatorial problem:

$$\begin{aligned} E^{s^*,\tau}_j \;=\; \operatorname*{arg\,min}_{E^{s,\tau}_j\subseteq I^+(j_\tau)} \Big\{ & \pi^{s,\tau}_j(X)\big[\varphi^{s,\tau,\tau',g}_j + \eta^g_3\big(\psi^{s,\tau,\tau',g}_j + (\varphi^{s,\tau,\tau',g}_j)^2\big)\big] \\ & + \sum_{k\in E^{s,\tau}_j-\{j_{\tau+1}\}} \pi^{s,\tau}_{jk}\big[\varphi^{s,\tau,g}_k + \eta^g_3\big(\psi^{s,\tau,g}_k + (\varphi^{s,\tau,g}_k)^2\big)\big] \\ & -\eta^g_3\big(\pi^{s,\tau}_j\varphi^{s,\tau,\tau',g}_j + \sum_{k\in E^{s,\tau}_j-\{j_{\tau+1}\}} \pi^{s,\tau}_{jk}\varphi^{s,\tau,g}_k\big)^2\Big\}. \end{aligned}$$

Determining the minimum effective strategy cost for (q, r, g):

i) The user-preference set $E^{s^*,\tau}_q$: For each $\tau \in \Omega$, compute

$$j^*_{\tau^*_c} = \operatorname*{arg\,min}_{j_{\tau_c}\in I^+(q_\tau)} \{\gamma_{travel}t_{qj} + EC^{s^*,\tau_c,\tau_c}_{(j,r,g)}(X)\},$$

where t_{qj} is the walking time of access arc (q_τ, j_{τ_c}), $\tau_c = \tau + t_{qj}$ and $EC^{s^*,\tau_c,\tau_c}_{(j,r,g)}(X)$ is the effective cost for reaching destination r from node j_{τ_c} assuming that the passengers using the optimal strategy, s^*, arrive at node j_{τ_c} at time τ_c. Then, set $E^{s^*,\tau}_q = \{j^*_{\tau^*_c}\}$ for all $\tau \in \Omega$.

ii) The effective strategy cost $EC^{s^*,\tau,\tau}_{(q,r,g)}(X)$: For each $\tau \in \Omega$,

$$EC^{s^*,\tau,\tau}_{(q,r,g)}(X) = \gamma_{travel}T_{qj^*} + EC^{s^*,\tau^*_c,\tau^*_c}_{(j^*,r,g)}(X).$$

iii) The optimal starting time τ^{s^*}:

$$\tau^{s^*} = \operatorname*{arg\,min}_{\tau\in\Omega}\{EC^{s^*,\tau,\tau}_{(q,r,g)}(X)\}.$$

iv) The effective cost for the optimal s^* is determined as follows:

$$EC^{s^*,\tau^{s^*}}_{(q,r,g)}(X) = EC^{s^*,\tau^{s^*},\tau^{s^*}}_{(q,r,g)}(X).$$

5.2 Solution algorithm

As in Hamdouch and Lawphongpanich (2008) and Hamdouch et al. (2011), we use the method of successive averages (MSA) that generates strategies one at time by solving a dynamic program. The convergence condition of the MSA was stated in Theorem 3 in Cantarella (1997). This theorem states that if the existence and uniqueness conditions mentioned in theorems 1 and 2 (including continuity and strictly monotonicity of cost function) hold and the link cost-flow functions have a symmetric continuous Jacobian with respect to link flows over the feasible solution set, then the MSA converges to the equilibrium link flow vector. Because the cost function EC may fail to meet the symmetric continuous Jacobian condition or strictly monotone condition, the convergence of the iterates towards an equilibrium solution is not guaranteed. In this context, the method must be viewed as a heuristic procedure.

The proposed algorithm first assumes that the TE network is not loaded with passengers (i.e., $z_j^{s,\tau} = 0$ for all nodes within the TE network). With the empty TE network, the corresponding optimal strategy for each OD pair ($s^*[0]$) is computed by the optimal strategy method described in Section 5.1 and is set to be the initial strategy set $S^{[0]}$ for network loading. Also, the initial strategic flow $X^{[0]}$ is set to be the travel demand of the corresponding OD pair and β is set to be zero. Then, the strategic flow $X^{[0]}$ is loaded using the stochastic loading process described in Section 4.2 for getting the corresponding flow of passengers within the TE network at iteration β, $z_j^{s,\tau}(X^{[\beta]})$, and the effective cost of the strategic flow, $EC(X^{[\beta]})$ is computed using the procedure illustrated in Section 4.3. Based on the current flow of passengers, an updated optimal strategy $s^*[\beta]$ can be found and the strategic assignment vector for this step, $Y^{[\beta]}$ with $y_{(q,r,g)}^{s^*[\beta]} = d_{(q,r)}^g$ and $y_{(q,r,g)}^s = 0, \forall s \neq s^*[\beta]$, can be determined. With the current strategic assignment vector, the convergence of the algorithm is checked by the following relative gap function (see e.g., Hamdouch and Lawphongpanich, 2008):

$$g(x) = \frac{EC(X^{[\beta]})^T(X^{[\beta]} - Y^{[\beta]})}{EC(X^{[\beta]})^T X^{[\beta]}} \tag{19}$$

If the value of the above gap function is less than some predetermined tolerance, the algorithm stops with $X^{[\beta]}$ and $S^{[\beta]}$ as the optimal strategic flow vector and strategy set, respectively. Otherwise, $X^{[\beta]}$ and $S^{[\beta]}$ are updated for the next MSA step by the following equation:

$$S^{[\beta+1]} = S^{[\beta]} \cup s^*[\beta] \tag{20}$$

$$X^{[\beta+1]} = \frac{1}{(\beta+1)}(\beta X^{[\beta]} + Y^{[\beta]}), \quad \beta = 0, 1, 2, \cdots \tag{21}$$

The value of β is also increased by one. The updated strategy set and strategic flow are then inputed to the dynamic loading process for getting the updated flow of passengers. A flowchart of the proposed algorithm is shown in Figure 3.

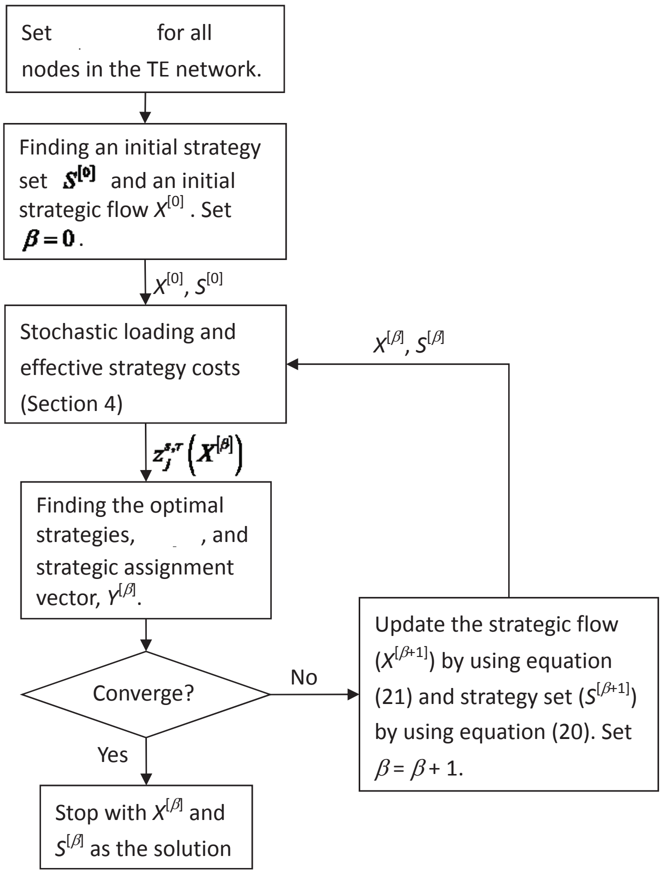

Figure 3: Flowchart of the proposed solution algorithm

At each iteration of the MSA, the stochastic loading process and the optimal strategy computation are performed within the TE network. For each of these two processes, the loading step is performed at most $|I^+(j_\tau)|$ times for each TE node j_τ, where $I^+(j_\tau)$ is the set of successor nodes for j_τ. It follows that the loading process is executed at most $\sum_{\tau\in\Omega}\sum_{j\in N}|I^+(j_\tau)| = |\Omega| \times |A|$ times and that the total running time of the solution algorithm (MSA) is polynomial.

