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AN OPTIMAL BINDING NUMBER CONDITION FOR BIPANCYCLISM*

ZHIQUAN HU†, KA HO LAW‡, AND WENAN ZANG§

Abstract. Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. The bipartite binding number of G, denoted by $B(G)$, is defined to be n if $G = K_{n,n}$ and $\min_{i\in\{1,2\}} \min_{\substack{\emptyset\neq S\subseteq V_i \\ |N(S)|<n}} |N(S)|/|S|$ otherwise. We call G bipancyclic if it contains a cycle of every even length m for $4 \le m \le 2n$. The purpose of this paper is to show that if $B(G) > 3/2$ and $n \ge 139$, then G is bipancyclic; the bound $3/2$ is best possible in the sense that there exist infinitely many balanced bipartite graphs G that have $B(G) = 3/2$ but are not Hamiltonian.

Key words. bipartite graph, Hamiltonian cycle, bipancyclism, binding number

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1. Introduction.

Let $G = (V, E)$ be a graph. The *binding number* of G, denoted by $b(G)$, is defined to be

$$\min_{\substack{\emptyset\neq S\subseteq V \\ |N(S)|<|V|}} |N(S)|/|S|,$$

where $N(S) = \{v \in V : uv \in E \text{ for some } u \in S\}$. This parameter was introduced by Woodall [8] to measure how well the vertices of G are bound together; in particular, if $b(G)$ is large, then G has lots of edges fairly well distributed. The binding number resembles some other graph invariants, such as the minimum degree, connectivity, and toughness, in certain ways while providing more global structural information. In the literature there are a number of results showing that various properties of G are consequences of assumptions on the value of $b(G)$, including the following theorem on Hamiltonian cycles.

THEOREM 1.1 (Woodall [8]). *Every graph G with $b(G) \ge 3/2$ is Hamiltonian.*

Call G *pancyclic* if it contains a cycle of every length m for $3 \le m \le |V|$. As conjectured by Woodall [8] and proved by Shi [6, 7], this assertion can be strengthened as follows.

THEOREM 1.2 (Shi [6, 7]). *Every graph G with $b(G) \ge 3/2$ is pancyclic.*

Observe that for bipartite graphs, the binding number does not give much information about their structures (or well-boundness) when compared to nonbipartite graphs. For instance, both $K_{n,n}$ (a complete bipartite graph) and nK_2 (union of n disjoint edges) have binding number 1 for $n \ge 1$; their structures, however, are dramatically different. Furthermore, for any bipartite graph $G = (V_1, V_2, E)$, we have

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†Faculty of Mathematics and Statistics, Central China Normal University, Wuhan, China (hu_zhiq@yahoo.com.cn). This author was supported in part by NSFC grants 11071096 and 11271149 and grant D20111110 of the Hubei Provincial Department of Education.

‡Corresponding author. Department of Mathematics, University of Hong Kong, Hong Kong, China (lawkaho@graduate.hku.hk).

§Department of Mathematics, University of Hong Kong, Hong Kong, China (wzang@maths.hku.hk). This author was supported in part by the Research Grants Council of Hong Kong.

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$b(G) \leq \min\{|V_2|/|V_1|,\ |V_1|/|V_2|\} \leq 1$. Hence neither Theorem 1.1 nor Theorem 1.2 applies to G. In graph theory it is common for results to have a "bipartite" version; such a typical example is Jackson's theorem [3], which asserts that every 2-connected k-regular graph with at most $3k$ vertices is Hamiltonian. Häggkvist [2] conjectured that every 2-connected k-regular bipartite graph G with at most $6k-6$ vertices is Hamiltonian, which was confirmed by Jackson and Li [4] when G contains at most $6k-38$ vertices. So a natural question to ask is, what are the counterparts of the above binding number theorems on bipartite graphs? To find the answer, clearly we need a new concept of binding number in order to better reflect the bipartiteness.

Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. The *bipartite binding number* of G, denoted by $B(G)$, is defined to be n if $G = K_{n,n}$ and

$$\min_{i \in \{1,2\}} \min_{\substack{\emptyset \neq S \subseteq V_i \\ |N(S)| < n}} |N(S)|/|S|$$

otherwise. We call G *bipancyclic* if it contains a cycle of every even length m for $4 \leq m \leq 2n$. The purpose of this paper is to establish the following bipartite version of the above two theorems.

THEOREM 1.3. *Let G be a balanced bipartite graph with $2n$ vertices. If $B(G) > 3/2$ and $n \geq 139$, then G is bipancyclic.*

We shall exhibit infinitely many balanced bipartite graphs G that have $B(G) = 3/2$ but are not Hamiltonian in section 2 (see Proposition 2.5). Thus the bound $3/2$ in our theorem is best possible. Moreover, the proof techniques of our theorem are substantially different from those of Theorems 1.1 and 1.2.

Let us introduce some notation before proceeding. Given a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For each $v \in V(G)$, we use $d(v)$ and $N(v)$ to denote its degree and neighborhood, respectively. For each $S \subseteq V(G)$, it is clear that $N(S) = \cup_{v \in S} N(v)$. For each subgraph H of G, let $G - H$ denote the subgraph of G induced by $V(G) - V(H)$ and set $N_H(S) := N(S) \cap V(H)$. When G is a bipartite graph with bipartition (V_1, V_2), we set $V_i(H) := V_i \cap V(H)$ for $i = 1, 2$.

Throughout this paper, we use C_n to denote a cycle of length n and assume that each cycle C has an implicit clockwise orientation. With this assumption, v_C^+ and v_C^- will stand for the successor and predecessor of a vertex v on C under this orientation, respectively; we shall drop the subscript C if there is no danger of confusion. We define v^{+i} recursively by $v^{+0} = v$ and $v^{+(i+1)} = (v^{+i})^+$ for $i \geq 0$ and define v^{-i} analogously. For any two vertices u and v on C, let $u\overrightarrow{C}v$ denote the path from u to v on C in the clockwise direction, and let $u\overleftarrow{C}v$ denote the path from u to v on C in the counterclockwise direction. Set $C[u,v] := V(u\overleftarrow{C}v)$ and $C(u,v] := C[u,v] - \{u\}$, etc. For each $X \subseteq V(C)$ and $i \geq 1$, define $X^{+i} := \{x^{+i} : x \in X\}$ and $X^{-i} := \{x^{-i} : x \in X\}$. If $X = N_C(v)$ for some vertex v, then we shall simply write $N_C^{+i}(v)$ and $N_C^{-i}(v)$ as opposed to the more cumbersome $(N_C(v))^{+i}$ and $(N_C(v))^{-i}$. We also define $X^{+0} := X =: X^{-0}$ for convenience.

The remainder of this paper is organized as follows. In section 2, we derive some basic properties satisfied by bipartite binding numbers. In section 3, we show the existence of certain nested cycle structures in G under some assumptions. In section 4, we first establish a bipartite version of the hopping lemma originally developed by Woodall [8] and then employ it to further grow the nested cycle structures obtained in section 3 under some other assumptions. In section 5, we prove that G contains a cycle of every even length based on the aforementioned nested cycle structures.

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2. Preliminaries. Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices such that $G \neq K_{n,n}$. Recall the definition of the bipartite binding number $B(G)$; a subset S of V_i, for $i = 1$ or 2, is called a *binding set* of G if $|N(S)| < n$ and $B(G) = |N(S)|/|S|$.

The following proposition asserts that the value of $B(G)$ is uniquely determined by G rather than its balanced bipartition, so the bipartite binding number is well defined.

PROPOSITION 2.1. *Let G be a balanced bipartite graph. Then the value of $B(G)$ is independent of the choice of balanced bipartition.*

Proof. If G is connected, then the choice of balanced bipartition is unique (up to permutation of V_1 and V_2), so the statement holds trivially. It remains to consider the case when G is disconnected.

Let (V_1, V_2) be a balanced bipartition of G such that the value of $B(G)$ is minimized (let c denote this minimum value) and, subject to this, a corresponding binding set S has smallest possible size. We claim that S is entirely contained in one component of G; for otherwise, let $G_1, G_2, \ldots, G_k$ be all components of G that intersect S, where $k \geq 2$, and set $S_i := S \cap V(G_i)$ for $1 \leq i \leq k$. From the minimality assumption on $|S|$, we deduce that $|N(S_i)| > c|S_i|$ for all i and hence $c|S| = \sum_{i=1}^{k} c|S_i| < \sum_{i=1}^{k} |N(S_i)| = |N(S)| = c|S|$; this contradiction justifies the claim. It follows that for any balanced bipartition (U_1, U_2) of G, either $S \subseteq U_1$ or $S \subseteq U_2$. Therefore, S is also a binding set of G with respect to bipartition (U_1, U_2). ∎

PROPOSITION 2.2. *Every balanced bipartite graph G with $B(G) > 1$ is connected.*

Let us now illustrate bipartite binding numbers using two special classes of graphs.

PROPOSITION 2.3. $B(C_{2n}) = \frac{n-1}{n-2}$ *for* $n \geq 3$.

Proof. Let (V_1, V_2) be the bipartition of C_{2n}, and let S be a nonempty subset of V_i, $i = 1$ or 2, with $|N(S)| < n$. From the structure of C_{2n}, we see that $|S| \leq n - 2$ and $|S| < |N(S)|$. Hence

$$\frac{|N(S)|}{|S|} \geq \frac{|S| + 1}{|S|} = 1 + \frac{1}{|S|} \geq 1 + \frac{1}{n-2} = \frac{n-1}{n-2}$$

with equality when $S = V_i - \{u, v\}$, where u and v are two vertices in V_i of distance 2 on C_{2n}. So the statement is established. ∎

Let s and t be two positive integers, and let $sK_2 \oplus t\overline{K}_2$ be the bipartite graph obtained from the union of s disjoint edges a_ib_i for $1 \leq i \leq s$ by adding $2t$ vertices $c_1, c_2, \ldots, c_t, d_1, d_2, \ldots, d_t$ and adding edges a_id_j and b_ic_j for all $1 \leq i \leq s$ and $1 \leq j \leq t$ (see Figure 2.1). For convenience, set $A := \{a_1, a_2, \ldots, a_s\}$, $B := \{b_1, b_2, \ldots, b_s\}$, $C := \{c_1, c_2, \ldots, c_t\}$, and $D := \{d_1, d_2, \ldots, d_t\}$. Clearly, $sK_2 \oplus t\overline{K}_2$ has a unique bipartition (V_1, V_2), where $V_1 = A \cup C$ and $V_2 = B \cup D$.

PROPOSITION 2.4. *Let s and t be two positive integers. Then*

$$B(sK_2 \oplus t\overline{K}_2) = \begin{cases} \dfrac{1}{t} & \text{if } s = 1, \\[2ex] \min\left\{\dfrac{s}{t}, \dfrac{s-1+t}{s-1}\right\} & \text{if } s > 1. \end{cases}$$

Proof. Let $G = sK_2 \oplus t\overline{K}_2$ and let S be a binding set of G. Symmetry allows us to assume that $S \subseteq V_1$. Thus $|N(S)| < |V_2|$ by definition.

If $s = 1$, then $a_1 \notin S$. So $S \subseteq C$ and $N(S) = \{b_1\}$. As S is a binding set of G, we must have $S = C$. Therefore, $B(G) = |N(S)|/|S| = 1/t$.

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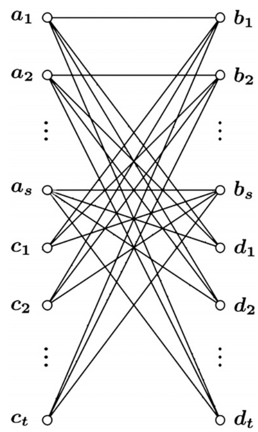

FIG. 2.1. $sK_2 \oplus t\overline{K}_2$.

If $s > 1$, then $A - S \neq \emptyset$. Furthermore, $S \cap A = \emptyset$ provided $S \cap C \neq \emptyset$, for otherwise we would have $N(S) = V_2$, a contradiction. It follows that S is either a proper subset of A or a subset of C. Thus $|N(S)|$ equals $|S| + t$ in the former case and s in the latter case. As S is a binding set of G, either $S = A - \{a_i\}$ for some $1 \leq i \leq s$ or $S = C$. From the definition we further deduce that $B(G) = |N(S)|/|S| = \min\{\frac{s}{t}, \frac{s-1+t}{s-1}\}$, completing the proof. ∎

The following proposition asserts that the bound 3/2 in Theorem 1.3 is indeed the threshold for a balanced bipartite graph to be Hamiltonian or bipancyclic.

