Model Order Reduction for Neutral Systems by Moment Matching

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Abstract Circuits with delay elements are very popular and important in the simulation of very-large-scale integration (VLSI) systems. Neutral systems (NSs) with multiple constant delays (MCDs), for example, can be used to model the partial element equivalent circuits (PEECs), which are widely used in high-frequency electromagnetic (EM) analysis. In this paper, the model order reduction (MOR) problem for the NS with MCDs is addressed by moment matching method. The nonlinear exponential terms coming from the delayed states and the derivative of the delayed states in the transfer function of the original NS are first approximated by a Padé approximation or a Taylor series expansion. This has the consequence that the transfer function of the original NS is exponential-free and the standard moment matching method for reduction is readily applied. The Padé approximation of exponential terms gives an expanded delay-free system, which is further reduced to a delay-free reduced-order model (ROM). A Taylor series expansion of exponential terms lets the inverse in the original transfer function have only powers-of-s terms, whose coefficient matrices are of the same size as the original NS, which results in a ROM modeled by a lower-order NS. Numerical examples are included to show the effectiveness of the proposed al-

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gorithms and the comparison with existing MOR methods, such as the linear matrix inequality (LMI)-based method.

Keywords Reduction · Moment · Time delay · Neutral system · Descriptor system

1 Introduction

To describe the behavior of complex physical systems accurately, high or even infinite order mathematical models are often required. However, direct simulation of the original high or even infinite order models is very difficult and sometimes prohibitive due to unmanageable levels of storage, high computational cost and long computation time. Therefore, model order reduction (MOR) , which replaces the original complex and high-order system by a reduced-order model (ROM), plays an important role in many areas of engineering, e.g., transmission lines in circuit packaging [31, 38], PCB (printed circuit board) design [11, 48, 58] and networked control systems [18, 28]. The obvious advantages of MOR include that the use of ROMs results in not only considerable savings in storage and computational time, but also fast simulation and verification leading to shortened design cycle [2, 3, 5, 12, 15, 26, 33, 34, 36, 52, 59].

A lot of MOR methods have been presented in the past few decades [4, 14, 16, 17, 20–22, 25, 30, 35, 40, 44, 49, 51]. Most of them fall into two categories. The first one are singular value decomposition (SVD)-based methods via constructing or optimizing the ROM according to a suitably chosen criterion, such as the H_∞ norm, energy-to-peak gain and Hankel norm [40, 44, 49, 51]. The second category are the moment matching-based methods. For linear time invariant (LTI) systems, the moment matching method [4, 5, 16, 20, 35] is to expand the transfer function by Taylor series, and then create a ROM for which the first few terms (also called moments) of its Taylor series expansion match those of the original model. The projection matrix to derive the ROM is usually obtained from Krylov subspace iterative schemes. Over the past years, moment matching methods are widely used due to the availability of efficient iterative schemes for constructing the projection matrix, in contrast to the SVD approach which usually involves solving expensive matrix equations or convex optimization problems.

In many physical, industrial and circuit systems, time delays occur due to the finite capability of information processing, data transmission among various parts of the systems and some essential simplification of the corresponding process models [9, 27, 29, 37, 39, 49, 55–57]. The delaying effect is often detrimental to the performance, and even renders instability. So, the presence of time delays substantially complicates analytical and theoretical aspects of system design. In the past few decades, researchers have paid great attention to the analysis and synthesis of time delay systems (TDSs). Most studies are involved with systems having delays in the system states only, which are often called retarded systems (RSs) [42, 46]. Another important and more general TDSs, called neutral systems (NSs), have dynamics governed by delays not only in the system states, but also in the derivative of system states. For example, in the context of circuit modeling and simulation, NSs can be

formulated for the partial element equivalent circuits (PEECs) widely used in electromagnetic (EM) simulation [24]. The NS has attracted a lot of research effort in recent years [23, 32, 39, 41, 45, 47, 53].

The MOR of TDSs is mainly based on the SVD-based method, which constructs the ROM such that the error norm between the original system and the ROM is less than some given tolerance. The H_∞ MOR problem for RSs is studied in [54] by solving linear matrix inequalities (LMIs) with a rank constraint. The problem in [54] for linear parameter-varying systems with both discrete and distributed delays is considered in [50] by solving parameterized LMIs. The MOR of a NS with multiple constant delays (MCDs), however, has received little attention despite its importance in theory and practice [43, 45]. The energy-to-peak MOR and H_∞ MOR for a NS are studied in [43] and [45], respectively, in terms of LMIs with inverse constraints. LMIs can be solved by interior-point method (IPM) together with Newton's method via minimizing a strictly convex function whereby all matrix variables are transformed into a high-order vector variable in [8]. In practice, the IPM fails to solve large scale LMIs as the storage of the Hessian matrix of the objective convex function used in Newton's method is memory-demanding and the computational cost for the Hessian matrix is also very high. Although cone complementarity linearization (CCL) algorithm [13] provides a way to transform the LMIs with inverse constraints to a minimization problem subject to original LMIs and additional LMIs coming from the inverse constraints, much higher computational cost is needed for expanded LMIs. Hence, though the methods in [43, 45] are theoretically correct, they are of little practical use in reducing high-order NSs due to the prohibitive computational cost.

As the SVD-based methods [43, 45] suffer from high computational cost, it is desirable to use moment matching method to approximate the NSs owing to its much faster computation. However, the major difficulty of applying moment matching method on a NS is the generation of moments from the NS transfer function, which are also the coefficient matrices of its Taylor series expansion. The reason is the appearance of nonlinear exponential terms in the transfer function from the delayed state and the derivative of the delayed state, making direct Taylor series expansion infeasible. In this paper, we propose two methods to approximate the nonlinear exponential terms and to generate their moments.

The major contribution of this paper is the reduction of the NS with MCDs by first approximating the nonlinear exponential terms via Padé approximation or Taylor series expansion of the exponential terms. The former results in an expanded-size, but exponential-free, state space which can then be reduced by standard moment matching method. Whereas the latter effectively replaces the exponential terms by truncated Taylor series, this allows the inverse in the transfer function computation to be again exponential-free, but contains only powers-of-s terms whose coefficient matrices are of the same size as those of the original NS. Subsequently, standard moment matching techniques for reduced-order modeling can be readily used. The proposed two methods with low computation cost make it applicable to the reduction of high-order NS.

The outline of this paper is as follows. In Sect. 1, the MOR problem for descriptor system (DS) by moment matching method is reviewed and the challenge of the MOR problem for the NS with MCDs is given. In Sect. 2, Padé approximation of

exponential terms results in a delay-free ROM modeled by DS. In Sect. 3, the ROM with the same structure of original NS is given by replacing exponential terms via their Taylor series expansions. Numerical examples to demonstrate the effectiveness of the proposed MOR results and the comparison with other methods are given in Sect. 4. Finally, Sect. 5 draws the conclusion.

1.1 Neutral Systems

Consider a NS with MCDs, denoted by Σ,

$$\Sigma: \quad E\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} A_{h_i} x(t - h_i) + \sum_{j=1}^{q} A_{d_j} \dot{x}(t - d_j) + Bu(t), \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

$$x(t) = \phi(t), \quad t \in [-\alpha, 0), \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^l$ is the output. E, A, A_{h_i}, A_{d_j}, B and C, $i = 1, \ldots, p$, $j = 1, \ldots, q$, are properly dimensioned real constant matrices. Here h_i and d_j, $i = 1, \ldots, p$, $j = 1, \ldots, q$, are the constant delays and $\alpha = \max\{h_i, d_j, i = 1, \ldots, p, j = 1, \ldots, q\}$. All derivations in this paper can straightforwardly be extended to time varying delays case by assuming h_i and d_j as the upper bounds of the time varying delays. The order of the NS Σ is defined as the number of states, i.e., n. Under the assumption $x(0) = \phi(0) = 0$, the transfer function from input $u(t)$ to state $x(t)$ is given by

$$G_X(s) = \left(sE - A - \sum_{i=1}^{p} A_{h_i} e^{-sh_i} - \sum_{j=1}^{q} A_{d_j} s e^{-sd_j}\right)^{-1} B, \quad (4)$$

by taking Laplace transform on the left and right sides of (1). The NS Σ is also characterized by its transfer function from input $u(t)$ to output $y(t)$

$$G(s) = C\left(sE - A - \sum_{i=1}^{p} A_{h_i} e^{-sh_i} - \sum_{j=1}^{q} A_{d_j} s e^{-sd_j}\right)^{-1} B. \quad (5)$$

1.2 MOR of Systems Without Delay

When the NS Σ does not have time delays h_i and d_j, i.e., $A_{h_i} = 0$ and $A_{d_j} = 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, it reduces to a DS Σ_{ds},

$$\Sigma_{\text{ds}}: \quad E\dot{x}(t) = Ax(t) + Bu(t), \qquad \qquad (6)$$
$$y(t) = Cx(t),$$

with transfer function

$$G_{\text{ds}}(s) = C(sE - A)^{-1} B.$$

The above DS becomes to an LTI system when $E = I$. The Taylor series expansion of $G_{ds}(s)$ around $s = 0$ is

$$G_{ds}(s) = -CB - C(A^{-1}E)A^{-1}Bs - C(A^{-1}E)^2 A^{-1}Bs^2 - \cdots, \quad (7)$$

by assuming that A is invertible. Coefficient matrices of its Taylor series expansion in (7) are called block moments or moments [35] of the DS Σ_{ds}.

