

ON CALABI-YAU THREEFOLDS WITH LARGE NONABELIAN FUNDAMENTAL GROUPS

LEV BORISOV AND ZHENG HUA

(Communicated by Ted Chinburg)

ABSTRACT. In this short note we construct Calabi-Yau threefolds with non-abelian fundamental groups of order 64 as quotients of the small resolutions of certain complete intersections of quadrics in \mathbb{P}^7 that were first considered by M. Gross and S. Popescu.

1. INTRODUCTION

In [GPo], M. Gross and S. Popescu constructed Calabi-Yau varieties that admit a free action of the group $(\mathbb{Z}/8\mathbb{Z})^2$. We observe that these varieties admit a larger automorphism group, and that two other subgroups of order 64 of this group act freely. Our construction is very simple, modulo the nontrivial calculations of [GPo]. The essential idea is that in the presence of only two nonisomorphic, minimal models every automorphism of the singular variety $V_{8,y}$ naturally extends to the models. We then construct the additional automorphisms explicitly as permutations of coordinates.

In the last section we very briefly describe what led to the discovery of these examples and how they fit into a more general question of finding free group actions on complete intersections of quadrics.

2. CONSTRUCTION

We recall the definition of the Calabi-Yau varieties of Gross and Popescu, [GPo, GPa]. Consider the projective space $\mathbb{C}\mathbb{P}^7$ with coordinates $(x_0 : \dots : x_7)$. For any $(y_0 : y_1 : y_2) \in \mathbb{C}\mathbb{P}^2$ in a general position, consider the intersection $V_{8,y}$ of four quadrics

$$\begin{aligned}y_1 y_3 (x_0^2 + x_4^2) - y_2^2 (x_1 x_7 + x_3 x_5) + (y_1^2 + y_3^2) x_2 x_6 &= 0, \\y_1 y_3 (x_1^2 + x_5^2) - y_2^2 (x_2 x_0 + x_4 x_6) + (y_1^2 + y_3^2) x_3 x_7 &= 0, \\y_1 y_3 (x_2^2 + x_6^2) - y_2^2 (x_3 x_1 + x_5 x_7) + (y_1^2 + y_3^2) x_4 x_0 &= 0, \\y_1 y_3 (x_3^2 + x_7^2) - y_2^2 (x_4 x_2 + x_6 x_0) + (y_1^2 + y_3^2) x_5 x_1 &= 0.\end{aligned}$$

This variety has 64 ODP singular points which are the orbit of $(0 : y_1 : y_2 : y_3 : 0 : -y_3 : -y_2 : -y_1)$ under the action of the group $G = (\mathbb{Z}/8\mathbb{Z})^2$ generated by τ and

Received by the editors October 16, 2006 and, in revised form, February 17, 2007.

2000 *Mathematics Subject Classification*. Primary 14J32.

The first author was partially supported by the National Science Foundation under grant No. DMS-0456801.

©2007 American Mathematical Society
Reverts to public domain 28 years from publication

σ where $\tau(x_i) = \zeta_8^{-i}x_i$ and σ is a cyclic permutation of the variables given by the cycle (01234567). This group G acts freely on $V_{8,y}$.

The variety $V_{8,y}$ admits two small resolutions, $V_{8,y}^1$ and $V_{8,y}^2$, both of which are Calabi-Yau threefolds with $h^{1,1} = h^{1,2} = 2$. These resolutions are related by a flop and are obtained from X by blowups of abelian surfaces in X of degrees 32 and 16, respectively.

The key to the current note is the following lemma that relies heavily on the results of [GPo].

Lemma 2.1. *Every automorphism γ of $V_{8,y}$ lifts to automorphisms of $V_{8,y}^1$ and $V_{8,y}^2$. If γ acts freely on $V_{8,y}$, then its lifts act freely on the small resolutions.*

Proof. We will show that γ lifts to $V_{8,y}^2$, with the other case being completely analogous.

As in [GPo], we can think of $V_{8,y}^2$ as the blowup of $V_{8,y}$ by the ideal of some Weil divisor A . Then there is a natural isomorphism $\hat{\gamma}$ from $V_{8,y}^2$ to the blowup of $V_{8,y}$ by the ideal of $\gamma(A)$. Denote the latter blowup by Z . It is clearly a minimal model of $V_{8,y}$, so by the observation of [GPo], Z is isomorphic to either $V_{8,y}^2$ or $V_{8,y}^1$ (over $V_{8,y}$). To prove the first assertion of the lemma, it remains to show that $V_{8,y}^2$ and $V_{8,y}^1$ are not isomorphic, since we could then compose $\hat{\gamma}$ with an isomorphism $Z \rightarrow V_{8,y}^2$ over $V_{8,y}$. From the description of their Kähler cones we can see that any such isomorphism between $V_{8,y}^i$ would have to respect their structures as fibrations over \mathbb{P}^1 as well as their maps to \mathbb{P}^7 . However, the degrees of the fibers are different in these two cases, hence $V_{8,y}^2 \not\cong V_{8,y}^1$.