PROPOSITION 2.5. *Let $G = sK_2 \oplus t\overline{K}_2$. Then $B(G) = 3/2$ if $s = 2t + 1$, and G is not Hamiltonian if $s \geq 2t + 1$.*

Proof. The first statement follows instantly from Proposition 2.4. If $s \geq 2t + 1$, then $G - (C \cup D)$ contains precisely s components (see Figure 2.1) with $s > |C \cup D|$. It follows that G contains no Hamiltonian cycle. ∎

The following lemma gives an alternative definition of the bipartite binding number.

LEMMA 2.6. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $G \neq K_{n,n}$, then $B(G)$ is the largest nonnegative number c such that*

$$c|N(S)| \geq (c-1)n + |S|$$

for every nonempty subset S of V_i $(i = 1, 2)$.

Proof. By definition, it suffices to show that for any given constant $c \geq 0$, the following two statements are equivalent:

(a) $c|N(S)| \geq (c-1)n + |S|$ for every nonempty $S \subseteq V_i$ and $i = 1, 2$;

(b) $|N(S)| \geq \min\{c|S|, n\}$ for every nonempty $S \subseteq V_i$ and $i = 1, 2$.

To this end, let S be a nonempty subset of V_i for $i = 1$ or 2, and let $T := V_{3-i} - N(S)$. Then $N(T)$ and S are disjoint subsets of V_i, so $|N(T)| + |S| \leq n$ and hence

(c) $|N(T)| \leq n - |S| \leq n - 1$.

If (a) holds, then (with T in place of S) either $c|N(T)| \geq (c-1)n + |T| = cn - |N(S)|$ or $T = \emptyset$. In the former case, $|N(S)| \geq \min\{c(n - |N(T)|), n\} \geq \min\{c|S|, n\}$ by (c). In the latter case, $V_{3-i} - N(S) = \emptyset$. So $|N(S)| = n \geq \min\{c|S|, n\}$. Combining these two cases, we obtain (b).

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Conversely, if (b) holds, then (with T in place of S) either $|N(T)| \geq \min\{c|T|, n\} = \min\{cn - c|N(S)|, n\}$ or $T = \emptyset$. In the former case, $c|N(S)| \geq cn - |N(T)| \geq (c-1)n + |S|$ by (c). In the latter case, $V_{3-i} - N(S) = \emptyset$. So $|N(S)| = n$ and hence $c|N(S)| = cn \geq (c-1)n + |S|$. Combining these two cases, we establish (a). ∎

As usual, we use $\delta(G)$ to denote the minimum degree of a graph G. The above lemma yields a lower bound on $\delta(G)$ when restricted to $|S| = 1$.

COROLLARY 2.7. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $B(G) \geq c > 0$, then*

$$\delta(G) \geq \frac{(c-1)n + 1}{c}.$$

LEMMA 2.8. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $B(G) > n/2$, then*

$$|N(S)| \geq \lceil (n + 2|S| + 1)/3 \rceil$$

for every nonempty proper subset S of V_i $(i = 1, 2)$.

Proof. As the statement holds trivially if $G = K_{n,n}$, we assume hereafter that $G \neq K_{n,n}$. Let $B(G) = c$ and let S be a nonempty proper subset of V_i for $i = 1, 2$. By Lemma 2.6, we have

$$|N(S)| \geq \frac{(c-1)n + |S|}{c} = n - \frac{n - |S|}{c}.$$

This together with $n - |S| > 0$ and $c > 3/2$ implies

$$|N(S)| > n - \frac{2(n - |S|)}{3},$$

and hence the desired statement holds. ∎

The following lemma will play an important role in the subsequent proofs.

LEMMA 2.9. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices and with $B(G) > 3/2$. Let $X \subseteq V_i$ and $Y \subseteq V_{3-i}$, with $i = 1$ or 2, be nonempty sets such that $|X|$, $|Y|$, $|N(X)|$, and $|N(Y)|$ are all less than n. If $|Y| \geq |N(X)| - t$ for some nonnegative integer t, then $|N(Y)| \geq |X| + (2n+4)/5 - t$.*

Proof. Symmetry allows us to assume that $i = 1$. For $S = X, Y$, by Lemma 2.8 and the definition of $B(G)$, we have

$$|N(S)| \geq \max\left\{\frac{n + 2|S| + 1}{3}, \frac{3|S| + 1}{2}\right\}. \tag{2.1}$$

It follows that

$$|Y| \geq |N(X)| - t \geq \max\left\{\frac{n + 2|X| + 1}{3} - t, \frac{3|X| + 1}{2} - t\right\}.$$

Plugging this inequality into (2.1) (with $S = Y$), we obtain

$$|N(Y)| \geq \max\left\{\frac{n + 2\left(\frac{3|X|+1}{2} - t\right) + 1}{3}, \frac{3\left(\frac{n+2|X|+1}{3} - t\right) + 1}{2}\right\}.$$

Consequently,

$$|N(Y)| \geq |X| + \max\{f(t), g(t)\} - t,$$

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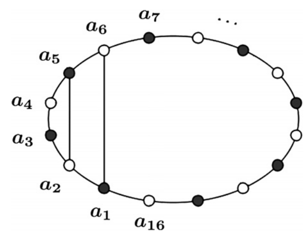

FIG. 3.1. C_{16}^2.

where $f(x) := (n+x+2)/3$ and $g(x) := (n-x+2)/2$. Observe that $f(x)$ is an increasing function of x, while $g(x)$ is a decreasing function of x, and that $f(x_0) = g(x_0) = (2n+4)/5$, with $x_0 = (n+2)/5$. Hence $\max\{f(x), g(x)\} \geq f(x_0)$ for all x. Therefore $|N(Y)| \geq |X| + f(x_0) - t = |X| + (2n+4)/5 - t$, as desired. $\square$

3. Nested cycle structures. Let k and m be two positive integers with $k \geq m+2$, let $C = a_1a_2\ldots a_{2k}a_1$ be a cycle of length k, where $a_{i+1} = a_i^+$ for each i (with $a_{2k+1} = a_1$), and let D be obtained from C by adding m chords a_ia_{2m+3-i} for $1 \leq i \leq m$. We write D as $\overline{a_1a_2\ldots a_{2m+2}}a_{2m+3}\ldots a_{2k}a_1$ and denote any graph isomorphic to D by C_{2k}^m. (See Figure 3.1 for C_{16}^2.) Observe that C_{2k}^m contains $m+1$ nested cycles $C_{2k}, C_{2k-2}, \ldots, C_{2k-2m}$ simultaneously. Intuitively, C_{2k}^m can be viewed as a ladder with m rungs; our proof will rely heavily on such ladders. For any vertex v on D, define $v^+ := v_C^+$ and $v^- := v_C^-$. For any two vertices u and v on D, define $u\overrightarrow{D}v := u\overrightarrow{C}v$ and $D[u,v] := C[u,v]$, etc.

To establish the main result, we first show the existence of C_4, C_6^1, and one of C_8^2, C_{10}^2, and C_{12}^2. The following statement and its proof are inspired by its counterparts on general graphs due to Reiman [5].

LEMMA 3.1. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $|E| > n/2\left(1+\sqrt{4n-3}\right)$, then G contains a C_4.*

Proof. Suppose G contains no C_4. Consider triples of the form $(x, \{y,z\})$ such that $x \in V_1$, $y, z \in V_2$ with $y \neq z$, and that x is adjacent to both y and z. Since G contains no C_4, each pair $\{y,z\}$ gives rise to at most one such triple. Hence the number of such triples is at most $\binom{n}{2}$.

On the other hand, since each $x \in V_1$ gives rise to exactly $\binom{d(x)}{2}$ such triples, the number of triples of the above form is equal to $\sum_{x\in V_1}\binom{d(x)}{2}$. Let $\sigma = \sum_{x\in V_1} d(x)/n$. Then $\sigma = |E|/n$. As the extended binomial coefficient $\binom{t}{2}$ is a convex function, by definition $\binom{\sigma}{2} \leq \frac{1}{n}\sum_{x\in V_1}\binom{d(x)}{2}$. So $\binom{\sigma}{2} \leq \frac{1}{n}\binom{n}{2}$ and hence $\sigma^2 - \sigma - (n-1) \leq 0$. Solving this inequality yields $\sigma \leq 1/2\left(1+\sqrt{4n-3}\right)$. Therefore $|E| \leq n/2\left(1+\sqrt{4n-3}\right)$, a contradiction. $\square$

LEMMA 3.2. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $B(G) > 3/2$ and $n > 3$, then G contains a C_4.*

Proof. By Lemma 2.8, we have $\delta(G) \geq \lceil (n+3)/3 \rceil$. This together with $n > 3$ implies

$$|E| \geq n\delta \geq n\lceil (n+3)/3 \rceil > \frac{n}{2}(1+\sqrt{4n-3}).$$

Thus the statement follows instantly from Lemma 3.1. $\square$

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By Propositions 2.3 and 2.4, C_6 and $3K_2 \oplus \overline{K}_2$ have bipartite binding numbers 2 and 3/2, respectively, yet neither of them contains a C_4. So the figures in the above lemma are both sharp.

LEMMA 3.3. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $B(G) > 3/2$ and $n \geq 10$, then G contains a C_6^1.*

Proof. By Lemma 3.2, G contains a cycle $x_1y_1x_2y_2x_1$ of length 4 with $X = \{x_1, x_2\} \subseteq V_1$ and $Y = \{y_1, y_2\} \subseteq V_2$. For $i = 1, 2$, define $X_i = N(y_i) - X$ and $Y_i = N(x_i) - Y$.

Assume on the contrary that G contains no C_6^1. Then there is no edge between $X_1 \cup X_2$ and $Y_1 \cup Y_2$. Furthermore, $x_1 \notin N(Y_2)$ or $y_1 \notin N(X_2)$. Symmetry allows us to assume that $x_1 \notin N(Y_2)$. Thus $Y_1 \cap Y_2 = \emptyset$. By Lemma 2.8, we obtain $|X_1| \geq \lceil (n+3)/3 \rceil - 2 = \lceil (n-3)/3 \rceil$, and the same is true for $|Y_1|$ and $|Y_2|$. Hence $|Y_1 \cup Y_2| \geq 2\lceil (n-3)/3 \rceil$.

As X_1 is nonempty and $X_1 \cap N(Y_1 \cup Y_2) = \emptyset$, we have

$$\begin{aligned} n &\geq |X_1| + |N(Y_1 \cup Y_2)| \\ &> \lceil (n-3)/3 \rceil + \frac{3}{2} \cdot 2\lceil (n-3)/3 \rceil, \end{aligned}$$

so $n \geq 4\lceil (n-3)/3 \rceil + 1$ and hence $n \geq 4(n-3)/3 + 1$, which implies $n \leq 9$, a contradiction. ∎

LEMMA 3.4. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices. If $B(G) > 3/2$ and $n \geq 14$, then G contains at least one of C_8^2, C_{10}^2, and C_{12}^2.*

Proof. Let $D = \overline{x_1y_1x_2y_2}x_3y_3x_1$ be a C_6^1 in G; the existence of D is guaranteed by Lemma 3.3. Recall the definition that x_1y_2 is an edge in D. Set $X := \{x_1, x_2, x_3\}$ and $Y := \{y_1, y_2, y_3\}$. Symmetry allows us to assume that $X \subseteq V_1$ and $Y \subseteq V_2$. Define $N_1(x_3) = N(x_3) - Y$, $N_2(x_3) = N(N_1(x_3)) - X$, and $N_3(x_3) = N(N_2(x_3)) - Y$. Define $N_i(y_3)$ symmetrically for $1 \leq i \leq 3$.

Assume on the contrary that G contains none of C_8^2, C_{10}^2, and C_{12}^2. We propose to show that

$$N(N_i(x_3)) \cap N_j(y_3) = \emptyset \text{ for all } 1 \leq i,\, j \leq 3. \tag{3.1}$$

Otherwise, let (i, j) be a pair such that $N(N_i(x_3)) \cap N_j(y_3) \neq \emptyset$ and, subject to this, $i + j$ is minimum. Then $i \equiv j \pmod 2$ and $G[\{x_3, y_3\} \cup (\cup_{s=1}^{i} N_s(x_3)) \cup (\cup_{t=1}^{j} N_t(y_3))]$ contains an (x_3, y_3)-path π of length $i + j + 1$. It follows that $\overline{y_3x_1y_1x_2y_2x_3}\pi y_3$ is a C_{6+i+j}^2 in G, and this contradiction establishes (3.1).