6 Numerical examples

To illustrate our approach, we consider passengers travelling in the morning peak within time period between 7:00 and 8:00 am. The network setting is the same as before. There are four OD pairs, namely (q-y), (q-r), (o-y), and (o-r). The desired arrival intervals for the passengers departing at q and o are [25, 35] and [30, 40], respectively. The travel demands for these four OD pairs are [40, 35, 10, 20]. The transit fares for arcs (a, b), (b, c), (c, d), (a, c), and (b, d) are 0.25, 0.50, 0.75, 1.00, and 0.50, respectively. The capacities for lines 1, 2, and 3 are 20, 30, and 10, respectively. In addition, the penalties for early and late arrivals are both taken as 0.1 per minute for all OD pairs, while the parameter for the discomfort penalty is 0.2. Based on this setting, five examples were conducted and are depicted in the following subsections. All the results were obtained by a laptop with a Core i7-3770 CPU @ 3.4GHz and a 32GB RAM. The memory required for each example is 115MB RAM.

6.1 Effects of value of travel time variance on the number of utilized strategies and convergence

To show the effect of the value of travel time variance, η_3 is set to be the same for all OD pairs and varied from 0.0 to 1.0. Table 7 displays the relative gap value of the method of successive averages (MSA) as well as the number of utilized strategies in selected iterations. It can be seen that the algorithm successfully achieved a relative gap of 0.1% or 0.001 for all three values of η_3. By comparing the three cases, it is noticed that a larger value of η_3 requires more iterations to converge. Moreover, the value of the relative gap fluctuated more significantly when $\eta_3 = 1.0$. Probably, this is because the effective cost function is not necessarily monotonic.

From the column for the number of utilized strategies, it can be seen that the algorithm generated new strategies in early few iterations, and the number of utilized strategies increased. Later, the algorithm updated the strategy set by removing unused strategies at later iterations and hence the number of utilized strategies dropped to the minimum and the number remained unchanged finally.

Iteration number	Number of utilized strategies			Relative gap (%)		
	$\eta_3 = 0.0$	$\eta_3 = 0.5$	$\eta_3 = 1.0$	$\eta_3 = 0.0$	$\eta_3 = 0.5$	$\eta_3 = 1.0$
1	8	8	8	7.81	5.05	7.24
2	9	9	8	8.38	8.94	13.53
3	10	11	10	4.35	3.59	2.92
4	10	11	10	5.21	3.81	6.91
5	10	11	12	2.97	2.94	2.34
10	9	10	11	1.83	1.81	1.98
20	8	10	12	0.81	1.34	1.24
35	7	10	10	0.48	0.45	8.26
41	6	9	9	0.08	1.36	1.00
200		7	9		0.34	0.28
292		7	9		0.10	0.18
323			9			0.10

Table 7: Iterates of MSA: utilized strategies and relative gap

6.2 Effects of the value of travel time variance on flow distributions

Tables 8, 9, and 10 show how strategies are utilized by passengers with various OD pairs and under different values of η_3, where all utilized strategies are depicted in Appendix B. Take Table 10 as an example, when $\eta_3 = 1.0$. Only two strategies, namely $\mathrm{s}^2(1.0)$ and $\mathrm{s}^7(1.0)$ are adopted, where $\mathrm{s}^2(1.0)$ and $\mathrm{s}^7(1.0)$, respectively, denote the second and the seventh strategies when $\eta_3 = 1.0$. There are 40 passengers for OD pair (q-y). 30.08 passengers use strategy $\mathrm{s}^2(1.0)$ and 9.92 passengers use strategy $\mathrm{s}^7(1.0)$. The other strategies are not used by these passengers.

For the same OD, the algorithm can generate different strategies that have the same preference set at some TE nodes but have different preference sets for at least one TE node. Take for example strategies $s_8(1.0)$ and $s_9(1.0)$ in Table 15, Appendix B. At nodes q_0, c_{14}-c_{16} and d_{17}-d_{23}, the two strategies have the same preference sets. However, the preference set at node a_5 is $[b_{l_1}, c_{l_2}, a_6]$ for strategy $s_8(1.0)$ while it is $[c_{l_2}, b_{l_1}, a_6]$ for strategy $s_9(1.0)$.

Although the total number of utilized strategies (after considering all OD pairs) is increasing with the increase of η_3 when passengers are more risk aversive, it is not the case for some OD pairs. For example, the number of utilized strategies of OD pairs (o-y) and (q-y) remain unchanged despite the change in η_3, while the number increases for the

OD	Strategies					
	$\mathrm{s}^1(0.0)$	$\mathrm{s}^2(0.0)$	$\mathrm{s}^3(0.0)$	$\mathrm{s}^4(0.0)$	$\mathrm{s}^5(0.0)$	$\mathrm{s}^6(0.0)$
q-y		19.52			20.48	
q-r	35.00					
o-y			10.00			
o-r				10.16		9.84

Table 8: Strategy utilization: $\eta_3 = 0.0$

OD	Strategies						
	$\mathrm{s}^1(0.5)$	$\mathrm{s}^2(0.5)$	$\mathrm{s}^3(0.5)$	$\mathrm{s}^4(0.5)$	$\mathrm{s}^5(0.5)$	$\mathrm{s}^6(0.5)$	$\mathrm{s}^7(0.5)$
q-y		12.85				27.15	
q-r	14.73		20.27				
o-y				10.00			
o-r					14.50		5.50

Table 9: Strategy utilization: $\eta_3 = 0.5$

OD	Strategies								
	$\mathrm{s}^1(1.0)$	$\mathrm{s}^2(1.0)$	$\mathrm{s}^3(1.0)$	$\mathrm{s}^4(1.0)$	$\mathrm{s}^5(1.0)$	$\mathrm{s}^6(1.0)$	$\mathrm{s}^7(1.0)$	$\mathrm{s}^8(1.0)$	$\mathrm{s}^9(1.0)$
q-y		30.08					9.92		
q-r			20.20					13.45	1.35
o-y				10.00					
o-r	3.83				13.21	2.96			

Table 10: Strategy utilization: $\eta_3 = 1.0$

	q-y	q-r	o-y	o-r
$\eta_3 = 0.0$	6.05	8.55	3.20	6.19
$\eta_3 = 0.5$	6.23	9.93	3.42	6.53
$\eta_3 = 1.0$	6.47	10.42	3.56	6.65

Table 11: Effective costs under different values of η_3

other OD pairs. This indicates that the effect of varying η_3 on the number of utilized strategies is different for various OD pairs. Moreover, the number of possible strategies is different for various OD pairs. For example, for OD pair (o-y), there is only one line connecting origin o and destination y. Comparing with OD pair (q-r), where there are two lines at the first boarding node, the total number of possible strategies for OD pair (o-y) is comparably smaller. Such issue is related to the design of transit networks and implies that the risk aversive passengers can experience higher travel cost because of limited choices of strategies (see Section 6.3).

6.3 Effects of the value of travel time variance on effective costs

Table 11 presents the optimal effective costs for all OD pairs when adjusting η_3. As expected, the effective cost increases with the increase of η_3, since the value of variance increases. However, the increment varies significantly for different OD pairs. For example, for OD pair (q-r), the effective cost grows by 1.38 when η_3 increases from 0.0 to 0.5, while it only increases by 0.49 when η_3 increases from 0.5 to 1.0. This is because passengers utilize more strategies (which can be verified from the previous tables) to minimize the effective cost when η_3 is larger. In contrast, for OD pair (q-y), the increment of effective cost is 0.18 when η_3 increases from 0.0 to 0.5 while it is 0.24 when η_3 increases from 0.5 to 1.0, since the number of utilized strategies remains unchanged when η_3 increases.

6.4 Effects of the value of travel time variance and capacity on departure and arrival times

Figures 4-9 demonstrate that the departure (and arrival) patterns are completely different under various values of η_3. Except for passengers of OD pair (o-y) constantly using one strategy, other passengers either advance or postpone their departure times to switch to a line with a lower variance when the value of travel time variance increases. For OD pair (q-y), some depart at 7:10 to take the second run of line 1, when $\eta_3 = 0.0$. When

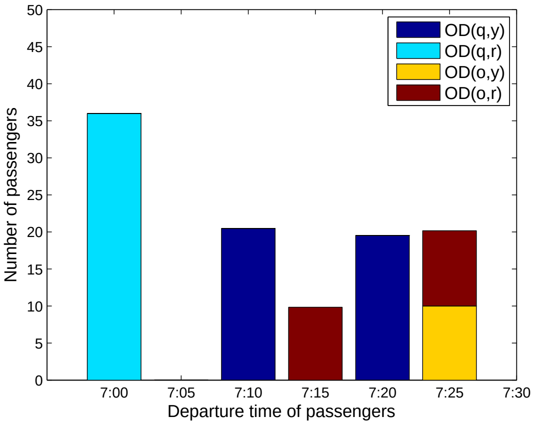

Figure 4: Departure time of passengers ($\eta_3 = 0$)

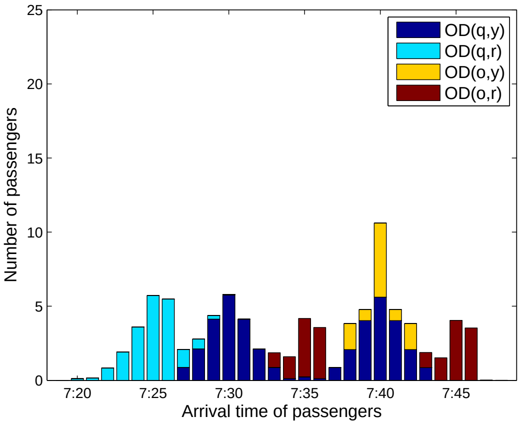

Figure 5: Arrival time of passengers ($\eta_3 = 0$)

considering variance, those passengers switch to a late departure strategy, for which they depart at 7:15 but board another transit line (the second run of line 2). This is because line 2 has a lower variance comparing with line 1. In contrast, some passengers of OD pair (o-r) tend to depart earlier when η_3 increases to 1.0, although such strategy induces more early arrival penalty, implying that passengers are more willing to have a higher arrival penalty to counteract the effect of the variance of travel time when they are more risk aversive.