The MOR by moment matching method for DS Σ_{ds} is to create a ROM for which the first few moments match those from the original model [4, 14, 16, 17, 20, 21, 35, 44, 49, 52]. The projection matrix $V \in \mathbb{R}^{n \times \hat{n}}$ to generate the ROM is from

$$\mathrm{colspan}(V) \supseteq K(0, \Sigma_{ds}, \hat{n}), \quad V^T V = I,$$

where $K(0, \Sigma_{ds}, \hat{n})$ is defined as the $\hat{n}$th Krylov subspace

$$K(0, \Sigma_{ds}, \hat{n})$$
$$= \mathrm{colspan}(A^{-1}B, (A^{-1}E)A^{-1}B, (A^{-1}E)^2 A^{-1}B, \ldots, (A^{-1}E)^{\hat{n}-1} A^{-1}B) \quad (8)$$

and the system matrices of the resulting ROM are

$$\hat{E} = V^T E V, \qquad \hat{A} = V^T A V, \qquad \hat{B} = V^T B, \qquad \hat{C} = CV \quad (9)$$

[7, 35]. Therefore, the key point in MOR by moment matching method is the generation of moments or coefficient matrices of the Taylor series expansion of the transfer function $G_{ds}(s)$.

1.3 MOR of NSs

For the MOR problem of the NS Σ, we also want to find a projection matrix V for constructing the ROM to match the first few moments of transfer function $G(s)$ in (5). However, when time delays are taken into account, $G(s)$ becomes much more complicated than $G_{ds}(s)$ due to exponential terms e^{-sh_i} and e^{-sd_j}, $i = 1, \ldots, p$, $j = 1, \ldots, q$, from the delayed states and the derivative of delayed states, respectively. As direct Taylor series expansion of $G(s)$ is impossible due to the appearance of nonlinear terms e^{-sh_i} and e^{-sd_j}, approximation of these e^{-sh_i} and e^{-sd_j} gives an exponential-free approximation of the Taylor series expansion of $G(s)$

$$G(s) \approx G_0 + G_1 s + G_2 s^2 + \cdots + G_n s^n + \cdots, \quad (10)$$

where G_i, $i = 0, 1, \ldots$, are constant matrices and called approximated moments of the NS Σ. Two kinds of approximation of exponential terms are used in this paper. One is the Padé approximation, which is the most frequently used method to approximate them by finite rational functions. The other is to expand exponential terms by their Taylor series expansions. The former gives rise to a ROM modeled by a DS in Sect. 2 and the latter results in a ROM modeled by a NS in Sect. 3.

2 ROM by Padé Approximation

The following lemma shows that the exponential term e^{-sh_i} is approximated by a transfer function of an LTI system in terms of Padé approximation. The most important advantage is that G_i in (10) are expressed by moments of an expanded DS. Then projection matrix proposed in [2, 7] is ready for the construction of the ROM.

Lemma 1 e^{-sh_i} *is approximated by the* β_{h_i} *th order transfer function of LTI system*

$$e^{-sh_i} \approx \bar{C}_{h_i}(sI - \bar{A}_{h_i})^{-1}\bar{B}_{h_i} + \bar{D}_{h_i}, \tag{11}$$

where

$$\bar{C}_{h_i} = \begin{cases} \left[b_0/b_{\beta_{h_i}} - a_0/a_{\beta_{h_i}} \quad b_1/b_{\beta_{h_i}} - a_1/a_{\beta_{h_i}} \quad \cdots \quad b_{\beta_{h_i}-1}/b_{\beta_{h_i}} - a_k/b_{\beta_{h_i}} \right], \\ \quad \alpha_{h_i} = \beta_{h_i}, \\ \left[b_0/a_{\beta_{h_i}} \quad b_1/a_{\beta_{h_i}} \quad \cdots \quad b_{\beta_{h_i}-1}/b_{\beta_{h_i}} \right], \quad \alpha_{h_i} \leq \beta_{h_i} - 1, \end{cases} \tag{12}$$

$$\bar{B}_{h_i} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T, \tag{13}$$

$$\bar{A}_{h_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0/a_{\beta_{h_i}} & -a_1/a_{\beta_{h_i}} & -a_2/a_{\beta_{h_i}} & \cdots & -a_{\beta_{h_i}-1}/a_{\beta_{h_i}} \end{bmatrix}, \tag{14}$$

$$\bar{D}_{h_i} = \begin{cases} \frac{b_{\beta_{h_i}}}{a_{\beta_{h_i}}}, & \alpha_{h_i} = \beta_{h_i}, \\ 0, & \alpha_{h_i} < \beta_{h_i}, \end{cases}$$

$$a_k = \frac{(\alpha_{h_i} + \beta_{h_i} - k)!\alpha_{h_i}!}{(\alpha_{h_i} + \beta_{h_i})!k!(\alpha_{h_i} - k)!}h_i^k, \tag{15}$$

$$b_k = \frac{(\alpha_{h_i} + \beta_{h_i} - k)!\alpha_{h_i}!}{(\alpha_{h_i} + \beta_{h_i})!k!(\alpha_{h_i} - k)!}(-h_i)^k, \tag{16}$$

and α_{h_i} *and* β_{h_i} *with* $\alpha_{h_i} \leq \beta_{h_i}$ *are positive integers.*

Proof From [19, p. 557], e^{-sh_i} can be approximated by β_{h_i}th order Padé approximation,

$$e^{-sh_i} \approx \frac{\sum_{k=0}^{\alpha_{h_i}} b_k s^k}{\sum_{k=0}^{\beta_{h_i}} a_k s^k},$$

where a_k and b_k are defined in (15) and (16). Firstly, we assume that $\alpha_{h_i} = \beta_{h_i}$. It is easy to get

$$e^{-sh_i} \approx \frac{\sum_{k=0}^{\beta_{h_i}} b_k s^k}{\sum_{k=0}^{\beta_{h_i}} a_k s^k} = \frac{b_{\beta_{h_i}}}{a_{\beta_{h_i}}} \frac{\sum_{k=0}^{\beta_{h_i}}(b_k/b_{\beta_{h_i}})s^k}{\sum_{k=0}^{\beta_{h_i}}(a_k/a_{\beta_{h_i}})s^k}$$

$$= \frac{b_{\beta_{h_i}}}{a_{\beta_{h_i}}} + \frac{\sum_{k=0}^{\beta_{h_i}-1} (b_k/b_{\beta_{h_i}} - a_k/a_{\beta_{h_i}})s^k}{\sum_{k=0}^{\beta_{h_i}} (a_k/a_{\beta_{h_i}})s^k}. \tag{17}$$

It follows by [1, Theorem 3.5.1] that the controllable canonical realization of the second term in (17) is equivalent to

$$\bar{C}_{h_i}(sI - \bar{A}_{h_i})^{-1}\bar{B}_{h_i},$$

which gives (11), with $\bar{C}_{h_i}$, $\bar{A}_{h_i}$ and $\bar{B}_{h_i}$ given in (12)–(14). In the case of $\alpha_{h_i} = \beta_{h_i} - 1$, i.e., $b_{\beta_{h_i}} = 0$, (17) becomes a β_{h_i}th order transfer function

$$e^{-sh_i} \approx \frac{\sum_{k=0}^{\beta_{h_i}-1} b_k s^k}{\sum_{k=0}^{\beta_{h_i}} a_k s^k} = \frac{\sum_{k=0}^{\beta_{h_i}-1} (b_k/a_{\beta_{h_i}})s^k}{s^{\beta_{h_i}} + \sum_{k=0}^{\beta_{h_i}-1} (a_k/a_{\beta_{h_i}})s^k}, \tag{18}$$

which results in (11) for the case $\alpha_{h_i} = \beta_{h_i} - 1$ by the controllable canonical realization again. The case $\alpha_{h_i} < \beta_{h_i} - 1$ can be obtained similar to the case $\alpha_{h_i} = \beta_{h_i} - 1$ by assuming $b_{\alpha_{h_i}+1} = \cdots = b_{\beta_{h_i}-1} = 0$. The conclusion holds. ∎

A proposition is followed from Lemma 1 related to the approximation of $G(s)$.

