On Modeling Credit Defaults: A Probabilistic Boolean Network Approach

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Abstract

One of the central issues in credit risk measurement and management is modeling and predicting correlated defaults. In this paper we introduce a novel model to investigate the relationship between correlated defaults of different industrial sectors and business cycles as well as the impacts of business cycles on modeling and predicting correlated defaults using the Probabilistic Boolean Network (PBN). The key idea of the PBN is to decompose a transition probability matrix describing correlated defaults of different sectors into several BN matrices which contain information about business cycles. An efficient estimation method based on entropy approach is used to estimate the model parameters. Using real default data, we build a PBN for explaining the default structure and make reasonably good prediction of joint defaults in different sectors.

Keywords: Business Cycles; Entropy; Correlated Defaults; Probabilistic Boolean Networks.

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1 Introduction

Modeling default risk is an important topic for credit risk measurement and management. Basically there are two major approaches to modeling default risk, namely, (i) the structural firm value approach pioneered by Black and Scholes (1973) [1] and Merton (1974) [23] and (ii) the reduced-form intensity-based approach introduced by Jarrow and Turnbull (1995) [12] and Madan and Unal (1998) [22]. The key idea of the structural firm value approach is to model explicitly the relationship between the asset value of a firm and the default of the firm. In particular, a default occurs if the asset value of the firm falls below a default barrier level. Consequently, under the structural firm value approach, default events are endogenous. Indeed, the KMV model developed by Moodys is a practical version of the Merton structural firm value model. One of the major shortcomings of structural firm value models is that the firm's assets are not traded or observable. Due to some empirical shortcomings of structural firm value models, Jarrow and Turnhull (1995) [12] introduced a reduced-form intensity-based model, where defaults are modeled by Poisson random processes and are exogenous events.

One of the central issues for credit risk measurement and management is the modeling and predicting correlated defaults. Different approaches have been proposed in the literature to model correlated defaults. One of the major approaches is based on copulas which are important tools in statistics, in particular in survival analysis, to model dependence of several random quantities. Li [14] and Embrechts [11] pioneered the use of copulas for modeling dependent credit risk. Other models for dependent defaults include the mixture model approach, in particular the Poisson mixture model in CreditRisk+ [7], Binomial Expansion Techniques [29], the infectious default models [8, 9], the multivariate Markov chain approach of Siu et al. (2005) [28] and Markov switching models [5, 6]. Besides, reduced-form intensity-based models have also been used to model dependent defaults. There are two major approaches along this direction, namely, the top-down approach and the bottom-up approach. The distinction between the two approaches is in the way that the default intensities are specified. The top-down approach models directly the aggregate default intensity of a credit portfolio, and a random thinning procedure is then used to specify individual default intensities. The bottom-up approach focuses on modeling individual default intensities. The intensity density of the credit portfolio is then determined by aggregating individual default intensities. For an excellent overview on both top-down and bottom-up approaches for portfolio credit risk modeling, interested readers may refer to Giesecke (2008) [16]. Duffie et al. (2009) [10] introduced a frailty-based intensity approach for modeling portfolio credit risk, which is a kind of bottom-up intensity-based approach. A related conditionally diversifiable default risk model was considered in Jarrow, Lando and Yu (2005) [13]. These models seem focused on modeling correlated defaults of firms or corporations in an industrial sectors. Relatively little attention has been paid to model correlated defaults among different industrial sectors.

It has been pointed out in Moody's reports on historical default rates of corporate bond issuers that the number of defaults, the number of credit rating downgrades and credit spreads are strongly correlated with the business cycle. Hackbarth, Miao and Morellec (2006) [17] established a modeling framework for analyzing the effect of macroeconomic conditions on credit risk and dynamic capital structures of corporations. The basic idea was based on the observation that firms may adopt their default and financing policies according to different phases of the business cycles when cash flows depend on current economic conditions.

A recent paper by Miao and Wang (2010) [24] developed an equilibrium approach to investigating the relationship between credit risk and business cycles. However, it seems that some of the existing literature mainly focus on modeling the impact of the business cycle on individual credit entities. The modeling of the impact of the business cycle on the joint defaults of corporations from different industrial sectors seems receiving relatively less attention. Intuitively, when the economy is in recession, it is likely that the profitabilities of firms in several industrial sectors decline, and they may default together in extreme scenarios. On the other hand, when the economy is booming, it is less likely that firms jointly default. Furthermore, the use of information about the relationship between joint defaults and the business cycle in predicting future joint defaults seems not fully explored yet.

