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On the Hermite-Hadamard type inequalities

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Abstract

In the present paper, we establish some new Hermite-Hadamard type inequalities involving two functions. Our results in a special case yield recent results on Hermite-Hadamard type inequalities.

MSC: 26D15

Keywords: Hermite-Hadamard inequality; Barnes-Godunova-Levin inequality; Minkowski integral inequality; Hölder inequality

1 Introduction

The following inequality is well known in the literature as Hermite-Hadamard's inequality [1].

Theorem 1.1 *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on an interval of real numbers. Then the following Hermite-Hadamard inequality for convex functions holds*:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

If the function f is concave, the inequality (1.1) *can be written as follows*:

$$f\left(\frac{a+b}{2}\right)\ge\frac{1}{b-a}\int_a^b f(x)\,dx\ge\frac{f(a)+f(b)}{2}. \tag{1.2}$$

Recently, many generalizations, extensions and variants of this inequality have appeared in the literature (see, *e.g.*, [2–10]) and the references given therein. In particular, in 2010, Özdemir and Dragomir [11] established some new Hermite-Hadamard inequalities and other integral inequalities involving two functions in $\mathbb{R}$. Following this work, the main purpose of the present paper is to establish some dual Hermite-Hadamard type inequalities involving two functions in $\mathbb{R}^2$. Our results provide some new estimates on such type of inequalities.

2 Preliminaries

A region $D\subset\mathbb{R}^2$ is called convex if it contains the close line segment joining any two of its points, or equivalently, if $\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2\in D$ whenever $x(x_1,y_1), y(x_2,y_2)\in D$ and $0\le\lambda\le 1$.

Let $z = f(x,y)$ be a duality function on the convex region $D \subset \mathbb{R}^2$. $z = f(x,y)$ is called a duality convex function on the convex region D if

$$f\big[\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2\big] \le \lambda f(x_1,y_1) + (1-\lambda)f(x_2,y_2), \tag{2.1}$$

whenever $(x_1,y_1),(x_2,y_2) \in D$ and $0 \le \lambda \le 1$.

If the function $f(x,y)$ is concave, the inequality (2.1) can be written as follows:

$$f\big[\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2\big] \ge \lambda f(x_1,y_1) + (1-\lambda)f(x_2,y_2). \tag{2.2}$$

Let $x = (x_{11},\ldots,x_{1n},\ldots,x_{m1},\ldots,x_{mn})$ and $p = (p_{11},\ldots,p_{1n},\ldots,p_{m1},\ldots,p_{mn})$ be two positive nm-tuples, and let $r \in \mathbb{R} \cup \{+\infty,-\infty\}$. Then, on putting $P_{mn} = \sum_{k_2=1}^{n}\sum_{k_1=1}^{m} p_{k_1k_2}$, it easy follows that if $-\infty \le r < s \le +\infty$, then

$$M_{mn}^{[r]} \le M_{mn}^{[s]} \tag{2.3}$$

(also see, *e.g.*, [1, p.15]). Here, the rth power mean of x with weights p is the following: $M_{mn}^{[r]} = (\frac{1}{P_{mn}}\sum_{k_2=1}^{n}\sum_{k_1=1}^{m} p_{k_1k_2}x_{k_1k_2}^{r})^{1/r}$ if $r \ne +\infty, 0, -\infty$; $M_{mn}^{[r]} = (\prod_{k_2=1}^{n}\prod_{k_1=1}^{m} x_{k_1k_2}^{p_{k_1k_2}})^{P_{mn}}$ if $r = 0$; $M_{mn}^{[r]} = \min(x_{11},\ldots,x_{1n},\ldots,x_{n1},\ldots,x_{mn})$ if $r = -\infty$ and $M_{mn}^{[r]} = \max(x_{11},\ldots,x_{1n},\ldots,x_{n1},\ldots,x_{mn})$ if $r = +\infty$.

