Electrohydrodynamic instability of a rotating couple stress dielectric fluid layer

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ABSTRACT

A linear stability analysis is performed to study the effect of a vertical alternating current (AC) electric field on the criterion for the onset of convection in a horizontal couple stress dielectric fluid layer heated uniformly either from below or from above. The lower and upper isothermal boundaries of the fluid layer are considered to be either rigid or free. The principle of exchange of stability is shown to be valid irrespective of the nature of velocity boundary conditions. The resulting eigenvalue problem is solved exactly for free–free boundaries and numerically using the Galerkin method for rigid–rigid boundaries. The stability characteristics of the system have been analyzed when the electric and buoyancy forces are acting together as well as in isolation and their coupling effects are highlighted in the presence of couple stresses. It is noted that there is a qualitative agreement between the results of free–free and rigid–rigid boundaries. The effect of increasing AC electric Rayleigh number is to increase the transfer of heat more effectively and hence to hasten the onset of convection. To the contrary, the effect of increasing couple stress parameter is to delay the onset of electrothermal convection. Moreover, the influence of couple stress on the stability characteristics of the system is found to be more pronounced for the case with rigid boundaries compared to the free boundaries case.

Key words: dielectric fluid; electrothermal convection; couple stress fluid

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Nomenclature

$a = \sqrt{\ell^2 + m^2}$ overall horizontal wave number

d thickness of the dielectric fluid layer

$D = d / dz$ differential operator

$\vec{E}$ root mean square value of the electric field

E_0 root mean square value of the electric field at $z = 0$

$\vec{g}$ acceleration due to gravity

$\hat{k}$ unit vector in z-direction

$\ell,\ m$ wave numbers in the x and y directions

p pressure

$Pr = \nu / \kappa$ Prandtl number

$R_t = \alpha\, g\, \Delta T d^3 / \nu\kappa$ thermal Rayleigh number

$R_{ea} = \gamma^2 \varepsilon_0 E_0^2 (\Delta T)^2 d^2 / \mu\kappa$ AC electric Rayleigh number,

t time

T temperature

T_L temperature of the lower boundary

T_U temperature of the upper boundary

$\vec{V} = (u, v, w)$ velocity vector

ϕ root mean square value of the electric potential

W amplitude of vertical component of perturbed velocity

$(x,\ y,\ z)$ Cartesian co-ordinates

Greek symbols

α thermal expansion coefficient

γ thermal expansion coefficient of dielectric constant

$\nabla_h^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ horizontal Laplacian operator

$\nabla^2 = \nabla_h^2 + \partial^2 / \partial z^2$ Laplacian operator

ε dielectric constant

ε_0 reference dielectric constant at T_L

κ thermal diffusivity of the fluid

σ	electrical conductivity of the fluid
μ	fluid viscosity
μ_c	couple stress viscosity
ω	growth rate
ρ	fluid density
ρ_e	free charge density
ρ_0	reference density at T_L
Φ	amplitude of perturbed ϕ
$\Lambda_c = \mu_c / \mu d^2$	couple stress parameter
Θ	amplitude of perturbed temperature

1. Introduction

Thermal convective instability in a horizontal viscous fluid layer heated from below is one of the classical problems and it is considered to be a simple model system for a wide variety of geophysical, astrophysical and industrial flows. However, many convective instability problems of practical importance involve electrically conducting fluids. In such cases, the effect of external fields like magnetic and electric fields become important. In particular, the magnetic field effects become dominant if the fluid is highly electrically conducting. To the contrary, if the fluid is dielectric with low electrical conductivity then the electric forces play a major role in driving the motion due to the effect of specific electroconvective instability mechanisms. The interplay between electric field and fluid motion has led to many complex and interesting instability phenomena in microchannels. It has been realized that utilizing electric force to manipulate fluids is the most efficient method in the achievement of a variety of purposes and functions in the applications of microfluidic devices. A brief discussion on the applications of electrohydrodynamic (EHD) instability has been recently presented by Lin (2009).

