Solitary wave solution to Aw-Rascle viscous model of traffic flow *

Chun-xiu WU(吴春秀)[1,2], Peng ZHANG(张鹏)[1], S.C. WONG(黄仕进)[3],
Dian-liang QIAO(乔殿梁)[1], Shi-qiang DAI(戴世强)[1]

(1. Shanghai Institute of Applied Mathematics and Mechanics,
Shanghai University, Shanghai, 200072, P.R. China;
2. College of Mathematics and Computer Science,
Quanzhou Normal University, Quanzhou 362000, P.R. China;
3. Department of Civil Engineering, The University of Hong Kong,
Pokfulam Road, Hong Kong SAR, P.R. China)

Abstract A traveling wave solution to the Aw-Rascle traffic flow model that includes the relaxation and diffusion terms is investigated. The model can be approximated by the well-known Kortweg-de Vries (KdV) equation. A numerical simulation is conducted by the first-order accurate Lax-Friedrichs scheme, which is known for its ability to capture the entropy solution to hyperbolic conservation laws. Periodic boundary conditions are applied to simulate a lengthy propagation, where the profile of the derived KdV solution is taken as the initial condition to observe the change of the profile. The simulation shows good agreement between the approximated KdV solution and the numerical solution.

Key words hyperbolic conservation laws; higher-order traffic flow model; traveling wave solution; conservative schemes

1 Introduction

As a remarkable improvement on the classical LWR model [1, 2], the higher-order traffic flow model [3–7] is able to reproduce more complex traffic flow phenomena, such as metastable states, phase transitions, and stop-and-go waves. In addition to the mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{1}$$

where $\rho(x,t)$ is the density and $v(x,t)$ is the average velocity, a higher-order model usually takes acceleration into account to derive more complete equations. In the Aw-Rascle (AR) model [8], the acceleration is assumed to equal the convective derivative of $-p(\rho)$, where $p(\rho)$ is the "pressure". A Riemann invariant $v + p(\rho)$ can be equivalently assumed, which gives $d(v + p(\rho))/dt = 0$, or $\partial(v + p(\rho))/\partial t + v\partial(v + p(\rho))/\partial x = 0$. As a further improvement, a relaxation term $\tau^{-1}(v_e(\rho) - v)$ has been added to the equation [9].

Here, we further include a diffusion term $\nu\rho^{-1}\partial^2 v/\partial x^2$ to study solitary waves in traffic flow, where the constant $\nu > 0$ is the viscosity coefficient. The resultant equation gives

$$\frac{\partial v}{\partial t} + [v - \rho p'(\rho)]\frac{\partial v}{\partial x} = \frac{1}{\tau}[v_e(\rho) - v] + \frac{\nu}{\rho}\frac{\partial^2 v}{\partial x^2}. \tag{2}$$

We use the same procedure to that in [8], in that we multiply Eq. (1) by $v + p(\rho)$ and Eq. (2) by ρ and then add the two together to obtain the conservation form of Eq. (2)

$$\frac{\partial(\rho(v + p(\rho)))}{\partial t} + \frac{\partial(\rho v(v + p(\rho)))}{\partial x} = \frac{\rho v_e(\rho) - \rho v}{\tau} + \nu\frac{\partial^2 v}{\partial x^2}, \tag{3}$$

* Project supported by the National Natural Science Foundation of China (11072141,11272199), the National Basic Research Program of China (2012CB725404), and the University Research Committee, HKU SPACE Research Fund and Faculty of Engineering Top-up Grant of the University of Hong Kong (201007176059).

Corresponding author Peng ZHANG, Professor, Ph.D., E-mail: pzhang@mail.shu.edu.cn

with a standard diffusion term.

Greenberg [10] discussed the traveling wave solution of this model for $\nu = 0$ using the Lagrangian coordinate, and found that the system was significantly related to a car-following model (see also discussions in [11]). Zhang and Wong and their collaborators focused on the study of a wide cluster solution with determined characteristic parameters in this and other inviscid higher-order models [11–14]. They found that for a small-viscosity coefficient $\nu > 0$, the asymptotic theory could be applied to study traveling wave solutions in these models. However, as indicated in [14], such solutions are asymptotic to those with a vanishing viscosity coefficient (see also discussions in [15, 16]).

For large viscosity coefficients $\nu >> 0$, such as $\nu \geq \nu_0$, where $\nu_0 > 0$ is a constant, this study adopts the technique used in Berg et al. [17] and Ge and Han [18] to obtain a solitary wave solution to the discussed model (Section 2). We also simulate the evolution of a derived solitary solution to show the stability of the solution and the small change in the wave profile (Section 3). Although the discussion and simulation are limited, we conclude the paper with some suggestions for future studies (Section 4).