Let $f(x,y) : [a,b] \times [c,d] \to \mathbb{R}$, and $p \ge 1$. Now, we define the p-norm of the function $f(x,y)$ on $[a,b] \times [c,d]$ as follows:

$$\big\|f(x,y)\big\|_p = \left(\int_a^b \int_c^d \big|f(x,y)\big|^p \,dx\,dy\right)^{1/p}, \quad 1 \le p < \infty,$$

and

$$\big\|f(x,y)\big\|_p = \sup\big|f(x,y)\big|, \quad p = \infty,$$

and $L^p([a,b] \times [c,d])$ is the set of all functions $f(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ such that $\|f(x,y)\|_p < \infty$.

Lemma 2.1 (see [12]) (Barnes-Godunova-Levin inequality) *Let $f(x,y)$, $g(x,y)$ be nonnegative concave functions on $[a,b] \times [c,d]$, then for $p,q > 1$ we have*

$$\big\|f(x,y)\big\|_p \big\|g(x,y)\big\|_q \le B(p,q) \int_a^b \int_c^d f(x,y)g(x,y)\,dx\,dy, \tag{2.4}$$

where

$$B(p,q) = \frac{6[(b-a)(d-c)]^{1/p+1/q-1}}{(p+1)^{1/p}(q+1)^{1/q}}.$$

Lemma 2.2 (see [1]) (Hermite-Hadamard inequality) *Let $f(x,y) : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be a convex function. Then the following dual Hermite-Hadamard inequality for convex functions holds*:

$$f\left(\frac{a+c}{2}, \frac{b+d}{2}\right) \le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dx\,dy \le \frac{f(a,b)+f(c,d)}{2}. \tag{2.5}$$

The inequality is reversed if the function $f(x,y)$ is concave.

Lemma 2.3 (see [13]) (A reversed Minkowski integral inequality) *Let $f(x,y)$ and $g(x,y)$ be positive functions satisfying*

$$0 < m \le \frac{f(x,y)}{g(x,y)} \le M, \quad (x,y) \in [a,b] \times [c,d]. \tag{2.6}$$

Then

$$\|f(x,y)\|_p + \|g(x,y)\|_p \le c\|f(x,y) + g(x,y)\|_p, \tag{2.7}$$

where $c = [M(m+1) + (M+1)]/[(m+1)(M+1)]$.

3 Main results

Our main results are established in the following theorems.

Theorem 3.1 *Let $p,q > 1$ and let $f(x,y), g(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ be nonnegative functions such that $f(x,y)^p$ and $g(x,y)^q$ are concave on $[a,b] \times [c,d]$. Then*

$$\begin{aligned} &\frac{f(a,b)+f(c,d)}{2} \times \frac{g(a,b)+g(c,d)}{2} \\ &\quad \le \frac{1}{[(b-a)(d-c)]^{1/p+1/q}} B(p,q) \int_a^b \int_c^d f(x,y)g(x,y)\,dx\,dy, \end{aligned} \tag{3.1}$$

where $B(p,q)$ is the Barnes-Godunova-Levin constant given by (2.4).

Proof Observe that whenever $f^p(x,y)$ is concave on $[a,b] \times [c,d]$, the nonnegative function $f(x,y)$ is also concave on $[a,b] \times [c,d]$. Namely,

$$f\big[\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\big]^p \ge \lambda f(a,b)^p + (1-\lambda)f(c,d)^p,$$

that is,

$$f\big[\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\big] \ge \big(\big(\lambda f(a,b)^p + (1-\lambda)f(c,d)^p\big)\big)^{1/p},$$

and $p > 1$, using the power-mean inequality (2.3), we obtain

$$f\big[\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d\big] \ge \lambda f(a,b) + (1-\lambda)f(c,d).$$

For $q > 1$, similarly, if $g^q(x,y)$ is concave on $[a,b] \times [c,d]$, the nonnegative function $g(x,y)$ is concave on $[a,b] \times [c,d]$.