In this paper, we introduce a novel model to investigate the relationship between correlated defaults of different industrial sectors and business cycles as well as the impacts of business cycles on modeling and predicting correlated defaults. The proposed model is built using the Probabilistic Boolean Network (PBN) [2, 3, 4]. Here we model the probabilistic behavior of joint defaults of corporations in different industrial sectors by a transition probability matrix. For example, suppose we have four industrial sectors. If there is a default in an industrial sector, we call the default state equal to "1"; otherwise, we call the default state equal to "0". Consequently, each sector has two default states, namely, "0" and "1", and the default states of the four sectors can then be described by a Markov chain with 2^4 state. In this case, the transition probability matrix of the chain is a $2^4 \times 2^4$-matrix. The key idea of the PBN is to decompose a transition probability matrix describing correlated, or joint, defaults of different sectors into a weighted average of several deterministic Boolean Network (BN) matrices which contain information about business cycles. Indeed, given an initial state, the BN will eventually enter into a cycle of state(s), called attractor cycle or limit cycle. We believe that if the concept of the business cycle is being taken seriously, it may be well-described by a limit cycle. Based on this belief, we attempt to explain the probabilistic behavior of joint defaults of corporations in different industrial sectors by various patterns of the business cycle described by different BN matrices. A weight is assigned to each of the BN matrices which describes how likely the probabilistic behavior of joint defaults among different sectors is explained by a particular BN matrix, or a particular pattern of the business cycle. Since the BN matrices are basically transition matrices of deterministic Markov chains, (i.e. with probability one of a particular transition), the only parameters in our proposed model appear to be the weights in the linear combination of the BN matrices. An efficient estimation method based on entropy is used to estimate these model parameters. Using real default data, we build a PBN for explaining the default structure and make reasonably good prediction of joint defaults in different sectors. To our best knowledge, this seems to be the first paper attempting to apply the PBN for credit default prediction.

The paper is organized as follows. The next section describes the basic concepts of BNs and PBNs. Section 3 presents a construction for a PBN and an algorithm for its estimation. Section 4 provides a real-data example for the proposed model. The final section gives concluding remarks.

2 Boolean Networks and Probabilistic Boolean Networks

Boolean Networks (BNs) were first introduced by Kauffman [18, 19, 20, 21]. In a BN, the vertices have two states represented as 1 and 0. The target vertex is determined by several genes called its input genes via a Boolean function. If the input vertices and also the corresponding Boolean functions are given, then a BN is defined. We note that a BN is essentially a deterministic model. Given an initial state, the BN will eventually enter into a cycle of state(s) called its attractor cycle. The idea of extending the concept of a BN (a deterministic model) to a PBN (a probabilistic model) is as follows. For each vertex, there can be more than one Boolean function and corresponding selection probabilities are assigned to the Boolean functions. The dynamics (transitions) of a PBN can be studied using Markov chain theory [2, 3, 25, 26, 27].

A Boolean Network (BN) $G(V, F)$ is represented by a set of vertices

$$V = \{v_1, v_2, \ldots, v_n\}$$

and also a set of Boolean functions

$$F = \{f_1, f_2, \ldots, f_n\}$$

where

$$f_i : \{0, 1\}^n \to \{0, 1\}.$$

States	$v_1(t)$	$v_2(t)$	$f^{(1)}$	$f^{(2)}$
1	0	0	0	1
2	0	1	1	0
3	1	0	0	0
4	1	1	1	0

Table 1: The Truth Table.

We define $v_i(t)$ to be the state (0 or 1) of vertex i at time t. Thus the rules of the interactions among the vertices can then be represented by Boolean functions:

$$v_i(t+1) = f_i(\mathbf{v}(t)),\ i = 1, 2, \ldots, n$$

where the Boolean vector

$$\mathbf{v}(t) = (v_1(t), v_2(t), \ldots, v_n(t))$$

can take any possible states from the set

$$S = \{(v_1, v_2, \ldots, v_n)^T : v_i \in \{0, 1\}\}$$

and it is easy to see that $|S| = 2^n$.

