JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume **9**, Number **2**, April **2013**

doi:10.3934/jimo.2013.9.487

pp. **487–504**

OPTIMAL PORTFOLIO IN A CONTINUOUS-TIME SELF-EXCITING THRESHOLD MODEL

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(Communicated by Xiaoqi Yang)

Abstract. This paper discusses an optimal portfolio selection problem in a continuous-time economy, where the price dynamics of a risky asset are governed by a continuous-time self-exciting threshold model. This model provides a way to describe the effect of regime switching on price dynamics via the self-exciting threshold principle. Its main advantage is to incorporate the regime switching effect without introducing an additional source of uncertainty. A martingale approach is used to discuss the problem. Analytical solutions are derived in some special cases. Numerical examples are given to illustrate the regime-switching effect described by the proposed model.

1. Introduction.

Asset allocation is one of the central topics in financial economics. The pioneering scientific contribution in this area was done by [7], where an elegant mathematical theory to an optimal asset allocation problem was first established. Markowitz considered a single period model and supposed that returns from individual assets are normally distributed. Under this assumption, he formulated the optimal asset allocation problem into a mean-variance optimization problem,

2010 *Mathematics Subject Classification.* Primary: 93E20.
Key words and phrases. Portfolio selection, self-exciting threshold model, regime switching, power utility, logarithmic utility.

where risk is described by the variance of the return of a portfolio. The mean-variance paradigm is a simple model for an optimal asset allocation problem, and this model leads to some nice theoretical results. The mean-variance asset allocation also laid down a theoretical foundation for the development of the capital asset pricing model of [17], [13], [6] and [10].

In practice asset allocation decisions are made dynamically over time. The single-period asset allocation model cannot describe this important feature. [12] considered an optimal asset allocation problem in a discrete-time, multi-period, economy using an utility maximization principle, and found that the solution in the discrete-time economy is qualitatively different from that of a static single-period model. [8] and [9] extended Samuelson's model to a continuous time case, pioneered the study of the optimal asset allocation problem in a continuous-time economy and adopted stochastic optimal control theory to investigate the problem. Under some simplifying assumptions on the preference structure and the asset price dynamics, Merton obtained a closed-form solution to an optimal asset allocation problem and devised that an optimal asset allocation strategy is to invest a constant proportion of wealth in a risky asset. This constant proportion is known as the Merton ratio. The work of Merton has stimulated a whole industry of research on optimal asset allocation in continuous time and the development of an important branch in modern finance, namely, the continuous-time finance.

Regime switching models are a popular class of models for economic and financial dynamics. The key idea of regime-switching models is to allow the model parameters to change dynamically over time according to the state of an economy whose evolution is described by a finite-state Markov chain. In practice, a two-state Markov chain is good enough to distinguish a normal market and one experiencing financial crisis. The history of regime switching models may be traced back to the early works of [11] and [2], where regime-switching regression models were introduced and applied to study nonlinear economic data. [15] and [16] introduced the idea of regime switching to parametric nonlinear time series analysis when the development of nonlinear time series analysis was still at its embryonic stage. In particular, they introduced the class of threshold models which is one of the oldest classes of nonlinear time series models. [3] pioneered the applications of regime-switching models in econometrics and economics. Many empirical studies reveal that regime-switching models can incorporate well important stylized features of financial time series, including time-varying conditional volatility, the asymmetry and heavy-tailed-ness of the unconditional distribution of asset returns, volatility clustering and regime switchings.

Recently, some attention have been paid to optimal asset allocation in regime-switching models. [19] and [18] considered the Markowitz mean-variance portfolio selection problem under regime-switching models in a continuous-time setting and a discrete-time setting, respectively. They formulated the problem as a linear-quadratic control problem. [4] studied an optimal consumption/investment problem in a Markovian regime switching model in the presence of transactions costs. From an economic perspective, regime-switching models can describe stochastic movements of investment opportunity sets where some interesting economic and financial insights can be gained. Another possible way to describe this is to consider the class of self-exciting threshold time series models by [15] and [16], where structural changes in investment opportunity sets depend on regions where the past prices of a financial assets fall into. These regions or regimes are specified by the threshold principle. The self-exciting threshold time series can be used to describe structural changes in price dynamics due to changes in market conditions, say bull or bear markets, which may be attributed to transitions in different stages of business cycles. Consequently, the self-exciting threshold time series can be used in different market conditions or regimes. Indeed, one of the key features of the self-exciting threshold time series by [15] and [16] is that it can be used to describe limit cycles which, in economics, manifest themselves as business cycles. Furthermore, a continuous-time version of the self-exciting threshold time series can also be used in the situation where price data are observed irregularly over time. This situation is typical when one considers high-frequency tick-by-tick price data.

In this paper, we discuss an optimal portfolio selection problem in a continuous-time economy with two primitive securities, a risk-free asset and a risky asset. We suppose that the price dynamics of the risky asset are governed by a continuous-time self-exciting threshold model. The key idea of the model is to incorporate the regime switching effect in the price dynamics of the risky asset via the "self-exciting" threshold principle. This principle was first proposed by [14] and [16], and [15] to nonlinear time series analysis, where one of the oldest classes of nonlinear time series models, namely, the, (self-exciting), threshold autoregressive (SETAR) model, was introduced. Here, using the self-exciting threshold principle, we introduce a set of threshold parameters to partition the state space of an asset's price. Then the parameters of the price dynamics depend on which region, or partition, a past value of the asset's price falls in. The idea resembles that of "self-exciting" in the SETAR model. Unlike Markovian regime-switching models, where the model parameters are modulated by an underlying state process, say a Markov chain, the self-exciting threshold asset price model considered here incorporates the regime switching effect without introducing an additional state process. In this case, the market in the self-exciting threshold asset price model is still complete. This simplifies an optimal portfolio selection problem especially when a martingale approach is used to discuss the problem. The object of the problem is to select an optimal portfolio process so as to maximize an expected utility of terminal wealth. We use a martingale approach to discuss the problem and derive analytical solutions of the problem in the cases of a power utility and a logarithmic utility. We also give some numerical examples to illustrate the impact of switching regimes described by the proposed model on the optimal portfolio.

