Spin Density Wave Fluctuations and \( p \)-Wave Pairing in \( \text{Sr}_2\text{RuO}_4 \)

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Recently, a debate has arisen over which of the two distinct parts of the Fermi surface of \( \text{Sr}_2\text{RuO}_4 \) is the active part for the chiral \( p \)-wave superconductivity exhibited. Early theories proposed \( p \)-wave pairing on the two-dimensional \( \gamma \) band, whereas a recent proposal focuses on the one-dimensional (\( \alpha, \beta \)) bands whose nesting pockets are the source of the strong incommensurate spin density wave (SDW) fluctuations. We apply a renormalization group theory to study quasi-one-dimensional repulsive Hubbard chains and explain the form of SDW fluctuations, reconciling the absence of long-range order with their nesting Fermi surface. The mutual exclusion of \( p \)-wave pairing and SDW fluctuations in repulsive Hubbard chains favors the assignment of the two-dimensional \( \gamma \) band as the source of \( p \)-wave pairing.

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\( \text{Sr}_2\text{RuO}_4 \) is generally believed to be a chiral \( p \)-wave superconductor, in analogy to the superfluidity of \( ^3\text{He} \). The field has attracted a lot of attention for its novel superconductivity \([1–4]\). The normal state electronic structure is well established. Near the Fermi level there are three \( \text{Ru} \) \( 4d \) bands, a two-dimensional (2D) \( \gamma \) band, mainly a \( d_{\text{yz}} \) orbital, and a pair of quasi-1D (\( \alpha, \beta \)) bands, mainly \( d_{\text{zx}} \) and \( d_{\text{yz}} \) orbitals, as illustrated in Fig. 1. There is also a general consensus that the pairing is likely of electronic origin. The recent debate is over which of the bands is the active source of the superconductivity, which is important to our basic understanding of the superconductivity in this material. The debate has been triggered by the failure to observe the persistent edge currents associated with the chirality, though this issue is also controversial \([3,5,6]\). Early theories propose the pairing predominantly arising from the 2D \( \gamma \) band. The 2D scenario predicts a chiral \( p \)-wave pairing and an edge current. Microscopic derivations of a chiral \( p \)-wave pairing state in the 2D scenario have been proposed based on a 2D Hubbard model by a \( T \)-matrix approach \([7]\), third-order perturbation theory \([8]\), and functional renormalization group (RG) calculations \([9]\). Very recently, Raghu et al. \([5,10]\) have argued the quasi-1D scenario as more compatible with the missing edge currents and have provided a microscopic justification for it by using an RG theory, but only in the limit of weak interactions.

A closely related and competing phenomenon in \( \text{Sr}_2\text{RuO}_4 \) is the spin density wave (SDW) fluctuations at an incommensurate nesting wave vector spanning the Fermi surfaces of the \( 4/3 \) filled (\( \alpha, \beta \)) bands \([11]\). SDW fluctuations at this wave vector \( \vec{Q} = (2\pi/3, 2\pi/3) \) \([12]\) were recently reported at room temperature and at energies as high as 80 meV \([13]\). The SDW peaks at \( \vec{Q} \), which combine nesting in both nearly 1D Fermi surfaces, grow as the temperature \( T \) is reduced and saturate at the crossover to 3D Fermi liquid behavior at \( T_{3D} = 60 \) K \([11]\) when the resistivity starts to show a \( T^2 \) behavior \([14]\). Smaller peaks were observed at the wave vectors \( (\pi, 2\pi/3) \) and \( (2\pi/3, \pi) \). To date, these have been discussed within a random phase approximation (RPA) scheme \([15–18]\). Because of the highly nesting character of the (\( \alpha, \beta \)) Fermi surface, it is necessary to choose a very weak interaction in RPA with a value typically an order of magnitude smaller than standard estimates \([19,20]\).

In this Letter, we show that treating the 1D character of the (\( \alpha, \beta \)) bands in an RG scheme can explain the strong SDW fluctuation and reconcile the absence of the SDW long-range order at \( T > T_{3D} \) using a standard value for the interaction as observed in \( \text{Sr}_2\text{RuO}_4 \). Furthermore, our RG scheme shows mutual exclusion of \( p \)-wave pairing and

![FIG. 1 (color online). Fermi surfaces of \( \text{Sr}_2\text{RuO}_4 \) (solid lines). The quasi-1D (\( \alpha, \beta \)) bands are derived from the \( d_{\text{zx}} \) and \( d_{\text{yz}} \) orbitals (dashed lines).](image-url)
SDW fluctuations in repulsive Hubbard chains and a sharp suppression of the SDW fluctuations at low frequency in the \( p \)-wave superconducting (SC) state. Such suppression has not been observed in early neutron scattering experiments, and its absence in more complete experiments would be a challenge to explain within the quasi-1D scenario.