The following is an example of a two-vertex BN with the truth table being given in Table 1. From the truth table, there are four states and they are $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$. One may label them by $1, 2, 3$ and 4 respectively. We note that if the current state of the network is 1, the network will go to State 2 in the next step (with probability one). Suppose the current state is 2, the network will go to State 3 in the next step (with probability one). If the current state is 4, the network will go to State 3 in the next step (with probability one). The transition probability matrix (Boolean network matrix) of the 2-gene BN is then given by

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

The truth table gives the one-step transition probability between any two states. The BN is a deterministic model, each column in B (the Boolean network matrix) has only one non-zero element. We observe that there is only one cycle (attractor) of period three given respectively as follows: $(0, 0) \to (0, 1) \to (1, 0) \to (0, 0)$. Moreover, $(1, 1)$ belongs to the basin of attraction of the three-period attractor cycle. We remark that there is an one-to-one relation between a BN and its corresponding BN matrix.

Here to extend the concepts of a BN to a stochastic model, for each vertex v_i in a PBN, instead of having only one Boolean function as in BN, there are a number of Boolean functions (predictor functions) $f_i^{(j)}(i = 1, 2, \ldots, l(j))$ to be chosen for determining the state of gene v_j. The probability of choosing $f_i^{(j)}$ as the predictor function is $c_i^{(j)}$,

$$0 \leq c_i^{(j)} \leq 1 \quad \text{and} \quad \sum_{i=1}^{l(j)} c_i^{(j)} = 1 \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{2}$$

We let f_i be the ith possible realization, where

$$f_i = (f_{i_1}^{(1)}, f_{i_2}^{(2)}, \ldots, f_{i_n}^{(n)}), \quad 1 \leq i_j \leq l(j), \quad j = 1, 2, \ldots, n. \tag{3}$$

Suppose that the selection of the Boolean function f_{i_j} for each gene j is an independent process, then the probability of choosing the corresponding BN with Boolean functions

$$(f_{i_1}, f_{i_2}, \ldots, f_{i_n})$$

is given by

$$q_{i_1 i_2 \cdots i_n} = \prod_{j=1}^{n} c_{i_j}^{(j)}.$$

There are at most $N = \prod_{j=1}^{n} l(j)$ different possible realizations of BNs. The transition process of the states in S actually forms a Markov chain process. Let $\mathbf{a}$ and $\mathbf{b}$ be any two column vectors in the set S. Then the transition probability

$$\begin{array}{l} \mathrm{P}\ \{\mathbf{v}(t+1) = \mathbf{a} \mid \mathbf{v}(t) = \mathbf{b}\} = \\ \sum_{i=1}^{N}\ \mathrm{P}\ \{\mathbf{v}(t+1) = \mathbf{a} \mid \mathbf{v}(t) = \mathbf{b}, \text{the } i\text{th network is selected}\} \cdot q_i. \end{array} \tag{4}$$

Here we let

$$q_i = q_{i_1 i_2 \cdots i_n} \quad \text{and} \quad i = i_1 + \sum_{j=2}^{n} \left((i_j - 1)(\prod_{k=1}^{j-1} l(k)) \right). \tag{5}$$

By letting $\mathbf{a}$ and $\mathbf{b}$ take all the possible states in S, one can get the transition probability matrix of the Markov chain. The transition probability matrix can be written as

$$A = \sum_{i=1}^{N} q_i A_i \tag{6}$$

with A_i being the corresponding transition probability matrix of the ith BN and q_i being the probability of choosing the ith BN matrix A_i, i.e., $\sum_{i=1}^{N} q_i = 1$ and $q_i \geq 0$.

3 Construction of PBN

Boolean networks (BNs) and Probabilistic Boolean networks (PBNs), are genetic regulatory networks in the computational systems biology. Ching et al. [4] proposed algorithms of generating PBNs from a given transition probability matrix A which can be written as the sum of the BN matrices A_i as in (6). This is an ill-posed inverse problem as there are many possible solutions. One possible approach to estimating $\{q_i\}$ is to formulate it as an entropy optimization problem as suggested in [4].