In view that $f^p(x,y)$ and $g^q(x,y)$ are concave functions on $[a,b] \times [c,d]$, from Lemma 2.2, we get

$$\begin{aligned} \left(\frac{f(a,b)^p + f(c,d)^p}{2}\right)^{1/p} &\le \frac{1}{[(b-a)(d-c)]^{1/p}} \left(\int_a^b \int_c^d f(x,y)^p\,dx\,dy\right)^{1/p} \\ &\le f\left(\frac{a+c}{2}, \frac{b+d}{2}\right), \end{aligned} \tag{3.2}$$

and

$$\left(\frac{g(a,b)^p + g(c,d)^q}{2}\right)^{1/q} \le \frac{1}{[(b-a)(d-c)]^{1/q}} \left(\int_a^b \int_c^d g(x,y)^q \, dx \, dy\right)^{1/q} \le g\left(\frac{a+c}{2}, \frac{b+d}{2}\right). \tag{3.3}$$

By multiplying the above inequalities, we obtain

$$\left(\frac{f(a,b)^p + f(c,d)^p}{2}\right)^{1/p} \left(\frac{g(a,b)^p + g(c,d)^q}{2}\right)^{1/q} \le \frac{1}{[(b-a)(d-c)]^{1/p+1/q}} \left(\int_a^b \int_c^d f(x,y)^p \, dx \, dy\right)^{1/p} \left(\int_a^b \int_c^d g(x,y)^q \, dx \, dy\right)^{1/q}. \tag{3.4}$$

If $p, q > 1$, then it is easy to show that

$$\left(\frac{f(a,b)^p + f(c,d)^p}{2}\right)^{1/p} \ge \frac{f(a,b) + f(c,d)}{2}, \tag{3.5}$$

and

$$\left(\frac{g(a,b)^q + g(c,d)^q}{2}\right)^{1/q} \ge \frac{g(a,b) + g(c,d)}{2}. \tag{3.6}$$

Thus, by applying Barnes-Godunova-Levin inequality to the right-hand side of (3.4) with (3.5), (3.6), we get (3.1).

The proof is complete. □

Remark 3.1 By multiplying inequalities (3.2), (3.3), we obtain

$$\frac{1}{[(b-a)(d-c)]^{1/p+1/q}} \left(\int_a^b \int_c^d f(x,y)^p \, dx \, dy\right)^{1/p} \left(\int_a^b \int_c^d g(x,y)^q \, dx \, dy\right)^{1/q} \le f\left(\frac{a+c}{2}, \frac{b+d}{2}\right) g\left(\frac{a+c}{2}, \frac{b+d}{2}\right). \tag{3.7}$$

By applying the Hölder inequality to the left-hand side of (3.7) with $(1/p) + (1/q) = 1$, we get

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) g(x,y) \, dx \, dy \le f\left(\frac{a+c}{2}, \frac{b+d}{2}\right) g\left(\frac{a+c}{2}, \frac{b+d}{2}\right). \tag{3.8}$$

Remark 3.2 Let $f(x,y)$ and $g(x,y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in Theorem 3.1 and Remark 3.1, we have the following.

Corollary 3.1 *Let $p, q > 1$ and let $f(x), g(x) : [a,b] \to \mathbb{R}$, $a < b$, be nonnegative functions such that $f(x)^p$ and $g(x)^q$ are concave on $[a,b]$. Then*

$$\frac{f(a) + f(b)}{2} \cdot \frac{g(a) + g(b)}{2} \le \frac{1}{(b-a)^{1/p+1/q}} B(p,q) \int_a^b f(x) g(x) \, dx,$$

and if $(1/p) + (1/q) = 1$, *then one has*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \le f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right).$$

This is just Theorem 2.1 established by Özdemir and Dragomir [11].