By taking $i = j = 1$ in (3.1), we see that $N_2(x_3) \cap N_1(y_3) = \emptyset$, so $y_3 \notin N(N_2(x_3))$. Repeated application of Lemma 2.8 yields

$$\begin{aligned} |N_1(x_3)| &\geq \lceil (n+3)/3 \rceil - |Y| \geq (n-6)/3, \\ |N_2(x_3)| &\geq \lceil (n + 2|N_1(x_3)| + 1)/3 \rceil - |X| \geq (5n - 36)/9, \\ |N_3(x_3)| &\geq \lceil (n + 2|N_2(x_3)| + 1)/3 \rceil - |Y - \{y_3\}| \geq (19n - 117)/27, \text{ and} \\ |N(N_3(x_3))| &\geq \lceil (n + 2|N_3(x_3)| + 1)/3 \rceil \geq (65n - 207)/81. \end{aligned}$$

Similarly, $|N_3(y_3)| \geq (19n - 117)/27$. In view of (3.1), $N(N_3(x_3))$ and $N_3(y_3)$ are disjoint subsets of V_1, so $|N(N_3(x_3))| + |N_3(y_3)| \leq n$, which implies $41n \leq 558$ and hence $n < 14$, contradicting the hypothesis. ∎

Let us digress briefly to introduce a term and make some simple observations, which will be used to show the existence of the aforementioned ladders.

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Let $D = \overline{a_1a_2\dots a_6}a_7\dots a_{2k}a_1$ be a C^2_{2k}, where $k \ge 4$ and $a_1 \in V_1(D)$. A family $(A_0, A_1, \dots, A_t)$, with $2 \le t \le 4$, of subsets of $V(D)$ is called *good* if the following two conditions are satisfied:

- $A_0 \cup A_1^{+} \cup \dots \cup A_t^{+t} \subseteq V_2(D)$ and
- $A_i^{+i} \cap A_j^{+j} \subseteq \{v \in V_2(D) : \{v^{-i}, v^{-j}\} \cap D[a_2, a_5] \ne \emptyset\}$ for all $0 \le i < j \le t$.

LEMMA 3.5. *Suppose $(A_0, A_1, \dots, A_t)$ is good. Then the following statements hold:*

(i) *If $t = 4$ and $a_6 \notin A_2^{+2} \cap A_3^{+3}$, then $|A_0| + \sum_{s=2}^{4} |A_s| \le k + 7$.*

(ii) *If $t \in \{2, 3\}$, then $\sum_{s=0}^{t} |A_s| \le k + \lceil 5t/2 \rceil$.*

Proof. Since $(A_0, A_1, \dots, A_t)$ is a good family of subsets of $V(D)$, it is a routine matter to check using the definition that (where A_i^{+i} exists only when $t \ge i$ for each i)

(1) $A_0 \cap A_1^{+}$ and $A_0 \cap A_2^{+2}$ are both subsets of $\{a_2, a_4, a_6\}$;
(2) $A_0 \cap A_3^{+3}$ and $A_0 \cap A_4^{+4}$ are both subsets of $\{a_2, a_4, a_6, a_8\}$;
(3) $A_1^{+} \cap A_2^{+2}$ is a subset of $\{a_4, a_6\}$;
(4) $A_1^{+} \cap A_3^{+3}$, $A_2^{+2} \cap A_3^{+3}$, $A_1^{+} \cap A_4^{+4}$, and $A_2^{+2} \cap A_4^{+4}$ are all subsets of $\{a_4, a_6, a_8\}$; and
(5) $A_3^{+3} \cap A_4^{+4}$ is a subset of $\{a_6, a_8\}$.

In the remainder of our proof, we use $f(v)$ to denote the number of sets in $\{A_0, A_2^{+2}, A_3^{+3}, A_4^{+4}\}$ if $t = 4$ and in $\{A_0, A_1^{+}, \dots, A_t^{+t}\}$ if $t \in \{2, 3\}$ that contain a vertex v.

(i) By (4) and (5), a_2 is contained in at most one set in $\{A_2^{+2}, A_3^{+3}, A_4^{+4}\}$, so $f(a_2) \le 2$. By (5), we have $f(a_4) \le 3$. By hypothesis, $a_6 \notin A_2^{+2} \cap A_3^{+3}$. So $f(a_6) \le 3$. From (1) we deduce that $f(a_8) \le 3$. For all vertices $v \in V_2(D) - \{a_2, a_4, a_6, a_8\}$, from (1), (2), (4), and (5) we see that $f(v) \le 1$. Combining the above observations, we obtain

$$|A_0| + |A_2^{+2}| + |A_3^{+3}| + |A_4^{+4}| = \sum_{v \in V_2(D)} f(v) \le |V_2(D)| + 7 = k + 7.$$

Thus (i) is established.

(ii) Let us consider the case when $t = 2$. By (3), we have $a_2 \notin A_1^{+} \cap A_2^{+2}$. So $f(a_2) \le 2$. Clearly, $f(a_4) \le 3$ and $f(a_6) \le 3$. Moreover, from (1) and (3) we deduce that $f(v) \le 1$ for all $v \in V_2(D) - \{a_2, a_4, a_6\}$. Hence

$$|A_0| + |A_1^{+}| + |A_2^{+2}| = \sum_{v \in V_2(D)} f(v) \le |V_2(D)| + 5 = k + \lceil 5t/2 \rceil.$$

It remains to consider the case when $t = 3$. By (3) and (4), a_2 is contained in at most one set in $\{A_1^{+}, A_2^{+2}, A_3^{+3}\}$. So $f(a_2) \le 2$. Clearly, $f(a_4) \le 4$ and $f(a_6) \le 4$. From (1) and (3), we see that a_8 is contained in at most one set in $\{A_0, A_1^{+}, A_2^{+2}\}$, so $f(a_8) \le 2$. Moreover, for all vertices $v \in V_2(D) - \{a_2, a_4, a_6, a_8\}$, from (1)–(4) we deduce that $f(v) \le 1$. Therefore,

$$\sum_{s=0}^{3} |A_s| = \sum_{v \in V_2(D)} f(v) \le |V_2(D)| + 8 = k + \lceil 5t/2 \rceil.$$

This completes the proof of the present lemma. ∎

LEMMA 3.6. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices, let $D = \overline{a_1a_2\dots a_6}a_7\dots a_{2k}a_1$ be a C^2_{2k} in G with $k \ge 4$, and let $X_0, X_1, \dots, X_t$ be disjoint subsets of $V(G - D)$ with $t \in \{3, 4\}$ such that*

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(i) $X_0 = \{x_0\}$, *where* $\{x_0, a_1\} \subseteq V_1$;
(ii) $|X_1| = 1$ *if* $t = 4$; *and*
(iii) $X_i \subseteq N(X_{i-1})$ *for* $1 \le i \le t$.

Suppose u *is a vertex in* $N_D(x_0) - \{a_2, a_4\}$. *Let* $A_0 := N_D(u^+) - \{u^{+2}, u^{+2(t-2)}\}$ *and* $A_i := N_D(X_i)$ *for* $1 \le i \le t$. *If* G *contains none of* C^2_{2k+2}, C^2_{2k+4}, *and* C^2_{2k+6}, *then* $(A_0, A_1, \ldots, A_t)$ *is a good family of subsets of* $V(D)$.

Proof. Assume the contrary: there exist $0 \le i < j \le t$ and $v \in A_i^{+i} \cap A_j^{+j}$ such that $\{v^{-i}, v^{-j}\} \cap D[a_2, a_5] = \emptyset$. Set $H := G - D$.

Let us first consider the case when $i = 0$. Now $v \in N_D(u^+) - \{u^{+2}, u^{+2(t-2)}\}$, $v^{-j} \in N_D(X_j)$, and $\{v, v^{-j}\} \cap D[a_2, a_5] = \emptyset$. Observe that both u and v are in V_2 and $u \notin D(v^{-j}, v)$ (for otherwise $v = u^{+2}$ and $j \in \{3, 4\}$, a contradiction). Let x_j be a neighbor of v^{-j} in X_j and let P be an (x_0, x_j)-path of length j in $H[\cup_{s=0}^{j} X_s]$. Since $u \in V_2(D)$ and $v \in A_0 \subseteq V_2(D) - \{u^{+2}, u^{+2(t-2)}\}$, we have $v^{-j} \ne u$ and hence $u \notin D[v^{-j}, v)$. This together with $\{v, v^{-j}\} \cap D[a_2, a_5] = \emptyset$ implies that either $D[v^{-j}, v] \subseteq D[u^+, a_1]$ or $D[v^{-j}, v] \subseteq D[a_6, u]$. Therefore

$$D' = \begin{cases} \overline{a_1 a_2 \ldots a_6}\overrightarrow{D} u x_0 \overrightarrow{P} x_j v^{-j} \overleftarrow{D} u^+ v \overrightarrow{D} a_1 & \text{if } D[v^{-j}, v] \subseteq D[u^+, a_1], \\ \overline{a_1 a_2 \ldots a_6}\overrightarrow{D} v^{-j} x_j \overleftarrow{P} x_0 u \overleftarrow{D} v u^+ \overrightarrow{D} a_1 & \text{if } D[v^{-j}, v] \subseteq D[a_6, u] \end{cases}$$

is a C^2_{2k+2} in G, contradicting the hypothesis.

Next, let us consider the case when $i \ge 1$. Now $v^{-i} \in N_D(X_i)$ and $v^{-j} \in N_D(X_j)$. Let x_i be a neighbor of v^{-i} in X_i and let y_j be a neighbor of v^{-j} in X_j. By (iii), $H[\cup_{s=t-3}^{i} X_s]$ contains a path $Q := x_{t-3} x_{t-2} \ldots x_i$ of length $i - (t-3)$, where $x_s \in X_s$ for $t - 3 \le s \le i$. Similarly, $H[\cup_{s=t-3}^{j} X_s]$ contains a path $R := y_{t-3} y_{t-2} \ldots y_j$ of length $j - (t-3)$, where $y_s \in X_s$ for $t - 3 \le s \le j$. Since $|X_{t-3}| = 1$, we have $x_{t-3} = y_{t-3}$. Let ℓ be the largest subscript with $t - 3 \le \ell \le i$ such that $x_\ell = y_\ell$. Then $0 \le i - \ell \le (t-1) - (t-3) \le 2$. Set $S := x_i \overleftarrow{Q} x_\ell \overrightarrow{R} y_j$. Clearly, S is a path in $H[\cup_{s=t-3}^{j} X_s]$ of length $j - i + 2(i - \ell)$. Thus we obtain a $C^2_{2k+2(i-\ell)+2}$ from D by replacing $v^{-j} \overrightarrow{D} v^{-i}$ with $v^{-j} y_j \overleftarrow{S} x_i v^{-i}$, contradicting the hypothesis again. ∎

Our next two lemmas show that if G contains a C^2_{2k}, denoted by D, such that $G - D$ has a path with length at least three, then we can find a C^2_{2t} in G based on the above two lemmas for some t with $k + 1 \le t \le k + 3$.

LEMMA 3.7. *Let* $G = (V_1, V_2, E)$ *be a balanced bipartite graph with* $2n$ *vertices, let* $D = \overline{a_1 a_2 \ldots a_6} a_7 \ldots a_{2k} a_1$ *be a* C^2_{2k} *in* G, *and let* $x_0 x_1 x_2 x_3 x_4$ *be a path in* $G - D$ *such that* $N_D(x_0) - \{a_2, a_3, a_4, a_5\} \ne \emptyset$. *If* $B(G) > 3/2$, $n \ge 139$, *and* $k \ge 4$, *then* G *contains at least one of* C^2_{2k+2}, C^2_{2k+4}, *and* C^2_{2k+6}.

Proof. Assume on the contrary that

$$G \text{ contains none of } C^2_{2k+2}, C^2_{2k+4}, \text{ and } C^2_{2k+6}. \tag{3.2}$$

By Proposition 2.2, G is connected. Symmetry allows us to assume that x_0 and a_1 are in the same color class of G, for otherwise rewrite D as $\overline{b_1 b_2 \ldots b_6} b_7 \ldots b_{2k} b_1$, where $b_i = a_{7-i}$ for $1 \le i \le 6$. Then x_0 and b_1 are in the same class, as desired. Renaming subscripts of V_i's if necessary, we may assume that $\{x_0, a_1\} \subseteq V_1$.