Here we consider such a problem. Let $\mathcal{T}$ be the time index set $\{0, 1, 2, \ldots, \}$ of our model. To model the uncertainty, we consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. Suppose that $X := \{X_t\}_{t \in \mathcal{T}}$ denote one stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$, where $\{X_t = (X_{1t}, X_{2t}, \ldots, X_{nt})\}$ denotes the default states of the n sectors at time t. X_{it} are quantized to only two levels: survival or default (represented as 0 and 1), $i = 1, 2, \ldots, n$ and $t \in \mathcal{T}$. The target state is predicted by previous state via a number of Boolean functions. Given the observable transition probability matrix $A_{2^n \times 2^n}$ of this stochastic process $\{X_t\}$, we admit the following representation

$$A = \sum_{i=1}^{M} q_i A_i + \epsilon,$$

where $\{A_i\}_{i=1}^{M}$ is a set of BNs and q_i is the probability of choosing A_i, and ϵ is the residual part of A. Here we regard A_i as the important part of the transition probability matrix A with its weight q_i, and the residual part ϵ is the noise of the transition probability matrix A with $||\epsilon||_F$ being sufficiently small.

3.1 The Algorithm

We consider the minimization problem

$$\min_{q_i} \left\{ - \sum_{i=1}^{N} q_i \log q_i \right\}$$

subject to

$$\sum_{1}^{N} q_i = 1 \quad \text{and} \quad q_i \geq 0.$$

Here we adopt the following algorithm proposed in [4] to solve the problem.

Step 0: Set $R_1 = A; k = 0$

Step 1: $k := k + 1$

Step 2: We assume in the ith column of R_k, there are m non-zero entries $[R_k]_{1i}, [R_k]_{2i}, \ldots, [R_k]_{mi}$.

Then we define the probability of choosing $[R_k]_{ji}$ to be $\frac{[R_k]_{ji}}{[R_k]_{1i}+[R_k]_{2i}+\ldots+[R_k]_{mi}}$.

After choosing entries based on the probability defined above,
suppose the concerned entries are given by $[R_k]_{k_1,1}, [R_k]_{k_2,2}, \ldots, [R_k]_{k_{2^n},2^n}$,
we choose the smallest entry q_k from $[R_k]_{k_i,i} (i = 1, \ldots, 2^n)$.

Then we define the following BN matrix:$A_k = [e_{k_1,1}, \ldots, e_{k_{2^n},2^n}]$.

Here $e_{j,i}$ is the unit column vector whose jth entry is 1 for $i = 1, \ldots, 2^n$.

Step 3: $R_{k+1} = R_k - q_k A_k$

Step 4: If R_{k+1} is a zero matrix then go to **Step 5** otherwise go to **Step 1**.

Step 5: $N = k$ and $A = \sum_{k=1}^{N} q_k A_k$.

This algorithm can be iterated for a number of times, where at each time we compute and record the entropy of solution **q** obtained. Finally, after a predetermined number of iterations (say 1000), we select the solution with the lowest entropy. The following proposition justifies the algorithm.

Proposition 1 *Suppose*

$$A = \sum_{i=1}^{N} q_i A_i$$

where

$$1 \geq q_1 \geq \ldots \geq q_N \geq 0$$

and

$$\sum_{i=1}^{N} q_i = 1.$$

Let M *be a positive integer and* E *be the entropy of* $(q_1, q_2, \ldots, q_N)$*, i.e.,*

$$E = -\sum_{i=1}^{N} q_i \log q_i \quad \text{and} \quad \epsilon = \sum_{i=M+1}^{N} q_i A_i.$$

We have

$$||\epsilon||_F^2 \leq 2^n \left(\frac{E}{\log(M+1)} \right)^2$$

where $||H||_F$ *is the Frobenius norm of the matrix* H*, defined by*

$$||H||_F^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} H_{ij}^2.$$

Proof: Since A_i is a BN matrix with its corresponding probability q_i and

$$\epsilon = \sum_{i=M+1}^{N} q_i A_i = (a_{ij})_{2^n \times 2^n},$$

then for any j, we have

$$\sum_{i=1}^{2^n} a_{ij} = \sum_{i=M+1}^{N} q_i.$$

Therefore one can get

$$\sum_{i=1}^{2^n} (a_{ij})^2 \leq \left(\sum_{i=1}^{2^n} a_{ij} \right)^2 = \left(\sum_{i=M+1}^{N} q_i \right)^2.$$