In view of a possible control of convection in liquid dielectrics and a control of heat transfer in high-voltage devices, the study on convective instability in a horizontal dielectric fluid layer subject to a vertical temperature gradient and a vertical AC or DC electric field has constituted a pole of attraction to researchers. In such situations, an applied temperature gradient produces non-uniformities in the electrical conductivity and/or the dielectric

permittivity. The variation of electrical conductivity of the fluid with temperature produces free charges in the bulk of the fluid. These free charges interacting with applied or induced electric field produce a force that eventually causes fluid motion. On the other hand, when there is variation in dielectric permittivity and the electric field is intense then the polarization force which is induced by the non-uniformity of the dielectric constant causes fluid motion. In either case, convection can occur in a dielectric fluid layer even if the temperature gradient is stabilizing and such an instability produced by an electric field is called electroconvection, which is analogous to Rayleigh-Benard convection. In addition, if the applied temperature gradient is also destabilizing then such an instability problem is called electrothermal convection.

Copious literature is available on the problem of the onset of convective instability in a horizontal dielectric fluid layer subject to a vertical temperature gradient and a vertical AC or DC electric field (Turnbull, 1968; Roberts, 1969; Takashima & Aldridge, 1976; Bradley, 1978; Jones, 1978; Maekawa, Abe & Tanasawa, 1992; Pontiga & Castellanos, 1994; Saville, 1997; Smorodin, 2001; Straughan, 2004). The combined effects of DC electric field and volumetric heat source on the onset of convection in a dielectric fluid layer heated from below is investigated by Shivakumara, Nagashree, and Hemalatha (2007), while the influences of a vertical AC electric field as well as internal heat generation on the onset of electrothermoconvection in a horizontal dielectric fluid layer is analyzed by Shivakumara, Rudraiah, and Hemalatha (2009). A more detailed analysis on EHD instability in a horizontal fluid layer with electrical conductivity gradient subject to a weak shear flow is presented by Chang, Ruo, and Chen (2009). The electrohydrodynamic instability of a horizontal rotating fluid layer with a vertical electrical conductivity gradient has been considered by Ruo, Chang, and Chen (2010). Recently, Shivakumara, Lee, Vajravelu and Akkanagamma (2012) have studied the effect of velocity and temperature boundary conditions on electrothermal convection in a rotating dielectric fluid layer.

Although EHD instability has been extensively investigated in a dielectric Newtonian fluid layer, relatively little attention is devoted to the problem concerning non-Newtonian fluids despite their occurrence and importance in many practical problems. There exist various types of non-Newtonian fluids, and obviously they do not lend themselves to a unified treatment like Newtonian ones. A few studies have addressed the effect of vertical AC electric field on the onset of convection for one such type of non-Newtonian fluid called

viscoelastic fluid. Takashima and Ghosh (1979) were the first to study the simultaneous action of a vertical AC electric field and a vertical temperature gradient on the onset of instability in a horizontal layer of viscoelastic dielectric fluid. Othman (2001) has studied the onset of convection in a viscoelastic dielectric fluid layer heated from below. Othman and Sweilam (2002) have examined the effect of vertical AC electric field on the stability of natural convection, which occurs in a dielectric viscoelastic fluid (Walters' liquid B') between two parallel horizontal plates with internal heat generation. Othman (2003) has considered the effect of thermal relaxation time on EHD instability in a viscoelastic dielectric fluid layer heated from below. The effect of rotation on EHD instability in a layer of viscoelastic dielectric fluid heated from below has been investigated by Othman (2004).

Nonetheless, consideration of non-Newtonian fluids with suspended particles is desirable as these fluids deform and produce a spin field due to the microrotation of suspended particles forming a micropolar fluid. The theory of micropolar fluids was developed by Eringen (1966) which takes care of local effects arising from the microstructure and as well as the intrinsic motions of microfluidics. The spin field, due to microrotation of freely suspended particles, sets up an anti-symmetric stress known as couple stress and thus forming couple stress fluid. The couple-stress fluid has distinct features, such as polar effects and whose microstructure is mechanically significant. Besides, couple stresses are found to appear in noticeable magnitude in fluids with large molecules. The constitutive equations for couple stress fluids are given by Stokes (1966). The theory proposed by Stokes is the simplest one for micro-fluids, which allows polar effects such as the presence of couple stress, body couple and non-symmetric tensors. These fluids have applications in a number of processes that occur in industry, such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, exotic lubrication and colloidal and suspension solutions. The onset of convection in a couple stress fluid layer has also attracted researchers in the recent past (Malashetty, Gaikwad & Swamy 2006; Sunil, 2011 and references therein). The effect of magnetic field and rotation on convective instability in a layer of couple stress fluid heated from below has been considered by Sharma and Sharma (2004).