With the adjustment of departure time, the arrival time changes accordingly. More importantly, when the level of risk aversion increases, more passengers arrive on time by adjusting their departure times or selecting a different strategy to reach their individual destination.

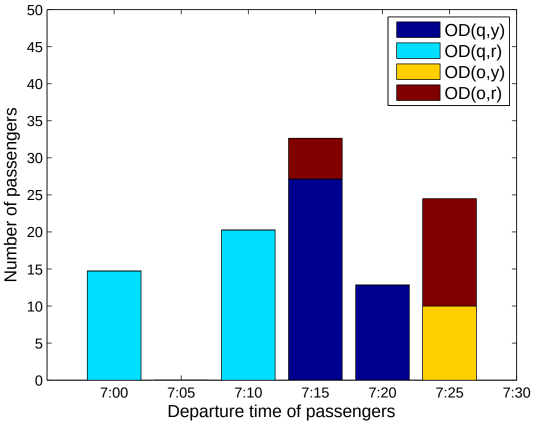

Figure 6: Departure time of passengers ($\eta_3 = 0.5$)

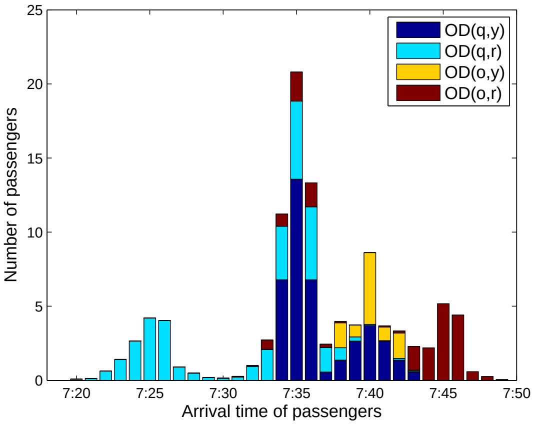

Figure 7: Arrival time of passengers ($\eta_3 = 0.5$)

To investigate the effects of early/late arrival penalties on the departure time choices, we set $\eta_3 = 0.5$ and both penalties for early and late arrivals are reduced by half. The departure and arrival patterns are plotted in Figures 10 and 11.

Figures 6-7 (base case) and Figures 10-11 (case with reduced early/late arrival penalties) illustrate that the penalties have different effects on different OD pairs. For example, the departure and arrival times of the passengers of OD pair (o, y) are not affected, implying that their choices are irrespective of the arrival penalty values. This is because these passengers can arrive at their destination within the desired arrival interval in the base case, where the arrival penalties are high. Therefore, they can use their original strategies, despite the reduction in the arrival penalties. For the other OD pairs, especially OD pairs (q, r) and (q, y), it is interesting to notice that these passengers choose to depart

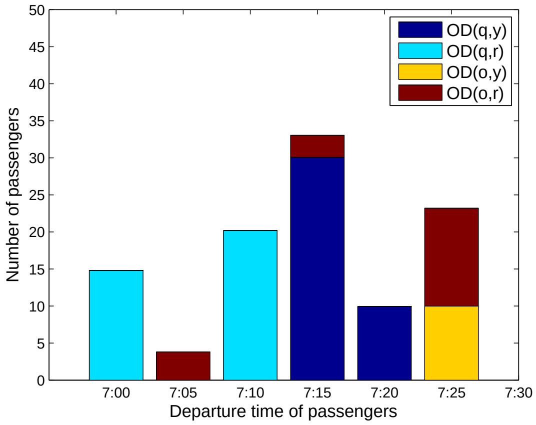

Figure 8: Departure time of passengers ($\eta_3 = 1$)

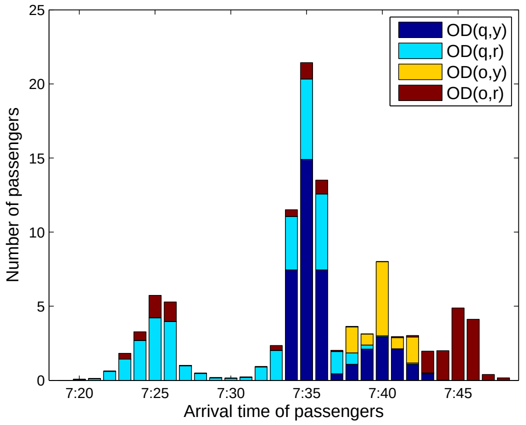

Figure 9: Arrival time of passengers ($\eta_3 = 1$)

earlier when the value of early/late arrival penalty is reduced by half. By investigating the arrival patterns, such departure time choice can be explained by the passengers' trade-off between the penalties and congestion effect. In the base case, more than 50 passengers arrive within 7:33-7:37, while in the case with reduced penalties, only around 30 passengers arrive within that time interval, implying that passengers incur a higher congestion cost in the base case. Therefore, when the values of early/arrival penalty is reduced, passengers select the departure time that allow them to board a less congested line. The trade-off between the congestion cost and arrival penalties can also be used to explain why the passengers of OD pair (q, r) select the early run instead of postponing their departure. The reason is that in the early run, they are the only passengers on the transit line who arrive within the time window 7:20-7:26. If they postpone their departure

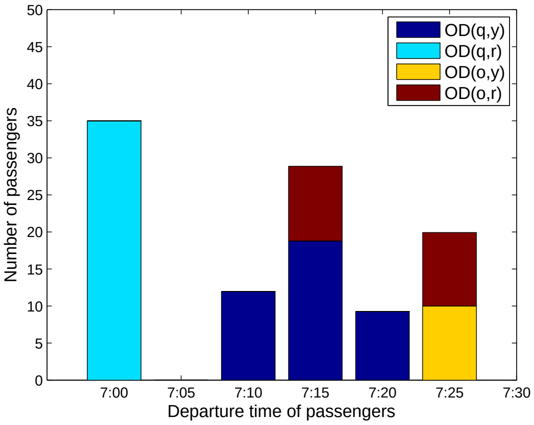

Figure 10: Departure time of passengers: $\eta_3 = 0.5$ and reduced early/late arrival penalties

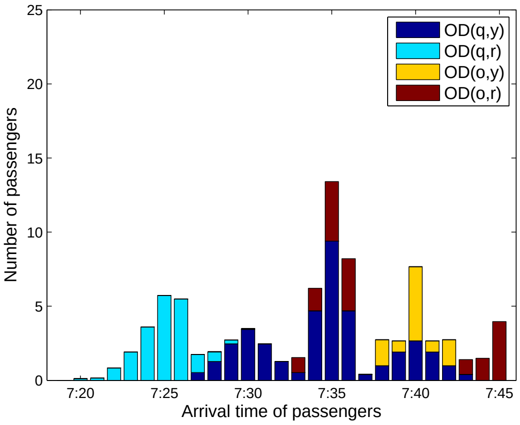

Figure 11: Arrival time of passengers: $\eta_3 = 0.5$ and reduced early/late arrival penalties

times, they must bear a higher congestion cost due to the boarding of the passengers of other OD pairs.