Proposition 1 *$G(s)$ is approximated by*

$$G(s) \approx \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$$
$$= -\mathcal{C}\mathcal{A}^{-1}\mathcal{B} - \mathcal{C}(\mathcal{A}^{-1}\mathcal{E})(\mathcal{A}^{-1}\mathcal{B})s - \mathcal{C}(\mathcal{A}^{-1}\mathcal{E})^2(\mathcal{A}^{-1}\mathcal{B})s^2 - \cdots, \tag{19}$$

where

$$\mathcal{E} = \text{diag}[E, I, \ldots, I, I, \ldots, I] \in \mathbb{R}^{\tilde{n} \times \tilde{n}}, \tag{20}$$

$$\mathcal{A} = \begin{bmatrix} A + \sum_{i=1}^{p} A_{h_i}\tilde{D}_{h_i} + \sum_{j=1}^{q} A_{d_j}\tilde{D}_{d_j} & A_{h_1}\tilde{C}_{h_1} & \cdots & A_{h_p}\tilde{C}_{h_p} & A_{d_1}\tilde{C}_{d_1} & \cdots & A_{d_q}\tilde{C}_{dq} \\ \tilde{B}_{h_1} & \tilde{A}_{h_1} & & & & & \\ \vdots & & \ddots & & & & \\ \tilde{B}_{h_p} & & & \tilde{A}_{h_p} & & & \\ \tilde{B}_{d_1} & & & & \tilde{A}_{d_1} & & \\ \vdots & & & & & \ddots & \\ \tilde{B}_{d_q} & & & & & & \tilde{A}_{d_q} \end{bmatrix}$$

$$\in \mathbb{R}^{\tilde{n} \times \tilde{n}}, \tag{21}$$

$$\mathcal{B} = [B^T \ 0 \ \cdots \ 0 \ 0 \ \cdots \ 0]^T \in \mathbb{R}^{\tilde{n} \times m}, \quad \mathcal{C} = [C \ 0 \ \cdots \ 0 \ 0 \ \cdots \ 0] \in \mathbb{R}^{p \times \tilde{n}}, \tag{22}$$

$$\tilde{C}_{h_i} = \text{diag}[\bar{C}_{h_i}, \ldots, \bar{C}_{h_i}] \in \mathbb{R}^{n \times nr_{h_i}}, \quad \tilde{A}_{h_i} = \text{diag}[\bar{A}_{h_i}, \ldots, \bar{A}_{h_i}] \in \mathbb{R}^{nr_{h_i} \times nr_{h_i}}, \tag{23}$$

$$\tilde{B}_{h_i} = \text{diag}[\bar{B}_{h_i}, \ldots, \bar{B}_{h_i}] \in \mathbb{R}^{nr_{h_i} \times n}, \quad \tilde{D}_{h_i} = \text{diag}[\bar{D}_{h_i}, \ldots, \bar{D}_{h_i}] \in \mathbb{R}^{n \times n},$$

$$i = 1, \ldots, p, \tag{24}$$

$$\tilde{C}_{d_j} = \text{diag}[\bar{C}_{d_j}, \ldots, \bar{C}_{d_j}] \in \mathbb{R}^{n \times nr_{d_j}}, \quad \tilde{A}_{d_j} = \text{diag}[\bar{A}_{d_j}, \ldots, \bar{A}_{d_j}] \in \mathbb{R}^{nr_{d_j} \times nr_{d_j}}, \quad (25)$$

$$\tilde{B}_{d_j} = \text{diag}[\bar{B}_{d_j}, \ldots, \bar{B}_{d_j}] \in \mathbb{R}^{nr_{d_j} \times n}, \quad \tilde{D}_{d_j} = \text{diag}[\bar{D}_{d_j}, \ldots, \bar{D}_{d_j}] \in \mathbb{R}^{n \times n},$$

$$j = 1, \ldots, q, \tag{26}$$

$$\tilde{n} = n \left(1 + \sum_{i=1}^{p} \beta_{h_i} + \sum_{j=1}^{q} \beta_{d_j} \right), \tag{27}$$

and $\bar{C}_{h_i}$, $\bar{A}_{h_i}$, $\bar{B}_{h_i}$, $\bar{D}_{h_i}$, r_{h_i}, $i = 1, \ldots, p$, and $\bar{C}_{d_j}$, $\bar{A}_{d_j}$, $\bar{B}_{d_j}$, $\bar{D}_{d_j}$ and r_{d_j}, $j = 1, \ldots, q$, are obtained from Lemma 1.

Proof From Lemma 1, e^{-sh_i} is approximated by the β_{h_i}th order transfer function

$$\tilde{C}_{h_i}(sI - \tilde{A}_{h_i})^{-1}\tilde{B}_{h_i} + \tilde{D}_{h_i}$$

which follows that

$$\mathcal{L}\big(x(t - h_i)\big) = e^{-sh_i} X(s) \approx \big(\tilde{C}_{h_i}(sI - \tilde{A}_{h_i})^{-1}\tilde{B}_{h_i} + \tilde{D}_{h_i}\big)X(s),$$

where $X(s)$ is the Laplace transform of $x(t)$ and $\tilde{C}_{h_i}$, $\tilde{A}_{h_i}$, $\tilde{B}_{h_i}$ and $\tilde{D}_{h_i}$, $i = 1, \ldots, p$, are defined in (23) and (24). So, $x(t - h_i)$, $i = 1, \ldots, p$, can be treated as the output of a β_{h_i}th order LTI system Σ_{h_i},

$$\Sigma_{h_i}: \quad \dot{x}_{h_i}(t) = \tilde{A}_{h_i} x_{h_i}(t) + \tilde{B}_{h_i} x(t),$$

$$x(t - h_i) = w_{h_i}(t) = \tilde{C}_{h_i} x_{h_i}(t) + \tilde{D}_{h_i} x(t).$$

Similarly, it is easy to get

$$\mathcal{L}\big(\dot{x}(t - d_j)\big) = se^{-sd_j} X(s) \approx \big(\tilde{C}_{d_j}(sI - \tilde{A}_{d_j})^{-1}\tilde{B}_{d_j} + \tilde{D}_{d_j}\big)X(s),$$

which can be treated as the output of a β_{d_j}th order LTI system Σ_{d_j},

$$\Sigma_{d_j}: \quad \dot{x}_{d_j}(t) = \tilde{A}_{d_j} x_{d_j}(t) + \tilde{B}_{d_j} x(t),$$

$$\dot{x}(t - d_j) = r_{d_j}(t) = \tilde{C}_{d_j} x_{d_j}(t) + \tilde{D}_{d_j} x(t),$$

where $\tilde{C}_{d_j}$, $\tilde{A}_{d_j}$, $\tilde{B}_{d_j}$ and $\tilde{D}_{d_j}$, $j = 1, \ldots, q$, are given in (25) and (26). Together with NS Σ, we have

$$\begin{aligned}
\Sigma: \quad E\dot{x}(t) &= Ax(t) + \sum_{i=1}^{p} A_{h_i} w_{h_i}(t) + \sum_{j=1}^{q} A_{d_j} r_{d_j}(t) + Bu(t), \\
\dot{x}_{h_i}(t) &= \tilde{A}_{h_i} x_{h_i}(t) + \tilde{B}_{h_i} x(t), \\
\dot{x}_{d_j}(t) &= \tilde{A}_{d_j} x_{d_j}(t) + \tilde{B}_{d_j} x(t), \\
w_{h_i}(t) &= \tilde{C}_{h_i} x_{h_i}(t) + \tilde{D}_{h_i} x(t), \\
r_{d_j}(t) &= \tilde{C}_{d_j} x_{d_j}(t) + \tilde{D}_{d_j} x(t), \quad i = 1, \ldots, p, j = 1, \ldots, q, \\
y(t) &= Cx(t).
\end{aligned} \tag{28}$$

By defining $x_s(t) = [x^T(t) \; x_{h_1}^T(t) \; \cdots \; x_{h_p}^T(t) \; x_{d_1}^T(t) \; \cdots \; x_{d_q}^T(t)]^T$, the system in (28) can be rewritten as an expanded DS Σ_s,

$$\Sigma_s: \quad \begin{aligned} \mathcal{E}\dot{x}_s(t) &= \mathcal{A}x_s(t) + \mathcal{B}u(t), \\ y_s(t) &= \mathcal{C}x_s(t), \end{aligned} \tag{29}$$

with $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ being defined in (20)–(22). Then we can say that the NS Σ can be approximated by a DS Σ_s in (29), i.e., $G(s) \approx \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$, which further gives (19). □

By matching approximated moments in (19), we get the ROM immediately.

Theorem 1 *The $\hat{n}$th reduced-order DS to approximate the NS Σ, is given by*

$$\hat{\Sigma}_s: \quad \begin{aligned} V^T\mathcal{E}V\dot{\hat{x}}_s(t) &= V^T\mathcal{A}V\hat{x}_s(t) + V^T\mathcal{B}u(t), \\ \hat{y}_s(t) &= \mathcal{C}V\hat{x}_s(t), \end{aligned}$$

where V is obtained by

$$\text{colspan}(V) \supseteq \text{Kr}\big((\mathcal{A}^{-1}\mathcal{E}), \mathcal{A}^{-1}\mathcal{B}, \hat{n}\big), \quad V^T V = I. \tag{30}$$

Proof The projection matrix V is defined in (30) by the same method in [2]. From (30) and by taking the same steps in [35], we have

$$\begin{aligned} \hat{\mathcal{M}}_i &= -\mathcal{C}V\big((V^T\mathcal{A}V)^{-1}V^T\mathcal{E}V\big)^i\big((V^T\mathcal{A}V)^{-1}V^T\mathcal{B}\big) \\ &= -\mathcal{C}\big(\mathcal{A}^{-1}\mathcal{E}\big)^i\big(\mathcal{A}^{-1}\mathcal{B}\big), \quad i = 0, \dots, \hat{n} - 1. \end{aligned}$$

Thus, the DS $\hat{\Sigma}_s$ matches the first $\hat{n}$ moments of the DS Σ_s, which gives an approximation to the NS Σ. □

Remark 1 By Padé approximation of e^{-sh_i} and e^{-sd_j}, the MOR problem of the NS Σ is transformed to the MOR problem of the DS Σ_s without delay. The main advantage is that the standard state space techniques can be applied to the ROM directly in the DS Σ_s for the analysis and simulation. Furthermore, practical application of a reduced-order DS is more convenient than ROM with delay as traditional software for simulation is all for delay-free systems. However, the order of the DS Σ_s, $\tilde{n} = n(1 + \sum_{i=1}^{p}\beta_{h_i} + \sum_{j=1}^{q}\beta_{d_j})$ in (27) is determined by the order of Padé approximation of e^{-sh_i} and e^{-sd_j}, and is higher than n, especially when p and q are large. So the result in Theorem 1 is more suitable for the NS with small p and q, or the case that the ROM should be delay-free system.