Let $H = G - D$ and $u \in N_D(x_0) - \{a_2, a_4\}$. Define $X_1 := \{x_1\}$, $X_2 := \{x_2\}$, $X_3 := N_H(x_2) - \{x_1\}$, $X_4 := N_H(X_3) - \{x_0, x_2\}$, and $X_5 := N_H(X_4) - (X_3 \cup \{x_1\})$ (see Figure 3.2). Note that X_1, X_2, X_3, X_4 and X_5 are disjoint subsets of $V(H) - \{x_0\}$. By (3.2), we have $N_H(u^+) \cap (X_1 \cup X_3 \cup X_5) = \emptyset$, so $N_H(u^+), X_1, X_3$, and X_5 are disjoint subsets of $V_2(H)$, which implies that $|N_H(u^+)| + |X_1| + |X_3| + |X_5| \le n - k$ and hence

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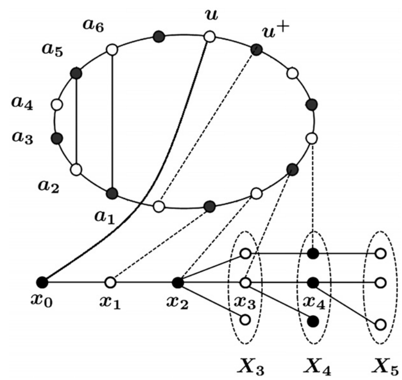

FIG. 3.2. *D and X_i's.*

$$|N_H(u^+)| + |N_H(x_2)| + |N_H(X_4) - (X_3 \cup \{v_1\})| \leq n - k. \tag{3.3}$$

Set $A_0 := N_D(u^+) - \{u^{+2}, u^{+4}\}$ and $A_i := N_D(X_i)$ for $1 \leq i \leq 4$. By Lemma 3.6,

$$(A_0, A_1, A_2, A_3, A_4) \text{ is a good family of subsets of } V(D). \tag{3.4}$$

Observe that

$$a_6 \notin A_2^{+2} \cap A_3^{+3}, \tag{3.5}$$

for otherwise a_4 is adjacent to x_2 and a_3 is adjacent to some vertex x_3' in X_3. It follows that $\overline{a_2 a_3 x_3' x_2 a_4 a_5} a_6 \overrightarrow{D} a_2$ is a C_{2k+2}^2 in G, and this contradiction to (3.2) establishes (3.5).

From (3.4), (3.5), and Lemma 3.5, we deduce that $|A_0|+|A_2|+|A_3|+|A_4| \leq k+7$. Hence

$$|N_D(u^+)| + |N_D(x_2)| + |N_D(X_3)| + |N_D(X_4)| \leq k + 9.$$

Adding this inequality to (3.3) yields

$$|N(u^+)| + |N(x_2)| + |N_D(X_3)| + |N(X_4)| - |X_3| \leq n + 10. \tag{3.6}$$

By (3.2), we have $N_D(X_3) \subseteq V_1(D) - \{u^+\}$ and $N_D(X_4) \subseteq V_2(D) - (\{u^{-2}, u^{+2}\} - \{a_2, a_4\})$, so $|N(X_i)| < n$ for $i = 3, 4$. As $|X_4| = |N_H(X_3) - \{x_0, x_2\}| \geq |N(X_3)| - (|N_D(X_3)| + 2)$, the triple $(X, Y, t) = (X_3, X_4, |N_D(X_3)| + 2)$ satisfies the hypothesis of Lemma 2.9 and hence

$$|N(X_4)| \geq |X_3| + (2n + 4)/5 - (|N_D(X_3)| + 2).$$

Combining this inequality with (3.6) gives $|N(u^+)| + |N(x_2)| + (2n - 6)/5 \leq n + 10$. Thus, by Lemma 2.8 we obtain $2(n + 3)/3 + (2n - 6)/5 \leq n + 10$, which implies $n \leq 138$, and this contradiction completes the proof of our lemma. ∎

LEMMA 3.8. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices, let $D = \overline{a_1 a_2 \ldots a_6} a_7 \ldots a_{2k} a_1$ be a C_{2k}^2 in G, and let $x_0 x_1 x_2 x_3$ be a path in $G - D$ such that $N_D(x_0) - \{a_2, a_3, a_4, a_5\} \neq \emptyset$. If $B(G) > 3/2$, $n \geq 139$, and $k \geq 4$, then G contains at least one of C_{2k+2}^2, C_{2k+4}^2, and C_{2k+6}^2.*

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Proof. Assume on the contrary that

$$G \text{ contains none of } C^2_{2k+2},\ C^2_{2k+4}, \text{ and } C^2_{2k+6}. \tag{3.7}$$

By symmetry, we may assume that $\{x_0, a_1\} \subseteq V_1$. (See the first paragraph of the proof of the preceding lemma.)

Let $H = G - D$ and $u \in N_D(x_0) - \{a_2, a_4\}$. Define $X_1 := N_H(x_0) - \{x_3\}$, $X_2 := N_H(X_1) - \{x_0\}$, and $X_3 := N_H(X_2) - X_1$. If there exists a vertex x_4 in $N_H(x_3) - \{x_0, x_2\}$, then $x_0x_1x_2x_3x_4$ would be a path in $G - D$ and thus we reach a contradiction to (3.7) by Lemma 3.7. Therefore

$$N_H(x_3) \subseteq \{x_0, x_2\}. \tag{3.8}$$

Similarly,

$$N_H(X_3) \subseteq \{x_0\} \cup X_2. \tag{3.9}$$

By (3.7), we have $N_H(u^+) \cap (X_1 \cup X_3) = \emptyset$, so $N_H(u^+)$, X_1, and X_3 are disjoint subsets of $V_2(H)$. It follows that $|N_H(u^+)| + |X_1| + |X_3| \leq n - k$ and hence

$$|N_H(u^+)| + |N_H(X_2)| \leq n - k. \tag{3.10}$$

From Lemma 3.6, we see that $(N_D(u^+) - \{u^{+2}\}, N_D(X_1), N_D(X_2), N_D(X_3))$ is a good family of subsets of $V(D)$. By Lemma 3.5, we thus obtain

$$(|N_D(u^+)| - 1) + |N_D(X_1)| + |N_D(X_2)| + |N_D(X_3)| \leq k + 8.$$

Adding this inequality to (3.10) yields

$$d(u^+) + |N_D(X_1)| + |N(X_2)| + |N_D(X_3)| \leq n + 9. \tag{3.11}$$

In view of (3.7), we get $N_D(X_1) \subseteq V_1(D) - \{u^+\}$ and $N_D(X_2) \subseteq V_2(D) - (\{u^{-2}, u^{+2}\} - \{a_2, a_4\})$. Hence $|N(X_i)| < n$ for $i = 1, 2$. As $|X_2| = |N_H(X_1) - \{x_0\}| \geq |N(X_1)| - (|N_D(X_1)| + 1)$, the triple $(X, Y, t) = (X_1, X_2, |N_D(X_1)| + 1)$ satisfies the hypothesis of Lemma 2.9 and hence

$$|N(X_2)| \geq |X_1| + (2n + 4)/5 - (|N_D(X_1)| + 1).$$

Combining this inequality with (3.11) gives

$$d(u^+) + |X_1| + (2n + 4)/5 - 1 + |N_D(X_3)| \leq n + 9.$$

Using (3.8), we obtain $|N_D(X_3)| \geq |N_D(x_3)| = |N(x_3)| - |N_H(x_3)| \geq d(x_3) - 2$, so $d(u^+) + (2n + 4)/5 + (d(x_3) - 2) \leq n + 9$. From Lemma 2.8, it follows that $(n + 3)/3 + (2n + 4)/5 + (n + 3)/3 - 2 \leq n + 9$. Therefore $n \leq 123$, and this contradiction completes the proof of our lemma. ∎

4. A generalized bipartite hopping lemma. The hopping lemma was first introduced by Woodall [8] in his proof of Theorem 1.1, which demonstrates that the approach of iterating cycle exchanges can be highly effective for finding long cycles. Variations of the lemma were subsequently developed by various authors for use in different works. In particular, Ash [1] developed a basic version of the hopping lemma for bipartite graphs.

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The following lemma is an extract of results from Ash [1] (see Lemmas 4.3, 4.4, 4.9, and 4.16; see also Jackson and Li [4]).

LEMMA 4.1 (Ash [1]). *Let $G = (V_1, V_2, E)$ be a bipartite graph, and let C be a longest cycle of G such that the number of components of $G-C$ is as small as possible and, subject to this, a smallest component H of $G-C$ is as small as possible. Suppose there exist $a \in V_1 - V(C)$ and $b \in V_2 - V(C)$ such that either a and b are both isolated vertices in $G - C$ or $V(H) = \{a, b\}$. For each vertex v in $G - C$, set $Y_0(v) := \emptyset$, and define recursively sets $X_i(v)$ and $Y_i(v)$ for $i \geq 1$ by $X_i(v) := N_C(Y_{i-1}(v) \cup \{v\})$ and $Y_i(v) := \{y \in C : y^-, y^+ \in X_i(v)\}$. Set $X_v := \cup_{i\geq 1} X_i(v)$ and $Y_v := \cup_{i\geq 1} Y_i(v)$. Then the following statements hold:*

(i) $N(Y_v) \subseteq X_v$ *for* $v \in \{a, b\}$*;*
(ii) $X_a \cap Y_b = \emptyset = X_b \cap Y_a$*;*
(iii) $|X_a^+ \cap X_b| \leq 1$ *and* $|X_a^- \cap X_b| \leq 1$*; and*
(iv) $X_a^+ \cap X_b = \emptyset = X_a^- \cap X_b$ *if* $ab \in E$.

For convenience, set $C_{2k}^0 := C_{2k}$ for all $k \geq 2$. Observe that in Ash's lemma C is assumed to be a longest cycle of G under certain restrictions, while in our proof we need a generalized version which can be used to deal with the case when G contains some C_{2k}^m (not necessarily a longest one) but no C_{2k+2}^m for $m \geq 0$ under some other restrictions. Let us now present this generalized bipartite hopping lemma, which ensures that the ladder structure can be preserved when growing a cycle.

LEMMA 4.2. *Let $G = (V_1, V_2, E)$ be a bipartite graph, and let $D = \overline{a_1 a_2 \dots a_{2m+2}} a_{2m+3} \dots a_{2k} a_1$ be a C_{2k}^m in G with $m \geq 0$ and $a_1 \in V_1$. Suppose G contains neither C_{2k+2}^m nor another C_{2k}^m, denoted by D', such that $G - D'$ has fewer components than $G - D$, and suppose there exist $a \in V_1 - V(D)$ and $b \in V_2 - V(D)$ such that both of them are isolated vertices in $G - D$. For each vertex v in $G - D$, set $Y_0(v) := \emptyset$, and define recursively sets $X_i(v)$ and $Y_i(v)$ for $i \geq 1$ by $X_i(v) := N_D(Y_{i-1}(v) \cup \{v\}) - D(a_1, a_{2m+2})$ and $Y_i(v) := \{y \in D : y^-, y^+ \in X_i(v)\}$, where $D(a_1, a_{2m+2}) = \emptyset$ if $m = 0$. Set $X_v := \cup_{i\geq 1} X_i(v)$ and $Y_v := \cup_{i\geq 1} Y_i(v)$. Then the following statements hold:*

(i) $N(Y_v) \subseteq X_v \cup D(a_1, a_{2m+2})$ *for* $v \in \{a, b\}$*;*
(ii) $X_a \cap Y_b = \emptyset = X_b \cap Y_a$*; and*
(iii) $|X_a^+ \cap X_b| \leq 1$ *and* $|X_a^- \cap X_b| \leq 1$.

Since the proof of this lemma is very tedious, we postpone it till section 6 so that the proof of our main theorem proceeds in a smoother and more coherent way. Clearly, the following monotonicity property holds for the objects defined in the above two lemmas:

$$X_1(v) \subseteq X_2(v) \subseteq X_3(v) \subseteq \cdots \subseteq X_v \quad \text{and} \quad Y_1(v) \subseteq Y_2(v) \subseteq Y_3(v) \subseteq \cdots \subseteq Y_v. \tag{4.1}$$

As an application of the above generalized bipartite hopping lemma, let us derive the following statement, which will be used later.