We organize the paper as follows. The next section presents the self-exciting threshold asset price model. We then formulate the optimal portfolio selection problem. Section 3 discusses the martingale approach to the optimization problem. The analytical solutions to the problem are then derived in Section 4. The numerical examples will be given in Section 5. The final section summarizes the paper.

2. **The self-exciting threshold asset price model.** We consider a continuous-time financial model with two primitive securities, namely, a risk-free bond B and a risky asset S. These securities are traded continuously over time on a finite time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$. To model uncertainty, we consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ describes the flow of market information generated by prices or rates and P is a real-world probability measure. In what follows we first describe the price dynamics of the two assets and then define the risk-neutral probability measure under which the optimal consumption-investment problem will be discussed.

We first define a partition $\{R_i\}_{i=0,1,\cdots,N}$ on the state space $[0,\infty]$ of the risky asset S, where $0 = R_0 < R_1 < \cdots < R_N = \infty$. For each $t \in \mathcal{T}$ and each $i = 1, 2, \cdots, N$, let $r_i(t)$ be the instantaneous continuously compounded rate of interest from the risk-free asset B at time t and in the regime i, where $r_i(t) > 0$. We suppose that, for each $i = 1, 2, \cdots, N$, the process of interest rate $\{r_i(t)|t \in \mathcal{T}\}$ is measurable, $\mathbb{F}$-adapted, and bounded. Write $\{B(t)|t \in \mathcal{T}\}$ and $\{S(t)|t \in \mathcal{T}\}$ for the price processes of the risk-free asset B and the risky asset S, respectively. Then the price of the risk-free asset B evolves over time as the following piecewise linear differential equation in the pathwise sense:

$$dB(t) = \sum_{i=1}^{N} I_{\{S(t)\in[R_{i-1},R_i)\}} r_i(t)B(t)dt \ , \quad B(0) = 1 \ .$$

Here I_E is the indicator of an event E. The existence and uniqueness of the piecewise linear differential equation follows from the boundedness condition for the processes $\{r_i(t)|t \in \mathcal{T}\}$ for $i = 1, 2, \cdots, N$.

Suppose, for each $t \in \mathcal{T}$ and $i = 1, 2, \cdots, N$, $\mu_i(t)$ and $\sigma_i(t)$ are the appreciation rate and the volatility of the risky asset S at time t and in the regime i, respectively, where $\mu_i(t) \in \Re$, $\sigma_i(t) > 0$ and N is a positive integer. Assume that, for each $i = 1, 2, \cdots, N$, $\{\mu_i(t)|t \in \mathcal{T}\}$ and $\{\sigma_i(t)|t \in \mathcal{T}\}$ are measurable, $\mathbb{F}$-adapted, and bounded processes. Suppose $\{W(t)|t \in \mathcal{T}\}$ is the standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Then the price dynamics of the risky asset are governed by:

$$\begin{aligned} dS(t) &= \sum_{i=1}^{N} \Big(\mu_i(t)S(t)dt + \sigma_i(t)S(t)dW(t) \Big) I_{\{S(t)\in[R_{i-1},R_i)\}} \ , \\ S(0) &= s > 0 \ . \end{aligned} \tag{1}$$

Write, for each $t \in \mathcal{T}$,

$$\begin{aligned} \mu(t, S(t)) &:= \sum_{i=1}^{N} \mu_i(t)S(t)I_{\{S(t)\in[R_{i-1},R_i)\}} \ , \\ \sigma(t, S(t)) &:= \sum_{i=1}^{N} \sigma_i(t)S(t)I_{\{S(t)\in[R_{i-1},R_i)\}} \ . \end{aligned}$$

Since $\{\mu_i(t)|t \in \mathcal{T}\}$ and $\{\sigma_i(t)|t \in \mathcal{T}\}$ are bounded processes for each $i = 1, 2, \cdots, N$, the functions $\mu(t, s)$ and $\sigma(t, s)$ satisfy the Lipschitz condition and the linear growth condition in s. Consequently, the stochastic differential equation (1) has a unique strong solution.

We now define the market price of risk as follows:

$$\theta(t) := \sum_{i=1}^{N} \left(\frac{\mu_i(t) - r_i(t)}{\sigma_i(t)} \right) I_{\{S(t)\in[R_{i-1},R_i)\}} \ , \quad t \in \mathcal{T} \ .$$

Note that $\{\theta(t)|t \in \mathcal{T}\}$ is bounded and $\mathbb{F}$-progressively measurable.

Consider an $\mathbb{F}$-adapted process $\Lambda := \{\Lambda(t)|t \in \mathcal{T}\}$ defined by:

$$\Lambda(t) := \exp\left(-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du \right) , \quad t \in \mathcal{T} \ .$$

Then

$$d\Lambda(t) = -\theta(t)\Lambda(t)dW(t) \ .$$

Note that $\{\theta(t)|t \in \mathcal{T}\}$ satisfies the Novikov condition, so $\{\Lambda(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, P)$-martingale. Consequently, $\mathrm{E}[\Lambda(T)] = 1$, where E is an expectation under P.

Define a new probability measure $P^\theta \sim P$ on $\mathcal{F}(T)$ by putting:

$$\left.\frac{dP^\theta}{dP}\right|_{\mathcal{F}(T)} := \Lambda(T) \ .$$

Then by Girsanov's theorem, the process $\{W^\theta(t)|t \in \mathcal{T}\}$ defined by:

$$W^\theta(t) := W(t) + \int_0^t \theta(u)du \ , \quad t \in \mathcal{T} \ ,$$

is an $(\mathbb{F}, P^\theta)$-standard Brownian motion.

Consequently, under P^θ, the price dynamics of the risky asset S are given by:

$$dS(t) = \sum_{i=1}^{N} \Big(r(t)S(t)dt + \sigma_i(t)S(t)dW^\theta(t) \Big) I_{\{S(t) \in [R_{i-1}, R_i)\}} \ ,$$

where

$$r(t) := \sum_{i=1}^{N} r_i(t) I_{\{S(t) \in [R_{i-1}, R_i)\}} \ .$$

For each $t \in \mathcal{T}$, let $\widetilde{S}(t) := \exp(-\int_0^t r(u)du)S(t)$. Then under P^θ,

$$d\widetilde{S}(t) = \sum_{i=1}^{N} \sigma_i(t)\widetilde{S}(t)dW^\theta(t) I_{\{S(t) \in [R_{i-1}, R_i)\}} \ .$$

Note that the market in the self-exciting threshold model is complete. So P^θ is the unique risk-neutral, or martingale, measure.