To illustrate the effect of capacity on departure and arrival times, the capacities of all the lines are doubled. The departure and arrival patterns are plotted in Figures 12 and 13. For OD pair (q, r), all the passengers depart at 7:10 when the capacity is doubled. By doing so, the total early arrival penalty of these passengers is reduced, because all these passengers arrive within the desired time interval as shown in Figure 13 . For OD pair (o, r), the passengers that depart at 7:15 switch from the second run to the first run of line 3, because such choice reduces their congestion cost. More importantly, it is worth mentioning the reason that these passengers do not depart at 7:05 before the capacity improvement. This is because some of the passengers of OD pair (q, r) that take line 1

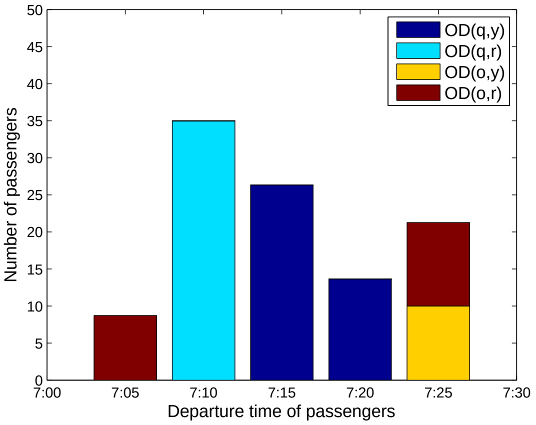

Figure 12: Departure time of passengers: $\eta_3 = 0.5$ and double capacity

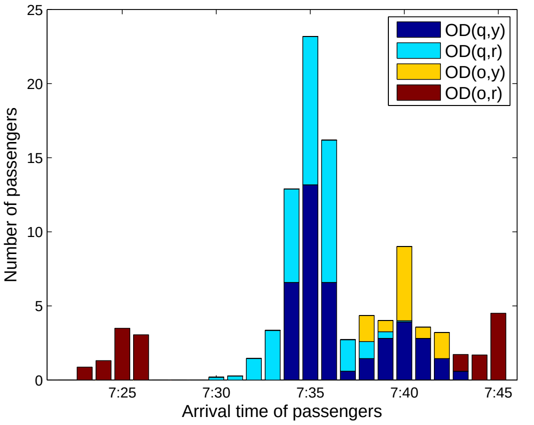

Figure 13: Arrival time of passengers: $\eta_3 = 0.5$ and double capacity

arrive at node b at an earlier time interval than the passengers between OD pair (o, r) and wait for line 3. Hence, these passengers have the priority to board line 3. Consequently, there is no residual capacity for the passengers of OD pair (o, r).

For the other two OD pairs, the increment in the capacity does not affect their departure choices. On one hand, this is because the demand (i.e., OD pair (o, y)) is low and can be accommodated before the capacity improvement. On the other hand, the stop that they board is the first stop of the transit line; thus boarding priority can be guaranteed. Moreover, it is observed that when the capacities are doubled, the first run of line 1 is not used. This implies that the operator can cancel certain runs of transit services by increasing vehicle capacity, resulting in a lower operational cost.

6.5 Effects of the value of travel time variance on equilibrium arc flows

To further illustrate the properties of the proposed model, three strategies adopted by the passengers of OD pair (q-r) are plotted in Figures 14, 15, and 16. Only the arcs with positive passenger flows are displayed and the number inside square blankets beside a selected arc denotes the passenger flow of OD pair (q-r) using a certain strategy. Due to the space limitation, node r (which can be easily reached from node d) is not shown in those figures.

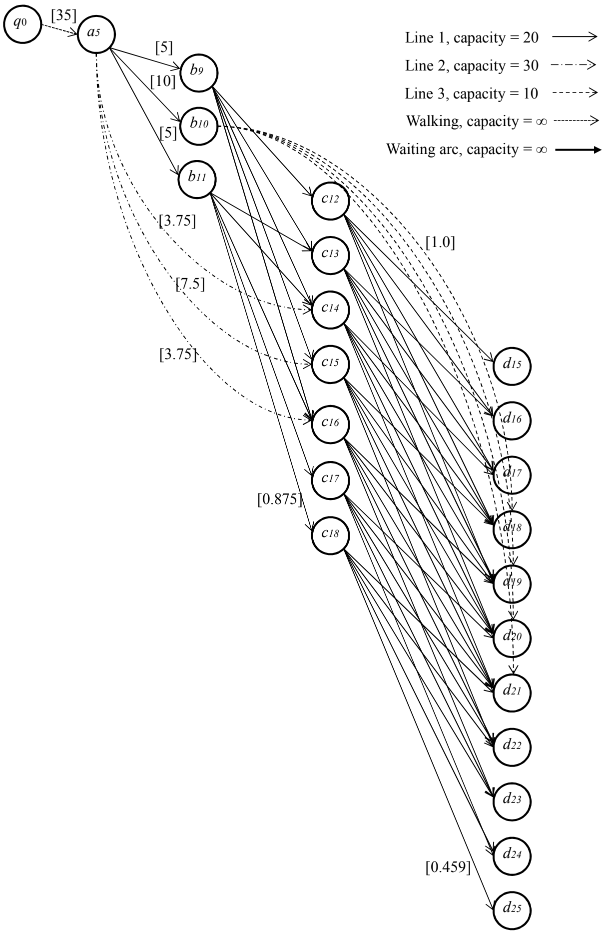

Figure 14: Equilibrium strategic flow of strategy $s^1(0.0)$

Figure 14 displays strategy $s^1(0.0)$ when $\eta_3 = 0.0$. All travellers depart from q at time interval 0 (i.e., 7:00 am) and arrive at stop a at time interval 5 via the walking arc. Afterwards, 20 passengers board the first run of line 1 and 15 passengers board line 2, respectively. It is worth to mention that, at node a_{15}, all 35 passengers have an identical optimal preference set, which is $[b_{l_1}, c_{l_2}, a_6]$. However, due to the capacity constraint of line 1, only 20 passengers can board line 1 (i.e., their first choice) while 15 passengers use their second choice. In addition, it is found that the uncertainty of travel time affects the passengers' travel choice. Take node b as an example, where the arrival time depends on the in-vehicle travel time distribution of arc (a-b) of line 1. From Table 1, the probability of arriving at node b_{10} is 0.5. In such case, passengers alight from line 1 and transfer to the first run of line 3. In contrast, if passengers arrive at b_9 or b_{11} with a probability of 0.25, they select to use line 1 continuously. This is because passengers arrive at b_{11} after the departure time of line 3, while the expected travel time from b_9 using line 3 is longer after considering the additional waiting time.

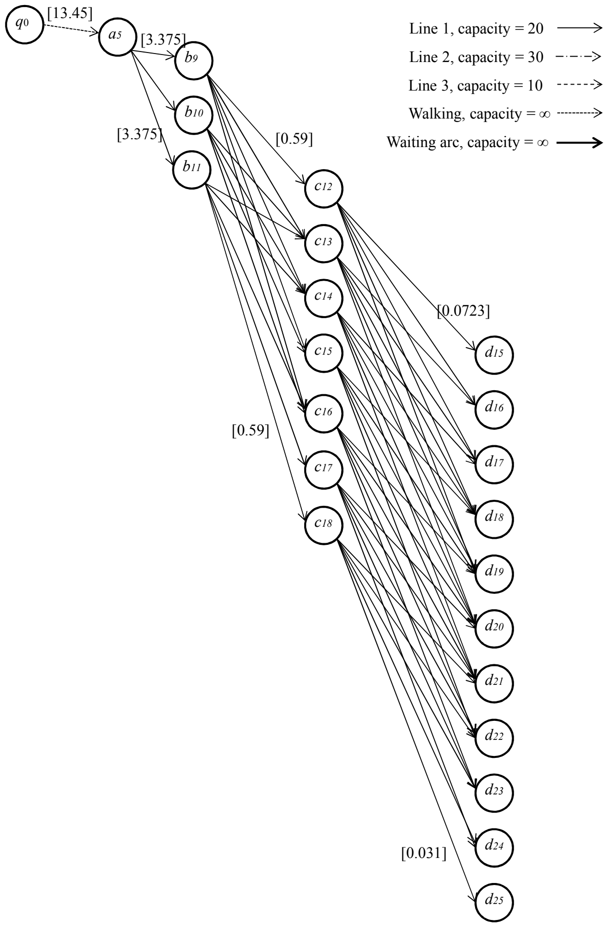

Figure 15: Equilibrium strategic flow of strategy $s^8(1.0)$

Figures 15 and 16 display strategies $s^8(1.0)$ and $s^9(1.0)$ utilized by OD pair (q-r) when $\eta_3 = 1.0$. When the passengers consider the effect of variance, only 14.8 passengers depart at node q_0, and others postpone their departure time. Those 14.8 passengers are divided into two groups. 13.45 passengers use strategy $s^8(1.0)$, while 1.35 passengers use strategy $s^9(1.0)$. At node a_5, the passengers utilizing strategy $s^8(1.0)$ select line 1 as their first choice, and the passengers utilizing strategy $s^9(1.0)$ select line 2 as their first choice. Comparing with the case of $\eta_3 = 0.0$, where all passengers have the same preference set $[b_{l_1}, c_{l_2}, a_6]$, the passengers have two different preference sets in the case of $\eta_3 = 1.0$, namely strategy $s^8(1.0)$ ($[b_{l_1}, c_{l_2}, a_6]$) and strategy $s^9(1.0)$ ($[c_{l_2}, b_{l_1}, a_6]$). These two strategies are different in terms of the order of arriving nodes (and the lines used) in their user-preference sets. Strategy $s^8(1.0)$ involves no transfer throughout the journey. Strategy $s^9(1.0)$ involves one transfer at node c and two transit lines. The implication is that there is a tradeoff between the variance of in-vehicle travel time and the variance of waiting time.