In this paper attention is focused on the study of EHD instability in a layer of couple stress dielectric fluid in the presence of a vertical AC electric field. Interest in such a problem stems from the studies on buoyancy driven convective instability in a layer of couple stress fluid, in which the presence of couple stresses shows a significant effect on the stability

characteristics of the system. The problem is discussed for two types of velocity boundary conditions namely, (i) both boundaries free (free-free), and (ii) both boundaries rigid (rigid-rigid) with prescribed different temperatures at the boundaries. The eigenvalue problem is solved exactly for free-free boundaries, while numerically using the Galerkin method for rigid-rigid boundaries. The coupling effects of electrical body force, buoyancy force and the couple stresses on the onset of electrothermal convection are explored in detail.

2. Formulation of the problem

We consider a horizontal layer of thickness d of an incompressible dielectric couple stress fluid in the absence of body couple with a uniform vertical AC electric field applied across the layer. A Cartesian coordinate system (x, y, z) is chosen with the origin at the bottom of the fluid layer and the z- axis normal to the fluid layer in the gravitational field. The lower surface at $z = 0$ and the upper surface at $z = d$ are maintained at temperatures T_L and T_U, respectively. Furthermore, the lower surface is grounded and the upper surface is kept at an alternating (60 Hz) potential whose root- mean- square value is f_1.

The relevant basic equations under the Oberbeck-Boussinesq approximation are:

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

$$\rho_0 \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V} \right] = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V} - \mu_c \nabla^4 \vec{V} - \frac{1}{2} \left(\vec{E} \cdot \vec{E} \right) \nabla \varepsilon \tag{2}$$

$$\frac{\partial T}{\partial t} + \left(\vec{V} \cdot \nabla \right) T = \kappa \nabla^2 T \tag{3}$$

$$\rho = \rho_0 \left\{ 1 - \alpha \left(T - T_L \right) \right\} \tag{4}$$

where $\vec{V} = (u, v, w)$ is the velocity vector, T is the temperature, $\vec{E}$ is the root-mean-square value of the electric field, $P = p - \frac{\rho}{2} \frac{\partial \varepsilon}{\partial \rho} \left(\vec{E} \cdot \vec{E} \right)$ is the modified pressure, ε is the dielectric constant, ρ is the fluid density, κ is the thermal diffusivity, μ is the fluid viscosity, μ_c is the material constant responsible for the couple-stress property known as the couple stress viscosity, $\vec{g}$ is the acceleration due to gravity, α is the thermal expansion coefficient, ρ_0 is the density at reference temperature $T = T_L$. The Coulomb force term $\rho_e \vec{E}$, where ρ_e is the free charge density, is of negligible order compared with the dielectrophoretic force term for most dielectric fluids in a 60-Hz AC electric field (Takashima and Ghosh, 1979). Therefore,

the Coulomb force term has been neglected in Eq. (2) and only the dielectrophoretic force term is retained i.e., the last term in Eq. (2). It is seen that the dielectrophoretic force term depends on ($\vec{E} \cdot \vec{E}$) rather than $\vec{E}$. Since the variation of $\vec{E}$ is very rapid, the root-mean-square value of $\vec{E}$ is used as the effective value in determining fluid motions.

Since there is no free charge, the relevant Maxwell equations are

$$\nabla \times \vec{E} = 0, \ \nabla \cdot (\varepsilon \vec{E}) = 0.$$ (5a,b)

In view of Eq.(5a), $\overset{\shortmid}{E}$ can be expressed as

$$\overset{\shortmid}{E} = -\ ?\, f$$ (6)

where f is the root mean square value of the electric potential. The dielectric constant is assumed to be a linear function of temperature in the form

$$\varepsilon = \varepsilon_0 [1 - \gamma (T - T_L)]$$ (7)

where γ (>0) is the thermal expansion coefficient of dielectric constant and is assumed to be small.

2.1 Basic state

The basic state is quiescent and is given by

$$\vec{V} = \vec{V}_b = 0, P = P_b(z), \ T = T_b(z) = T_L - \mathsf{D}T\, z/d$$

$$\varepsilon = \varepsilon_b(z) = \varepsilon_0 [1 + \gamma\, \Delta T\, z/d]$$

$$\overset{r}{E} = \overset{r}{E}_b(z) = \frac{E_0}{(1 + g\mathsf{D}T\, z/d)}\hat{k}$$ (8)

where $\mathsf{D}T = T_L - T_U$ and the subscript b denotes the basic state. The pressure distribution is of no consequence as we are going to eliminate the same. We note that

$$f_b(z) = -\ \frac{E_0 d}{g\mathsf{D}T} log\left(1 + g\mathsf{D}Tz/d\right)$$ (9a)

where

$$E_0 = -\ \frac{f_1 g\mathsf{D}T/d}{log\left(1 + g\mathsf{D}T\right)}$$ (9b)

is the root mean square value of the electric field at $z = 0$.