LEMMA 4.3. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices, and let $D = \overline{a_1 a_2 \dots a_{2m+2}} a_{2m+3} \dots a_{2k} a_1$ be a C_{2k}^m in G with $m \geq 0$. Suppose G does not contain another C_{2k}^m, denoted by D', such that $G - D'$ has fewer components than $G - D$, and suppose there exist $a \in V_1 - V(D)$ and $b \in V_2 - V(D)$ such that both of them are isolated vertices in $G - D$. If $B(G) > 3/2$ and $m + 2 \leq k \leq n - 3m - 1$, then G contains a C_{2k+2}^m.*

Proof. Assume the contrary: G contains no C_{2k+2}^m. Recall Lemma 4.2 and consider the sets $X_i(a)^+$ and $X_i(a)^-$ for $i \geq 1$. By (4.1) and Lemma 4.2(iii), each of $X_i(a)^+$ and $X_i(a)^-$ contains at most one vertex in $X_i(b)$. Hence $|X_i(a)^+ \cap X_i(a)^-| = |X_i(a)^+| +$

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$|X_i(a)^-|-|X_i(a)^+\cup X_i(a)^-| \geq |X_i(a)^+|+|X_i(a)^-|-|V_1(D)-X_i(b)|-2 = (2|X_i(a)|-2)-(k-|X_i(b)|)$. By the definition of $Y_i(a)$, we have

$$|Y_i(a)| \geq |X_i(a)^+ \cap X_i(a)^-| \geq (2|X_i(a)|-2)-(k-|X_i(b)|).$$

Similarly,

$$|Y_i(b)| \geq |X_i(b)^+ \cap X_i(b)^-| \geq (2|X_i(b)|-2)-(k-|X_i(a)|).$$

Adding these two inequalities yields

$$|Y_i(a)|+|Y_i(b)| \geq 3|X_i(a)|+3|X_i(b)|-2k-4.$$

From the definition, (4.1), and Lemma 4.2(ii), it is clear that $Y_i(a) \subseteq V_1(D) - D(a_1, a_{2m+2}) - X_1(b)$. As $D(a_1, a_{2m+2}) \cap V_1(D) \neq \emptyset$ if $m \geq 1$ and $X_1(b) \neq \emptyset$ if $m = 0$, we have $Y_i(a) \neq V_1(D)$. Therefore $Y_i(a) \cup \{a\}$ is a proper subset of V_1 for all $i \geq 0$. Since a is an isolated vertex of $G - D$, from the definition, (4.1), and Lemma 4.2(i), we deduce that $N(Y_{i-1}(a) \cup \{a\}) \subseteq X_i(a) \cup (V_2 \cap D(a_1, a_{2m+2}))$. This together with Lemma 2.8 implies that

$$|X_i(a)|+m \geq |N(Y_{i-1}(a) \cup \{a\})| \geq \frac{n+2|Y_{i-1}(a)|+3}{3},$$

so $3|X_i(a)| \geq n+2|Y_{i-1}(a)|-3(m-1)$. Similarly, $3|X_i(b)| \geq n+2|Y_{i-1}(b)|-3(m-1)$. Hence

$$\begin{aligned}|Y_i(a)|+|Y_i(b)| &\geq 2n+2|Y_{i-1}(a)|+2|Y_{i-1}(b)|-6(m-1)-2k-4\\ &= 2\left(|Y_{i-1}(a)|+|Y_{i-1}(b)|\right)+2(n-k)-6m+2\\ &\geq 2\left(|Y_{i-1}(a)|+|Y_{i-1}(b)|\right)+4 \ \ (\text{as } k \leq n-3m-1),\end{aligned}$$

which implies

$$|Y_i(a)|+|Y_i(b)|+4 \geq 2\left(|Y_{i-1}(a)|+|Y_{i-1}(b)|+4\right).$$

Since $Y_0(a) = Y_0(b) = \emptyset$, it follows that $|Y_i(a)|+|Y_i(b)|+4 \geq 2^{i+2}$ for all $i \geq 1$, and hence $|Y_i(a)|+|Y_i(b)| \to \infty$ as $i \to \infty$, which is absurd. ∎

5. Proof of Theorem 1.3. The proof of our theorem comes in three steps, and different steps require different counting techniques. Actually we have already carried out step 1 in section 3 by showing the existence of C_4, C_6^1 and one of C_8^2, C_{10}^2, and C_{12}^2 in G. Based on such a ladder and Lemma 4.3, we can now proceed to step 2, which aims to prove that G contains a C_{2k} for every k with $2 \leq k \leq n-6$.

LEMMA 5.1. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices and with a C_{2k}^2. If $B(G) > 3/2$, $n \geq 139$, and $4 \leq k \leq n-7$, then G contains at least one of C_{2k+2}^2, C_{2k+4}^2, and C_{2k+6}^2.*

Proof. By hypothesis, G contains a subgraph $D = \overline{a_1a_2\ldots a_6}a_7\ldots a_{2k}a_1$ (with $a_1 \in V_1$), which is a C_{2k}^2. Assume on the contrary that G contains none of C_{2k+2}^2, C_{2k+4}^2, and C_{2k+6}^2. Let us make some simple observations about $G-D$.

Claim 1. No component H of $G-D$ satisfies $\min\{|V_1(H)|, |V_2(H)|\} \geq 2$.

Suppose for a contradiction that $\min\{|V_1(H)|, |V_2(H)|\} \geq 2$ for some component H of $G-D$. Then H contains a path $x_0x_1x_2x_3$ with $x_0 \in V_1(H)$. By Lemma 3.8, we have $N_D(x_0) \cup N_D(x_3) \subseteq D[a_2, a_5]$, so $N_D(x_0) \subseteq \{a_2, a_4\}$ and $N_D(x_3) \subseteq \{a_3, a_5\}$.

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Using Lemma 2.8, we obtain $|N_H(x_i)| = d(x_i) - |N_D(x_i)| \geq (n-3)/3$ for $i = 0, 3$. Thus $|V_i(H)| \geq (n-3)/3 > 45$ for $i = 1, 2$.

Symmetry allows us to assume that $|V_1(H)| \geq |V_2(H)|$. Let us show that there exist two distinct vertices v_1 and v_2 in $V_1(H)$ such that

$$N_D(v_i) - \{a_2, a_4\} \neq \emptyset \text{ for } i = 1, 2. \tag{5.1}$$

Otherwise, there is a subset X of $V_1(H)$ with $|X| \geq |V_1(H)| - 1$ such that $N_D(X) \subseteq \{a_2, a_4\}$. So $|N(X)| < n$ and hence $|N_H(X)| = |N(X)| - |N_D(X)| > 3|X|/2 - 2 \geq (3|V_1(H)| - 7)/2 \geq (3|V_2(H)| - 7)/2 > |V_2(H)|$ as $V_2(H) \geq 45$, a contradiction. Therefore (5.1) is true.

Set $A := N_H(v_1)$ and $B := N_H(A) - \{v_1\}$. If $N_H(B) - A$ contains a vertex z, then letting $y \in N_B(z)$ and $x \in N_A(y)$, the path v_1xyz is fully contained in $G - D$, contradicting Lemma 3.8. So $N_H(B) \subseteq A$, which in turn implies $V_1(H) = B \cup \{v_1\}$ and $V_2(H) = A$. Hence $|A| \geq (n-3)/3 > 45$ and $v_2 \in B$. Let $u_2 \in N_A(v_2)$ and $u_1 \in A - \{u_2\}$. Then $v_2u_2v_1u_1$ is a path in $G - D$, which contradicts Lemma 3.8. So Claim 1 is justified.

Claim 2. Each component of $G - D$ contains at most two vertices.

Suppose the contrary: some component H of $G - D$ has at least three vertices. By Claim 1, we have $\min\{|V_1(H)|, |V_2(H)|\} \leq 1$. So H is a star. By symmetry, we may assume that $V_1(H) = \{x_1, x_2, \ldots, x_r\}$ and $V_2(H) = \{y\}$, where $r \geq 2$. Set $X := V_1(H) - \{x_1\}$. Since G contains no C^2_{2k+2},

$$\begin{aligned} &N_D(x_1) \cap N_D^+(y) \subseteq \{a_2, a_6\}, \\ &N_D(x_1) \cap N_D^{+2}(X) \subseteq \{a_2, a_4, a_6\}, \quad \text{and} \\ &N_D^+(y) \cap N_D^{+2}(X) \subseteq \{a_4, a_6\}. \end{aligned}$$

So each of a_2 and a_4 is contained in at most two sets in $\{N_D(x_1), N_D^{+2}(X), N_D^+(y)\}$. Consequently, $|N_D(x_1)| + |N_D^+(y)| + |N_D^{+2}(X)| \leq |V_2(D) - \{a_2, a_4, a_6\}| + 7 = k + 4$. By Lemma 2.8, we have $|N_D(x_1)| = |N(x_1) - \{y\}| \geq n/3$, $|N_D(y)| = |N(y)| - r \geq (n+3)/3 - r$, and $|N_D(X)| = |N(X) - \{y\}| \geq (n + 2|X| + 1)/3 - 1 = (n + 2r - 4)/3$. Therefore, $n/3 + (n+3)/3 - r + (n + 2r - 4)/3 \leq k + 4$, which implies that $3n \leq 3k + r + 13 \leq n + 2k + 13$ and hence $k > n - 7$, contradicting the hypothesis. This proves Claim 2.

Claim 3. $G - D$ contains at most one isolated edge.

To justify this, we assume that both x_1y_1 and x_2y_2 are two isolated edges of $G-D$ with $\{x_1, x_2\} \subseteq V_1$ and $\{y_1, y_2\} \subseteq V_2$. We propose to show that

$$\begin{aligned} &N_D(x_1) \cap N_D^{+2}(x_2) \subseteq \{a_2, a_4, a_6\} \\ \text{or} \quad &N_D(y_1) \cap N_D^{+2}(y_2) \subseteq \{a_3, a_5, a_7\}. \end{aligned} \tag{5.2}$$

Suppose not. Then there exist two vertices $v_1 \in N_D(x_1) \cap N_D^{+2}(x_2) - \{a_2, a_4, a_6\}$ and $v_2 \in N_D(y_1) \cap N_D^{+2}(y_2) - \{a_3, a_5, a_7\}$. By symmetry, we may assume that $v_1 \in D(v_2, a_1)$. If $v_2 = v_1^-$, then by replacing $v_1^- v_1$ with $v_1^- y_1 x_1 v_1$ in D, we get a C^2_{2k+2} in G, and this contradiction implies that $v_2 \neq v_1^-$. So $v_2 \in D(v_2^{-2}, v_1^{-3}] \subseteq D(a_7, v_1^{-3}]$. It follows that

$$\overline{a_1a_2\ldots a_6}a_7\overrightarrow{D}v_2^{-2}y_2x_2v_1^{-2}\overleftarrow{D}v_2y_1x_1v_1\overrightarrow{D}a_1$$

is a C^2_{2k+2} in G, a contradiction. So (5.2) holds.

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By symmetry, we may assume that

$$N_D(x_1) \cap N_D^{+2}(x_2) \subseteq \{a_2, a_4, a_6\}. \tag{5.3}$$

Since G contains no C_{2k+2}^2, clearly we have

$$N_D(x_1) \cap N_D^{+}(y_1) \subseteq \{a_2, a_6\}. \tag{5.4}$$

Moreover,

$$|N_D(y_1)^+ \cap N_D^{+2}(x_2) - \{a_4, a_6\}| \le 1, \tag{5.5}$$

for otherwise, let $\{u_1, u_2\} \subseteq N_D(y_1) \cap N_D^+(x_2)$ such that $\{u_1, u_2\} \cap \{a_3, a_5\} = \emptyset$, where $u_2 \in D(u_1, a_1]$. Then

$$\overline{a_1 a_2 \dots a_6} \overrightarrow{D} u_1^- x_2 u_2^- \overleftarrow{D} u_1 y_1 u_2 \overrightarrow{D} a_1$$

is a C_{2k+2}^2 in G, a contradiction. Let z be the vertex in $N_D(y_1)^+ \cap N_D^{+2}(x_2) - \{a_4, a_6\}$, if any. Then each of a_2 and a_4 is contained in at most two of the sets in $\{N_D(x_1), N_D(y_1)^+, N_D^{+2}(x_2) - \{z\}\}$. Consequently, $|N_D(x_1)| + |N_D^+(y_1)| + (|N_D^{+2}(x_2)| - 1) \le |V_2(D) - \{a_2, a_4, a_6\}| + 7 = k + 4$. By Lemma 2.8, we have $|N_D(y_1)| = |N(y_1) - \{x_1\}| \ge n/3$, and the same holds for $|N_D(x_i)|$ for $i = 1, 2$. Thus $3(n/3) - 1 \le k + 4$ and hence $k \ge n - 5$, contradicting the hypothesis. So Claim 3 is established.