We start with a single chain Hamiltonian for \((\alpha, \beta)\) bands by neglecting the \( \gamma \) band,

\[
H_0 = H_{\text{kin}}^0 + H_U,
\]

\[
H_{\text{kin}}^0 = \sum_{m\sigma} \epsilon_{m\sigma} c_{m\alpha}^{\dagger} c_{m\sigma},
\]

\[
H_U = \sum_{imn} n_{im} n_{in},
\]

In the above equations, \( \epsilon_{x(y)z} \) is the electron annihilation operator with orbital \( m = d_{xz}, d_{yz} \) and momentum \( \vec{k} \) and spin \( \sigma, H_U \) is the Hubbard term for the on-site intraorbital interaction and \( n_{im\sigma} = c_{ima}^{\dagger} c_{im\sigma} \).

The properties of single chains in a one-loop RG were derived in an early application of RG to condensed-matter systems [21]. This includes the important cancellation scenario.

To have a better modeling for this, we examine the effects on the SDW fluctuations arising from (1) the hybridization and spin-orbit coupling between the two orbitals \( d_{xz} \) and \( d_{yz} \), which give rise to coupled perpendicular chains and (2) the on-site interorbital interactions. As we shall see below, the former introduces a low-energy cutoff on the response function in Eq. (2), and the latter enhances SDW fluctuations at the wave vector \((2k_F, 2k_F)\).

The angular momentum operators and spin operators are represented in terms of the totally antisymmetric tensor \( \epsilon_{mn} = i\epsilon_{ann} \) and Pauli matrices \( \sigma \), respectively. For the system we are interested in, the strengths of the mixing and spin-orbit coupling are \( t'' = 0.1t \) and \( \eta = 0.1t \), respectively [24,25].

When the above perturbation \( \delta H \) is taken into account, the kinetic energy term becomes \( H_{\text{kin}}^\lambda = H_{\text{kin}}^0 + \delta H \), and the quasiparticle spectrum opens a gap \( 2\Delta \) near \(( \pm k_F, \pm k_F)\) with \( \Delta = \sqrt{3t''/2} + \eta^2 \). Therefore, the dispersion for the \( d_{xz} \) orbital is modified to

\[
e'_{xz,k} = v_F |k_x| - k_F| + \text{sgn}(|k_x| - k_F)|L(|k_x|)|\lambda.
\]

with \( L(x) \) the Lorentzian function centered at \( k_F \). The bare spin susceptibility or the SDW response function of \( H_{\text{kin}}^\lambda \) for both \( d_{xz} \) and \( d_{yz} \) orbitals is found to be

\[
\chi_{\text{bare}}(\vec{q}, \omega) = \frac{1}{\pi v_F} \left[ \frac{4T}{E_0 + 2\lambda} \ln \frac{T + 2\lambda}{E_0} + \int_0^\infty \ln \left( x + \frac{\lambda}{2x} \right) \text{sech}^2 x dx \right].
\]

with \( \vec{q} = (2k_F, q_x) \) and \((q_x, 2k_F)\) for \( d_{xz} \) and \( d_{yz} \) orbitals, respectively, where we have set \( h = 1 \) and the lattice spacing as the length unit. At low \( T \) and in the limit \( \lambda \ll E_0 \), we have

\[
\chi_{\text{bare}}(\vec{q}, \omega) = \frac{1}{\pi v_F} \ln \frac{T + 2\lambda}{E_0}.
\]

A standard RG calculation, including particle-particle and particle-hole graphs, gives the dressed susceptibility when intraorbital interactions \( H_U \) are included,

\[
\chi_{\text{RG}}(\vec{q}, \omega) = \frac{1}{\pi v_F} \left( \frac{E_0}{T + 2\lambda} \right)^\omega.
\]

From the expression above, one can see that due to the hopping between the two orbitals, a finite low-energy cutoff \( \lambda \) appears, killing the divergence at \( T \to 0 \).
where use Gutzwiller renormalized values of with $N$ holes \[26\].

indices RPA-like method, leading to with $U$ the interorbital Coulomb repulsion and $J_H$ the Hund’s rule coupling.