Consider a utility function, $U : (0, \infty) \to \Re$ is a real-valued function in C^1 such that it satisfies the following conditions:

1. $U(\cdot)$ is strictly increasing and strictly concave;
2. the derivative $U'(c)$ is such that

$$\lim_{c \to \infty} U'(c) = 0 \ , \quad \lim_{c \to 0^+} U'(c) = \infty \ .$$

For each $t \in \mathcal{T}$, let $\pi(t)$ be the number of units of the risky asset S invested by an investor at time t. We suppose that the portfolio process $\{\pi(t)|t \in \mathcal{T}\}$ is a measurable, $\mathbb{F}$-adapted, process such that

$$\int_0^T |\pi(t)|^2 dt < \infty \ , \quad P\text{-a.s.}$$

We suppose that the portfolio process is self-financing. Then the wealth process $\{V^\pi(t)|t \in \mathcal{T}\}$ of the investor adopting the portfolio process $\{\pi(t)|t \in \mathcal{T}\}$ evolves over time as:

$$\begin{aligned} dV^\pi(t) \quad = \quad & \sum_{i=1}^{N} \Big([r_i(t)V^\pi(t) + \pi(t)(\mu_i(t) - r_i(t))]dt + \pi(t)\sigma_i(t)dW(t) \Big) \\ & \times I_{\{S(t) \in [R_{i-1}, R_i)\}} \ . \end{aligned}$$

From now on, to simplify the notation, we suppress the superscript π, and write $V(t)$ for $V^\pi(t)$ unless otherwise stated.

A portfolio process $\{\pi(t)|t \in \mathcal{T}\}$ is said to belong to $\mathcal{A}(v)$ if for initial wealth $v \geq 0$, the corresponding wealth process satisfies

$$V(t) := V(t,\omega) \geq 0 \ , \quad \text{for almost all } (t,\omega) \in \mathcal{T} \times \Omega.$$

We now consider the problem of maximizing the expected utility of terminal wealth. Define

$$\mathcal{A}_1(v) := \{\pi \in \mathcal{A}(v) | \mathrm{E}[U^-(V^\pi(T))] < \infty\} \ ,$$

where $U^-(\cdot)$ is the negative part of the function $U(\cdot)$.

Then for each $\pi \in \mathcal{A}_1(v)$, we define the performance function associated with the strategy π and the initial wealth v by:

$$J^\pi(v) := \mathrm{E}[U(V^\pi(T))] \ .$$

The goal of the investor is to select π so as to maximize $J^\pi(v)$. So we define the value function of this optimization problem as:

$$\Phi(v) := \sup_{\pi \in \mathcal{A}(v)} J^\pi(v) \ .$$

3. **The martingale approach.** In this section we use the martingale approach to discuss the optimization problem presented in the last section. The key idea of the martingale approach is to first translate the "dynamic" portfolio selection problem into a static optimization problem with constraint in a complete market by exploiting the market completeness and the martingale representation.

Firstly, it is not difficult to see that under P^θ,

$$dV(t) = \sum_{i=1}^{N} \Big(r_i(t)V(t)dt + \pi(t)\sigma_i(t)dW^\theta(t) \Big) I_{\{S(t) \in [R_{i-1}, R_i)\}} \ .$$

Note that

$$B(t) = \exp\left(\sum_{i=1}^{N} \int_0^t r_i(u) I_{\{S(u) \in [R_{i-1}, R_i)\}} du \right) \ .$$

Define the discounting process $\{\beta(t)|t \in \mathcal{T}\}$ by

$$\beta(t) := B^{-1}(t) \ .$$

Then

$$M(t) := \beta(t)V(t) = v + \sum_{i=1}^{N} \int_0^t \beta(u)\pi(u)\sigma_i(u) I_{\{S(u) \in [R_{i-1}, R_i)\}} dW^\theta(u) \ .$$

Consequently the discounted wealth process $\{M(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, P^\theta)$-(local)-martingale.

Since $\pi \in \mathcal{A}(v)$, the process $\{M(t)|t \in \mathcal{T}\}$ is bounded from below. Consequently, by the Fatou lemma (see Karatzas and Shreve [5], Page 92 therein), it is an $(\mathbb{F}, P^\theta)$-supermartingale. Using the optional stopping theorem on $\{M(t)|t \in \mathcal{T}\}$, if $\pi \in \mathcal{A}(v)$, then

$$\mathrm{E}^\theta[\beta(\tau)V(\tau)] \leq v \ .$$

Here E^θ is an expectation under P^θ and τ is any stopping time taking a value in $\mathcal{T}$. The financial interpretation for this inequality is that the expected wealth at time τ is bounded by the initial wealth v.

In particular,

$$\mathrm{E}^{\theta}[\beta(T)V(T)] \leq v \ .$$

Since the utility function U is strictly increasing and it only depends on the terminal wealth $V(T)$, one must select a portfolio process π so as to increase the terminal wealth $V(T)$ up to the limit imposed by the bound v. Consequently, one must only consider portfolio processes for which

$$\mathrm{E}^{\theta}[\beta(T)V(T)] = \mathrm{E}[\Lambda(T)\beta(T)V(T)] = v \ .$$

Then the optimal portfolio selection problem discussed in the last section becomes:

$$\Phi(v) := \sup_{\pi \in \mathcal{A}_1(v)} \mathrm{E}[U(V^{\pi}(T))] \ ,$$

subject to the constraint:

$$\mathrm{E}[\Lambda(T)\beta(T)V(T)] = v \ .$$

This constrained maximization problem has the following Lagrangian:

$$\Gamma(\pi, \gamma) := \mathrm{E}[U(V(T))] + \gamma(v - \mathrm{E}[\Lambda(T)\beta(T)V(T)]) \ .$$

Note that the derivative U' of the utility function U is strictly decreasing. There is an inverse map K of U'. The first-order conditions imply that the optimal terminal wealth level V^* must satisfy:

$$V^* = K(\gamma\Lambda(T)\beta(T)) \ ,$$

and

$$\mathrm{E}[\Lambda(T)\beta(T)V^*] = v \ .$$

Note that γ can be determined by the condition that $\mathrm{E}[\Lambda(T)\beta(T)V^*] = v$.