2 Solitary wave solution of the model

We assume a smooth travel wave solution $\rho(x,t) = \rho(z)$, $v(x,t) = v(z)$, and $q(x,t) = q(z)$, where the flow $q = \rho v$, $z = x - ct$, and the constant c is the wave speed. The application of the solution variables ρ and q to Eqs (1) and (2) yields

$$-c\frac{d\rho}{dz} + \frac{dq}{dz} = 0, \tag{4}$$

and

$$\frac{[q - \rho^2 p'(\rho) - c\rho](c\rho - q)}{\rho^2}\frac{d\rho}{dz} = \frac{1}{\tau}[q_e(\rho) - q] + \frac{\nu(c\rho - q)}{\rho^2}[\frac{d^2\rho}{dz^2} - \frac{2}{\rho}(\frac{d\rho}{dz})^2], \tag{5}$$

where $q_e(\rho) = \rho v_e(\rho)$ is the fundamental diagram. A similar derivation is given in [3, 6, 12–14] to study traveling wave solutions in a higher-order model.

As we propose to derive a KdV equation, we assume that q is sufficiently close to $q_e(\rho)$ and that there exists Δz such that $q(\rho(z)) = q_e(\rho(z + \Delta z))$, which is approximated by the following second-order Taylor expansion.

$$q = q_e(\rho) + a_1\frac{d\rho}{dz} + a_2\frac{d^2\rho}{dz^2}. \tag{6}$$

Here, the coefficients a_1 and a_2 depend on ρ and Δz. However, we choose

$$a_1 = \tau[c - v_e(\rho) + \rho p'(\rho)][c - v_e(\rho)], \tag{7}$$

$$a_2 = \frac{\tau\nu}{\rho}[c - v_e(\rho)],$$

to balance the lower-order terms in Eq. (5). The solution ρ is assumed to be a small perturbation $\hat{\rho}$ to a constant state ρ^*, that is, $\rho(z) = \rho^* + \hat{\rho}(z)$. The coefficients in Eq. (6) are thus approximated by $a_1 = a_1(\rho^*)$ and $a_2 = a_2(\rho^*)$ and the function by

$$q_e(\rho) = q_e(\rho^*) + q_e'(\rho^*)\hat{\rho} + \frac{1}{2}q_e''(\rho^*)\hat{\rho}^2.$$

The substitution of q in Eq. (4) by Eq. (6) yields the following ODE.

$$[q_e'(\rho^*) + q_e''(\rho^*)\hat{\rho} - c]\frac{d\hat{\rho}}{dz} + a_1(\rho^*)\frac{d^2\hat{\rho}}{dz^2} + a_2(\rho^*)\frac{d^3\hat{\rho}}{dz^3} = 0. \tag{8}$$

By setting $a_1(\rho^*) = 0$ and $a_2(\rho^*) < 0$, by which Eq. (7) gives the traveling wave speed

$$c = v_e(\rho^*) - \rho^* p'(\rho^*), \tag{9}$$

Eq. (8) can be transformed into the standard KdV equation

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + \frac{\partial^3 U}{\partial X^3} = 0,$$

with

$$X = \sqrt{\frac{-1}{a_2}}x,\ T = -\sqrt{\frac{-1}{a_2}}t,\ U = -[q_e'(\rho^*) + q_e''(\rho^*)\hat{\rho}(x - ct)].$$

This suggests a solitary wave solution $U = 3|c|sech^2[\sqrt{|c|}(X + cT)/2]$, which turns out to be

$$\rho = \rho^* - \frac{q_e'(\rho^*)}{q_e''(\rho^*)} - \frac{3|c|}{q_e''(\rho^*)}sech^2[\frac{1}{2}\sqrt{\frac{-|c|\rho^*}{\tau\nu(c - v_e(\rho^*))}}(x - ct)]. \tag{10}$$

3 Numerical Simulation

Even with a diffusion term, the numerical solutions of the model equations are mostly discontinuous, which means that the conservative equations of (1) and (3) must be adopted for the numerical simulation [14]. The system is rewritten as

$$u_t + f(u)_x = R(u, v_{xx}), \tag{11}$$

where $u = (\rho, s)^T$, $s = \rho(v + p(\rho))$, $f(u) = (s - \rho p(\rho), \rho^{-1}s^2 - sp(\rho))^T$, and $R(u, v_{xx}) = (0, \tau^{-1}(q_e(\rho) - s + \rho p(\rho)) + \nu v_{xx})^T$. In Eq. (11), ρ and s are taken as two conservative solution variables and v is the function of ρ and s, which is given by $v = s/\rho - p(\rho)$. A first-order conservative scheme of Eq. (11) can be generally written as

$$u_i^{(n+1)} = u_i^{(n)} - \frac{\hat{f}(u_i^{(n)}, u_{i+1}^{(n)}) - \hat{f}(u_{i-1}^{(n)}, u_i^{(n)})}{\Delta x} + R(u_i^{(n)}, v_{xx}|_i^{(n)}).$$