Theorem 3.2 *Let* $p \ge 1$ *and let* $\int_a^b \int_c^d f(x,y)^p\,dx\,dy < \infty$ *and* $\int_a^b \int_c^d g(x,y)^p\,dx\,dy < \infty$, *and let* $f(x,y), g(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ *be positive functions with*

$$0 < m \le \frac{f(x,y)}{g(x,y)} \le M, \quad \forall (x,y) \in [a,b] \times [c,d].$$

Then

$$\|f(x,y)\|_p^2 + \|g(x,y)\|_p^2 \ge \left(\frac{(M+1)(m+1)}{M} - 2\right)\|f(x,y)\|_p \|g(x,y)\|_p. \tag{3.9}$$

Proof Since $f(x,y)$, $g(x,y)$ are positive, as in the proof of Lemma 2.3 (see [13, p.2]), we have

$$\left(\int_a^b \int_c^d f(x,y)^p\,dx\,dy\right)^{1/p} \le \frac{M}{M+1}\left(\int_a^b \int_c^d \big(f(x,y)+g(x,y)\big)^p\,dx\,dy\right)^{1/p}$$

and

$$\left(\int_a^b \int_c^d g(x,y)^p\,dx\,dy\right)^{1/p} \le \frac{1}{m+1}\left(\int_a^b \int_c^d \big(f(x,y)+g(x,y)\big)^p\,dx\,dy\right)^{1/p}.$$

By multiplying the above inequalities and in view of the Minkowski inequality, we get

$$\begin{aligned}
&\left(\int_a^b \int_c^d f(x,y)^p\,dx\,dy\right)^{1/p}\left(\int_a^b \int_c^d g(x,y)^p\,dx\,dy\right)^{1/p} \\
&\quad\le \frac{M}{(M+1)(m+1)}\left(\int_a^b \int_c^d \big(f(x,y)+g(x,y)\big)^p\,dx\,dy\right)^{2/p} \\
&\quad\le \frac{M}{(M+1)(m+1)}\left(\left(\int_a^b \int_c^d f(x,y)^p\,dx\,dy\right)^{1/p}\right. \\
&\qquad\left.+\left(\int_a^b \int_c^d g(x,y)^p\,dx\,dy\right)^{1/p}\right)^2.
\end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned}
&\left(\int_a^b \int_c^d f(x,y)^p\,dx\,dy\right)^{2/p} + \left(\int_a^b \int_c^d g(x,y)^p\,dx\,dy\right)^{2/p} \\
&\quad\ge \left(\frac{(M+1)(m+1)}{M} - 2\right)\left(\int_a^b \int_c^d f(x,y)^p\,dx\,dy\right)^{1/p}\left(\int_a^b \int_c^d g(x,y)^p\,dx\,dy\right)^{1/p}.
\end{aligned}$$

This proof is complete. □

Remark 3.3 Let $f(x,y)$ and $g(x,y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in (3.9), (3.9) reduces to an inequality established by Özdemir and Dragomir [11].

Theorem 3.3 *If $f^p(x,y)$ and $g^q(x,y)$ are as in Theorem* 3.1, *then the following inequality holds*:

$$\frac{1}{(b-a)(d-c)}\|f(x,y)\|_p^p\cdot\|g(x,y)\|_q^q\geq\frac{(f(a,b)+f(c,d))^p(g(a,b)+g(c,d))^q}{2^{p+q}}. \tag{3.11}$$

Proof If $f^p(x,y)$ and $g^q(x,y)$ are concave on $[a,b]\times[c,d]$, then from Lemma 2.2, we get

$$\frac{f(a,b)^p+f(c,d)^p}{2}\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)^p\,dx\,dy$$

and

$$\frac{g(a,b)^q+g(c,d)^q}{2}\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)^q\,dx\,dy,$$

which imply that

$$\begin{aligned}&\frac{[f(a,b)^p+f(c,d)^p][g(a,b)^q+g(c,d)^q]}{4}\\&\quad\leq\frac{1}{[(b-a)(d-c)]^2}\int_a^b\int_c^d f(x,y)^p\,dx\,dy\int_a^b\int_c^d g(x,y)^q\,dx\,dy.\end{aligned} \tag{3.12}$$