Remark 2 When $A_{d_j} = 0$, $j = 1, \dots, q$, the NS Σ becomes a RS Σ_{rs},

$$\Sigma_{rs}: \quad \begin{aligned} E\dot{x}(t) &= Ax(t) + \sum_{i=1}^{p} A_{h_i}x(t - h_i) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{31}$$

and the result in Theorem 1 is also applicable to the RS Σ_{rs} by deleting columns, rows and elements related to A_{d_j} from (20)–(22). Theorem 1 is also true when $A_{h_i} = A_{d_j} = 0$, which is the MOR result by moment matching for the DS in [35].

3 ROM by Taylor Series Expansion

3.1 Moment Matching Around $s_0 = 0$

The inverse formula shown below is needed for later development.

Lemma 2 (See [12], p. 679) *The inverse of $X_0 + sX_1 + s^2X_2 + \cdots + s^{r-1}X_{r-1}$ is given by*

$$
\begin{aligned}
\left(X_0 + sX_1 + s^2X_2 + \cdots + s^{r-1}X_{r-1}\right)^{-1} \\
= \left(I + X_0^{-1}\left(sX_1 + s^2X_2 + \cdots + s^{r-1}X_{r-1}\right)\right)^{-1}X_0^{-1} \\
= \left(\sum_{k=0}^{\infty} M_k s^k\right)X_0^{-1},
\end{aligned}
$$

where

$$
M_k = -\sum_{j=0}^{k-1} X_0^{-1}X_{k-j}M_j, \quad M_0 = I.
$$

In order to approximate $G_X(s)$ around $s_0 = 0$,

$$
\Gamma_0 = -A - \sum_{i=1}^{p} A_{h_i}, \tag{32}
$$

is assumed to be invertible. When the NS Σ reduces to the DS Σ_{ds} in (6), the nonsingular assumption of $\Gamma_0 = -A$ becomes the standard assumption for moments' computation of the DS in (7). In the following proposition, we approximate $G_X(s)$ around $s_0 = 0$, by combining the idea of approximating exponential terms e^{-sh_i} and e^{-sd_j}, $i = 1, \ldots, p$, $j = 1, \ldots, q$, by their Taylor series expansions in [42, p. 834] and the inverse formula in Lemma 2. G_i in (10) are expressed by the matrices having the same dimension as the original NS Σ. A lower-order NS is obtained by matching these G_i in (10).

Proposition 2 *$G_X(s)$ is approximated around $s_0 = 0$ by*

$$
G_X(s) \approx G_{\text{app}}(s) = \left(\sum_{k=0}^{\infty} L_k s^k\right)\Gamma_0^{-1}B, \tag{33}
$$

where

$$L_k = -\sum_{j=0}^{k-1} \Gamma_0^{-1} \Gamma_{k-j} L_j, \quad L_0 = I, \tag{34}$$

$$\Gamma_1 = E - \sum_{i=1}^{p}(-h_i)A_{h_i} - \sum_{j=1}^{q} A_{d_j}, \tag{35}$$

$$\Gamma_k = -\sum_{i=1}^{p}\frac{(-h_i)^k}{k!}A_{h_i} - \sum_{j=1}^{q}\frac{(-d_j)^{k-1}}{(k-1)!}A_{d_j}, \quad k \geq 2. \tag{36}$$

Proof We first expand exponential terms e^{-sh_i} and e^{-sd_j}, $i = 1, \ldots, p$, $j = 1, \ldots, q$, by their Taylor series expansions around $s_0 = 0$ [42, p. 834],

$$e^{-sh_i} = \sum_{k=0}^{\infty}\frac{(-h_i)^k}{k!}s^k \quad \text{and} \quad e^{-sd_j} = \sum_{k=0}^{\infty}\frac{(-d_j)^k}{k!}s^k,$$

which render

$$sE - A - \sum_{i=1}^{p}A_{h_i}e^{-sh_i} - \sum_{j=1}^{q}A_{d_j}se^{-sd_j} = \Gamma_0 + s\Gamma_1 + s^2\Gamma_2 + \cdots + s^n\Gamma_n + \cdots, \tag{37}$$

where Γ_k, $k = 0, 1, \ldots$, are defined in (32), (35) and (36). If we truncate the right hand side of (37) to only the first $r \geq \hat{n}$ terms, we get

$$G_X(s) \approx G_{\text{app}}(s) = \left(\Gamma_0 + s\Gamma_1 + s^2\Gamma_2 + \cdots + s^{r-1}\Gamma_{r-1}\right)^{-1}B. \tag{38}$$

Then (33) can be get directly by applying the inverse formula in Lemma 2 to $G_{\text{app}}(s)$. □

From Proposition 2, the Krylov subspace for the NS Σ is approximated by coefficient matrices of $G_{\text{app}}(s)$ in (38) in the following proposition.

Proposition 3 *The Krylov subspace around* 0 *for the NS* Σ *is approximated by*

$$K(0, \Sigma, \hat{n}) = \text{colspan}\left\{L_0\Gamma_0^{-1}B, L_1\Gamma_0^{-1}B, \ldots, L_{\hat{n}-1}\Gamma_0^{-1}B\right\}.$$

Remark 3 In the case of $A_{h_i} = A_{d_j} = 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, $K(0, \Sigma, \hat{n})$ becomes $K(0, \Sigma_{\text{ds}}, \hat{n})$ in (8). So, Proposition 3 not only provides an approximation to the Krylov subspace for NS, but also gives an extension of Krylov subspace from the DS [7] to the NS.

Similar to the idea of generating the projection matrix V (colspan$(V) \supseteq K(0, \Sigma_{\text{ds}}, \hat{n})$, $V^T V = I$) for the MOR of the DS [35] or the single parameter linear system [12], the projection matrix V is obtained by

$$\text{colspan}(V) \supseteq K(0, \Sigma, \hat{n}), \quad V^T V = I, \tag{39}$$

to get the reduced-order system $\hat{\Sigma}$ in (40).

Theorem 2 *The $\hat{n}$th reduced-order NS is given by*

$$\hat{\Sigma}: \quad \dot{\hat{x}}(t) = V^T A V \hat{x}(t) + \sum_{i=1}^{p} V^T A_{h_i} V \hat{x}(t - h_i)$$

$$+ \sum_{j=1}^{q} V^T A_{d_j} V \dot{\hat{x}}(t - d_j) + V^T \hat{B} u(t), \tag{40}$$

$$\hat{y}(t) = C V \hat{x}(t).$$

Proof From Proposition 2, $X(s)$ can be approximated by

$$\begin{aligned}
X(s) &= G_X(s) U(s) \\
&\approx \left(\Gamma_0 + s \Gamma_1 + s^2 \Gamma_2 + \cdots + s^{r-1} \Gamma_{r-1} \right)^{-1} B U(s) \\
&= \left(\sum_{k=0}^{\infty} L_k s^k \right) \Gamma_0^{-1} B U(s) = \theta(s),
\end{aligned}$$

where $U(s)$ is the Laplace transform of $u(t)$. Inspiring from [12, 35], we want to find a projection matrix V in (39) by matching the first $\hat{n}$ terms of the approximated $G_X(s)$, which are also first $\hat{n}$ coefficients of $\theta(s)$. By assuming $\theta(s) = V\hat{\theta}(s)$, and considering

$$\left(\Gamma_0 + s \Gamma_1 + s^2 \Gamma_2 + \cdots + s^{r-1} \Gamma_{r-1} \right) \theta(s) = B U(s)$$

we obtain

$$V^T \left(\Gamma_0 + s \Gamma_1 + s^2 \Gamma_2 + \cdots + s^{\hat{n}-1} \Gamma_{\hat{n}-1} \right) V \hat{\theta}(s) = V^T B U(s),$$

$$Y(s) \approx C V \hat{\theta}(s),$$

where $Y(s)$ is the Laplace transform of $y(t)$. From the expression of Γ_k, $k = 0, 1, \ldots, \hat{n} - 1$, in (32), (35) and (36), it is easy to show that