Define, for each $\gamma \in (0, \infty)$,

$$L(\gamma) := \mathrm{E}[\Lambda(T)\beta(T)K(\gamma\Lambda(T)\beta(T))] \ .$$

Suppose, for $\gamma \in (0, \infty)$, $L(\gamma) < \infty$. Then from the properties of K, L is a continuous and strictly decreasing function of γ with

$$L(0+) = \infty \ , \quad L(\infty) = 0 \ .$$

Consequently, there is an inverse map G of L. This implies that for each $v > 0$, there is a unique γ such that

$$\gamma = G(v) \ .$$

Then

$$V^* = K(G(v)\Lambda(T)\beta(T)) \ ,$$

so

$$\mathrm{E}^{\theta}[\beta(T)V^*] = \mathrm{E}[\beta(T)\Lambda(T)K(G(v)\Lambda(T)\beta(T))] = v \ . \tag{2}$$

The following theorem is standard, (see, for example, [1] or [5]). We state the result without giving the proof.

Theorem 3.1. *Suppose $\mathcal{V}$ satisfies the condition that $E^{\theta}[\beta(T)\mathcal{V}] \leq v$. Then there is a portfolio process $\pi \in \mathcal{A}(v)$ such that the corresponding wealth process $\{V(t)|t \in \mathcal{T}\}$ satisfies $V(T) = \mathcal{V}$, P-a.s. If $E^{\theta}[\beta(T)\mathcal{V}] = v$, the discounted wealth process $\{\beta(t)V(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, P^{\theta})$-martingale.*

From Theorem 3.1 and Equation (2), there is a portfolio process $\pi^* \in \mathcal{A}(v)$ such that the corresponding wealth process $\{V^*(t)|t \in \mathcal{T}\}$ satisfies $V^*(T) = V^*$, P-a.s., (i.e. the optimal terminal wealth V^* is attained). Furthermore, the discounted wealth process $\{\beta(t)V^*(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, P^\theta)$-martingale. Then by the martingale representation theorem, there is a measurable, $\mathbb{F}$-adapted process $\{\psi(t)|t \in \mathcal{T}\}$ satisfying

$$\mathrm{E}\left[\int_0^T \psi^2(t)dt\right] < \infty ,$$

such that

$$\begin{aligned}\beta(t)V^*(t) &= \mathrm{E}^\theta[\beta(T)V^*(T)|\mathcal{F}(t)] \\ &= v + \int_0^t \psi(u)dW^\theta(t) , \quad P\text{-a.s.}\end{aligned}$$

Take, for each $t \in \mathcal{T}$,

$$\pi^*(t) := \sum_{i=1}^N \frac{\psi(t)}{\beta(t)\sigma_i(t)} I_{\{S(t)\in[R_{i-1},R_i)\}} .$$

Consequently,

$$\beta(t)V^*(t) = v + \sum_{i=1}^N \int_0^t \pi^*(u)\beta(u)\sigma_i(u) I_{\{S(u)\in[R_{i-1},R_i)\}} dW^\theta(u) , \quad P\text{-a.s.}$$

4. Analytical solution.

In this section, we give analytical solutions to the optimal portfolio selection problem in the cases of a logarithmic utility and a power utility. We first consider a logarithmic utility given by:

$$U(V(T)) := \ln[V(T)] .$$

Then analytical forms of the optimal portfolio and the value function are given in the following theorem.

Theorem 4.1. *Suppose U is a logarithmic utility and the initial wealth is $v \geq 0$. Then the optimal portfolio $\{\pi^*(t)|t \in \mathcal{T}\}$ and the value function $\Phi(v)$ are given, respectively, by:*

$$\pi^*(t) = \sum_{i=1}^N \frac{V^*(t)\theta(t)}{\sigma_i(t)} I_{\{S(t)\in[R_{i-1},R_i)\}} ,$$

and

$$\Phi(v) \quad = \quad \ln v + E\left[\frac{1}{2}\int_0^T \left[\left(\frac{\mu_i(t) - r_i(t)}{\sigma_i(t)}\right)^2 + 2r_i(t)\right] I_{\{S(t)\in[R_{i-1},R_i)\}} dt\right] .$$

Proof. Firstly, note that the inverse K in this case is:

$$K(\gamma) = \frac{1}{\gamma} .$$

Then

$$L(\gamma) = \mathrm{E}[\Lambda(T)\beta(T)K(\gamma\Lambda(T)\beta(T))] = \frac{1}{\gamma} .$$

Consequently,

$$V^* = V^*(T) = K(G(v)\Lambda(T)\beta(T)) = (G(v)\Lambda(T)\beta(T))^{-1} ,$$

and

$$G(v) = \frac{1}{v} \ .$$

Then

$$\begin{aligned}
\beta(T)V^*(T) &= v\Lambda^{-1}(T) \\
&= v \exp\left(\int_0^T \theta(t) dW(t) + \frac{1}{2} \int_0^T \theta^2(t) dt \right) \\
&= v \exp\left(\int_0^T \theta(t) dW^\theta(t) - \frac{1}{2} \int_0^T \theta^2(t) dt \right) \ .
\end{aligned}$$

This is the terminal value of an $(\mathbb{F}, P^\theta)$-martingale. Consequently,

$$\begin{aligned}
\beta(t)V^*(t) &= \mathrm{E}^\theta[\beta(T)V^*(T)|\mathcal{F}(t)] \\
&= v \exp\left(\int_0^t \theta(u) dW^\theta(u) - \frac{1}{2} \int_0^t \theta^2(u) du \right) \\
&= v + \int_0^t \theta(u)\beta(u)V^*(u) dW^\theta(u) \ .
\end{aligned}$$

Note also that

$$\beta(t)V^*(t) = v + \sum_{i=1}^N \int_0^t \pi^*(u)\beta(u)\sigma_i(u) I_{\{S(u)\in[R_{i-1},R_i)\}} dW^\theta(u) \ .$$

Then, by the unique decomposition of a special semimartingale,

$$\sum_{i=1}^N \pi^*(t)\beta(t)\sigma_i(t) I_{\{S(t)\in[R_{i-1},R_i)\}} = \theta(t)\beta(t)V^*(t) \ .$$