On the other hand, if $p,q\geq 1$, from (2.3) we get

$$\frac{f(a,b)^p+f(c,d)^p}{2}\leq 2^{-p}\bigl[f(a,b)+f(c,d)\bigr]^p$$

and

$$\frac{g(a,b)^q+g(c,d)^q}{2}\leq 2^{-q}\bigl[g(a,b)+g(c,d)\bigr]^q,$$

which imply that

$$\begin{aligned}&\frac{[f(a,b)^p+f(c,d)^p][g(a,b)^p+g(c,d)^q]}{4}\\&\quad\geq 2^{-p-q}\bigl[f(a,b)+f(c,d)\bigr]^p\bigl[g(a,b)+g(c,d)\bigr]^q.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain the desired inequality as

$$\begin{aligned}&2^{-p-q}\bigl[f(a,b)+f(c,d)\bigr]^p\bigl[g(a,b)+g(c,d)\bigr]^q\\&\quad\leq\frac{1}{[(b-a)(d-c)]^2}\|f(x,y)\|_p^p\cdot\|g(x,y)\|_q^q.\end{aligned}$$

This proof is complete. □

Remark 3.4 Let $f(x,y)$ and $g(x,y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in (3.11), (3.11) reduces to an inequality established by Özdemir and Dragomir [11].

Theorem 3.4 *Let $f(x,y), g(x,y) : [a,b] \times [c,d] \to \mathbb{R}^+$ be functions such that $f(x,y)^p$, $g(x,y)^q$ and $f(x,y)g(x,y)$ are in $L_1([a,b] \times [c,d])$, and*

$$0 < m \le \frac{f(x,y)}{g(x,y)} \le M, \quad \forall (x,y) \in [a,b] \times [c,d], a,b,c,d \in [0,\infty).$$

Then

$$\begin{aligned}&\int_a^b \int_c^d f(x,y)g(x,y)\,dx\,dy\\ &\quad\le c_1\left(\frac{\|f(x,y)\|_p^p + \|g(x,y)\|_p^p}{2}\right) + c_2\left(\frac{\|f(x,y)\|_q^q + \|g(x,y)\|_q^q}{2}\right),\end{aligned} \tag{3.14}$$

where

$$c_1 = \frac{2^p}{p}\left(\frac{M}{M+1}\right)^p, \qquad c_2 = \frac{2^q}{q}\left(\frac{1}{m+1}\right)^q,$$

and $(1/p) + (1/q) = 1$ with $p > 1$.

Proof Since $0 < m \le \frac{f(x,y)}{g(x,y)} \le M$, $\forall (x,y) \in [a,b] \times [c,d]$, we have

$$f(x,y) \le \frac{M}{M+1}\bigl(f(x,y) + g(x,y)\bigr)$$

and

$$g(x,y) \le \frac{1}{m+1}\bigl(f(x,y) + g(x,y)\bigr).$$

In view of the Young-type inequality and using the elementary inequality

$$(a+b)^p \le 2^{p-1}\bigl(a^p + b^p\bigr), \quad p > 1, a, b \in \mathbb{R}^+,$$

we have

$$\begin{aligned}&\int_a^b \int_c^d f(x,y)g(x,y)\,dx\,dy\\ &\quad\le \frac{1}{p}\left(\frac{M}{M+1}\right)^p \int_a^b \int_c^d \bigl(f(x,y) + g(x,y)\bigr)^p\,dx\,dy\\ &\qquad+ \frac{1}{q}\left(\frac{1}{m+1}\right)^q \int_a^b \int_c^d \bigl(f(x,y) + g(x,y)\bigr)^q\,dx\,dy\\ &\quad\le \frac{1}{p}\left(\frac{M}{M+1}\right)^p 2^{p-1} \int_a^b \int_c^d \bigl[f(x,y)^p + g(x,y)^p\bigr]\,dx\,dy\\ &\qquad+ \frac{1}{q}\left(\frac{1}{m+1}\right)^q 2^{q-1} \int_a^b \int_c^d \bigl[f(x,y)^q + g(x,y)^q\bigr]\,dx\,dy.\end{aligned}$$

This completes the proof. □

Remark 3.5 Let $f(x,y)$ and $g(x,y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in (3.14), (3.14) reduces to an inequality established by Özdemir and Dragomir [11].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

C-JZ, W-SC and X-YL jointly contributed to the main results Theorems 3.1-3.4. All authors read and approved the final manuscript.

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