To show the last assertion, observe that if the lift of γ had a fixed point, then the image of that point in $V_{8,y}$ would be fixed by γ . □

We can now describe our construction.

Proposition 2.2. *We define the group G_1 generated by τ and the permutation $\sigma_1 = (07214365)$ of the coordinates x_i . Then G_1 is a nonabelian group isomorphic to a semidirect product of two copies of $\mathbb{Z}/8\mathbb{Z}$. We define the group G_2 generated by τ and the permutations $\sigma_2 = (0246)(1357)$ and $\sigma_3 = (0145)(3276)$. Then G_2 is a nonabelian group isomorphic to a semidirect product of the normal subgroup $\mathbb{Z}/8\mathbb{Z}$ generated by τ and the quaternion group H generated by σ_2 and σ_3 . Both G_1 and G_2 act freely on $V_{8,y}$.*

Proof. The structure of G_1 and G_2 is immediate from their definition, and it is straightforward to see that σ_i acts on $V_{8,y}$. To show that they act freely, it is enough to check the action of all involutions in G_i . In view of $1 \rightarrow \langle \tau \rangle \rightarrow G_1 \rightarrow \mathbb{Z}/8\mathbb{Z} \rightarrow 1$, all involutions of G_1 lie in the subgroup generated by τ and σ_1^4 . Similarly, all involutions of G_2 lie in the subgroup generated by τ and σ_2^2 . It remains to observe that these subgroups are contained in G , which is known to act freely [GPo]. □

Corollary 2.3. *For $i = 1, 2, j = 1, 2$, the quotients of $V_{8,y}^i$ by (the lift of) the group G_j are smooth Calabi-Yau threefolds with fundamental groups G_j of order 64.*

Proof. By [GPa], $V_{8,y}^i$ are simply connected. It remains to combine Lemma 2.1 and Proposition 2.2. □

We remark that G_2 contains the quaternion group in its regular representation, and $V_{8,y}$ can be thought of as a singular member of the family constructed by

Beauville in [B]. Also, the quotients $V_{s,y}^i/G_j$ have Hodge numbers $h^{1,1} = h^{1,2} = 2$ and the structures of fibration with abelian surface fibers.

3. COMMENTS

We have stumbled upon these examples largely by chance. We originally set out to investigate free actions of finite groups on complete intersections of four quadrics in hopes of extending the construction of [B]. Every such group action naturally leads to a projective representation of dimension eight, which can then be thought of as a linear representation of a Schur cover S of the finite group. If the action is free, the holomorphic Lefschetz formula essentially dictates what character of S one needs to consider, moreover, it leads to strong restrictions on possible group actions. We have used the GAP software package extensively in our search.

It turns out that the maximum possible order of the group (if one allows the variety to have ODP singularities) is 64. We have found five possible groups of order 64. Upon closer consideration, it turns out that three of them act on the same family of varieties and they are precisely the groups G , G_1 and G_2 that appear in this note. The other two, together with the classification of groups of smaller order will be addressed by the second author in [H].

ACKNOWLEDGMENTS

We have extensively used the GAP software package in our project, although no computer calculations are necessary to check the results of this paper.

REFERENCES

- [B] A. Beauville, *A Calabi-Yau threefold with non-Abelian fundamental group*, New Trends in Algebraic Geometry (EuroConference Warwick, July 1996), 13-17. Cambridge University Press (1999). MR1714819 (2000f:14060)
- [GP_a] M. Gross, S. Pavanelli, *A Calabi-Yau threefold with Brauer group $(\mathbb{Z}/8\mathbb{Z})^2$* , preprint math.AG/0512182.
- [GP_o] M. Gross, S. Popescu, *Calabi-Yau threefolds and moduli of abelian surfaces. I*, Compositio Math. **127** (2001), 169-228. MR1845899 (2002f:14057)
- [H] Z. Hua, in preparation.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: borisov@math.wisc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: hua@math.wisc.edu