We then take the Lax-Friedrichs flux

$$\hat{f}(u_i^{(n)}, u_{i+1}^{(n)}) = \frac{1}{2}(f(u_i^{(n)}) + f(u_{i+1}^{(n)}) - \alpha^{(n)}(u_{i+1}^{(n)} - u_i^{(n)})),$$

where $\alpha^{(n)} = \underset{u}{max}max(|\lambda_1(u)|, |\lambda_2(u)|)$, λ_1 and λ_2 are two eigenvalues of the system, and the maximum is taken over $u_i^{(n)}$. The viscous term is approximated by

$$v_{xx}|_i^{(n)} = \frac{v_{i+1}^{(n)} - 2v_i^{(n)} + v_{i-1}^{(n)}}{\Delta x^2},\ v_i^{(n)} = s_i^{(n)}/\rho_i^{(n)} - p(\rho_i^{(n)}).$$

We apply the formula $\Delta t^{(n)} = \mathrm{CFL}\Delta x^2(\alpha^{(n)}\Delta x + 2\nu/\rho_{jam})^{-1}$ for numerical stability, and take $\mathrm{CFL} = 0.3$ and $\Delta x/L = 4 \times 10^{-3}$.

The initial condition $\rho(x, 0)$ is taken as a KdV solution of Eq. (10) by setting $ct = 0.5L$ with $L = 10000m$, where $[0, L]$ is the computational interval. The initial states are assumed to be at equilibrium when $s(x, 0) = \rho(x, 0)(v_e(\rho(x, 0)) - p(\rho(x, 0)))$. The periodic boundary conditions are applied to observe the evolution of the profile. For certainty, the pressure and the velocity-density relationship are given by

$$p(\rho) = 4v_f(\rho/\rho_{jam})^{0.4},\ v_e(\rho) = v_f((1 + e^{\frac{\rho/\rho_m - 0.25}{0.06}})^{-1} - (1 + e^{\frac{0.75}{0.06}})^{-1}),$$

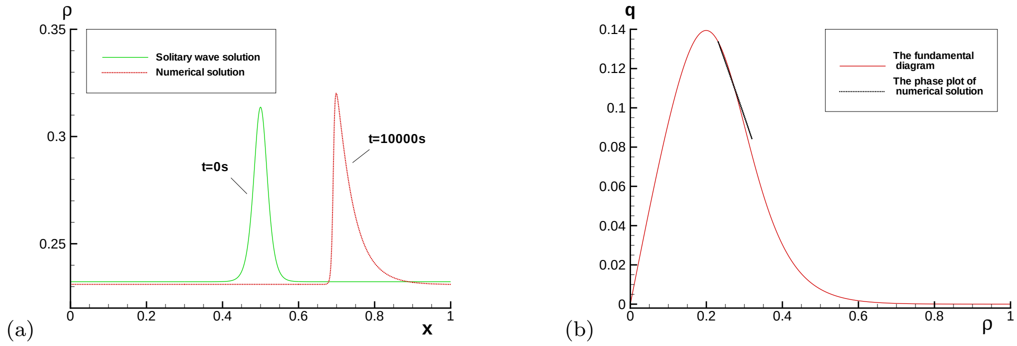

Fig. 1 Evolution of the KdV solution, (a) comparison between the original profile and the evolved profile; (b) comparison between the solution phase-plot and the fundamental diagram.

with $v_f = 30m/s$. In Eqs (10) and (11), we set $\tau = 10s$, $\nu = 0.00015Lv_f\rho_{jam}$, and $\rho^* = 0.13\rho_{jam}$. Because the density ρ is scaled by its maximum ρ_{jam} and x by the length L in the computation and illustration, the value of ρ_{jam} is not needed in the aforementioned formulas.

Fig. 1 shows the numerical result at $t = 10000s$. Fig. 1(a), indicates that the KdV solution is stable and that the profile remains approximately the same even after a long propagation time. Fig. 1(b) shows that the phase plot of the numerical solution is almost a straight line, which can also be approximately described by Eq. (4) or by $q = c\rho + q_0$, where q_0 is the integral constant. The phase plot is also very close to the fundamental diagram. However, the difference explains the change in solution profile in Fig. 1(a) according to the discussion in the context of Eq. (6).