2.2 Perturbed state

To study the stability of the basic state, we superimpose infinitesimally small perturbations on the basic state in the form

$$\vec{V} = \vec{V}', \ P = P_b + P', \ \vec{E} = \vec{E}_b + \vec{E}', \ T = T_b + T', \ \rho = \rho_b + \rho', \ \varepsilon = \varepsilon_b + \varepsilon' \tag{10}$$

where $\vec{V}', P', \vec{E}', T', r$ and e are the perturbed quantities over their equilibrium counterparts. Substituting Eq.(10) into Eqs. (1) - (7), linearizing the equations, eliminating the pressure from the momentum equation by operating curl twice and retaining the vertical component and non-dimensionalizing the resulting equations by scaling (x, y, z) by d, t by d^2/κ, $\vec{V}'$ by κ/d, T' by ΔT and $f\phi$ by $\gamma E_0 \Delta T d$, we obtain the linear stability equations (after neglecting the primes for simplicity) in the form

$$\left(\frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 + \Lambda_c \nabla^4 \right) \nabla^2 w = R_t \nabla_h^2 T + R_{ea} \nabla_h^2 \left(T - \frac{\partial \phi}{\partial z} \right) \tag{11}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2 \right) T = w \tag{12}$$

$$\nabla^2 \phi = \frac{\partial T}{\partial z} \tag{13}$$

where $R_t = \alpha \, g \, \Delta T d^3 / \nu \kappa$ is the thermal Rayleigh number, $R_{ea} = \gamma^2 \varepsilon_0 E_0^2 (\Delta T)^2 d^2 / \mu \kappa$ is the AC electric Rayleigh number, $\Lambda_c = \mu_c / \mu d^2$ is the couple stress parameter and $Pr = \nu / \kappa$ is the Prandtl number.

The isothermal boundaries of the fluid layer are considered to be either stress-free or rigid with vanishing couple stress. The boundary conditions are:

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = \frac{\partial \phi}{\partial z} = 0, \ T = 0 \tag{14}$$

on the stress-free boundary and

$$w = \frac{\partial w}{\partial z} = \frac{\partial^3 w}{\partial z^3} = \phi = 0, \ T = 0 \tag{15}$$

on the rigid boundary.

Equations (11)-(13) are simplified in the usual manner by decomposing the solution in terms of normal modes, so that

$$(w, T, \phi) = (W, \Theta, \Phi)(z) \exp(i\ell x + imy + \omega t) \tag{16}$$

where $\omega (= \omega_r + i\omega_i)$ is the growth rate, ℓ and m are the horizontal wave numbers in the x and y directions respectively. Substituting Eq. (16) into Eqs. (11) - (13), we obtain the following stability equations:

$$\left\{\frac{\omega}{Pr}-(D^2-a^2)+\Lambda_c(D^2-a^2)^2\right\}(D^2-a^2)W=-(R_t+R_{ea})a^2\Theta+R_{ea}a^2D\Phi \qquad (17)$$

$$\left\{\omega-(D^2-a^2)\right\}\Theta=W \qquad (18)$$

$$(D^2-a^2)\Phi=D\Theta \qquad (19)$$

where $D=d/dz$ and $a=\sqrt{\ell^2+m^2}$ is the overall horizontal wave number.