Thus we have

$$||\epsilon||_F^2 = \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} a_{ij}^2 \leq \sum_{j=1}^{2^n} \left(\sum_{i=M+1}^{N} q_i \right)^2 \leq 2^n \left(\sum_{i=M+1}^{N} q_i \right)^2.$$

Since

$$q_1 \geq q_2 \geq \ldots, \geq q_N \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} q_i = 1,$$

we have $q_i \leq \frac{1}{i}$. Hence we have

$$\begin{aligned} E &= -\sum_{i=1}^{N} q_i \log q_i \\ &= -\sum_{i=1}^{M} q_i \log q_i - \sum_{i=M+1}^{N} q_i \log q_i \\ &\geq -\sum_{i=1}^{M} q_i \log q_i + \log(M+1) \sum_{i=M+1}^{N} q_i \\ &\geq \log(M+1) \sum_{i=M+1}^{N} q_i. \end{aligned}$$

Thus we have

$$\sum_{i=M+1}^{N} q_i \leq \frac{E}{\log(M+1)}$$

and hence

$$||\epsilon||_F^2 \leq 2^n \left(\frac{E}{\log(M+1)} \right)^2.$$

□

Corollary 1 *For $q = (q_1, q_2, \ldots, q_N)$ such that $A = \sum_{i=1}^{N} q_i A_i$, and a given positive integer M, if $E \to 0$ then $||\epsilon||_F \to 0$ and $q_1 \to 1$.*

4 Empirical Results for the Proposed Model

In this section we present the empirical results of applying the algorithms stated in Section 3 to solve our proposed problem, using real default data extracted from the figures in [15]. The default data come from four different sectors. They include consumer/service sector, energy and natural resources sector, leisure time/media sector and transportation sector. Table 2 shows the default data taken from [15]. The data sets are time series (quarterly) of number of defaults in the captured sectors. From the table, the proportions of defaults for Consumer, Energy, Media and Transport are 24.11%, 16.90%, 20.46% and 21.00%, respectively.

Sectors	**Total**	**Defaults**
Consumer	1041	251
Energy	420	71
Media	650	133
Transport	281	59

Table 2: The default data (Taken from [15]).

To construct a PBN, here we only consider binary data (0 if there is no default observed and 1 if there is at least one default). To build the model, we first choose all the four sectors to write a transition probability matrix. However, the matrix is of size $2^4 \times 2^4$=16×16, while we just have 88 quarterly default data extracted from [15]. The relative inadequacy of the data source will lead to the inaccuracy of our numerical study. This encourages us to reduce the number of sectors in our experiment. Here we consider the first three sectors, consumer, energy and media. There are eight default states (0 represents no default, 1 represents default observed) as shown in Table 3.

State	**Consumer** v_1	**Energy** v_2	**Media** v_3
1 (000)	0	0	0
2 (100)	1	0	0
3 (010)	0	1	0
4 (001)	0	0	1
5 (110)	1	1	0
6 (101)	1	0	1
7 (011)	0	1	1
8 (111)	1	1	1

Table 3: The default states of the three sectors.

Using the real default data extracted from [15], we first construct the transition frequency matrix F_{ij} by using the observed transition frequency from State j to State i. Then the transition probability matrix $[A_{ij}]$ can be obtained by making a column normalization i.e.,

$$A_{ij} = \begin{cases} \dfrac{F_{ij}}{\sum_{j=1}^{N} F_{ij}} & \text{if } \sum_{j=1}^{N} F_{ij} \neq 0 \\ \delta_{ij} & \text{if } \sum_{j=1}^{N} F_{ij} = 0. \end{cases}$$