Now let D be a C_{2k}^2 in G such that the number of components of $G - D$ is as small as possible. Recall that $4 \le k \le n - 7$. By Claims 2 and 3, $G - D$ contains two isolated vertices $a \in V_1 - V(D)$ and $b \in V_2 - V(D)$. From Lemma 4.3 (with $m = 2$), we see that G contains a C_{2k+2}^2. This contradiction completes the proof of our lemma. ∎

The objective of step 3 is to show that every C_{2k}, with $n - 6 \le k \le n - 1$, can be extended to a C_{2k+2} in G.

LEMMA 5.2. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph with $2n$ vertices and with a C_{2k}. If $B(G) > 3/2$, $n \ge 139$, and $n - 6 \le k \le n - 1$, then G contains a C_{2k+2}.*

Proof. Assume on the contrary that G contains no C_{2k+2}. Let C be a C_{2k} in G such that the number of components of $G - C$ is as small as possible. Let us make some simple observations about $G - C$.

Claim 1. $G - C$ contains no path of length 3.

Suppose the contrary: $x_0 x_1 x_2 x_3$ is a path in $G - C$. By symmetry, we may assume that $x_0 \in V_2$. Since G contains no C_{2k+2}, we deduce that $N_C(x_0)$, $N_C^+(x_1)$, $N_C^{+2}(x_2)$, and $N_C^{+3}(x_3)$ are disjoint subsets of $V_1(C)$. Hence

$$|N_C(x_0)| + |N_C^+(x_1)| + |N_C^{+2}(x_2)| + |N_C^{+3}(x_3)| \le k.$$

By Lemma 2.8, we have $|N_C(x_i)| \ge d(x_i) - (n-k) \ge (n+3)/3 - (n-k) = (3k-2n+3)/3$ for $0 \le i \le 3$. It follows that $4(3k - 2n + 3)/3 \le k$, so $8n \ge 9k + 12 \ge 9(n - 6) + 12$ and hence $n \le 42$, contradicting the hypothesis. Thus Claim 1 is justified.

Claim 2. Each component of $G - C$ contains at most two vertices.

Otherwise, some component H of $G - C$ has at least three vertices. By Claim 1, H contains no path of length 3. Hence at least one of $V_1(H)$ and $V_2(H)$ contains only one vertex. Symmetry allows us to assume that $V_2(H) = \{u\}$. Then all vertices in $V_1(H)$ are adjacent to u. Let v be a vertex in $V_1(H)$ and set $X := V_1(H) - \{v\}$. Since

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G contains no C_{2k+2}, we see that $N_C^+(u)$, $N_C(v)$, and $N_C^{+2}(X)$ are disjoint subsets of $V_2(C)$. So $|N_C^+(u)|+|N_C(v)|+|N_C^{+2}(X)| \le k$ and hence

$$(d(u)-r)+(d(v)-1)+(|N(X)|-1) \le k, \tag{5.6}$$

where $r := |V_1(H)|$. By Lemma 2.8, we have $|N(X)| \ge (n+2|X|+1)/3 = (n+2r-1)/3$ and $\min\{d(u), d(v)\} \ge (n+3)/3$. This together with (5.6) implies $3n \le 3k+r+1 \le n+2k+1$, so $k > n-1$ and hence $k=n$, and this contradiction justifies Claim 2.

Claim 3. $G-C$ contains at most one isolated edge.

Assume on the contrary that x_1y_1 and x_2y_2 are two isolated edges of $G-C$ with $\{x_1,x_2\} \subseteq V_1$ and $\{y_1,y_2\} \subseteq V_2$. Then $k \le n-2$. Since G contains no C_{2k+2}, we have $N_C(x_i) \cap N_C^+(y_i) = \emptyset$ for $i=1,2$. It is easy to see that at least one of $N_C(x_1) \cap N_C^{+2}(x_2)$ and $N_C(y_1) \cap N_C^{+2}(y_2)$ is empty, for otherwise G would contain a C_{2k+2}, a contradiction. Symmetry allows us to assume that $N_C(x_1) \cap N_C^{+2}(x_2) = \emptyset$. Then $k \ge |N_C(x_1) \cup N_C^{+2}(x_2) \cup N_C^+(y_1)| = |N_C(x_1)| + |N_C^{+2}(x_2) \cup N_C^+(y_1)| = |N_C(x_1)| + |N_C^{+2}(x_2)| + |N_C^+(y_1)| - |N_C^{+2}(x_2) \cap N_C^+(y_1)|$. By Lemma 2.8, each of $|N_C(x_1)|$, $|N_C^+(y_1)|$, and $|N_C^{+2}(x_2)|$ is at least $n/3$. Hence $|N_C^{+2}(x_2) \cap N_C^+(y_1)| \ge n-k \ge 2$, which again implies the existence of C_{2k+2} in G. This contradiction establishes Claim 3.

Claim 4. $G-C$ contains no isolated vertex.

Otherwise, by Claim 2, there exist $a \in V_1 - V(C)$ and $b \in V_2 - V(C)$ such that both of them are isolated vertices in $G-C$ (as G is balanced). From Lemma 4.3 (with $m=0$), it follows instantly that G contains a C_{2k+2}, and this contradiction proves Claim 4.

From Claims 1–4, we deduce that $G-C$ contains only two vertices, say, a and b, with $a \in V_1$ and $b \in V_2$, and that $ab \in E$. This in turn implies that C is a longest cycle in G. Thus Lemma 4.1 is applicable to the triple $(C; a, b)$. For each $i \ge 1$ and $v \in \{a,b\}$, let $X_i(v)$ and $Y_i(v)$ be as defined in this lemma. By definition, (4.1) and Lemma 4.1(ii), $Y_i(a) \subseteq V_1(C) - X_1(b)$. Hence $Y_i(a) \cup \{a\}$ is a proper subset of V_1 for all $i \ge 0$. Similarly, $Y_i(b) \cup \{b\}$ is a proper subset of V_2 for all $i \ge 0$. Therefore, for $i \ge 1$ and $v \in \{a,b\}$, Lemma 2.8 applies to $S = Y_{i-1}(v) \cup \{v\}$. As each of a and b has exactly one neighbor outside C, we have

$$|X_i(v)| \ge \frac{n+2(|Y_{i-1}(v)|+1)+1}{3} - 1 = \frac{n+2|Y_{i-1}(v)|}{3}.$$

Thus $3|X_i(v)| \ge n+2|Y_{i-1}(v)|$. By (4.1) and Lemma 4.1(iv), both $X_i(a)^+$ and $X_i(a)^-$ are subsets of $V_1(C) - X_i(b)$. So $|X_i(a)^+ \cap X_i(a)^-| = |X_i(a)^+| + |X_i(a)^-| - |X_i(a)^+ \cup X_i(a)^-| \ge 2|X_i(a)| - (k - |X_i(b)|)$. As $X_i(a)^+ \cap X_i(a)^- \subseteq Y_i(a)$, we have

$$|Y_i(a)| \ge 2|X_i(a)| - (k - |X_i(b)|).$$

Similarly, $|Y_i(b)| \ge 2|X_i(b)| - (k - |X_i(a)|)$. Adding these two inequalities yields

$$|Y_i(a)| + |Y_i(b)| \ge 3|X_i(a)| + 3|X_i(b)| - 2k.$$

Hence

$$\begin{aligned} |Y_i(a)| + |Y_i(b)| &\ge (2n + 2|Y_{i-1}(a)| + 2|Y_{i-1}(b)|) - 2k \\ &= 2\left(|Y_{i-1}(a)| + |Y_{i-1}(b)|\right) + 2(n-k) \\ &\ge 2\left(|Y_{i-1}(a)| + |Y_{i-1}(b)|\right) + 2, \end{aligned}$$

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which implies

$$|Y_i(a)| + |Y_i(b)| + 2 \geq 2\left(|Y_{i-1}(a)| + |Y_{i-1}(b)| + 2\right).$$

Therefore, $|Y_i(a)| + |Y_i(b)| + 2 \geq 2^{i+1} \to \infty$ as $i \to \infty$. This contradiction completes the proof of our lemma. $\square$

Now we are ready to establish the main result of this paper.

Proof of Theorem 1.3. By Lemmas 3.2–3.4, G contains a C_4, a C_6^1, and at least one of C_8^2, C_{10}^2, and C_{12}^2. By Lemma 5.1, if G contains a C_{2k}^2 for any k with $4 \leq k \leq n-7$, then G contains at least one of C_{2k+2}^2, C_{2k+4}^2, and C_{2k+6}^2. Recall that every C_{2t}^2 with $t \geq 4$ contains cycles C_{2t}, C_{2t-2}, and C_{2t-4} simultaneously. From all these observations, we conclude that G contains a cycle C_{2k} for every k with $2 \leq k \leq n-6$. This together with Lemma 5.2 implies that G contains a C_{2k} for every k with $2 \leq k \leq n$. Therefore G is bipancyclic. $\square$

6. Proof of Lemma 4.2. As stated before, Lemma 4.1 aims to deal with a longest cycle under certain restrictions, while Lemma 4.2 is intended for a ladder (not necessarily a longest one) under some other restrictions. Nevertheless, the basic ideas underlying their proofs are essentially similar, and their origin can be traced back to Woodall [8].

The proof of Lemma 4.2 is based on the four claims $A(i,j)$, $B(i,j)$, $B^*(i,j)$, and $C(i)$ for all natural numbers i and j.

Claim $A(i,j)$. There do not exist two disjoint paths $P_{ij} = u_1u_2\ldots u_f$ and $Q_{ij} = u_{f+1}u_{f+2}\ldots u_g$ with the following properties:

(P1) $a_1\overrightarrow{D}a_{2m+2}$ is a subpath of either P_{ij} or Q_{ij} when $m \geq 1$;
(P2) $\{u_1, u_{f+1}\} \subseteq X_i(a)$ and $\{u_f, u_g\} \subseteq X_j(b)$;
(P3) if $u_s \in Y_h(a)$ for some $h < i$ and $s \notin \{f, g\}$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(a)$;
(P4) if $u_s \in Y_h(b)$ for some $h < j$ and $s \notin \{1, f+1\}$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(b)$; and
(P5) either $V(P_{ij}) \cup V(Q_{ij}) = V(D)$ or $V(P_{ij}) \cup V(Q_{ij}) = V(D) - \{a_0, b_0\}$ for some $a_0 \in V_1(D) - Y_{i-1}(a)$ and $b_0 \in V_2(D) - Y_{j-1}(b)$ such that $a_0b_0 \in E(G)$.

Claim $B(i,j)$. There does not exist a path $R_{ij} = u_1u_2\ldots u_f$ with the following properties:

(R1) $a_1\overrightarrow{D}a_{2m+2}$ is a subpath of R_{ij} when $m \geq 1$;
(R2) $\{u_1, u_f\} \subseteq X_i(a)$;
(R3) if $u_s \in Y_h(a)$ for some $h < i$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(a)$;
(R4) if $u_s \in Y_h(b)$ for some $h < j$ and $s \notin \{1, f\}$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(b)$; and
(R5) $V(R_{ij}) = V(D) - \{a_0\}$ for some $a_0 \in X_j(b)$.

Claim $B^*(i,j)$. There does not exist a path $R^*_{ij} = u_1u_2\ldots u_f$ with the following properties:

(r1) $a_1\overrightarrow{D}a_{2m+2}$ is a subpath of R^*_{ij} when $m \geq 1$;
(r2) $\{u_1, u_f\} \subseteq X_j(b)$;
(r3) if $u_s \in Y_h(b)$ for some $h < j$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(b)$;
(r4) if $u_s \in Y_h(a)$ for some $h < i$ and $s \notin \{1, f\}$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(a)$; and
(r5) $V(R^*_{ij}) = V(D) - \{b_0\}$ for some $b_0 \in X_i(a)$.

Claim $C(i)$. For each $v \in \{a, b\}$, there does not exist a path $T_i = u_1u_2\ldots u_f$ with the following properties:

(T1) $a_1\overrightarrow{D}a_{2m+2}$ is a subpath of T_i when $m \geq 1$;
(T2) $\{u_1, u_f\} \subseteq X_i(v)$;
(T3) if $u_s \in Y_h(v)$ for some $h < i$ and $s \notin \{1, f\}$, then $\{u_{s-1}, u_{s+1}\} \subseteq X_h(v)$; and

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(T4) $V(T_i) = V(D) - \{v_0\}$ for some $v_0 \in V(D) - Y_{i-1}(v)$ with $N(v_0) \not\subseteq V(D) \cup \{v\}$.