Consequently,

$$\pi^*(t) = \sum_{i=1}^N \frac{V^*(t)\theta(t)}{\sigma_i(t)} I_{\{S(t)\in[R_{i-1},R_i)\}} \ .$$

Note that

$$\begin{aligned}
\Phi(v) &= \mathrm{E}[\ln V^*(T)] \\
&= \ln v + \mathrm{E}\left[\frac{1}{2} \int_0^T \left[\left(\frac{\mu_i(t) - r_i(t)}{\sigma_i(t)} \right)^2 + 2r_i(t) \right] I_{\{S(t)\in[R_{i-1},R_i)\}} dt \right] \ .
\end{aligned}$$

The second equality is due to the fact that

$$\mathrm{E}\left[\int_0^T \theta(t) dW(t) \right] = 0 \ .$$

$\square$

We now consider a power utility given by:

$$U(V(T)) := \frac{(V(T))^{1-\alpha}}{1-\alpha} \ ,$$

where $\alpha > 0$, $\alpha \neq 1$.

Then an analytical form of the value function is given in the following theorem.

Theorem 4.2. *Suppose U is a power utility and the initial wealth is $v \geq 0$. Then the value function $\Phi(v)$ is given by*

$$\Phi(v) = \frac{v^{1-\alpha}}{1-\alpha}\left(E(\Lambda(T)\beta(T))^{\frac{\alpha-1}{\alpha}}\right)^{\alpha}.$$

Proof. In this case,

$$K(\gamma) = \gamma^{-1/\alpha} .$$

Then

$$L(\gamma) = \gamma^{-1/\alpha}\mathrm{E}[(\Lambda(T)\beta(T))^{(\alpha-1)/\alpha}] .$$

Consequently,

$$V^* = V^*(T) = (G(v)\Lambda(T)\beta(T))^{-1/\alpha} ,$$

and

$$G(v) = v^{-\alpha}(\mathrm{E}[(\Lambda(T)\beta(T))^{(\alpha-1)/\alpha}])^{\alpha} .$$

So

$$V^*(T) = \frac{v(\Lambda(T)\beta(T))^{-1/\alpha}}{\mathrm{E}[(\Lambda(T)\beta(T))^{(\alpha-1)/\alpha}]} ,$$

which results in

$$\Phi(v) = \mathrm{E}\left[\frac{(V^*(T))^{1-\alpha}}{1-\alpha}\right] = \frac{v^{1-\alpha}}{1-\alpha}\left(\mathrm{E}(\Lambda(T)\beta(T))^{\frac{\alpha-1}{\alpha}}\right)^{\alpha}.$$

□

5. **Numerical examples.** In this section we provide numerical examples for the optimal portfolio strategy derived in the previous section. In the numerical examples, we investigate how the optimal portfolio strategies and the corresponding value functions are influenced when the initial value of the share price varies in both the logarithmic utility and power utility cases. In what follows, we simply call the continuous-time TAR model for the asset price as the threshold model and the geometric Brownian motion as the non-threshold model. For illustration, we consider a two-regime threshold model with a single threshold parameter R_1.

5.1. **Optimal strategy for power utility.** The optimal strategy for logarithmic utility function under threshold model can be studied separately in each of the threshold because of the property of the logarithmic function. This helps a lot in simplifying the solution. Due to the threshold effect, the interest rate and market price of risk change with the threshold and thus the stock price itself can affect its process parameter. By the Markovian property of $S(t)$, we propose another approach to find the optimal strategy for power utility under threshold model. We let

$$\Phi(t, S(t), V(t)) := \sup_{\pi} \mathrm{E}[U(V^{\pi}(T))|S(t), V(t)],$$

and we can obtain

$$\sup_{\pi_t}\Big[\Phi_t + \mu S\Phi_S + \frac{1}{2}\sigma^2 S^2\Phi_{SS} + (rV + \pi_t(\mu - r))\Phi_V + \frac{1}{2}\pi_t^2\sigma^2\Phi_{VV} + \pi_t\sigma^2 S\Phi_{SV}\Big] = 0$$

Because $\Phi_{VV} < 0$, the optimal strategy is

$$\pi_t^* = -\frac{(\mu - r)\Phi_V + \sigma^2 S \Phi_{SV}}{\sigma^2 \Phi_{VV}}.$$

Now we focus on the function Φ, let $\Lambda_t(T) := \Lambda(T)/\Lambda(t)$ and $\beta_t(T) := \beta(T)/\beta(t)$. First, we know,

$$\begin{aligned}
\frac{V^*(T)}{V^*(t)} &= \frac{\big(\Lambda(T)\beta(T)\big)^{-1/\alpha}\Lambda(t)\beta(t)}{\mathrm{E}\Big[\Big(\Lambda(T)\beta(T)\Big)^{(\alpha-1)/\alpha}|\mathcal{F}(t)\Big]} \\
V^*(T) &= \frac{V^*(t)\big(\Lambda_t(T)\beta_t(T)\big)^{-1/\alpha}}{\mathrm{E}\Big[\big(\Lambda_t(T)\beta_t(T)\big)^{(\alpha-1)/\alpha}|\mathcal{F}(t)\Big]}
\end{aligned}$$

Then,

$$\begin{aligned}
&\Phi(t, S(t), V(t)) = \sup_{\pi} \mathrm{E}[U(V^{\pi}(T))|S(t), V(t)] \\
= \;& \frac{(V(t))^{1-\alpha}}{1-\alpha}\mathrm{E}\Bigg[\frac{\big(\Lambda_t(T)\beta_t(T)\big)^{(-1/\alpha)(1-\alpha)}}{\{\mathrm{E}\big[\big(\Lambda_t(T)\beta_t(T)\big)^{(\alpha-1)/\alpha}|\mathcal{F}(t)\big]\}^{1-\alpha}}\Bigg|\mathcal{F}(t)\Bigg] \\
= \;& \frac{(V(t))^{1-\alpha}}{1-\alpha}\Big[\mathrm{E}\big[\big(\Lambda_t(T)\beta_t(T)\big)^{(\alpha-1)/\alpha}|\mathcal{F}(t)\big]\Big]^{\alpha} \\
= \;& \frac{(V(t))^{1-\alpha}}{1-\alpha}\exp\big(\alpha A(t, S)\big)
\end{aligned}$$

where $A(t, S(t)) := \ln(\mathrm{E}[f(T)/f(t)|S(t)])$, $f(t) := [\Lambda(t)\beta(t)]^{\frac{\alpha-1}{\alpha}}$, and

$$A(t, S(t)) = \ln\Bigg[\mathrm{E}\Bigg[\exp\Big(-\int_t^T \frac{\alpha-1}{\alpha^2}\frac{\theta^2(s)}{2} + \frac{\alpha-1}{\alpha}r(s)ds\Big)\Bigg|S(t)\Bigg]\Bigg].$$