$$\begin{aligned}
&V^T \left(\Gamma_0 + s \Gamma_1 + s^2 \Gamma_2 + \cdots + s^{\hat{n}-1} \Gamma_{\hat{n}-1} \right) V \hat{\theta}(s) \\
&= V^T \left(-A - \sum_{i=1}^{p} A_{h_i} \right) V + s V^T \left(E - \sum_{i=1}^{p} (-h_i) A_{h_i} - \sum_{j=1}^{q} A_{d_j} \right) V \\
&\quad + \cdots + s^{\hat{n}-1} V^T \left(-\sum_{i=1}^{p} \frac{(-h_i)^{\hat{n}-1}}{(\hat{n}-1)!} A_{h_i} - \sum_{j=1}^{q} \frac{(-d_j)^{\hat{n}-2}}{(\hat{n}-2)!} A_{d_j} \right) V \\
&= s V^T E V - V^T A V - \sum_{i=1}^{p} V^T A_{h_i} V - s \sum_{i=1}^{p} V^T (-h_i) A_{h_i} V \\
&\quad - \cdots - s^{\hat{n}-1} \sum_{i=1}^{p} \frac{(-h_i)^{\hat{n}-1}}{(\hat{n}-1)!} V^T A_{h_i} V
\end{aligned}$$

$$- s\left(\sum_{j=1}^{q} V^T A_{d_j} V - s\sum_{j=1}^{q}(-d_j)V^T A_{d_j} V\right.$$

$$\left. - \cdots - s^{\hat{n}-2}\sum_{j=1}^{q}\frac{(-d_j)^{\hat{n}-2}}{(\hat{n}-2)!}V^T A_{d_j} V\right)$$

$$\approx s V^T E V - V^T A V - \sum_{i=1}^{p} V^T A_{h_i} V e^{-\mathrm{s}h_i} - s\sum_{j=1}^{q} V^T A_{d_j} V e^{-\mathrm{s}d_j}.$$

Consequently, it follows that

$$V^T\left(\Gamma_0 + s\Gamma_1 + s^2\Gamma_2 + \cdots + s^{\hat{n}-1}\Gamma_{\hat{n}-1}\right)V\hat{\theta}(s)$$

$$\approx \left(s V^T E V - V^T A V - \sum_{i=1}^{p} V^T A_{h_i} V e^{-\mathrm{s}h_i} - s\sum_{j=1}^{q} V^T A_{d_j} V e^{-\mathrm{s}d_j}\right)\hat{\theta}(s)$$

$$= V^T B U(s)$$

which is equivalent to the reduced-order NS $\hat{\Sigma}$, where $\hat{x}(t)$ is the inverse Laplace transform of $\hat{\theta}(s)$. $\qquad\qquad\square$

Corollary 1 *The result in Theorem* 2 *is still true for the RS* Σ_{rs} *in* (31) *by taking* $A_{d_j} = 0$, $j = 1,\ldots,q$, *from* (35) *and* (36). *In the case of* $p = q = 1$, *the projection matrix V in* (39) *is given by*

$$\mathrm{colspan}(V) \supseteq K(0, \Sigma, \hat{n}), \quad V^T V = I,$$

$$L_k = -\sum_{j=0}^{k-1}\Gamma_0^{-1}\Gamma_{k-j}L_j, \qquad L_0 = I,$$

$$\Gamma_0 = -A - A_{h_1}, \qquad \Gamma_1 = E + h_1 A_{h_1} - A_{d_1},$$

$$\Gamma_k = -\frac{(-h_1)^k}{k!}A_{h_1} - \frac{(-d_1)^{k-1}}{(k-1)!}A_{d_1}, \quad k \geq 2.$$

Remark 4 The MOR result in [42] considers the reduction of a special kind of RS with $C = B^T$. The moments of this special RS are approximated by the moments of a large scale DS with system matrices given by

$$\bar{\mathcal{C}} = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \cdots & \Gamma_r \\ I_n & & & \\ & \ddots & & \\ & & I_n & 0 \end{bmatrix} \in \mathbb{R}^{rn \times rn}, \qquad \bar{\mathcal{G}} = \begin{bmatrix} \Gamma_0 & & & \\ & -I_n & & \\ & & \ddots & \\ & & & -I_n \end{bmatrix} \in \mathbb{R}^{rn \times rn},$$

$$\bar{\mathcal{L}} = \begin{bmatrix} B^T & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^{rn \times m},$$

where r is defined in (38). This will result in two obvious shortcomings. One is that the dimension of ROM may be higher than the original RS as low-order ROM may

cause large error. The other is that high-order system matrices $\bar{C}$, $\bar{G}$ and $\bar{L}$ make this method fail to reduce higher-order RS as the storage of $\bar{C}$, $\bar{G}$ and $\bar{L}$ can be memory-demanding. Fortunately, Proposition 2 avoids this by using the inverse formula in Lemma 2 to produce moments having the same dimension as the original NS. The comparison with the result in [42, p. 834] is shown in Examples 2 and 3 in Sect. 4.

Remark 5 The MOR problems with ROM in NS form are also investigated in [43] and [45], respectively, by guaranteeing that the H_∞ norm or energy-to-peak gain of the error system is less than a given scalar in terms of LMIs with inverse constraints. It is solved by CCL algorithm [13] by transforming it to a minimization problem subject to original LMIs and additional LMIs coming from the inverse constraints, which are further solved by IPM. However, IPM requires that all matrix variables are transformed to a very huge vector variable in [8]. Obviously, this may render an out of memory problem due to large size matrix variables, and this further results in IPM not being able to handle large scale LMIs. Moreover, the computational cost of solving LMIs with inverse constraints is very high because of solving a minimization problem. Although LMI-based method provides a good approximation of the original NS by ensuring global accuracy, as a trade-off, high computational cost makes it inapplicable to reduce high-order NS with MCDs. The comparison with the methods in [43] and [45] is given in Example 4 in Sect. 4.

3.2 Extension to the Point $s_0 \neq 0$ and multi-point moment matching

The result in Theorem 2 is extended to a nonzero point s_0 in the following theorem. Assume that

$$\Upsilon_0(s_0) = s_0 E - A - \sum_{i=1}^{p} e^{-s_0 h_i} A_{h_i} - \sum_{j=1}^{q} s_0 e^{-s_0 d_j} A_{d_j}, \qquad (41)$$

is nonsingular in order to approximate $G_X(s)$ around s_0.

Theorem 3 *The projection matrix V is obtained from the approximated Krylov subspace around s_0*

$$\mathrm{colspan}(V) \supseteq K(s_0, \Sigma, \hat{n}), \quad V^T V = I, \qquad (42)$$

the $\hat{n}$th reduced-order NS is given by

$$\hat{\Sigma}: \quad \dot{\hat{x}}(t) = V^T A V \hat{x}(t) + \sum_{i=1}^{p} V^T A_{h_i} V \hat{x}(t - h_i)$$

$$+ \sum_{j=1}^{q} V^T A_{d_j} V \dot{\hat{x}}(t - d_j) + V^T \hat{B} u(t),$$

$$\hat{y}(t) = C V \hat{x}(t),$$

where

$$K(s_0, \Sigma, \hat{n}) = \text{colspan}\big\{ J_0(s_0) \Upsilon_0^{-1}(s_0) B,\ J_1(s_0) \Upsilon_0^{-1}(s_0) B,\ \ldots,\ J_{\hat{n}-1}(s_0) \Upsilon_0^{-1}(s_0) B \big\},$$
(43)

$$J_k(s_0) = -\sum_{j=0}^{k-1} \Upsilon_0^{-1}(s_0) \Upsilon_{k-j}(s_0) J_j(s_0), \quad J_0(s_0) = I,$$
(44)

$$\Upsilon_1(s_0) = E - \sum_{i=1}^{p} (-h_i) e^{-s_0 h_i} A_{h_i} - \sum_{j=1}^{q} \big(s_0(-d_j) + 1\big) e^{-s_0 d_j} A_{d_j},$$
(45)

$$\Upsilon_k(s_0) = -\sum_{i=1}^{p} \frac{(-h_i)^k}{k!} e^{-s_0 h_i} A_{h_i} - \sum_{j=1}^{q} \left(s_0 \frac{(-d_j)^k}{k!} + \frac{(-d_j)^{k-1}}{(k-1)!} \right) e^{-s_0 d_j} A_{d_j},$$
$$k \geq 2.$$
(46)