We obtain

$$A = \begin{pmatrix} 0.57 & 0.00 & 0.10 & 0.00 & 0.00 & 0.04 & 0.00 & 0.00 \\ 0.14 & 0.31 & 0.00 & 0.50 & 0.13 & 0.13 & 0.33 & 0.06 \\ 0.00 & 0.08 & 0.40 & 0.25 & 0.25 & 0.00 & 0.67 & 0.00 \\ 0.00 & 0.15 & 0.00 & 0.00 & 0.00 & 0.08 & 0.00 & 0.00 \\ 0.00 & 0.15 & 0.30 & 0.00 & 0.00 & 0.13 & 0.00 & 0.00 \\ 0.29 & 0.31 & 0.20 & 0.00 & 0.25 & 0.29 & 0.00 & 0.39 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.38 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.25 & 0.00 & 0.33 & 0.00 & 0.56 \end{pmatrix}.$$

We apply the algorithm for 1000 iterations, and get the estimation of $\{q_i\}$ with the lowest entropy. There are 23 positive q_i in $q = (q_1, q_2, \ldots, q_N)$. Without loss of generality, we reorder $\{q_i\}$ from the largest down to the smallest and we use the same notation $\{q_i\}$ and assume $N = 23$ while $M = 6$. Thus we have

$$1 \geq q_1 \geq q_2 \geq \ldots \geq q_N \geq 0,$$

$$\text{and} \quad A = \sum_{i=1}^{M} q_i A_i + \epsilon \quad \text{and} \quad \sum_{i=M+1}^{N} q_i < 0.111.$$

Thus we have $||\epsilon||_F^2 < 2^3(0.111)^2 = 0.099$. The estimation results are given as follows:

$$(q_1, q_2, \ldots, q_6) = (0.29, 0.20, 0.13, 0.12, 0.09, 0.05).$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&1&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{pmatrix} \quad A_6 = \begin{pmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Table 4: The truth table of A_1.

state	v_1	v_2	v_3	f_1	f_2	f_3
1	0	0	0	0	0	0
2	1	0	0	1	0	1
3	0	1	0	1	1	0
4	0	0	1	1	0	0
5	1	1	0	0	1	1
6	1	0	1	1	0	1
7	0	1	1	0	1	0
8	1	1	1	1	1	1

Table 5: The truth table of A_2.

state	v_1	v_2	v_3	f_1	f_2	f_3
1	0	0	0	0	0	0
2	1	0	0	1	0	0
3	0	1	0	1	0	1
4	0	0	1	1	0	0
5	1	1	0	0	1	0
6	1	0	1	1	1	1
7	0	1	1	1	0	0
8	1	1	1	1	0	1

Suppose we regard the six major BNs $A_i, i = 1, 2, \ldots, 6$ as the most important parts of the original transition probability matrix A, then we may assume the default data is explained by the six Boolean networks. We can then summarize the rules in Table 11 (a column stochastic matrix), where we drop ϵ from A and change q_i to $q_i/\sum_{i=1}^{M} q_i$. Note that an attractor cycle, or a limit cycle, in each of the six BNs represents a particular pattern of the business cycle which manifests itself in the joint default pattern of the three industrial sectors. For example, in the BN matrix A_1, there is a limit cycle of three periods, namely, $(010) \rightarrow (110) \rightarrow (011) \rightarrow (010)$. This reflects how the business cycle influences the joint default of the three sectors. For example, in the recession, there are

Table 6: The truth table of A_3.

state	v_1	v_2	v_3	f_1	f_2	f_3
1	0	0	0	1	0	0
2	1	0	0	1	1	0
3	0	1	0	0	1	0
4	0	0	1	0	1	0
5	1	1	0	1	0	1
6	1	0	1	1	1	1
7	0	1	1	0	1	0
8	1	1	1	1	1	1

Table 7: The truth table of A_4.

state	v_1	v_2	v_3	f_1	f_2	f_3
1	0	0	0	1	0	1
2	1	0	0	0	0	1
3	0	1	0	0	1	0
4	0	0	1	1	1	1
5	1	1	0	1	0	0
6	1	0	1	1	1	0
7	0	1	1	0	1	0
8	1	1	1	1	1	1

Table 8: The truth table of A_5.

state	v_1	v_2	v_3	f_1	f_2	f_3
1	0	0	0	1	0	1
2	1	0	0	1	0	0
3	0	1	0	0	1	0
4	0	0	1	1	1	1
5	1	1	0	1	0	1
6	1	0	1	1	0	0
7	0	1	1	1	0	0
8	1	1	1	1	0	1

Table 9: The truth table of A_6.

state	v_1	v_2	v_3	f_1	f_2	f_3
1	0	0	0	1	0	1
2	1	0	0	0	1	0
3	0	1	0	0	0	0
4	0	0	1	0	1	0
5	1	1	0	0	1	0
6	1	0	1	0	0	1
7	0	1	1	0	1	0
8	1	1	1	1	0	1

Table 10: Attractor cycles of the six BNs.