Then, if we use subscript to denote partial derivatives of Φ with respect to V and S, we have

$$\begin{aligned}
\Phi_V &= (1-\alpha)\Phi/V \\
\Phi_{SV} &= (1-\alpha)\alpha\Phi A_S/V \\
\Phi_{VV} &= (1-\alpha)(-\alpha)\Phi/V^2
\end{aligned}$$

Put them back to the previous result, we obtain the form of the optimal strategy

$$\pi_t^*(S(t)) = \left[\frac{1}{\alpha}\frac{\mu(S(t)) - r(S(t))}{\sigma^2(S(t))} + A_S(t, S(t))S\right]V^*(t). \tag{3}$$

If there is no threshold effect, the parameters of the stock price process does not depend on the stock price and they are constant. $A(t, S(t)) = A(t)$ and $A_S(t, S(t)) = 0$. We can obtain the optimal strategy for the non threshold case. Using the above result, we recognise the linkage between the optimal stragety in a threshold market and the non threshold model. An additional factor $A(t, S(t))$ is used to summaried the threshold effect derived from the stock price movement.

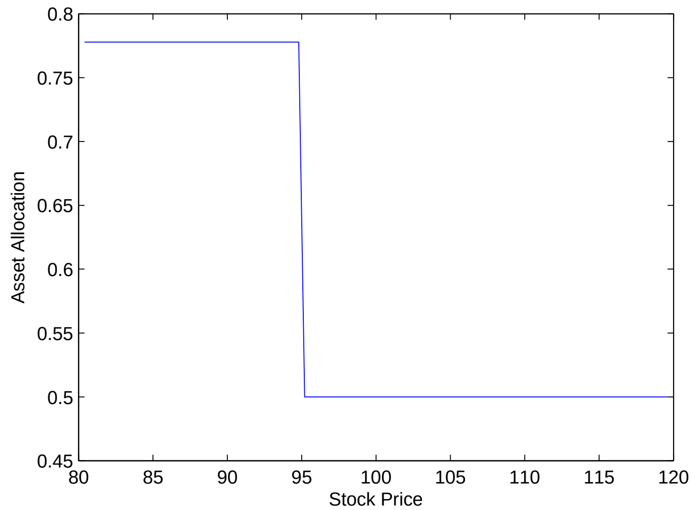

FIGURE 1. Optimal portfolio strategy for logarithmic utility

5.2. **Numerical example.** In our numerical example, we adopt the following configurations of the hypothetical parameter values:

$$
\begin{aligned}
&\mu_1 = 10\%; \quad \mu_2 = 5\%; \\
&r_1 = 3\%; \quad r_2 = 3\%; \\
&\sigma_1 = 30\%; \quad \sigma_2 = 20\%; \\
&R_1 = 95\ .
\end{aligned}
$$

Suppose the economic agent has a logarithmic utility function with an unit initial wealth. Then the optimal portfolio strategies for a one-year period investment arising in the threshold model and the non-threshold model are depicted in Figure 1 and it is the same as the optimal strategies in the corresponding non threshold cases.

For the case of a power utility function, we assume that $\alpha = 0.5$ and that the initial wealth of the agent is equal to 1. The optimal strategy for an non-threshold model is $\pi^* = 14/9$ with the set of parameters corresponding to "Threshold 1" $\pi^* = 1$ with the set of parameters corresponding to "Threshold 2". The optimal portfolio strategies for a one-year period investment arising in the threshold model are depicted in Figure 2. The threshold effect is strong when the stock price passes through the threshold level which may be interpreted as a technical indictor. The effect of the threshold also extends in the other stock price levels to reflect the possibility of a threshold change.

The value functions for a one-year investment horizon arising in the threshold model with logarithmic utility and power utility functions are depicted in Figures 3 and 4.

From the two figures, we can see that in both the logarithmic utility and power utility cases, the value functions are relatively stable when the initial share price is far away from the threshold level. The market is relatively stable and the effect of the share price on the value function is not significant. When the share price is close to the threshold level, there is a high chance that the share price might go to the other regime and the effect on the value function is largely attributed to potential