The simulation futher indicates that the profile will become unstable when ν increases to a certain larger value. This agrees with the analytical conclusion in [19] that in the KK model ν/τ must be bounded by a constant to maintain the stability of the traveling wave solution.

4 Conclusions

We derive a KdV solution by including the relaxation and viscous terms in the AR model. A numerical example shows that the KdV solution gives a good approximation of a traveling wave in the model. However, there are multiple choices for the three parameters ρ^*, τ, and ν in the solution, and future studies should seek to determine the most appropriate choices for these parameters based upon sound theoretical analysis to give a better approximation. The numerical schemes could also be improved by incorporating the relaxation and viscous terms into a convective term give a better resolution.

References

[1] Lighthill, M.J. and Whitham, G.B. On kinematic waves: II. A theory of traffic flow on long crowded roads. *Procrrdings of the Royal Society of London, Series A*, **229**(1178), 317-345 (1955).

[2] Richards, P.I. Shockwaves on the highway. *Operations Research*, **4**(1), 42-51 (1956).

[3] Kerner, B.S. and Konhäuser, P. Structure and parameters of clusters in traffic flow. *Physical Review E*, **50**, 54-83 (1994).

[4] Kühne, R.D. Macroscopic freeway model for dense traffic-stop-start waves and incident detection, *P*roc. 9th International Symposium on Transportation and Traffic Theory (eds. J. Volmuller and R. Hamerslag), VNU Science Press, Utrecht, 21-42 (1984).

[5] Payne H.J. Models of freeway traffic and control, *M*athematical Models of Public Systems (ed. A.G. Bekey), Simulation Council Proc., La Jola, 1, 51-61 (1971).

[6] Whitham G.B. *L*inear and Nonlinear Waves, John Wiley and Sons, New York (1974).

[7] Zhang P., Wong S.C., and Dai S.Q. *A* conserved higher-order anisotropic traffic flow model: description of equilibrium and non-equilibrium flows. **T**ransportation Research Part B, **43**, 562-574 (2009).

[8] Aw A. and Rascle M. *R*esurrection of "second order" models of traffic flow. **S**IAM J. Appl. Math., **60**, 916-938 (2000).

[9] Rascle M. *A*n improved macroscopic model of traffic flow: Derivation and links with the Lighthill-Whitham Model. **M**athematical and Computer Modelling **35**, 581-590 (2002).

[10] Greenberg J.M. *C*ongestion redux. **S**IAM Journal on Applied Mathematics **64**(4), 1175-1185 (2004).

[11] Zhang P., Wu C.X., and Wong S.C. *A* semi-discrete model and its approach to a solution for wide moving jam in traffic flow. **P**hysica A, **391**(3), 456-463 (2012).

[12] Xu R.Y., Zhang P., Dai S.Q., and Wong S.C. *A*dmissibility of a wide cluster solution in anisotropic higher-order traffic flow models. **S**IAM Journal on Applied Mathematics **68**(2), 562-573 (2007).

[13] Zhang P., Wong S.C., and Dai S.Q. *C*haracteristic parameters of a wide cluster in a higher-order traffic flow model. **C**hinese Physics Letters, **232**, 516-519 (2006).

[14] Zhang P. and Wong S.C. *E*ssence of conservation forms in the traveling wave solutions of higher-order traffic flow models. **P**hysical Review E, **74**(2), 026109 (2006).

[15] Kerner B.S., Klenov S.L., and Konhäuser P. *A*symptotic theory of traffic jams. **P**hys. Rev. E, **56**(4), 4199-4216 (1997).

[16] Wu C.X., Zhang P., Dai S.Q., and Wong S.C. *A*symptotic solution of a wide cluster in Kühne's higher-order traffic flow model. **P**roceedings of the 5th International Conference on Nonlinear Mechanics (eds. W.Z. Chien et al.), Shanghai University Press, 1132-1136 (2007).

[17] Berg P. and Woods A. *O*n-ramp simulation and solitary waves of a car-following model. **P**hysical Review E, **64**, 035602 (2001).

[18] Ge H.X. and Han X.L. *D*ensity viscous continuum traffic flow model. **P**hysica A, **371**, 667-673 (2006).

[19] Li T. *S*tability of traveling waves in quasi-linear hyperbolic systems with relaxation and diffusion. **S**IAM J. Math. Anal., **40**(3), 1058-1075 (2008).