Proof By expanding e^{-sh_i} and e^{-sd_j}, $i = 1, \ldots, p$, $j = 1, \ldots, q$, by their Taylor series expansions around s_0,

$$e^{-sh_i} = \sum_{k=0}^{\infty} e^{-s_0 h_i} \frac{(-h_i)^k}{k!} (s - s_0)^k \quad \text{and} \quad e^{-sd_j} = \sum_{k=0}^{\infty} e^{-s_0 d_j} \frac{(-d_j)^k}{k!} (s - s_0)^k,$$

we have

$$sE - A - \sum_{i=1}^{p} A_{h_i} e^{-sh_i} - \sum_{j=1}^{q} A_{d_j} s e^{-sd_j}$$

$$= sE - A - \sum_{i=1}^{p} A_{h_i} e^{-sh_i} - \sum_{j=1}^{q} s_0 A_{d_j} e^{-sd_j} - \sum_{j=1}^{q} (s - s_0) A_{d_j} e^{-sd_j}$$

$$= (s - s_0)E + s_0 E - A - \sum_{i=1}^{p} A_{h_i} \sum_{k=0}^{\infty} e^{-s_0 h_i} \frac{(-h_i)^k}{k!} (s - s_0)^k$$

$$\quad - \sum_{j=1}^{q} s_0 A_{d_j} \sum_{k=0}^{\infty} e^{-s_0 d_j} \frac{(-d_j)^k}{k!} (s - s_0)^k$$

$$\quad - \sum_{j=1}^{q} A_{d_j} \sum_{k=0}^{\infty} e^{-s_0 d_j} \frac{(-d_j)^k}{k!} (s - s_0)^{k+1}$$

$$= s_0 E - A - \sum_{i=1}^{p} e^{-s_0 h_i} A_{h_i} - \sum_{j=1}^{q} s_0 e^{-s_0 d_j} A_{d_j}$$

$$\quad + (s - s_0) \left(E - \sum_{i=1}^{p} (-h_i) e^{-s_0 h_i} A_{h_i} - \sum_{j=1}^{q} \big(s_0(-d_j) + 1\big) e^{-s_0 d_j} A_{d_j} \right)$$

$$+ (s - s_0)^2 \left(-\sum_{i=1}^{p} \frac{(-h_i)^2}{2!} e^{-s_0 h_i} A_{h_i} - \sum_{j=1}^{q} \left(s_0 \frac{(-d_j)^2}{2!} + \frac{(-d_j)}{1!} \right) e^{-s_0 d_j} A_{d_j} \right)$$

$$+ \cdots + (s - s_0)^n \left(-\sum_{i=1}^{p} \frac{(-h_i)^n}{n!} e^{-s_0 h_i} A_{h_i} \right.$$

$$\left. - \sum_{j=1}^{q} \left(s_0 \frac{(-d_j)^n}{n!} + \frac{(-d_j)^{n-1}}{(n-1)!} \right) e^{-s_0 d_j} A_{d_j} \right) + \cdots$$

$$= \Upsilon_0(s_0) + (s - s_0)\Upsilon_1(s_0) + (s - s_0)^2 \Upsilon_2(s_0) + \cdots + (s - s_0)^n \Upsilon_n(s_0) + \cdots,$$

where Υ_k, $k = 0, 1, \ldots$, are defined in (41), (45) and (46). Then the proof can be finished similarly to the proof in Proposition 2 and Theorem 2. □

Theorem 3 can be further extended to the multi-point case.

Corollary 2 *The projection matrix V in* (42) *obtained from the approximated Krylov subspace around multiple points,* $s_1, s_2, \ldots, s_g$ *is given by*

$$\text{colspan}(V) \supseteq \bigcup_{i=1}^{g} K(s_i, \Sigma, \hat{n}), \quad V^T V = I,$$

where $K(s_i, \Sigma, \hat{n})$, $i = 1, \ldots, g$, *are defined in* (43).

4 Numerical Examples

All the computation described in this section is performed in Intel Core 2 Quad processors with CPU 2.66 GHz and 2.87 GB memory. The first example is to show that with the same order of the ROM, Taylor series expansion method is better than Padé approximation method although it provides a delay-free ROM with better simulation and analysis than systems with delay via the standard state space techniques.

Example 1 An artificial NS Σ of order 24 with a single delay in the state and a single delay in the derivative of the state is considered to be reduced. It has two inputs and two outputs. The 8th order Padé approximation is chosen to approximate exponential terms in the original NS, which gives a transformed DS with order 408 by Proposition 1. A fourth-order ROM modeled by the DS is derived from Theorem 1. If exponential terms are approximated by their Taylor series expansions via truncating them to the first 4 terms, a fourth-order NS from Theorem 2 is obtained. The time domain responses, time domain errors and relative errors are compared in Fig. 1 with an input $u(t) = e^{-1.5t} \sin(2t)[1\ 1]^T$. The frequency domain response comparison in terms of the maximal singular value (MSV) of the transfer functions is shown in Fig. 2. The above figures show that ROM from Taylor series expansion method captures the original NS better than the ROM by Padé approximation method. Higher-order ROM
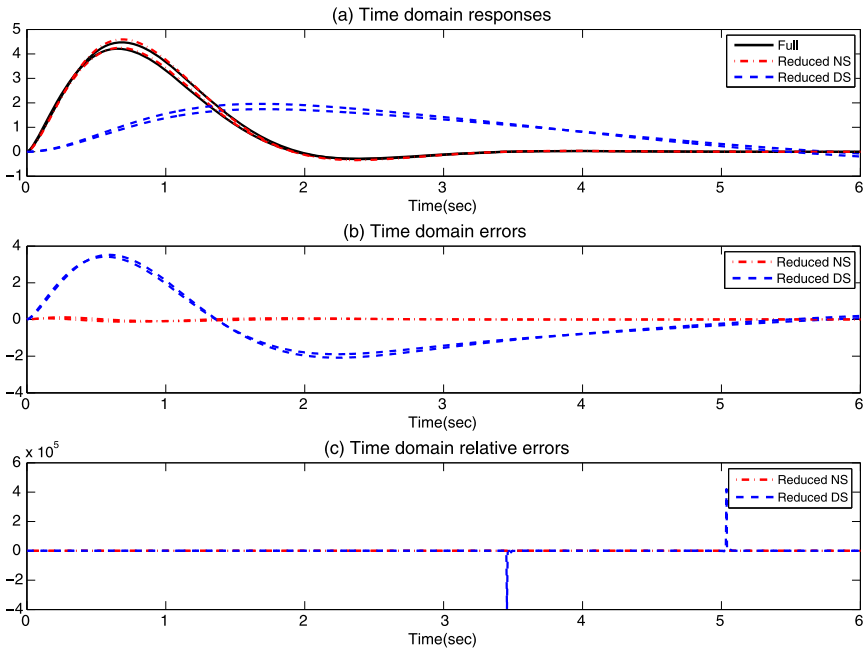
Fig. 1 Time domain responses of Example 1. (**a**) Time domain responses. (**b**) Time domain errors. (**c**) Time domain relative errors
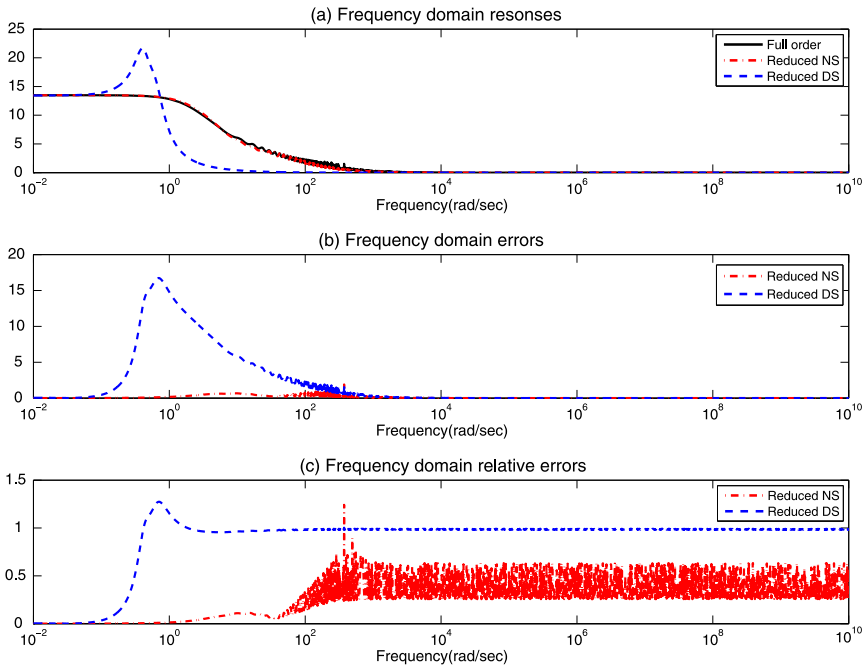

Fig. 2 Frequency domain responses of Example 1. (**a**) Frequency domain responses. (**b**) Frequency domain errors. (**c**) Frequency domain relative errors
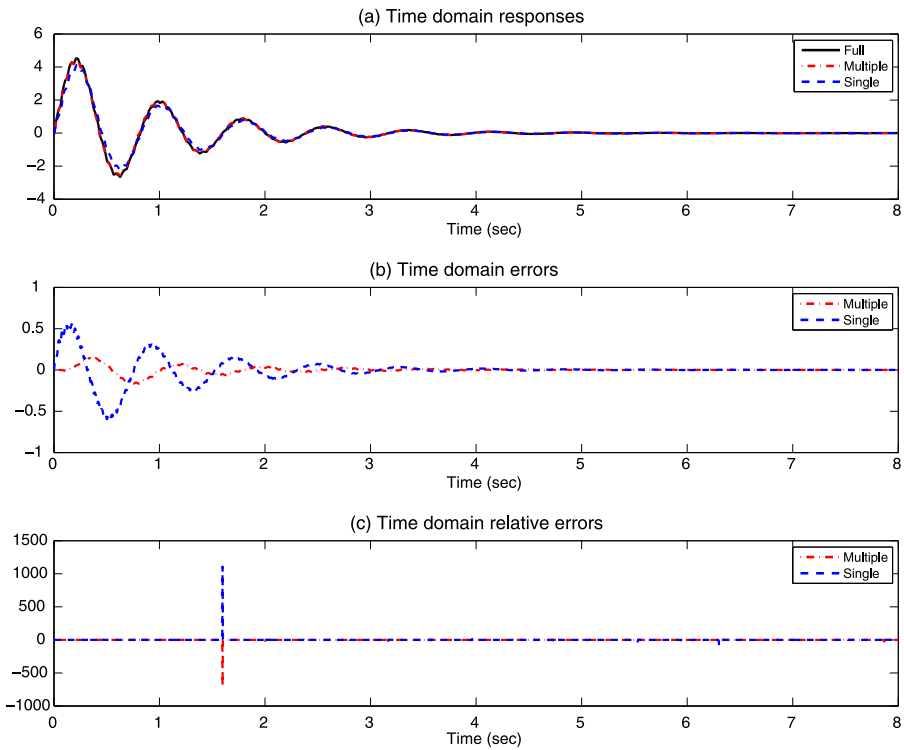
Fig. 3 Time domain responses of Example 2. (**a**) Time domain responses. (**b**) Time domain errors. (**c**) Time domain relative errors

by Padé approximation method may result in better matching of the original system (see Remark 1 for the details).