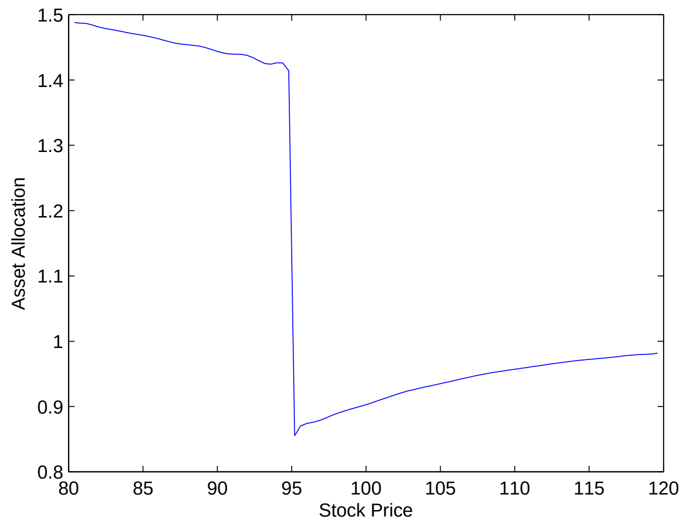

FIGURE 2. Optimal portfolio strategy for power utility

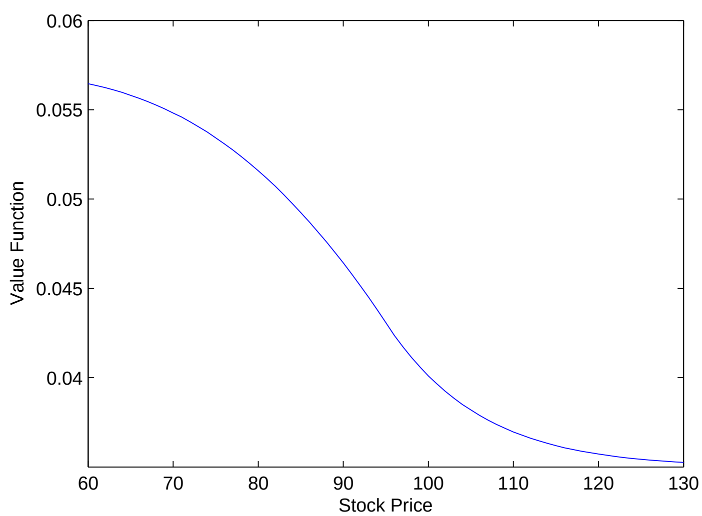

FIGURE 3. Value function of logarithmic utility under different initial stock prices

changes in market conditions. That is, the regime switching effect of the share price is even stronger than the effect of the share price itself on the value function. Indeed, the threshold effect on the value function is significant, so the change in the market state has a significant impact on how good an utility maximizer can achieve in portfolio selection.

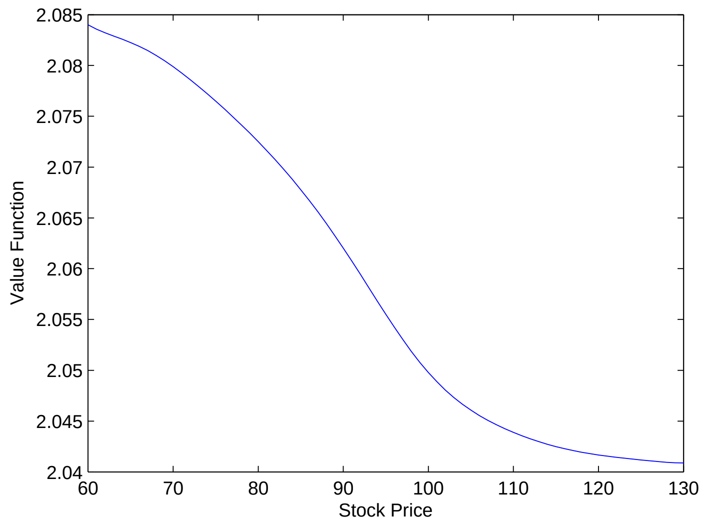

FIGURE 4. Value functions of power utility under different initial stock prices

6. **Empirical analysis.** In this section we present an empirical analysis of the optimal portfolio strategy and the value function by estimating a discretized version of the continuous-time TAR model using real financial data. In particular, we use daily logarithmic returns of S&P 500 index for the period from 3 January 2000 to 2 December 2011, a total of 3000 observations, to estimate the model parameters of the discretized version of the continuous-time TAR model. The same set of data is used to estimate the discretized version of the geometric Brownian motion which is used as a control in our comparison. For illustration, we consider a two-regime continuous-time TAR model as in the last section. The maximum likelihood estimation method is used to estimate the model parameters in both the threshold and non-threshold case. For the estimation of the threshold parameter R_1 in the threshold model, we employ both the likelihood function and the Akakie Information Criterion (AIC). The estimation method is briefly described as follows:

Let $\{Y_1, Y_2, \cdots, Y_N\}$ be the time series of observed daily returns from the S&P 500 index. Write $\{Y_{(1)}, Y_{(2)}, \cdots, Y_{(N)}\}$ for the ordered sample of the daily returns. Suppose $\{R_{11}, R_{12}, \cdots, R_{1K}\}$ is a set of candidate values of the threshold parameter R_1. For each $k = K$, we consider the candidate threshold value R_{1k} and divide the return series into two parts, namely, $S_1 := \{Y_{(1)}, Y_{(2)}, \cdots, Y_{(N_1)}\}$ and $S_2 := \{Y_{(N_1+1)}, Y_{(N_1+2)}, \cdots, Y_{(N)}\}$, where $N_1 + N_2 = N$, so that all observations in S_1 are strictly less than R_{1k} while all observations in S_2 are greater than or equal to R_{1k}.

For the threshold model, we discretize it using the Euler forward scheme as follows:

$$Y(t) = \left(\mu_1 - \frac{1}{2}\sigma_1^2 + \sigma_1\xi(t)\right) I_{\{S(t-1)<R_1\}} + \left(\mu_2 - \frac{1}{2}\sigma_2^2 + \sigma_2\xi(t)\right) I_{\{S(t-1)\geq R_1\}} \ .$$

Here $Y(t) := \ln(S(t)/S(t-1))$ is the logarithmic return in the t^{th} trading day, for each $t = 1, 2, \cdots, N$; $\{\xi(t)|t = 1, 2, \cdots\}$ is a sequence of independent and identically distributed random variables with common distribution $N(0,1)$; S_{t-1} is the closing value of the index by the end of the $(t-1)^{st}$ trading day.

For the non-threshold model, we also discretize it using the Euler forward scheme:

$$Y(t) = \mu - \frac{1}{2}\sigma^2 + \sigma\xi(t) \ .$$

Based on the discretized processes of the threshold model and the non-threshold model, it is not difficult to evaluate the log likelihood functions for the two models from which the maximum likelihood estimates of the unknown parameters of the two models are obtained.

For the threshold model, for each candidate parameter R_{1k}, the log likelihood function $L_k(\Theta)$ can be decomposed into two parts as follows:

$$L_k(\Theta) = L_k(\Theta_1) + L_k(\Theta_2) \ ,$$

where

1. for each $i = 1, 2$, $\Theta_i := (\mu_i, \sigma_i)$;
2. $\Theta := (\Theta_1, \Theta_2)$;
3. $L_k(\Theta_i)$ is the log-likelihood function evaluated by all of the observations in S_i.

Then we select the threshold parameter R_1 as the value R_{1k^*} such that

$$k^* := arg \max_{k\in\{1,2,\cdots,K\}} L_k(\hat{\Theta}) \ ,$$

where $\hat{\Theta}$ is the maximum likelihood estimate of Θ.