Observe that if $m \geq 1$, then

$$Y_j(v) \cap D[a_1, a_{2m+2}] = \emptyset \quad \text{for each } v \in \{a, b\} \text{ and } j \geq 1, \tag{6.1}$$

because $X_j(v) \cap D[a_2, a_{2m+1}] = \emptyset$. We shall repeatedly use this simple observation in the subsequent proofs.

Proof of Lemma 4.2. (assuming Claims $A(i,j)$, $B(i,j)$, $B^*(i,j)$, and $C(i)$ for all i and j).

(i) Suppose the contrary: $N(Y_v)$ is not a subset of $X_v \cup D(a_1, a_{2m+2})$ for $v = a$ or b. By definition, we have $N(Y_i(v)) \not\subseteq X_{i+1}(v) \cup D(a_1, a_{2m+2})$ for some $i \geq 1$. So $N(Y_i(v)) \not\subseteq V(D)$ and hence $N(v_0) \notin V(D)$ for some $v_0 \in Y_i(v)$. In view of (6.1), we obtain $v_0 \notin D[a_1, a_{2m+2}]$ when $m \geq 1$. Note that Y_v and v are both contained in V_j for $j = 1$ or 2, so $N(v_0) \not\subseteq V(D) \cup \{v\}$. As $Y_0(v) = \emptyset$, there exists a subscript h with $1 \leq h \leq i$ such that $v_0 \in Y_h(v) - Y_{h-1}(v)$. Thus $\{v_0^-, v_0^+\} \subseteq X_h(v)$, and $v_0 \in V(D) - Y_{h-1}(v)$. Setting $T_h := v_0^+ \overrightarrow{D} v_0^-$, we see that conditions (T1)–(T4) (with h in place of i) are all satisfied by T_h, and hence Claim $C(h)$ is violated. This contradiction implies that $N(Y_v) \subseteq X_v \cup D(a_1, a_{2m+2})$ for $v = a$ and b.

(ii) Suppose $u \in X_j(b) \cap Y_i(a)$ for some subscripts i and j. Then $\{u^-, u^+\} \subseteq X_i(a)$. By (6.1), $u \notin D[a_1, a_{2m+2}]$ if $m \geq 1$. Setting $a_0 := u$ and $R_{ij} := u^+ \overrightarrow{D} u^-$, we see that conditions (R1)–(R5) are all satisfied by R_{ij}, so Claim $B(i,j)$ is violated. Hence $X_j(b) \cap Y_i(a) = \emptyset$ for all subscripts i and j, which implies that $X_b \cap Y_a = \emptyset$. Similarly, from Claim $B^*(i,j)$ we can deduce that $X_i(a) \cap Y_j(b) = \emptyset$ for all subscripts i and j, and hence $X_a \cap Y_b = \emptyset$.

(iii) Suppose u and v are two distinct vertices in $X_a^+ \cap X_b$. Then there exist subscripts i and j such that $\{u^-, v^-\} \subseteq X_i(a)$ and $\{u, v\} \subseteq X_j(b)$. As none of u^-, u, v^-, and v is contained in $D[a_2, a_{2m+1}]$ when $m \geq 1$, either $D[a_1, a_{2m+2}] \subseteq D[u, v^-]$ or $D[a_1, a_{2m+2}] \subseteq D[v, u^-]$. Setting $P_{ij} := u^- \overleftarrow{D} v$ and $Q_{ij} := v^- \overleftarrow{D} u$, we see that conditions (P1)–(P5) are all satisfied by P_{ij} and Q_{ij}, and hence Claim $A(i,j)$ is violated. This contradiction implies that $|X_a^+ \cap X_b| \leq 1$. Similarly, we have $|X_a^- \cap X_b| \leq 1$. So the lemma is established. $\square$

From the preceding proof, we conclude that

(Z1) Claim $A(i,j)$ implies that $|X_i(a)^+ \cap X_j(b)| \leq 1$ and $|X_i(a)^- \cap X_j(b)| \leq 1$;
(Z2) Claim $B(i,j)$ implies that $X_j(b) \cap Y_i(a) = \emptyset$;
(Z3) Claim $B^*(i,j)$ implies that $X_i(a) \cap Y_j(b) = \emptyset$; and
(Z4) Claims $C(h)$, for all h with $1 \leq h \leq i$, imply that $N(Y_i(v)) \subseteq X_{i+1}(v) \cup D(a_1, a_{2m+2})$ for $v \in \{a, b\}$.

We shall appeal to these observations in the following inductive proof of the above claims for all possible subscripts.

Proof of Claims $A(1,1)$, $B(1,1)$, $B^*(1,1)$, and $C(1)$. Suppose such paths P_{11} and Q_{11} exist. Then a is adjacent to u_1 and u_{f+1}, and b is adjacent to u_f and u_g. If $V(P_{11}) \cup V(Q_{11}) = V(D)$, then we can obtain a C_{2k+2}^m from D by adding a and b, a contradiction. If $V(P_{11}) \cup V(Q_{11}) = V(D) - \{a_0, b_0\}$ for some $a_0 \in V_1(D) - Y_0(a)$ and $b_0 \in V_2(D) - Y_0(b)$ such that $a_0 b_0 \in E(G)$. Then we can get another C_{2k}^m, denoted by D', on the vertex set $(V(D) - \{a_0, b_0\}) \cup \{a, b\}$ such that $G - D'$ has at least one component fewer than $G - D$, because both a and b are isolated vertices in $G - D$, while $a_0 b_0 \in E(G)$. This contradiction justifies Claim $A(1,1)$.

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Suppose such a path R_{11} exists. Then a is adjacent to u_1 and u_f, and $V(R_{11}) = V(D) - \{a_0\}$ for some $a_0 \in X_1(b)$ (so $a_0 b \in E(G)$). Hence we can obtain another C_{2k}^m, denoted by D', on the vertex set $(V(D) - \{a_0\}) \cup \{a\}$ such that $G - D'$ has at least one component fewer than $G - D$, because both a and b are isolated vertices in $G - D$, while $a_0 b \in E(G)$. This contradiction justifies Claim $B(1,1)$. Similarly, Claim $B^*(1,1)$ also holds.

Suppose such a path T_1 exists. Then v is adjacent to u_1 and u_f, and $V(T_1) = V(D) - \{v_0\}$ for some $v_0 \in V(D)$ with $N(v_0) \nsubseteq V(D) \cup \{v\}$. Hence we can obtain another C_{2k}^m, denoted by D', on the vertex set $(V(D) - \{v_0\}) \cup \{v\}$ such that $G - D'$ has at least one component fewer than $G - D$, because v is an isolated vertex in $G - D$ while $N(v_0) \nsubseteq V(D) \cup \{v\} = V(D') \cup \{v_0\}$. This contradiction justifies Claim $C(1)$.

Proof of Claims $A(i,j)$, $B(i,j)$, and $B^*(i,j)$ for $i+j > 2$. We proceed by induction on $i + j$. Suppose $i + j > 2$ and $A(i_0, j_0)$, $B(i_0, j_0)$, $B^*(i_0, j_0)$ hold for all subscripts i_0 and j_0 with $i_0 + j_0 < i + j$.

(1) To prove $A(i,j)$, suppose on the contrary that such paths P_{ij} and Q_{ij} exist. By symmetry, we may assume that $i \geq j$ (so $i > 1$). Let us distinguish among three cases.

Case A1. $\{u_1, u_{f+1}\} \subseteq X_{i-1}(a)$.

In this case let $P_{i-1,j} := P_{ij}$ and $Q_{i-1,j} := Q_{ij}$. Then the existence of such two paths contradicts Claim $A(i-1, j)$.

Case A2. Precisely one of u_1 and u_{f+1} is in $X_{i-1}(a)$.

In this case symmetry allows us to assume that $u_1 \in X_{i-1}(a)$, while $u_{f+1} \notin X_{i-1}(a)$. Then u_{f+1} is adjacent to some $y \in Y_{i-1}(a) - Y_{i-2}(a)$. If $y \notin V(P_{ij} \cup Q_{ij})$, then, by (P5), we have $V(P_{ij} \cup Q_{ij}) = V(D) - \{a_0, b_0\}$ and $y \in \{a_0, b_0\}$, where $a_0 \in V_1(D) - Y_{i-1}(a)$ and $b_0 \in V_2(D) - Y_{j-1}(b)$. It follows that $y = a_0 \notin Y_{i-1}(a)$, a contradiction. Hence $y = u_s$ for some s with $1 \leq s \leq g$. By (6.1), $u_s \notin D[a_1, a_{2m+2}]$ when $m \geq 1$. By Claim $B(i-1, j)$ and (Z2), $X_j(b) \cap Y_{i-1}(a) = \emptyset$, so $s \notin \{f, g\}$. As $\{u_1, u_{f+1}\} \subseteq X_i(a)$, we see that $u_1 \in V_2$ and $u_s \in N(u_{f+1}) \subseteq V_1$, and hence $s \notin \{1, f+1\}$. Consequently, either $1 < s < f$ or $f + 1 < s < g$. By (P3), we have $\{u_{s-1}, u_{s+1}\} \subseteq X_{i-1}(a)$. Set

$$P_{i-1,j} := \begin{cases} u_1 \overrightarrow{P_{ij}} u_s u_{f+1} \overrightarrow{Q_{ij}} u_g & \text{if } 1 < s < f, \\ P_{ij} & \text{if } f+1 < s < g, \end{cases}$$

$$Q_{i-1,j} := \begin{cases} u_{s+1} \overrightarrow{P_{ij}} u_f & \text{if } 1 < s < f, \\ u_{s-1} \overleftarrow{Q_{ij}} u_{f+1} u_s \overrightarrow{Q_{ij}} u_g & \text{if } f+1 < s < g. \end{cases}$$

Let us show that (P1)–(P5) (with $i-1$ in place of i) are all satisfied by $P_{i-1,j}$ and $Q_{i-1,j}$. Suppose $1 < s < f$. Then the details of the proof are given below.

- (P1) As $u_s \notin D[a_1, a_{2m+2}]$, it is clear that $a_1 \overrightarrow{D} a_{2m+2}$ remains a subpath of either $P_{i-1,j}$ or $Q_{i-1,j}$ when $m \geq 1$.
- (P2) By assumption, $u_1 \in X_{i-1}(a)$ and $\{u_f, u_g\} \subseteq X_j(b)$. As remarked above, $u_{s-1} \in X_{i-1}(a)$.
- (P3) Since P_{ij} and Q_{ij} satisfy (P3), the only possible vertex on $P_{i-1,j} \cup Q_{i-1,j}$ that can violate (P3) is u_s. However, since $u_s \notin Y_{i-2}(a)$, we have $u_s \notin Y_h(a)$ for all $h < i-1$ by (4.1).
- (P4) Since P_{ij} and Q_{ij} satisfy (P4), the only possible vertex on $P_{i-1,j} \cup Q_{i-1,j}$ that can violate (P4) is u_{f+1}. However, since Claim $B(i-1, j)$ implies $X_j(b) \cap Y_{i-1}(a) = \emptyset$ (recall (Z2)), we have $u_s \notin X_j(b)$. This together with $u_{f+1} u_s \in$

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$E(G)$ and $u_s \in V(D) - D(a_1, a_{2m+2})$ implies $u_{f+1} \notin Y_{j-1}(b)$, and hence $u_{f+1} \notin Y_h(b)$ for all $h < j$ by (4.1).

(P5) This follows from the fact that $V(P_{ij} \cup Q_{ij}) = V(P_{i-1,j} \cup Q_{i-1,j})$ and that P_{ij} and Q_{ij} satisfy (P5). Also, if P_{ij} and Q_{ij} miss $a_0 \in V_1(D) - Y_{i-1}(a)$, then we have $a_0 \in V_1(D) - Y_{i-2}(a)$ as well by (4.1).

The proof goes along the same line when $f+1 < s < g$.

Case A3. $\{u_1, u_{f+1}\} \cap X_{i-1}(a) = \emptyset$.