To incorporate the interorbital interactions, we define the joint SDW response function by including the orbital indices $m$,

$$
\chi_H(\vec{q},i\Omega) = -\int_0^\beta e^{i\Omega\tau} \langle \mathcal{O}(\vec{q}, \tau) \mathcal{O}(\vec{q}, 0) \rangle d\tau,
$$

where

$$
\mathcal{O}(\vec{q}, \tau) = \frac{1}{N} \sum_{\vec{k},m} \left[ c_{\vec{k},m}^\dagger(\tau) c_{\vec{k}+\vec{q},m}(\tau) - c_{\vec{k},m}^\dagger(\tau) c_{\vec{k}+\vec{q},m}^\dagger(\tau) \right],
$$

with $N$ the total number of sites and $\vec{q} = (2k_F, q_y)$ and $(q_x, 2k_F)$ for $d_{xz}$ and $d_{yz}$ orbitals, respectively.

To first order in the interorbital interactions, the only nonvanishing term is

$$
J_H \frac{T^2}{N^2} \sum_{\vec{k},\vec{k}',m,m'} G_{xz}(\vec{k} + \vec{Q}, i\omega_n + i\Omega) G_{xz}(\vec{k}, i\omega_n)
\times G_{yz}(\vec{k} + \vec{Q}, i\omega', i\Omega) G_{yz}(\vec{k}, i\omega'),
$$

with a corresponding diagram shown in Fig. 2(a). Note that the wave vector for the response function $\vec{Q} = (2k_F, 2k_F)$ is the same for both $d_{xz}$ and $d_{yz}$ orbitals due to the conservation of momentum in the scattering process in Fig. 2(a).

Another important consequence is that only the Hund’s rule coupling contributes to the SDW response function, while other on-site interaction terms in Eq. (8) are not involved. This result originates from the spin configuration in Fig. 2(a). In this sense, the Hund’s rule coupling assists the spin-flip processes between different orbitals. An intuitive physical picture is that the spin-flip processes are coherent even in different orbitals, due to the ferromagnetic Hund’s rule coupling between the two orbitals. Dynamical mean-field theory found that the Hund’s rule coupling is important in Sr$_2$RuO$_4$ [19]. In our calculations below we use Gutzwiller renormalized values of $J_H = 0.13$ eV to take into account the strong on-site repulsion between holes [26].

The full dressed joint SDW response function in Eq. (9) is obtained by first including the intraorbital interaction $U$ in an RG scheme, which means that the bare bubbles in Fig. 2(a) are replaced with the dressed ones in Eq. (7). However, due to the 2D perpendicular scattering nature of the Hund’s rule coupling term, $J_H$ can be treated in an RPA-like method, leading to

\[ H_I = \sum_{i,m<n,\sigma} \left\{ U^\dagger n_{i m \sigma} n_{i n \bar{\sigma}} + (U^\dagger - J_H) n_{i m \sigma} n_{i n \bar{\sigma}} \right\}
- J_H^\dagger c_{i m \sigma}^\dagger c_{i n \bar{\sigma}} c_{i n \bar{\sigma}}^\dagger c_{i m \sigma}
- J_H^\dagger c_{i m \sigma}^\dagger c_{i n \bar{\sigma}} c_{i n \bar{\sigma}}^\dagger c_{i m \sigma} + \text{H.c.}, \tag{8} \]

with $U^\dagger$ the interorbital Coulomb repulsion and $J_H$ the Hund’s rule coupling.

FIG. 2 (color online). (a) The lowest order Feynman diagram for the spin-spin correlation function connecting propagators from different orbitals via the Hund’s rule coupling. The red and blue lines stand for electrons in the $xz$ and $yz$ orbitals, respectively. The solid and dashed lines correspond to electrons belonging to the branches containing $+k_F$ and $-k_F$ in the 1D model, respectively. Because of momentum conservation at the vertex, $\vec{Q}$ is locked to be $(2k_F, 2k_F)$. (b) SDW vs paramagnetic phase diagram in the parameter space of $\lambda$ and $J_H$ at $T = 0$ from Eq. (13). The estimated parameter region for Sr$_2$RuO$_4$ is indicated in the paramagnetic phase. In this region, $J_H$ ranges from 0.13 to 0.4 eV, the renormalized and bare values of the Hund’s rule coupling.

\[ \chi_H(\vec{Q}, T) = \frac{2\chi_{RG}(\vec{Q}, T)}{1 - J_H^\dagger \chi_{RG}(\vec{Q}, T)} \tag{12} \]

The divergence of $\chi_H(\vec{Q}, T)$ in Eq. (12) gives an estimate of the mean-field transition temperature to long-range SDW order,

\[ T_{c, \text{SDW}}^{\text{SDW}} = E_0 \left( \frac{J_H}{2\pi\nu_F^2} \right)^{1/\nu^*} - 2\lambda. \tag{13} \]