BN	Attractor cycles
A_1	(000), (101), (111), (010) $\rightarrow$ (110) $\rightarrow$ (011) $\rightarrow$ (010)
A_2	(000), (100), (101) $\leftrightarrow$ (111)
A_3	(010), (111)
A_4	(010), (111)
A_5	(100), (010)
A_6	(000) $\rightarrow$ (101) $\rightarrow$ (001) $\rightarrow$ (010) $\rightarrow$ (000)

Table 11: The prediction rules from the six BNs.

	(000)	(100)	(010)	(001)	(110)	(101)	(011)	(111)
(000)	0.55	0.00	0.06	0.00	0.00	0.00	0.00	0.00
(100)	0.15	0.32	0.00	0.55	0.14	0.10	0.32	0.00
(010)	0.00	0.06	0.39	0.21	0.28	0.00	0.68	0.00
(001)	0.00	0.14	0.00	0.00	0.00	0.06	0.00	0.00
(110)	0.00	0.15	0.33	0.00	0.00	0.14	0.00	0.00
(101)	0.30	0.33	0.22	0.00	0.25	0.33	0.00	0.38
(011)	0.00	0.00	0.00	0.00	0.33	0.00	0.00	0.00
(111)	0.00	0.00	0.00	0.24	0.00	0.37	0.00	0.62

defaults in two sectors, (i.e. either (110) or (011)), whereas in an economic boom, there is only one default, say (010). This piece of information can be used to predict the pattern of correlated defaults among different sectors.

From the prediction table we have the following observations. If current state is (000) (no default in all sectors) then either there will be no default in all sectors or there must be default in the consumer sector in the next quarter. If the current state is (100) (default only found in the consumer sector) then default will be observed in at least one of the sectors. If the current state is (010) (default only found in the energy sector) then either there will be no more default in all the sectors or default will be observed in at least one of the sectors. If the current state is (001) (default only found in the media sector) then default will be observed in at least one of the sectors or even all the sectors. We see that both the consumer and media sectors have strong infectious effect when compared to the energy sector. Finally, if the current state is (111) (all sectors have default cases) then default will be observed in at least two of the sectors or even all the sectors.

We have further conducted some numerical on the other groups of default data consisting three sectors. Let G_1 denote the group consist the consumer, media and transport sector; G_2 denote the group of energy, media and transport sector; G_3 denote the group of the consumer, energy and transport sector. There are eight default states the same as shown in Table 3, where the order of sector are listed as above, e.g., v_1 stands for consumer, v_2 stands for media and v_3 stands for transport in G_1. We report the attractor cycles of the major BNs in these groups of default data in Tables 12, 14 and 16, and the corresponding prediction rules in Tables 13, 15 and 17. We remark that $||\epsilon||_F^2 < 0.142$ for all the groups and the PBN approach seems to give reasonable results in all the groups.

Table 12: Attractor cycles of the seven major BNs for G_1.

BN	Attractor cycles
A_1	(011) , (100) $\leftrightarrow$ (010), (110) $\leftrightarrow$ (111)
A_2	(000), (110), (011), (111)
A_3	(100), 011, (111)
A_4	(011), (000) $\leftrightarrow$ (001), (110) $\leftrightarrow$ (111)
A_5	(100), (011), 111
A_6	(011), (000) $\rightarrow$ (111) $\rightarrow$ (110) $\rightarrow$ (101) $\rightarrow$ (000)
A_7	(011), (100) $\leftrightarrow$ (001)

Table 13: The prediction rules from the seven major BNs for G_1.