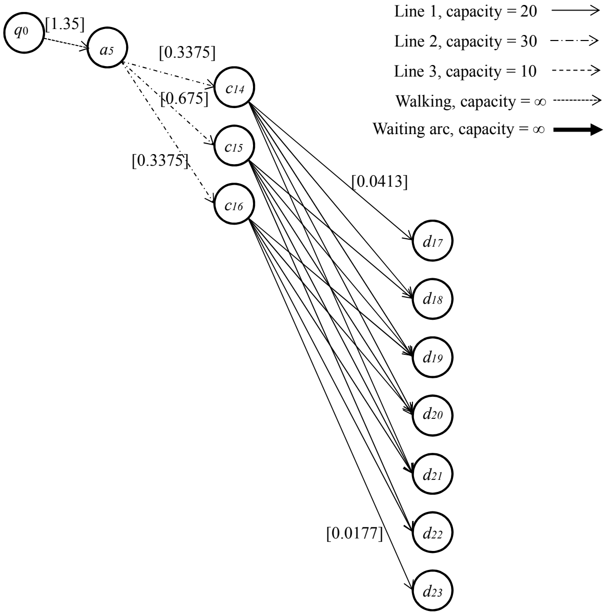

Figure 16: Equilibrium strategic flow of strategy $s^9(1.0)$

The first run of line 1 is fully occupied when the variance effect is ignored by the passengers. (The number of passengers in the first run is equal to the vehicle capacity of 20.) However, when the passengers consider the effect of variance, the first run of line 1 is not fully occupied. Only 13.45 passengers board line 1. This implies that the route choice behavior affects the level of service inside transit vehicles. In addition, it is observed that at node b_{10}, the choice of using line 3 is removed from the optimal strategy set unlike to the case when $\eta_3 = 0.0$. Passengers prefer to continuously stay on line 1, implying that passengers may avoid transfer when they are more risk aversive. This may occur because the waiting time uncertainty can be high at the next stop or the in-vehicle travel time of

	Nb. of utilized strategies			
	q-y	q-r	o-y	o-r
$\eta_3 = 0.5$, $\phi = 0.3$	2	2	1	2
$\eta_3 = 0.5$, $\phi = 0.6$	2	2	1	3
$\eta_3 = 1.0$, $\phi = 0.3$	2	3	1	3
$\eta_3 = 1.0$, $\phi = 0.6$	2	2	1	3

Table 12: Effects of ϕ on the number of utilized strategies

the next service is high, and hence it is better to stay in transit vehicles so as to arrive on time. Hence, it can be concluded that passengers can adjust their choices of next transit stations, in addition to just their adjusting departure times, to counteract the effect of travel time variance. Consequently, the resultant flow pattern is significantly different.

6.6 Effects of coefficient ϕ on the number of utilized strategies

Table 12 illustrates the effects of coefficient ϕ on the number of utilized strategies for each OD pair. In general, a larger value of ϕ indicates that the mean and variance of a link cost are affected more by its previous arc. Consequently, by increasing the value of ϕ , the strategy cost as well as the number of utilized strategies can be changed. For OD pairs q-y and o-y, the numbers of utilized strategies are unaffected by the value of ϕ, because the transit lines used in these strategies only traverse one arc. For the other two OD pairs, the effects of ϕ also depend on the value of η_3^g. Surprisingly, all possible trends for the number of utilized strategies are observed including increasing (i.e., OD pair o-r, when $\eta_3^g = 0.5$), remaining stable (i.e., OD pair o-r, when $\eta_3^g = 1.0$) or decreasing (i.e., OD pair q-r, when $\eta_3^g = 1.0$). These observations imply that the value of ϕ can induce various effects on the number of utilized strategies; thus it is important to have an accurate estimation of the value of ϕ, which is left for future study.

7 Considerations in real life applications

To apply the proposed methodology to real-life applications, three issues are required to consider: computational resource requirement, convergence conditions, and computational time. Because some variables in the proposed model are indexed by at least transit station, time period, strategy, and transit line, the size of the matrices (computer storage) grows exponentially when the TE network becomes larger or more transit lines are modelled. In such cases, additional effort should be made on the effective allocation of computer storage as most of the matrices are sparse matrices. A good data structure (for example, using vectors and pointers) can be developed to reduce the computation storage. Moreover, when solving for optimal solutions, it is not necessary to explicitly construct and maintain the whole TE network in computer memory. Time-expanded nodes and arcs can be generated as needed when solving the problem, e.g., to find a strategy with the least effective cost.

The convergence condition of the solution algorithm is another important issue for the applicability of our proposed methodology. The proposed MSA requires that the cost function EC satisfies the symmetric continuous Jacobian condition or strictly monotone condition for convergence. However, this condition may not be satisfied, especially for realistic, large transit networks. Canteralla (1997) proposed the cost averaging algorithm. Compared with the MSA, the cost averaging algorithm is a method of successive cost averages instead of successive flow averages. To ensure convergence, this algorithm does not rely on that the cost function EC satisfies the symmetric continuous Jacobian condition or strictly monotone condition for convergence. Instead, the algorithm only requires some milder assumptions for convergence (see Theorem 4 in his paper). Some assumptions are used to ensure that a link flow solution exists to the problem and is unique (see Theorem 2 in his paper).

Computational time is also a crucial issue for the applicability of our methodology in real transit network applications. Compared to the other transit assignment models, the proposed model is more suitable to adopt parallel computing for reducing computational times. It is because the loading process and the computation of optimal strategy are performed on a node basis. Thus, for each of these processes, they can be started simultaneously from different nodes given that the specific criteria (reverse T&C order for stochastic loading and optimal strategy computation) are satisfied. Moreover, the convergence speed of the MSA may be slow even for solving medium-size network transit assignment problems, because of the step size used. The self-regulated averaging method proposed by Liu et al. (2009) was shown to converge to the equilibrium solution faster than the MSA. The self-regulated averaging method adjusts the step size to speed up the convergence and has been applied to solve other traffic assignment problems (e.g., Szeto et al., 2011a; Long et al., 2014). The convergence requirements are basically the same as those for the MSA. This algorithm can be one of the candidate solution methods for real-life applications. Furthermore, the cost averaging version of the self-regulated averaging method proposed by Long et al. (2014) can be another choice. It has the advantages of both the cost averaging method and the self-regulated averaging method.

8 Conclusion

In this paper, we propose a new schedule-based transit assignment model in which passengers adopt strategies to travel from their origins to their destinations. While this strategy concept has been successfully used in previous transit assignment studies with fixed timetables, the new proposed model captures explicitly the stochastic nature of the transit schedules and in-vehicle travel times due to road conditions, incidents, or adverse weather. No such analytical schedule-based model has been developed in the literature to consider both travel strategies and supply uncertainties. When loading passengers on a first-come-first-serve basis, the model takes into account the transit capacities explicitly. Using a mean-variance approach, the equilibrium conditions for this schedule-based transit assignment problem are stated as a variational inequality involving a vector-valued function of effective strategy costs. To find an equilibrium solution, we adopt the method of successive averages in which the optimal strategy of each iteration is generated by solving a dynamic program.

Numerical studies are included to illustrate the effect of supply uncertainties, vehicle capacity and early/late arrival penalty parameters on travel strategies and/or departure times of passengers. In particular, we show that

i) When the value of travel time variance increases, people may decide to leave later.

ii) Increasing/reducing vehicle capacity may have no effect on departure time choice.

iii) Early/late arrival penalties may have no effect on departure time choice.

iv) Passengers may make a tradeoff between the variance of in-vehicle travel time and the variance of waiting time.

v) Passengers can adjust their choices of next transit stations, in addition to just adjusting their departure times, to counteract the effect of travel time variance.

vi) For the same OD, the algorithm can generate different strategies that have the same preference set at some TE nodes but have different preference sets for at least one TE node.

vii) The number of utilized strategies for an OD pair does not necessary increase with the value of travel time variance.

This study opens up many future research directions. One direction is to extend our model to consider stochastic user equilibrium (SUE) and different nonlinear and non-additive fare structures. Given that a fixed point formulation can easily cope with these types of fare structures and SUE (see Cantarella, 1997) simultaneously, a resultant fixed point formulation may be developed in the future. Moreover, the resultant formulation can be solved by the cost averaging algorithm proposed by Cantarella (1997) or its extension such as the cost averaging version of self-regulated averaging method proposed by Long et al. (2014). Other future research of this study include the consideration of demand uncertainties (Ng et al., 2011), the extra considerations of other dynamics such as the year-to-year dynamic (e.g., Szeto and Lo, 2008; Lo and Szeto, 2009) and the day-to-day dynamic (Watling and Cantarella, 2013), and the development of efficient solution algorithms (e.g., Long et al., 2010; Szeto and Wu, 2011) for the large scale implementation of the proposed model for transit assignment and vehicle scheduling.