The first term gives an upper limit on $T_{c, \text{SDW}}$ due to the Hund’s rule coupling, which is about 50 K, similar to $T_{c, \text{SDW}}$. The presence of the second term, of order $10^3$ K, guarantees that the ground state is paramagnetic. To illustrate the underlying physics, we construct the phase diagram in the $J_H - \lambda$ parameter space as shown in Fig. 2(b). The parameter set for Sr$_2$RuO$_4$ is in the paramagnetic region, but near the phase boundary to SDW order. Therefore, the strongly enhanced response function generated by Hund’s rule coupling at $\vec{Q}$ naturally explains the strong enhancement of the SDW signal near $\vec{Q}$ in the experiments [11].
Next, we briefly comment on how interchain tunneling between parallel chains affects the SDW in the quasi-1D $(\alpha, \beta)$ bands of Sr$_2$RuO$_4$. The interchain hopping $t_\perp$ alone would give rise to a singular SDW response function at the wave vector $(2k_F, \pi)$ since the tight-binding approximation preserves the perfect nesting for quasi-1D systems $[21,27]$. However, in the case of Sr$_2$RuO$_4$, the Fermi surfaces of the $(\alpha, \beta)$ bands are distorted due to hybridization and spin-orbit coupling between orbitals, as discussed previously. Therefore, the nesting property at $(2k_F, \pi)$ is lost, and a strong enhancement for the SDW fluctuation is not expected.

Another mechanism to affect the SDW response function at $(2k_F, \pi)$ is the superexchange interaction $J_{\text{ex}} = 4t_\perp^2/U$ between two neighboring parallel chains. But a rough estimation yields $J_{\text{ex}} \ll J_H$, since $t_\perp$ is only about 0.026 eV $[2]$. Combining the above two effects for the parallel chains, the spin fluctuation response at $(2k_F, \pi)$ should be much weaker than that at $(2k_F, 2k_F)$, as observed in the experiment $[13]$.

Finally, we consider the effects that follow from SC pairing order in 1D bands on the magnetic response. In Sr$_2$RuO$_4$, there is a crossover to 3D Fermi liquid with enhanced SDW fluctuations from $(\alpha, \beta)$ bands, which we do not treat in detail here. But the transition to the ordered SC state at $T^*_c = 1.5$ K can be treated in mean field, and if the $(\alpha, \beta)$ bands are the active bands, this should be observable in neutron scattering experiments. Early measurements did not find a change in the magnetic response at $T^*_c$ upon cooling through the SC transition at $T^*_c$ $[11]$.

To make our analysis more transparent, we restrict the discussion to one dimension and, thus, consider the following Hamiltonian $H = H_{\text{SC}} + H_{\text{int}}$, where $H_{\text{int}} = U \sum n_i n_j$ is the Hubbard on-site interaction term, which can be reduced to the standard form describing different scattering processes with $g_1 = g_2 = U/\pi v_F$ $[21,28]$, and $H_{\text{SC}}$ is to incorporate the SC pairing, $H_{\text{SC}} = \sum_k v_F (|k| - k_F) c_k^\dagger \sigma c_{k\sigma} + \sum_k [\Delta(k) c_k^\dagger \sigma^c c_{-k}^{-\sigma} + \text{H.c.}],$ $(14)$

which models 1D electrons with $p$-wave SC pairing $\Delta(k)$ and can be solved in the mean-field approximation. We assume that the mean-field results are stabilized via the interchain couplings. The assumption spin-orbit coupling locks the $\hat{d}$ vector along the crystal $c$ axis has been made, consistent with the polarized neutron scattering experiment in Sr$_2$RuO$_4$ $[29]$. While there is a crossover to 3D Fermi liquid above the SC transition temperature in experiment, here our aim is to show the effect of SC order on the SDW fluctuations by using a quasi-1D model.