This example will show that multi-point moment matching is better than single-point moment matching especially when the frequency domain response of the original system has multiple local minima and maxima.

Example 2 A RS is constructed by borrowing $E = I_{1006}$, A, B and C from FOM example in [10] and assuming $A_{h_1} = 0.1 I_{1006}$ and $h_1 = 1$. The method in [42] fails due to out of memory (see Remark 4 for the details). From Theorems 2 and 3, and Corollary 2, the time domain comparison in Fig. 3 with an input $u(t) = e^{-t} \sin(8t)$ shows that the 16th-order RS by multi-point moment matching gives a better approximation than the 16th-order RS by matching moments around zero only. The frequency domain responses, frequency domain errors and frequency domain relative error in Fig. 4 also draw the same conclusion.

Example 3 In this example, a network composing of five lands and 50 conductors transmission line, representing the delay element, and a circuit with RLC components, has been considered for reduction. The network is modeled by a RS with order
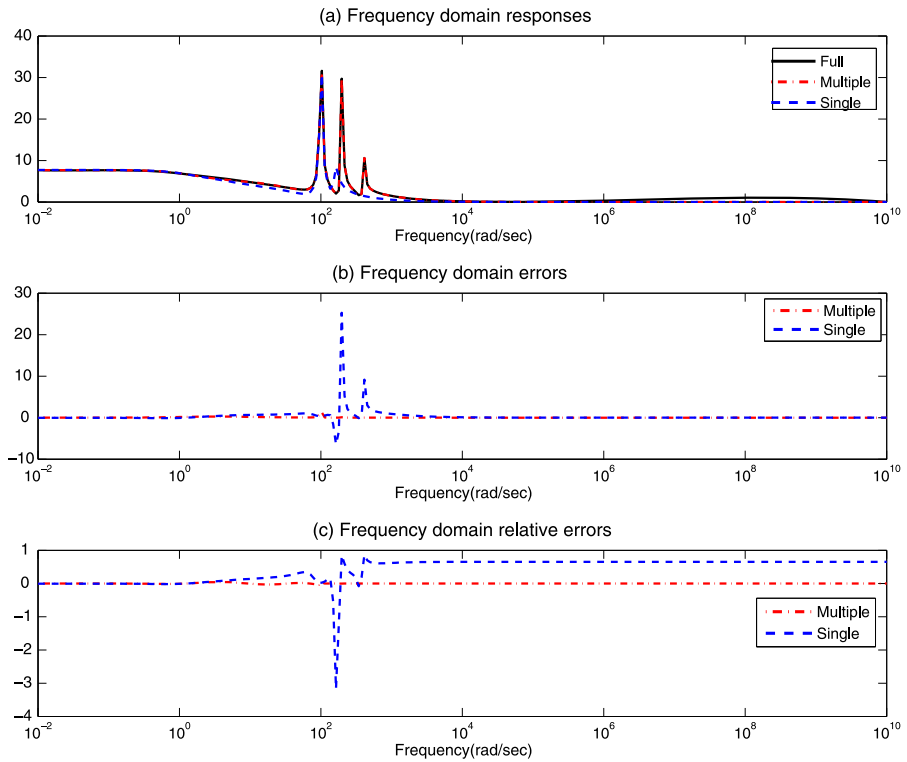
Fig. 4 Frequency domain responses of Example 2. (**a**) Frequency domain responses. (**b**) Frequency domain errors. (**c**) Frequency domain relative errors

602 with four delays in the states. The method in [42] cannot be applied to this example due to out of memory, too. By the proposed method in Theorem 2, a reduced-order RS with order 30 is given by approximating exponential terms by the first 30 terms of their Taylor series expansions. The time domain responses, time domain error and relative error are plotted in Figs. 5 with $u(t) = \sin(10^{12}t)$. Moreover, Figs. 6 and 7 show the frequency domain responses, frequency errors and relative errors in low and high frequencies, respectively. It is clear that the reduced-order NS matches the original system very well.

The comparison with the existing LMI-based method in [43, 45] for reduction of NSs and method in [42] for reducing RS is given in the following example.

Example 4 Four examples are used to for comparison. The above three examples are used and another example is from a small PEEC model borrowed from [6, 24]. The system matrices A, A_{h_1} and A_{d_1} and the delays h_1 and d_1 are given in [6]. B and C are chosen to be

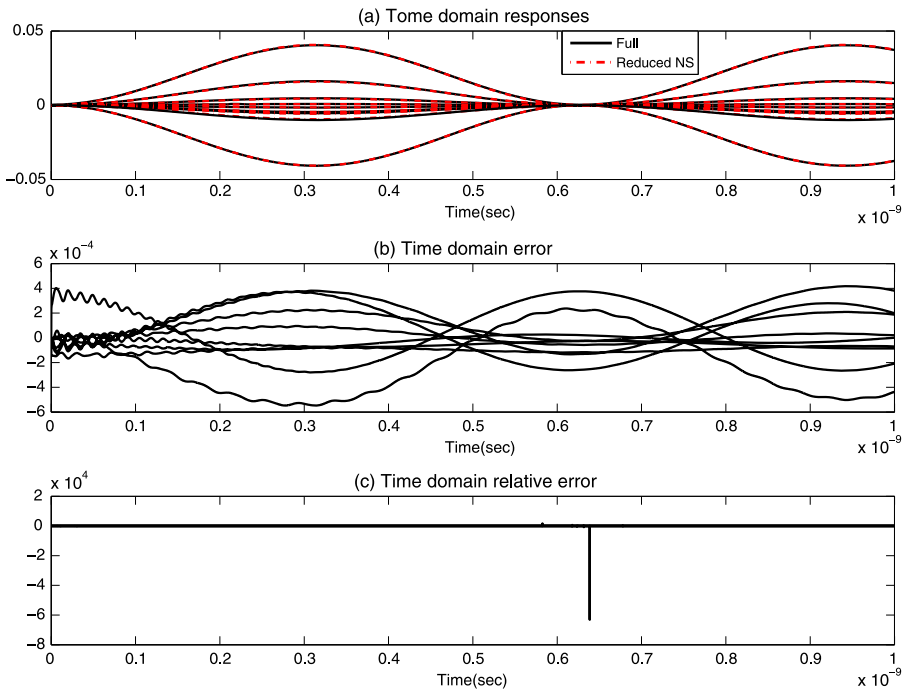$$B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Fig. 5 Time domain responses of Example 3. (**a**) Time domain responses. (**b**) Time domain errors. (**c**) Time domain relative errors

The time domain and frequency domain comparisons are given in Figs. 8 and 9 with time domain input $u(t) = \cos(5t)$. Clearly the ROM from the proposed method matches the original system better than the ROM from LMI-based method. Moreover, the proposed method in Theorem 2 by constructing a projection matrix to match the approximated moments uses less time than the LMI-based method shown in Table 1. We also conclude from Table 1 that LMI-based method cannot reduce the NS with more than order 24 due to out of memory problem (see the details in Remark 5). The method in [42] has the same problem in reducing RS with order 1006. The conclusion is that the proposed method in this paper has more practical use especially in reducing large scale NSs.