Acknowledgements

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Appendix A

i) We first show that $\sigma^2_{jk} = \phi\sigma^2_{ij} + (1-\phi)Var(Y_{jk}) + \phi(1-\phi)\big(\mu_{ij} - E(Y_{jk})\big)^2$

$$
\begin{aligned}
\sigma^2_{jk} &= \sum_t t^2 P(T_{jk}=t) - \big(\sum_t tP(T_{jk}=t)\big)^2 \\
&= \sum_t t^2\big(\phi P(T_{jk}=t) + (1-\phi)P(Y_{jk}=t)\big) \\
&\quad -\big(\sum_t t\big(\phi P(T_{jk}=t) + (1-\phi)P(Y_{jk}=t)\big)\big)^2 \\
&= \phi\sum_t t^2 P(T_{ij}=t) + (1-\phi)\sum_t t^2 P(Y_{jk}=t) - \phi^2\big(\sum_t tP(T_{ij}=t)\big)^2 \\
&\quad -(1-\phi)^2\big(\sum_t tP(Y_{jk}=t)\big)^2 - 2\phi(1-\phi)\sum_t tP(T_{ij}=t)\sum_t tP(Y_{jk}=t) \\
&= \phi E(T^2_{ij}) - \phi^2(E(T_{ij}))^2 + (1-\phi)E(Y^2_{jk}) - (1-\phi)^2(E(T_{jk}))^2 \\
&\quad -2\phi(1-\phi)E(T_{ij})E(Y_{jk}) \\
&= \phi\big(E(T^2_{ij}) - (E(T_{ij}))^2\big) + (1-\phi)\big(E(Y^2_{jk}) - (E(Y_{jk}))^2\big) \\
&\quad +\phi(1-\phi)(E(T_{ij}))^2 + \phi(1-\phi)(E(Y_{jk}))^2 - 2\phi(1-\phi)E(T_{ij})E(Y_{jk}) \\
&= \phi\sigma^2_{ij} + (1-\phi)Var(Y_{jk}) + \phi(1-\phi)\big((E(T_{ij}))^2 + (E(Y_{jk}))^2 - 2E(T_{ij})E(Y_{jk})\big) \\
&= \phi\sigma^2_{ij} + (1-\phi)Var(Y_{jk}) + \phi(1-\phi)\big(E(T_{ij}) - E(Y_{jk})\big)^2 \\
&= \phi\sigma^2_{ij} + (1-\phi)Var(Y_{jk}) + \phi(1-\phi)\big(\mu_{ij} - E(Y_{jk})\big)^2.
\end{aligned}
$$

ii) For each $1 \le n \le N_l - 2$, we will show by induction on n', $1 \le n' \le N_l - 1 - n$ that: $Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+n'}(l)i_{n+n'+1}(l)}) = \phi^{n'}\sigma^2_{i_n(l)i_{n+1}(l)}$.

- $n' = 1$

$$
\begin{aligned}
& Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+1}(l)i_{n+2}(l)}) \\
=\ & Cov(T_{i_n(l)i_{n+1}(l)}, \phi T_{i_n(l)i_{n+1}(l)} + (1-\phi)Y_{i_{n+1}(l)i_{n+2}(l)}) \\
=\ & Cov(T_{i_n(l)i_{n+1}(l)}, \phi T_{i_n(l)i_{n+1}(l)}) \text{ (since } T_{jk} \text{ and } Y_{jk} \text{ are independent)} \\
=\ & E(\phi T^2_{i_n(l)i_{n+1}(l)}) - E(T_{i_n(l)i_{n+1}(l)})E(\phi T_{i_n(l)i_{n+1}(l)}) \\
=\ & \phi\big(E(T^2_{i_n(l)i_{n+1}(l)}) - (E(T_{i_n(l)i_{n+1}(l)}))^2\big) \\
=\ & \phi\,\sigma^2_{i_n(l)i_{n+1}(l)}.
\end{aligned}
$$

- Assume $Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+n'}(l)i_{n+n'+1}(l)}) = \phi^{n'}\sigma^2_{i_n(l)i_{n+1}(l)}$

– $n' + 1$

$$\begin{aligned}
& Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+n'+1}(l)i_{n+n'+2}(l)}) \\
= \;& Cov(T_{i_n(l)i_{n+1}(l)}, \phi T_{i_{n+n'}(l)i_{n+n'+1}(l)} + (1-\phi)Y_{i_{n+n'+1}(l)i_{n+n'+2}(l)}) \\
= \;& Cov(T_{i_n(l)i_{n+1}(l)}, \phi T_{i_{n+n'}(l)i_{n+n'+1}(l)}) \\
= \;& \phi\, Cov(T_{i_n(l)i_{n+1}(l)}, T_{i_{n+n'}(l)i_{n+n'+1}(l)}) \\
= \;& \phi\, \phi^{n'} \sigma^2_{i_n(l)i_{n+1}(l)} \\
= \;& \phi^{n'+1} \sigma^2_{i_n(l)i_{n+1}(l)}.
\end{aligned}$$

Appendix B

<table>
<tr><td colspan="10">Strategy: $s^1(0.0)$</td></tr>
<tr><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td></tr>
<tr><td>q_0</td><td>$[a_5]$</td><td>a_5</td><td>$[b_{l_1}, c_{l_2}, a_6]$</td><td>b_9</td><td>$[c_{l_1}, b_{10}]$</td><td>c_{12}</td><td>$[d_{l_1}, c_{13}]$</td><td>d_{15}</td><td>$[r_{20}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{10}</td><td>$[d_{l_3}, c_{l_1}, b_{11}]$</td><td>$c_{13}$</td><td>$[d_{l_1}, c_{14}]$</td><td>$d_{16}$</td><td>$[r_{21}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{11}</td><td>$[c_{l_1}, b_{12}]$</td><td>c_{14}</td><td>$[d_{l_1}, c_{15}]$</td><td>d_{17}</td><td>$[r_{22}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{15}</td><td>$[d_{l_1}, c_{16}]$</td><td>d_{18}</td><td>$[r_{23}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{16}</td><td>$[d_{l_1}, c_{17}]$</td><td>d_{19}</td><td>$[r_{24}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{17}</td><td>$[d_{l_1}, c_{18}]$</td><td>d_{20}</td><td>$[r_{25}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{18}</td><td>$[d_{l_1}, c_{19}]$</td><td>d_{21}</td><td>$[r_{26}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{22}</td><td>$[r_{27}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{23}</td><td>$[r_{28}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{24}</td><td>$[r_{29}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{25}</td><td>$[r_{30}]$</td></tr>
<tr><td colspan="10">Strategy: $s^2(0.0)$</td></tr>
<tr><td>q_{20}</td><td>$[a_{25}]$</td><td>a_{25}</td><td>$[b_{l_1}, a_{26}]$</td><td>b_{29}</td><td>$[c_{l_1}, b_{30}]$</td><td>c_{32}</td><td>$[y_{37}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{30}</td><td>$[c_{l_1}, b_{31}]$</td><td>c_{33}</td><td>$[y_{38}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{31}</td><td>$[c_{l_1}, b_{32}]$</td><td>c_{34}</td><td>$[y_{39}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{35}</td><td>$[y_{40}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{36}</td><td>$[y_{41}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{37}</td><td>$[y_{42}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{38}</td><td>$[y_{43}]$</td><td></td><td></td></tr>
<tr><td colspan="10">Strategy: $s^3(0.0)$</td></tr>
<tr><td>o_{25}</td><td>$[b_{30}]$</td><td>b_{30}</td><td>$[c_{l_1}, b_{31}]$</td><td>c_{33}</td><td>$[y_{38}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{34}</td><td>$[y_{39}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{35}</td><td>$[y_{40}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{36}</td><td>$[y_{41}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{37}</td><td>$[y_{42}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="10">Strategy: $s^4(0.0)$</td></tr>
<tr><td>o_{25}</td><td>$[b_{30}]$</td><td>b_{30}</td><td>$[d_{l_3}, c_{l_1}, b_{31}]$</td><td>$c_{33}$</td><td>$[d_{l_1}, c_{34}]$</td><td>$d_{36}$</td><td>$[r_{41}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{34}</td><td>$[d_{l_1}, c_{35}]$</td><td>d_{37}</td><td>$[r_{42}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{35}</td><td>$[d_{l_1}, c_{36}]$</td><td>d_{38}</td><td>$[r_{43}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{36}</td><td>$[d_{l_1}, c_{37}]$</td><td>d_{39}</td><td>$[r_{44}]$</td><td></td><td></td></tr>
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<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{43}</td><td>$[r_{48}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{44}</td><td>$[r_{49}]$</td><td></td><td></td></tr>
<tr><td colspan="10">Strategy: $s^5(0.0)$</td></tr>
<tr><td>q_{10}</td><td>$[a_{15}]$</td><td>a_{15}</td><td>$[b_{l_1}, a_{16}]$</td><td>b_{19}</td><td>$[c_{l_1}, b_{20}]$</td><td>c_{22}</td><td>$[y_{27}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>a_{16}</td><td>$[a_{17}]$</td><td>b_{20}</td><td>$[c_{l_1}, b_{21}]$</td><td>c_{23}</td><td>$[y_{28}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>a_{17}</td><td>$[a_{18}]$</td><td>b_{21}</td><td>$[c_{l_1}, b_{22}]$</td><td>c_{24}</td><td>$[y_{29}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>a_{18}</td><td>$[a_{19}]$</td><td></td><td></td><td>c_{25}</td><td>$[y_{30}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>a_{19}</td><td>$[a_{20}]$</td><td></td><td></td><td>c_{26}</td><td>$[y_{31}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>a_{20}</td><td>$[c_{l_2}, a_{21}]$</td><td></td><td></td><td>c_{27}</td><td>$[y_{32}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{28}</td><td>$[y_{33}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{29}</td><td>$[y_{34}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{30}</td><td>$[y_{35}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{31}</td><td>$[y_{36}]$</td><td></td><td></td></tr>
<tr><td colspan="10">Strategy: $s^6(0.0)$</td></tr>
<tr><td>o_{15}</td><td>$[b_{20}]$</td><td>b_{20}</td><td>$[d_{l_3}, c_{l_1}, b_{21}]$</td><td>$d_{28}$</td><td>$[r_{33}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>d_{29}</td><td>$[r_{34}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>d_{30}</td><td>$[r_{35}]$</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>d_{31}</td><td>$[r_{36}]$</td><td></td><td></td><td></td><td></td></tr>
</table>