	(000)	(100)	(010)	(001)	(110)	(101)	(011)	(111)
(000)	0.23	0.00	0.40	0.48	0.00	0.22	0.00	0.00
(100)	0.23	0.27	0.46	0.05	0.27	0.23	0.00	0.00
(010)	0.00	0.31	0.00	0.00	0.05	0.00	0.00	0.06
(001)	0.35	0.05	0.00	0.00	0.00	0.11	0.00	0.00
(110)	0.11	0.23	0.00	0.31	0.23	0.00	0.00	0.45
(101)	0.00	0.14	0.00	0.16	0.08	0.00	0.00	0.00
(011)	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
(111)	0.08	0.00	0.14	0.00	0.37	0.44	0.00	0.49

Table 14: Attractor cycles of the nine major BNs for G_2.

BN	Attractor cycles
A_1	(000), (101), (111)
A_2	(010) $\leftrightarrow$ (011)
A_3	(000) $\leftrightarrow$ (001), (100) $\leftrightarrow$ (010)
A_4	(101), (110) $\leftrightarrow$ (011)
A_5	(010), (111)
A_6	(000) $\leftrightarrow$ (100), (011) $\leftrightarrow$ (111)
A_7	(000), (100) $\leftrightarrow$ (010)
A_8	(100), (111)
A_9	(111), (100) $\leftrightarrow$ (110)

Table 15: The prediction rules from the nine major BNs for G_2.

	(000)	(100)	(010)	(001)	(110)	(101)	(011)	(111)
(000)	0.33	0.19	0.26	0.38	0.10	0.00	0.00	0.00
(100)	0.08	0.05	0.24	0.06	0.16	0.47	0.00	0.00
(010)	0.39	0.18	0.10	0.38	0.19	0.00	0.43	0.10
(001)	0.14	0.00	0.10	0.00	0.11	0.00	0.00	0.00
(110)	0.00	0.22	0.08	0.00	0.00	0.00	0.23	0.19
(101)	0.06	0.26	0.00	0.00	0.00	0.36	0.00	0.00
(011)	0.00	0.10	0.19	0.18	0.10	0.17	0.00	0.27
(111)	0.00	0.00	0.03	0.00	0.34	0.00	0.34	0.44

Table 16: Attractor cycles of the ten major BNs for G_3.

BN	Attractor cycles
A_1	(100), (111)
A_2	(100) $\leftrightarrow$ (110), (101) $\leftrightarrow$ (111)
A_3	(111), (100) $\leftrightarrow$ (101)
A_4	(010), (101), (110) $\leftrightarrow$ (111)
A_5	(100), (010) $\leftrightarrow$ (101)
A_6	(000) $\rightarrow$ (111) $\rightarrow$ (100) $\rightarrow$ (000)
A_7	(000), (010), (111), (100) $\leftrightarrow$ (001), (110) $\leftrightarrow$ (011)
A_8	(010), (100) $\rightarrow$ (111) $\rightarrow$ (110) $\rightarrow$ (101) $\rightarrow$ (100)
A_9	(000), (010)
A_{10}	(000) $\leftrightarrow$ (100)

Table 17: The prediction rules from the ten major BNs for G_3.

	(000)	(100)	(010)	(001)	(110)	(101)	(011)	(111)
(000)	0.09	0.12	0.08	0.32	0.00	0.05	0.00	0.00
(100)	0.41	0.32	0.25	0.68	0.32	0.29	0.00	0.05
(010)	0.11	0.00	0.22	0.00	0.25	0.07	0.51	0.00
(001)	0.25	0.05	0.00	0.00	0.00	0.00	0.00	0.00
(110)	0.00	0.25	0.13	0.00	0.00	0.25	0.24	0.17
(101)	0.00	0.21	0.07	0.00	0.14	0.09	0.00	0.35
(011)	0.00	0.00	0.25	0.00	0.05	0.00	0.00	0.00
(111)	0.14	0.05	0.00	0.00	0.24	0.25	0.25	0.43

5 Conclusion

A framework for modeling and predicting correlated defaults based on information about the business cycle was proposed. The modeling framework was built using the concept of the Probabilistic Boolean Network (PBN). A transition probability matrix describing joint defaults was decomposed into a weighted average of Boolean Network matrices giving information about the impact of different patterns of the business cycle on different patterns of joint defaults among different industrial sectors. This piece of information was used to predict joint defaults of different sectors. We provided a real data example to illustrate the practical implementation of the model and its use for predicting joint default behavior of different industrial sectors.

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