Alternatively, we can select R_1 as the value of R_{1k^*} which minimizes the AIC defined as follows:

$$AIC_k(\Theta) := -2L_k(\hat{\Theta}) + 2p \ ,$$

where p is the number of parameters in the model.

By applying the above estimation scheme, the annualized estimated model parameters in both the threshold model are shown in the following.

$$\mu_1 = 20.8189\%, \ \mu_2 = -1.6592\%, \ \sigma_1 = 33.9570\%, \ \sigma_2 = 17.7554\%, \ R = 994.6,$$

and for the non-threshold model, we have

$$\mu = -0.2665\%, \ \sigma = 21.2458\%.$$

Using the estimated model parameters we evaluate the optimal portfolio strategies as well as the value functions arising in both the threshold model and the non-threshold one in both the logarithmic utility and the power utility cases. Yield of US 1-year Treasury Bond is taken to be 0.09%.

Figures 5 and 6 depict the plots of the optimal portfolio strategies against the initial values of the stock price arising from the threshold model and the non-threshold one in the logarithmic utility and power utility cases. The solid line represents the optimal strategy in the threshold case while the dotted line represents the optimal strategy in non threshold case.

From the data, we observe a strong mean reverting property of the S&P 500 index in the recent 10 years. The market will demonstrate a strong rebound when it drops below the threshold with a drift return rate higher than 20%. It gives some support for the investment wisdom that we should be greedy when the others are fearful. However, we have relatively little support that we should short sell asset in a "tranquil market regime" as the drift return rate in the "tranquil market regime"

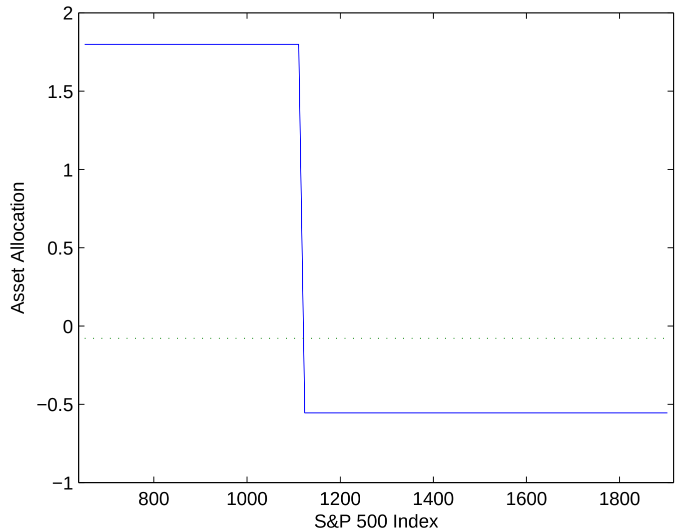

Figure 5. Asset allocation of logarithmic utility under different initial stock prices

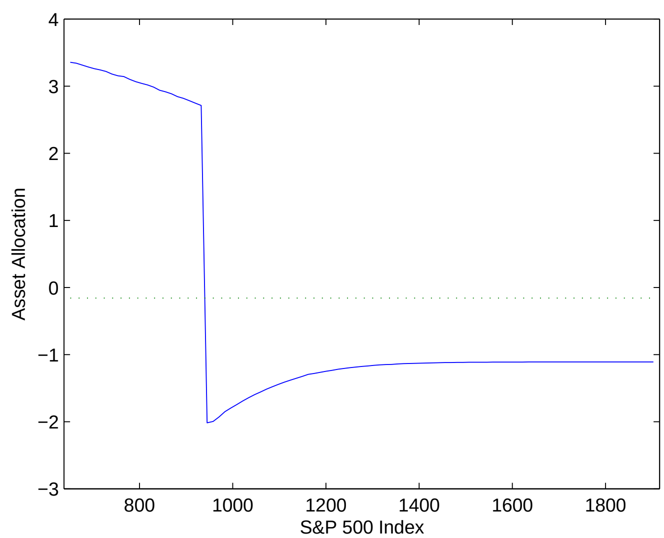

Figure 6. Asset allocation of power utility under different initial stock prices

is just a little bit lower than the risk free rate. The negative risk premium is not significant.

From the perspective of standard portfolio theory, the investment opportunity sets which consist of all available risk-return combinations of investment securities have a significant impact on optimal portfolio selection. These changes in the investment opportunity sets can be described by our continuous-time self-exciting asset price process and the impact of these changes on the optimal portfolio strategy is reflected on the optimal portfolio given in Theorem 4.1. For example, we consider a simple case where there are two states in the continuous-time self-exciting asset price process. In this situation, the optimal portfolio in Theorem 4.1 becomes:

$$\pi^*(t) = \frac{V^*(t)\theta(t)}{\sigma_1(t)} I_{\{S(t)<R\}} + \frac{V^*(t)\theta(t)}{\sigma_2(t)} I_{\{S(t)\geq R\}} \ .$$

In this situation, we may interpret the regime described by "$\{S(t) < R\}$" as a "volatile market regime" and that described by "$\{S(t) \geq R\}$" as a "tranquil market regime". We also suppose that $\sigma_1(t) > \sigma_2(t)$, so that the market is more volatile in the "volatile market regime" than in the "tranquil market regime". Consequently the optimal portfolio strategy arising from our continuous-time self-exciting asset price process devises that we should allocate more money in the risky asset in the "volatile market regime" as it has a higher expected return than in the "tranquil market regime". This, of course, makes intuitive sense.

7. **Conclusion.** We present a continuous-time self-exciting threshold model and study the optimal portfolios as well as the value functions in the cases of a logarithmic utility and a power utility. A martingale approach is used to obtain the optimal investment strategy in the logarithmic utility case, and we find that the strategy depends on the current market condition, (i.e. state dependency). Numerical results for the case of a power utililty function reveal that the regime switching effect of the stock price in this model is more significant than the derived effect of the stock price itself. The model also suggests that the US market has a mean reverting property in the recent 10 years.

Acknowledgments. We would like to thank the Associate Editor and the reviewers for their helpful comments. H. Meng is supported by the Nature Science Foundation of China (11271385) and the MOE Project of Key Research Institute of Humanities and Social Science in universities (12JJD790017). H. Yang would like to thank the Research Grants Council of the Hong Kong Special Administrative Region, China (project No. HKU 706611P).

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Received February 2011; 1st revision February 2012; final revision October 2012.

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