Table 13: Utilized strategies ($\eta_3 = 0.0$)

Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.
Strategy: $s^1(0.5)$									
q_0	$[a_5]$	a_5	$[b_{l_1}, c_{l_2}, a_6]$	b_9	$[c_{l_1}, b_{10}]$	c_{12}	$[d_{l_1}, c_{13}]$	d_{15}	$[r_{20}]$
				b_{10}	$[d_{l_3}, c_{l_1}, b_{11}]$	c_{13}	$[d_{l_1}, c_{14}]$	d_{16}	$[r_{21}]$
				b_{11}	$[c_{l_1}, b_{12}]$	c_{14}	$[d_{l_1}, c_{15}]$	d_{17}	$[r_{22}]$
						c_{15}	$[d_{l_1}, c_{16}]$	d_{18}	$[r_{23}]$
						c_{16}	$[d_{l_1}, c_{17}]$	d_{19}	$[r_{24}]$
						c_{17}	$[d_{l_1}, c_{18}]$	d_{20}	$[r_{25}]$
						c_{18}	$[d_{l_1}, c_{19}]$	d_{21}	$[r_{26}]$
								d_{22}	$[r_{27}]$
								d_{23}	$[r_{28}]$
								d_{24}	$[r_{29}]$
								d_{25}	$[r_{30}]$
Strategy: $s^2(0.5)$									
q_{20}	$[a_{25}]$	a_{25}	$[b_{l_1}, a_{26}]$	b_{29}	$[c_{l_1}, b_{30}]$	c_{32}	$[y_{37}]$		
				b_{30}	$[c_{l_1}, b_{31}]$	c_{33}	$[y_{38}]$		
				b_{31}	$[c_{l_1}, b_{32}]$	c_{34}	$[y_{39}]$		
						c_{35}	$[y_{40}]$		
						c_{36}	$[y_{41}]$		
						c_{37}	$[y_{42}]$		
						c_{38}	$[y_{43}]$		
Strategy: $s^3(0.5)$									
q_{10}	$[a_{15}]$	a_{15}	$[b_{l_1}, a_{16}]$	b_{19}	$[c_{l_1}, b_{20}]$	c_{22}	$[d_{l_1}, c_{23}]$	d_{25}	$[r_{30}]$
		a_{16}	$[a_{17}]$	b_{20}	$[d_{l_3}, c_{l_1}, b_{21}]$	c_{23}	$[d_{l_1}, c_{24}]$	d_{26}	$[r_{31}]$
		a_{17}	$[a_{18}]$	b_{21}	$[c_{l_1}, b_{22}]$	c_{24}	$[d_{l_1}, c_{25}]$	d_{27}	$[r_{32}]$
		a_{18}	$[a_{19}]$			c_{25}	$[d_{l_1}, c_{26}]$	d_{28}	$[r_{33}]$
		a_{19}	$[a_{20}]$			c_{26}	$[d_{l_1}, c_{27}]$	d_{29}	$[r_{34}]$
		a_{20}	$[c_{l_2}, a_{21}]$			c_{27}	$[d_{l_1}, c_{28}]$	d_{30}	$[r_{35}]$
						c_{28}	$[d_{l_1}, c_{29}]$	d_{31}	$[r_{36}]$
						c_{29}	$[c_{30}]$	d_{32}	$[r_{37}]$
						c_{30}	$[c_{31}]$	d_{33}	$[r_{38}]$
						c_{31}	$[c_{32}]$	d_{34}	$[r_{39}]$
						c_{32}	$[d_{l_1}, c_{33}]$	d_{35}	$[r_{40}]$
								d_{36}	$[r_{41}]$
								d_{37}	$[r_{42}]$
								d_{38}	$[r_{43}]$
								d_{39}	$[r_{44}]$
Strategy: $s^4(0.5)$									
o_{25}	$[b_{30}]$	b_{30}	$[c_{l_1}, b_{31}]$	c_{33}	$[y_{38}]$				
		b_{31}	$[c_{l_1}, b_{32}]$	c_{34}	$[y_{39}]$				
				c_{35}	$[y_{40}]$				
				c_{36}	$[y_{41}]$				
				c_{37}	$[y_{42}]$				
				c_{38}	$[y_{43}]$				
Strategy: $s^5(0.5)$									
o_{25}	$[b_{30}]$	b_{30}	$[d_{l_3}, c_{l_1}, b_{31}]$	c_{33}	$[d_{l_1}, c_{34}]$	d_{36}	$[r_{41}]$		
		b_{31}	$[c_{l_1}, b_{32}]$	c_{34}	$[d_{l_1}, c_{35}]$	d_{37}	$[r_{42}]$		
				c_{35}	$[d_{l_1}, c_{36}]$	d_{38}	$[r_{43}]$		
				c_{36}	$[d_{l_1}, c_{37}]$	d_{39}	$[r_{44}]$		
				c_{37}	$[d_{l_1}, c_{38}]$	d_{40}	$[r_{45}]$		
				c_{38}	$[d_{l_1}, c_{39}]$	d_{41}	$[r_{46}]$		
						d_{42}	$[r_{47}]$		
						d_{43}	$[r_{48}]$		
						d_{44}	$[r_{49}]$		
						d_{45}	$[r_{50}]$		
Strategy: $s^6(0.5)$									
q_{15}	$[a_{20}]$	a_{20}	$[c_{l_2}, a_{21}]$	c_{29}	$[y_{34}]$				
				c_{30}	$[y_{35}]$				
				c_{31}	$[y_{36}]$				
Strategy: $s^7(0.5)$									
o_{15}	$[b_{20}]$	b_{20}	$[d_{l_3}, c_{l_1}, b_{21}]$	c_{23}	$[d_{l_1}, c_{24}]$	d_{26}	$[r_{31}]$		
				c_{24}	$[d_{l_1}, c_{25}]$	d_{27}	$[r_{32}]$		
				c_{25}	$[d_{l_1}, c_{26}]$	d_{28}	$[r_{33}]$		
				c_{26}	$[d_{l_1}, c_{27}]$	d_{29}	$[r_{34}]$		
				c_{27}	$[d_{l_1}, c_{28}]$	d_{30}	$[r_{35}]$		
						d_{31}	$[r_{36}]$		
						d_{32}	$[r_{37}]$		
						d_{33}	$[r_{38}]$		
						d_{34}	$[r_{39}]$		

Table 14: Utilized strategies ($\eta_3 = 0.5$)

Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.	Node	Pref.
Strategy: $s^1(1.0)$									
o_5	$[b_{10}]$	b_{10}	$[d_{l_3}, c_{l_1}, b_{11}]$	c_{13}	$[d_{l_1}, c_{14}]$	d_{16}	$[r_{21}]$		
				c_{14}	$[d_{l_1}, c_{15}]$	d_{17}	$[r_{22}]$		
				c_{15}	$[d_{l_1}, c_{16}]$	d_{18}	$[r_{23}]$		
				c_{16}	$[d_{l_1}, c_{17}]$	d_{19}	$[r_{24}]$		
				c_{17}	$[d_{l_1}, c_{18}]$	d_{20}	$[r_{25}]$		
						d_{21}	$[r_{26}]$		
						d_{22}	$[r_{27}]$		
						d_{23}	$[r_{28}]$		
						d_{24}	$[r_{29}]$		
Strategy: $s^2(1.0)$									
q_{15}	$[a_{20}]$	a_{20}	$[c_{l_2}, a_{21}]$	b_{29}	$[c_{l_1}, b_{30}]$	c_{33}	$[y_{38}]$		
		a_{21}	$[a_{22}]$	b_{30}	$[c_{l_1}, b_{31}]$	c_{34}	$[y_{39}]$		
		a_{22}	$[a_{23}]$	b_{31}	$[c_{l_1}, b_{32}]$	c_{35}	$[y_{40}]$		
		a_{23}	$[a_{24}]$			c_{36}	$[y_{41}]$		
		a_{24}	$[a_{25}]$			c_{37}	$[y_{42}]$		
		a_{25}	$[b_{l_1}, a_{26}]$			c_{38}	$[y_{43}]$		
Strategy: $s^3(1.0)$									
q_{10}	$[a_{15}]$	a_{15}	$[b_{l_1}, a_{16}]$	b_{19}	$[c_{l_1}, b_{20}]$	c_{22}	$[d_{l_1}, c_{23}]$	d_{25}	$[r_{30}]$
		a_{16}	$[a_{17}]$	b_{20}	$[d_{l_3}, c_{l_1}, b_{21}]$	c_{23}	$[d_{l_1}, c_{24}]$	d_{26}	$[r_{31}]$
		a_{17}	$[a_{18}]$	b_{21}	$[c_{l_1}, b_{22}]$	c_{24}	$[d_{l_1}, c_{25}]$	d_{27}	$[r_{32}]$
		a_{18}	$[a_{19}]$			c_{25}	$[d_{l_1}, c_{26}]$	d_{28}	$[r_{33}]$
		a_{19}	$[a_{20}]$			c_{26}	$[d_{l_1}, c_{27}]$	d_{29}	$[r_{34}]$
		a_{20}	$[c_{l_2}, a_{21}]$			c_{27}	$[d_{l_1}, c_{28}]$	d_{30}	$[r_{35}]$
						c_{28}	$[d_{l_1}, c_{29}]$	d_{31}	$[r_{36}]$
						c_{29}	$[c_{30}]$	d_{32}	$[r_{37}]$
						c_{30}	$[c_{31}]$	d_{33}	$[r_{38}]$
						c_{31}	$[c_{32}]$	d_{34}	$[r_{39}]$
						c_{32}	$[d_{l_1}, c_{33}]$	d_{35}	$[r_{40}]$
								d_{36}	$[r_{41}]$
								d_{37}	$[r_{42}]$
								d_{38}	$[r_{43}]$
								d_{39}	$[r_{44}]$
Strategy: $s^4(1.0)$									
o_{25}	$[b_{30}]$	b_{30}	$[c_{l_1}, b_{31}]$	c_{33}	$[y_{38}]$				
				c_{34}	$[y_{39}]$				
				c_{35}	$[y_{40}]$				
				c_{36}	$[y_{41}]$				
				c_{37}	$[y_{42}]$				
Strategy: $s^5(1.0)$									
o_{25}	$[b_{30}]$	b_{30}	$[d_{l_3}, c_{l_1}, b_{31}]$	c_{33}	$[d_{l_1}, c_{34}]$	d_{36}	$[r_{41}]$		
				c_{34}	$[d_{l_1}, c_{35}]$	d_{37}	$[r_{42}]$		
				c_{35}	$[d_{l_1}, c_{36}]$	d_{38}	$[r_{43}]$		
				c_{36}	$[d_{l_1}, c_{37}]$	d_{39}	$[r_{44}]$		
				c_{37}	$[d_{l_1}, c_{38}]$	d_{40}	$[r_{45}]$		
						d_{41}	$[r_{46}]$		
						d_{42}	$[r_{47}]$		
						d_{43}	$[r_{48}]$		
						d_{44}	$[r_{49}]$		

Table 15: Utilized strategies ($\eta_3 = 1.0$, first five strategies)

<table>
<tr><td colspan="10">Strategy: $s^6(1.0)$</td></tr>
<tr><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td><td>Node</td><td>Pref.</td></tr>
<tr><td>o_{15}</td><td>$[b_{20}]$</td><td>b_{20}</td><td>$[d_{l_3}, c_{l_1}, b_{21}]$</td><td>$c_{23}$</td><td>$[d_{l_1}, c_{24}]$</td><td>$d_{26}$</td><td>$[r_{31}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{24}</td><td>$[d_{l_1}, c_{25}]$</td><td>d_{27}</td><td>$[r_{32}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{25}</td><td>$[d_{l_1}, c_{26}]$</td><td>d_{28}</td><td>$[r_{33}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{26}</td><td>$[d_{l_1}, c_{27}]$</td><td>d_{29}</td><td>$[r_{34}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{27}</td><td>$[d_{l_1}, c_{28}]$</td><td>d_{30}</td><td>$[r_{35}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{31}</td><td>$[r_{36}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{32}</td><td>$[r_{37}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{33}</td><td>$[r_{38}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{34}</td><td>$[r_{39}]$</td><td></td><td></td></tr>
<tr><td colspan="10">Strategy: $s^7(1.0)$</td></tr>
<tr><td>q_{20}</td><td>$[a_{25}]$</td><td>a_{25}</td><td>$[b_{l_1}, a_{26}]$</td><td>b_{29}</td><td>$[c_{l_1}, b_{30}]$</td><td>c_{32}</td><td>$[y_{37}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{30}</td><td>$[c_{l_1}, b_{31}]$</td><td>c_{33}</td><td>$[y_{38}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{31}</td><td>$[c_{l_1}, b_{32}]$</td><td>c_{34}</td><td>$[y_{39}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{35}</td><td>$[y_{40}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{36}</td><td>$[y_{41}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{37}</td><td>$[y_{42}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{38}</td><td>$[y_{43}]$</td><td></td><td></td></tr>
<tr><td colspan="10">Strategy: $s^8(1.0)$</td></tr>
<tr><td>q_0</td><td>$[a_5]$</td><td>a_5</td><td>$[b_{l_1}, c_{l_2}, a_6]$</td><td>b_9</td><td>$[c_{l_1}, b_{10}]$</td><td>c_{12}</td><td>$[d_{l_1}, c_{13}]$</td><td>d_{15}</td><td>$[r_{20}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{10}</td><td>$[d_{l_3}, c_{l_1}, b_{11}]$</td><td>$c_{13}$</td><td>$[d_{l_1}, c_{14}]$</td><td>$d_{16}$</td><td>$[r_{21}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td>b_{11}</td><td>$[c_{l_1}, b_{12}]$</td><td>c_{14}</td><td>$[d_{l_1}, c_{15}]$</td><td>d_{17}</td><td>$[r_{22}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{15}</td><td>$[d_{l_1}, c_{16}]$</td><td>d_{18}</td><td>$[r_{23}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{16}</td><td>$[d_{l_1}, c_{17}]$</td><td>d_{19}</td><td>$[r_{24}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{17}</td><td>$[d_{l_1}, c_{18}]$</td><td>d_{20}</td><td>$[r_{25}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>c_{18}</td><td>$[d_{l_1}, c_{19}]$</td><td>d_{21}</td><td>$[r_{26}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{22}</td><td>$[r_{27}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{23}</td><td>$[r_{28}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{24}</td><td>$[r_{29}]$</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{25}</td><td>$[r_{30}]$</td></tr>
<tr><td colspan="10">Strategy: $s^9(1.0)$</td></tr>
<tr><td>q_0</td><td>$[a_5]$</td><td>a_5</td><td>$[c_{l_2}, b_{l_1}, a_6]$</td><td>c_{14}</td><td>$[d_{l_1}, c_{15}]$</td><td>d_{17}</td><td>$[r_{22}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{15}</td><td>$[d_{l_1}, c_{16}]$</td><td>d_{18}</td><td>$[r_{23}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td>c_{16}</td><td>$[d_{l_1}, c_{17}]$</td><td>d_{19}</td><td>$[r_{24}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{20}</td><td>$[r_{24}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{21}</td><td>$[r_{26}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{22}</td><td>$[r_{27}]$</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>d_{23}</td><td>$[r_{28}]$</td><td></td><td></td></tr>
</table>

Table 16: Utilized strategies ($\eta_3 = 1.0$, last four strategies)

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