5 Conclusions

In this paper, the moment matching method is used to get two different kinds of ROM to approximate a NS with MCDs depending on ways of approximating exponential terms in the transfer function of the original NS. The Padé approximation of exponential terms renders a delay-free system modeled by the high-order DS, with the obvious price to be paid of higher storage and computational complexity. However, delay-free ROM facilitates the analysis and application of the standard state space techniques as
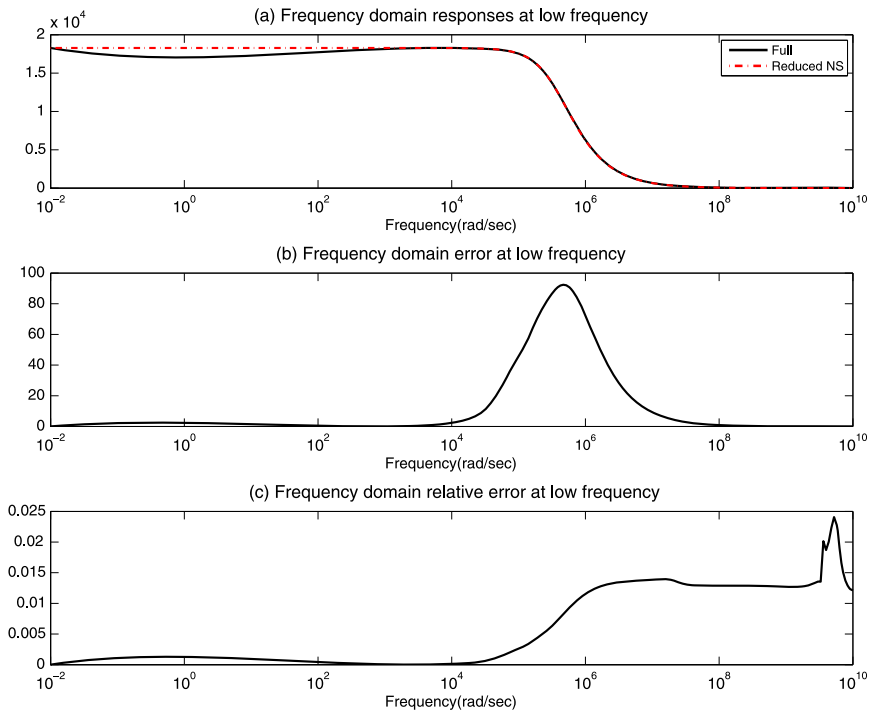
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Fig. 6 Frequency domain responses at low frequency of Example 3. (**a**) Frequency domain responses at low frequency. (**b**) Frequency domain errors at low frequency. (**c**) Frequency domain relative errors at low frequency

Table 1 Comparison with other methods

Examples	n	$\hat{n}$	Proposed (Thm. 2)	LMI-based method in [43, 45]	Method in [42]
Example in [6]	3	1	0.0007 sec	5.5628 sec	Not applicable
Example 1	24	4	0.0046 sec	Out of memory	Not applicable
Example 2 (single)	1006	16	33.0147 sec	Out of memory	Out of memory
Example 2 (multiple)	1006	16	135.2990 sec	Out of memory	Out of memory
Example 3	602	30	136 sec	Out of memory	Out of memory

most of the traditional software for simulation is for delay-free systems. The other ROM has the same structure of the original NS by using Taylor series expansion to replace exponential terms. The most important advantage is that approximated moments have the same dimension as the original NS, which makes this method capable of reducing higher-order NSs. The proposed results can be applied to RSs directly by deleting the matrices related to the derivative of the delayed state. Numerical examples have demonstrated that the Taylor series expansion-based MOR method is much more suitable to reduce high-order NSs than existing MOR methods. Further research will focus on the ROM with stability and passivity preservation by adding additional constraints.
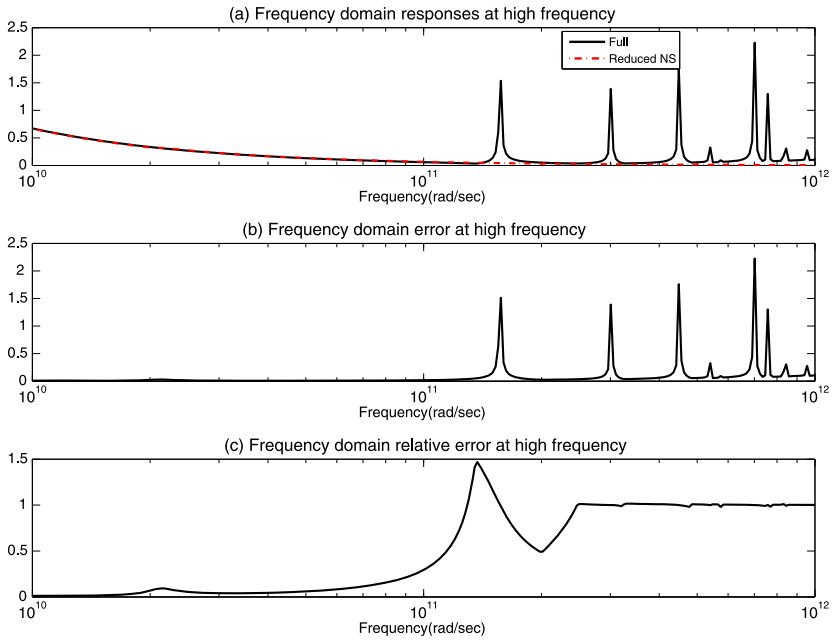
Fig. 7 Frequency domain responses at low frequency of Example 3. (**a**) Frequency domain responses at high frequency. (**b**) Frequency domain errors at high frequency. (**c**) Frequency domain relative errors at high frequency.
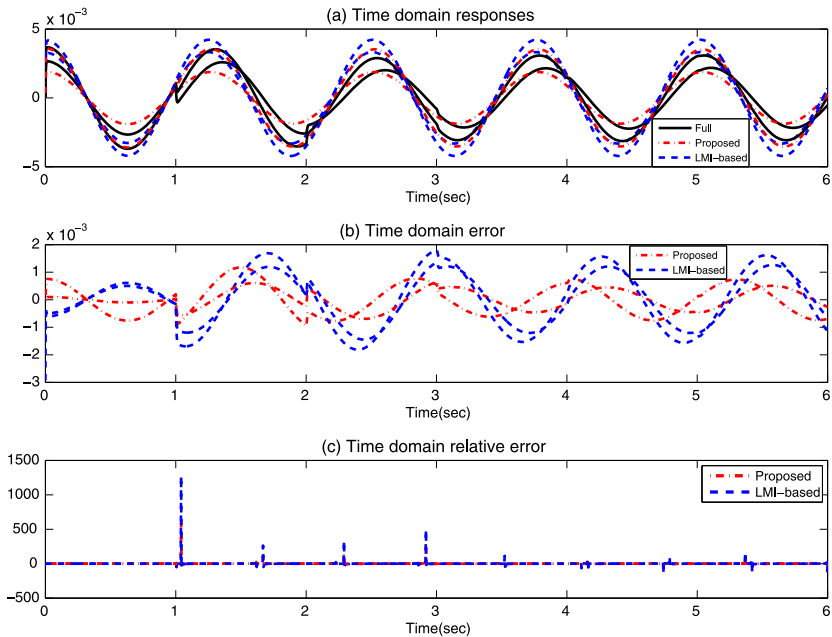

Fig. 8 Time domain responses of Example 4. (**a**) Time domain responses. (**b**) Time domain errors. (**c**) Time domain relative errors
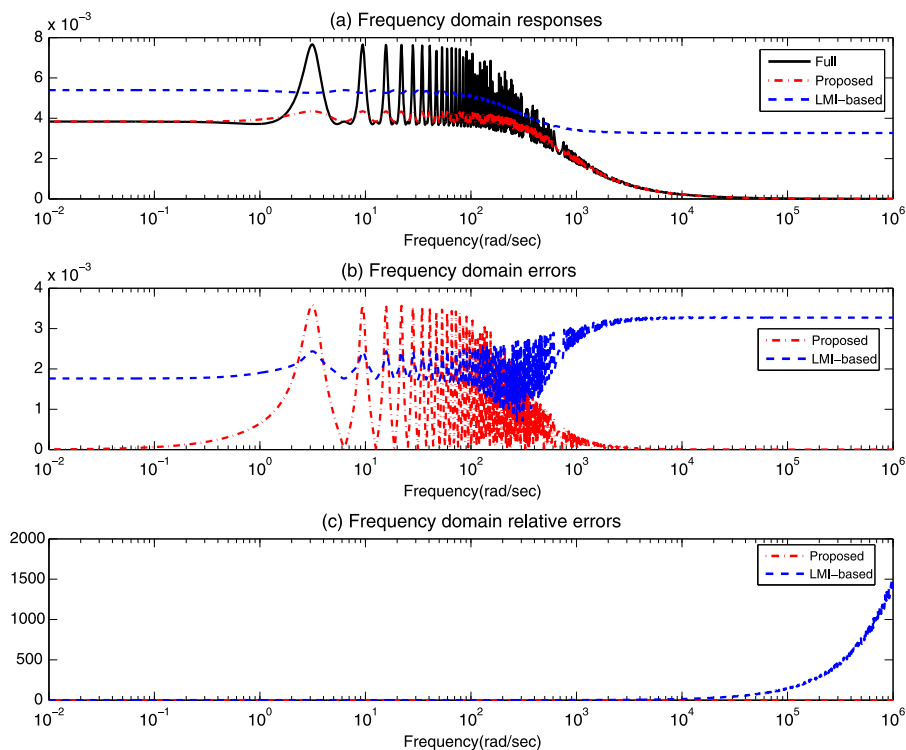
Fig. 9 Frequency domain responses of Example 4. (**a**) Frequency domain responses. (**b**) Frequency domain errors. (**c**) Frequency domain relative errors

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