As in Case A2, we can now deduce that u_1 is adjacent to some $u_r \in Y_{i-1}(a) - Y_{i-2}(a)$, and u_{f+1} is adjacent to some $u_s \in Y_{i-1}(a) - Y_{i-2}(a)$, where $2 \le r, s \le g-1$ and $\{r, s\} \cap \{f, f+1\} = \emptyset$. By (P3), we have $\{u_{r-1}, u_{r+1}, u_{s-1}, u_{s+1}\} \subseteq X_{i-1}(a)$. By (6.1), we obtain $\{u_r, u_s\} \cap D[a_1, a_{2m+2}] = \emptyset$ when $m \ge 1$. Symmetry allows us to assume that $u_r \in P_{ij}$ whenever $u_r = u_s$ and $r < s$ whenever u_r and u_s are two distinct vertices both on P_{ij} or both on Q_{ij}. Thus there are four possibilities for r and s altogether: (i) $1 < r < f < f+1 < s < g$; (ii) $1 < s < f < f+1 < r < g$; (iii) $1 < r \le s < f$; or (iv) $f+1 < r < s < g$. Set

$$P_{i-1,j} := \begin{cases} u_{r-1}\overleftarrow{P_{ij}}u_1u_r\overrightarrow{P_{ij}}u_f & \text{if } 1 < r < f < f+1 < s < g, \\ u_{r-1}\overleftarrow{Q_{ij}}u_{f+1}u_s\overrightarrow{P_{ij}}u_f & \text{if } 1 < s < f < f+1 < r < g, \\ u_{s+1}\overrightarrow{P_{ij}}u_f & \text{if } 1 < r \le s < f, \\ u_{s-1}\overleftarrow{Q_{ij}}u_ru_1\overrightarrow{P_{ij}}u_f & \text{if } f+1 < r < s < g, \end{cases}$$

$$Q_{i-1,j} := \begin{cases} u_{s-1}\overleftarrow{Q_{ij}}u_{f+1}u_s\overrightarrow{Q_{ij}}u_g & \text{if } 1 < r < f < f+1 < s < g, \\ u_{s-1}\overleftarrow{P_{ij}}u_1u_r\overrightarrow{Q_{ij}}u_g & \text{if } 1 < s < f < f+1 < r < g, \\ u_{r-1}\overleftarrow{P_{ij}}u_1u_r\overrightarrow{P_{ij}}u_su_{f+1}\overrightarrow{Q_{ij}}u_g & \text{if } 1 < r \le s < f, \\ u_{r-1}\overleftarrow{Q_{ij}}u_{f+1}u_s\overrightarrow{Q_{ij}}u_g & \text{if } f+1 < r < s < g. \end{cases}$$

Again, it is a routine matter to check that $(P_{i-1,j}, Q_{i-1,j})$ satisfies (P1)–(P5) (with $i-1$ in place of i). This contradiction to Claim $A(i-1, j)$ completes the proof for the present case. Therefore Claim $A(i, j)$ is established.

(2) Let us now justify Claims $B(1, j)$ for $j > 1$. Assume such a path R_{1j} exists with corresponding $a_0 \in X_j(b)$. Then $a_0 \notin X_{j-1}(b)$, for otherwise Claim $B(1, j-1)$ is violated. Hence a_0 is adjacent to some $u_r \in Y_{j-1}(b) - Y_{j-2}(b)$, where $1 \le r \le f$. By Claim $B^*(1, j-1)$ and (Z3), we have $Y_{j-1}(b) \cap X_1(a) = \emptyset$, so $u_r \notin X_1(a)$. This together with $\{u_1, u_f\} \subseteq X_1(a)$ implies $1 < r < f$. By (R4), $\{u_{r-1}, u_{r+1}\} \subseteq X_{j-1}(b)$. By (6.1), we have $u_r \notin D[a_1, a_{2m+2}]$ when $m \ge 1$. Let $P_{1,j-1} := u_1u_2 \dots u_{r-1}$, $Q_{1,j-1} := u_fu_{f-1} \dots u_{r+1}$, and $b_0 := u_r$. Then it is easy to see that (P1)–(P5) (with $(1, j-1)$ in place of (i, j)) are all satisfied by $(P_{1,j-1}, Q_{1,j-1})$. This contradiction to Claim $A(1, j-1)$ establishes Claim $B(1, j)$ for all $j > 1$.

Similarly, we can justify Claim $B^*(1, j)$ for all $j > 1$.

(3) Next, let us justify Claim $B(i, j)$ for $i > 1$. Assume such a path R_{ij} exists with corresponding $a_0 \in X_j(b)$. We consider three cases.

Case B1. $\{u_1, u_f\} \subseteq X_{i-1}(a)$.

In this case set $R_{i-1,j} := R_{ij}$. Then the existence of this path contradicts Claim $B(i-1, j)$.

Case B2. Precisely one of u_1 and u_f is in $X_{i-1}(a)$.

In this case symmetry allows us to assume that $u_1 \in X_{i-1}(a)$ while $u_f \notin X_{i-1}(a)$. Then u_f is adjacent to some $y \in Y_{i-1}(a) - Y_{i-2}(a)$. By Claim $B(i-1, j)$ and (Z2), we have $X_j(b) \cap Y_{i-1}(a) = \emptyset$, so $y \ne a_0$, for otherwise $a_0 \in X_j(b) \cap Y_{i-1}(a)$, a contradiction. In view of (R5), we have $V(D) = V(R_{ij}) \cup \{a_0\}$, so $y \in V(R_{ij})$ and

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hence $y = u_r$ for some r with $1 \le r \le f$. By (6.1), we get $u_r \notin D[a_1, a_{2m+2}]$ when $m \ge 1$. As $\{u_1, u_f\} \subseteq X_i(a) \subseteq V_2$ and $u_r \in N(u_f) \subseteq V_1$, we obtain $1 < r < f$. It follows from (R3) that $u_{r+1} \in X_{i-1}(a)$. Let $R_{i-1,j} := u_1 u_2 \dots u_r u_f u_{f-1} \dots u_{r+1}$. Then (R1)–(R5) (with $i-1$ in place of i) are all satisfied by $R_{i-1,j}$; the details of the proof are given below.

- (R1) Since $u_r \notin D[a_1, a_{2m+2}]$, it is clear that $a_1 \overrightarrow{D} a_{2m+2}$ remains a subpath of $R_{i-1,j}$ when $m \ge 1$.
- (R2) By assumption, $u_1 \in X_{i-1}(a)$. As remarked above, $u_{r+1} \in X_{i-1}(a)$.
- (R3) Since R_{ij} satisfies (R3), the only possible vertex on $R_{i-1,j}$ that can violate (R3) is u_r. However, since $u_r \notin Y_{i-2}(a)$, we have $u_r \notin Y_h(a)$ for all $h < i-1$ by (4.1).
- (R4) Since R_{ij} satisfies (R4), the only possible vertex on $R_{i-1,j}$ that can violate (R4) is u_f. However, by Claim $B(i-1, j)$ and (Z2), we have $X_j(b) \cap Y_{i-1}(a) = \emptyset$, so $u_r \notin X_j(b)$. This together with $u_f u_r \in E(G)$ and $u_r \in V(D) - D(a_1, a_{2m+2})$ implies $u_f \notin Y_{j-1}(b)$, and hence $u_f \notin Y_h(b)$ for all $h < j$ by (4.1).
- (R5) This follows from the fact that $V(R_{ij}) = V(R_{i-1,j})$ and that R_{ij} satisfies (R5).

Therefore the existence of $R_{i-1,j}$ contradicts Claim $B(i-1, j)$.

Case B3. $\{u_1, u_f\} \cap X_{i-1}(a) = \emptyset$.

As in Case B2, we can now deduce that u_1 is adjacent to some u_s, and u_f is adjacent to some u_r, where $\{u_s, u_r\} \subseteq Y_{i-1}(a) - Y_{i-2}(a)$ and $1 < s, r < f$. By (R3), we have $\{u_{r-1}, u_{r+1}, u_{s-1}, u_{s+1}\} \subseteq X_{i-1}(a)$. By (6.1), we obtain $\{u_r, u_s\} \cap D[a_1, a_{2m+2}] = \emptyset$ when $m \ge 1$. Set

$$R_{i-1,j} := \begin{cases} u_{s-1} \overleftarrow{R_{ij}} u_1 u_s \overrightarrow{R_{ij}} u_r u_f \overleftarrow{R_{ij}} u_{r+1} & \text{if } r \ge s, \\ u_{r-1} \overleftarrow{R_{ij}} u_1 u_s \overrightarrow{R_{ij}} u_f u_r \overrightarrow{R_{ij}} u_{s-1} & \text{if } r < s. \end{cases}$$

It is then a routine matter to check that (R1)–(R5) (with $i-1$ in place of i) are all satisfied by $R_{i-1,j}$. Thus the existence of $R_{i-1,j}$ contradicts Claim $B(i-1, j)$.

Similarly, we can justify Claim $B^*(i, j)$ for all $i > 1$.

Proof of Claim $C(i)$ for $i > 1$. We proceed by induction on i. Suppose $i > 1$ and $C(i_0)$ holds for all i_0 with $1 \le i_0 < i$. To prove $C(i)$, assume on the contrary that such a path T_i exists with corresponding $v_0 \notin Y_{i-1}(v)$ such that $N(v_0) \nsubseteq V(D) \cup \{v\}$, where $v \in \{a, b\}$. We consider three cases.

Case C1. $\{u_1, u_f\} \subseteq X_{i-1}(v)$.

In this case set $T_{i-1} := T_i$. Then the existence of this path contradicts Claim $C(i-1)$.

Case C2. Precisely one of u_1 and u_f is in $X_{i-1}(v)$.

In this case symmetry allows us to assume that $u_1 \in X_{i-1}(v)$ while $u_f \notin X_{i-1}(v)$. Then u_f is adjacent to some $y \in Y_{i-1}(v) - Y_{i-2}(v)$. As $v_0 \notin Y_{i-1}(v)$, we have $y \ne v_0$. Using (T4), we see that $y \in V(T_i)$, so $y = u_r$ for some r with $1 \le r \le f$. In view of (6.1), we obtain $u_r \notin D[a_1, a_{2m+2}]$ when $m \ge 1$. Since both u_1 and u_f are in V_i for $i = 1$ or 2, we deduce that $1 < r < f$. Using (T3), we get $\{u_{r-1}, u_{r+1}\} \subseteq X_{i-1}(v)$. Let $T_{i-1} := u_1 u_2 \dots u_r u_f u_{f-1} \dots u_{r+1}$. We can now show that (T1)–(T4) (with $i-1$ in place of i) are all satisfied by T_{i-1}; the details of the proof are given below.

- (T1) Since $u_r \notin D[a_1, a_{2m+2}]$, it is clear that $a_1 \overrightarrow{D} a_{2m+2}$ must remain a subpath of T_{i-1} when $m \ge 1$.
- (T2) By assumption, $u_1 \in X_{i-1}(v)$. As noted above, $u_{r+1} \in X_{i-1}(v)$.

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(T3) Since T_i satisfies (T3), the only possible vertex on T_{i-1} that can violate (T3) is u_r. However, since $u_r \notin Y_{i-2}(v)$, we have $u_r \notin Y_h(v)$ for all $h < i-1$ by (4.1).

(T4) This follows from the fact that $V(T_i) = V(T_{i-1})$ and that T_i satisfies (T4). As $v_0 \notin Y_{i-1}(v)$, we have $v_0 \notin Y_{i-2}(v)$ as well by (4.1).

Hence the existence of T_{i-1} contradicts Claim $C(i-1)$.

Case C3. $\{u_1, u_f\} \cap X_{i-1}(v) = \emptyset$.

As in Case C2, we can now deduce that u_1 is adjacent to some u_s, and u_f is adjacent to some u_r, where $\{u_s, u_r\} \subseteq Y_{i-1}(v) - Y_{i-2}(v)$ and $1 < s, r < f$. By (T3), we have $\{u_{r-1}, u_{r+1}, u_{s-1}, u_{s+1}\} \subseteq X_{i-1}(a)$. By (6.1), we obtain $\{u_r, u_s\} \cap D[a_1, a_{2m+2}] = \emptyset$ when $m \geq 1$. Set

$$T_{i-1} := \begin{cases} u_{s-1}\overleftarrow{T_i}u_1u_s\overrightarrow{T_i}u_ru_f\overleftarrow{T_i}u_{r+1} & \text{if } r \geq s, \\ u_{r-1}\overleftarrow{T_i}u_1u_s\overrightarrow{T_i}u_fu_r\overrightarrow{T_i}u_{s-1} & \text{if } r < s. \end{cases}$$

It is then a routine matter to check that (T1)–(T4) (with $i-1$ in place of i) are all satisfied by T_{i-1}. Thus the existence of T_{i-1} contradicts Claim $C(i-1)$.

This completes the proof of all the claims. □

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