In Nambu’s spinor representation, the normal and anomalous Green’s functions are given by $[30]$

$$G_{\sigma \sigma'}(k, i\omega_n) = -\delta_{\sigma \sigma'} \frac{i\omega_n + \xi_k}{\omega_n^2 + \xi_k^2 + \Delta_0^2}$$ $(15)$

and

$$F_{\sigma \sigma'}(k, i\omega_n) = \frac{\Delta_{\sigma \sigma'}(k)}{\omega_n^2 + \xi_k^2 + \Delta_0^2}.$$ $(16)$

Here, $\xi_k = v_F (|k| - k_F)$ and $\Delta_{\sigma \sigma'}(k) = \Delta(k)\sigma^c_\sigma \sigma^c_\gamma$. Near the Fermi surface, we have, due to the odd parity, $\Delta(k) = \text{sgn}(k)\Delta_0$, with $\Delta_0$ the SC gap near the Fermi surface. The calculation of SDW response function in the SC state is straightforward. It is interesting that the contributions from the particle-particle and particle-hole diagrams cancel each other, similar to the case in the Luttinger liquid case. The particle-hole bubble diagram can be expressed as

$$T \sum_{k, i\omega_n} G_{\uparrow\uparrow}(k, i\omega_n) G_{\downarrow\uparrow}(k + 2k_F, i\omega_n + i\Omega)$$

$$- F_{\uparrow\downarrow}(k, i\omega_n) F_{\downarrow\uparrow}(k + 2k_F, i\omega_n + i\Omega).$$ $(17)$

![FIG. 3 (color online). (Upper panels) Structure of the vertex diagrams. (a) and (b) are for the particle-particle channel, whereas (c) and (d) are for the particle-hole channel. The solid and dashed lines correspond to electrons belonging to the branches containing $+k_F$ and $-k_F$ in the one-dimensional model, respectively. The wavy lines stand for bare on-site interactions. (Lower panel) Response function as a function of $\omega$ at $T = 0$ is shown in (e). Here, $\Delta_{\text{RG}}(2k_F, \omega)$ is scaled by $\Delta_0 = \text{Re} \chi_{\text{RG}}(2k_F, \omega = 0)$, and $\theta^*$ is about 0.41 in the system of interest.](167003-4)
To the leading order in the logarithmic accuracy, this expression is reduced to

$$\frac{1}{2\pi v_F} \left[ \ln \sqrt{\omega^2 - 4\Delta_0^2} - i \frac{\pi}{2} \Theta(\omega - 2\Delta_0) \right],$$

with $\Theta(x)$ the Heaviside function, and we have performed an analytic continuation to real frequency $\omega$ at zero temperature. The structure of this expression is also similar to its counterpart in the normal state. Because of this analogy, the RG flow equations for the interaction constants $g_1$ and $g_2$ should be the same as those for the non-SC case \[21,28\]. Therefore, in the case of $\theta^* > 0$, the SDW fluctuation is expected as usual.

A standard RG analysis yields the final results for the SDW response function in the SC state as follows:

$$\text{Re}_{\Gamma_{\text{RG}}}^{\text{SC}}(2k_F, \omega) = \frac{1}{\pi v_F \theta^*} \left( \frac{E_0}{\sqrt{\omega^2 - 4\Delta_0^2}} \right)^{\theta^*}$$

$$\text{Im}_{\Gamma_{\text{RG}}}^{\text{SC}}(2k_F, \omega) = \frac{\Theta(\omega - 2\Delta_0)}{2v_F} \left( \frac{E_0}{\sqrt{\omega^2 - 4\Delta_0^2}} \right)^{\theta^*}.$$  \hfill (19)

As shown in Fig. 3(e), if one looks at the low-energy properties $\omega \rightarrow 2\Delta_0$, the response function diverge as $\chi \sim |\omega - 2\Delta_0|^{-\theta^*/2}$. This result indicates that the transition to superconductivity in the 1D bands will open a gap in the low-energy spectra at wave vector $\vec{Q}$. While early neutron scattering experiments by Braden et al. \[11\] did not show a change in low-energy spectra at $\vec{Q}$, a more complete investigation would be worthwhile to definitively decide if an SC gap opens up in the 1D $(\alpha, \beta)$ bands at the onset of superconductivity at $T^*$. In summary, we have applied an RG scheme starting from the 1D analysis for single chains to explain the strong SDW fluctuations and the absence of SDW order at temperatures above the crossover to 3D Fermi liquid behavior with the strong on-site Hubbard repulsion estimated for Sr$_2$RuO$_4$. The mutual exclusion in the 1D RG theory of enhancement in the SDW and simultaneously in the $p$-wave pairing channel is in favor of the 2D $\gamma$ band as the source of the superconductivity.

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[12] This wave vector is equivalent to $(4\pi/3, 4\pi/3)$ in the electron notion.


[26] The effect of strong on-site repulsion suppresses the probability of having two electrons from distinct orbitals at the same site to be 1/3. Thus, $J_H$ is reduced by a Gutzwiller factor of 1/3, compared with the bare value $J_H = 0.4$ eV in Ref. \[19\].