On using Eq. (16) in the boundary conditions (14) and (15), we get respectively

$$W=D^2W=D^4W=D\Phi=0, \ \Theta=0 \qquad (20)$$

and

$$W=DW=D^3W=\Phi=0, \ \Theta=0. \qquad (21)$$

3. Validity of principle of exchange of stability

To establish this, we first eliminate Φ from Eq. (17) by operating (D^2-a^2) throughout and then use Eq. (19) to obtain

$$\left[\frac{\omega}{Pr}-(D^2-a^2)+\Lambda_c(D^2-a^2)^2\right](D^2-a^2)^2W=-R_ta^2D^2\Theta+(R_t+R_{ea})a^4\Theta. \qquad (22)$$

Also, we have

$$\left[\omega-(D^2-a^2)\right]\Theta=W. \qquad (23)$$

Multiplying Eq. (22) by W^* (the complex conjugate of W) and integrating from z = 0 to 1, we obtain

$$<W^*\left[\frac{\omega}{Pr}-(D^2-a^2)+\Lambda_c(D^2-a^2)^2\right](D^2-a^2)^2W>$$
$$=-R_ta^2<W^*D^2\Theta>+(R_t+R_{ea})a^4<W^*\Theta> \qquad (24)$$

where, $<\cdots>=\int_0^1(\cdots)dz$. On using the velocity boundary conditions and repeatedly applying integration by parts and simplifying, the LHS of Eq.(24) can be written as

$$\frac{\omega}{Pr}<|D^2W|^2+2a^2|DW|^2+a^4|W|^2>+<\Lambda_c|D^4W|^2+(1+4a^2\Lambda_c)|D^3W|^2>$$
$$+<3a^2(1+2a^2\Lambda_c)|D^2W|^2+a^4(3+4a^2\Lambda_c)|DW|^2+a^6(1+a^2\Lambda_c)|W|^2>. \qquad (25)$$

From Eq. (23), we have

$$\left[\omega^* - \left(D^2 - a^2 \right) \right] \Theta^* = W^*. \tag{26}$$

The right hand side of Eq. (24) on using Eq. (26) becomes

$$R_t a^2 \left[\omega^* < \left| D\Theta \right|^2 > + < \left| D^2\Theta \right|^2 + a^2 \left| D\Theta \right|^2 > \right]$$

$$+ \left(R_t + R_{ea} \right) a^4 \left[\omega^* < \left| \Theta \right|^2 > + < \left| D\Theta \right|^2 + a^2 \left| \Theta \right|^2 > \right]. \tag{27}$$

Use of Eq.(25) and Eq.(27) in Eq.(24) gives

$$\frac{\omega}{Pr} < \left| D^2W \right|^2 + 2a^2 \left| DW \right|^2 + a^4 \left| W \right|^2 > + < \Lambda_c \left| D^4W \right|^2 + \left(1 + 4a^2\Lambda_c \right) \left| D^3W \right|^2 >$$

$$+ < 3a^2 \left(1 + 2a^2\Lambda_c \right) \left| D^2W \right|^2 + a^4 \left(3 + 4a^2\Lambda_c \right) \left| DW \right|^2 + a^6 \left(1 + a^2\Lambda_c \right) \left| W \right|^2 >$$

$$= R_t a^2 < \left| D^2\Theta \right|^2 + a^2 \left| D\Theta \right|^2 > + \left(R_t + R_{ea} \right) a^4 < \left| D\Theta \right|^2 + a^2 \left| \Theta \right|^2 > \tag{28}$$

$$+ \omega^* \left[R_t a^2 < \left| D\Theta \right|^2 > + \left(R_t + R_{ea} \right) a^4 < \left| \Theta \right|^2 > \right].$$

The real and imaginary parts of the above equation are equated separately. The real part, in effect, determines whether the system returns to the equilibrium position or not depending on the sign of the real part of ω. The imaginary part gives the information about the possibility of occurring oscillatory instability. Equating the imaginary parts of Eq. (28), we obtain

$$\frac{\omega_i}{Pr} < \left| D^2W \right|^2 + 2a^2 \left| DW \right|^2 + a^4 \left| W \right|^2 >$$

$$= -\omega_i \left[R_t a^2 < \left| D\Theta \right|^2 > + \left(R_t + R_{ea} \right) a^4 < \left| \Theta \right|^2 > \right]. \tag{29}$$

Thus oscillatory convection occurs only if,

$$\frac{1}{Pr} < \left| D^2W \right|^2 + 2a^2 \left| DW \right|^2 + a^4 \left| W \right|^2 > + R_t a^2 < \left| D\Theta \right|^2 > + \left(R_t + R_{ea} \right) a^4 < \left| \Theta \right|^2 > = 0. \tag{30}$$

But the above condition will never be satisfied as $R_{ea} \geq 0$ and the left hand side of Eq.(30) is positive definite. This establishes that oscillatory instability cannot occur in a layer of couple stress dielectric fluid heated uniformly from below under the influence of a vertical AC electric field. Therefore, we restrict ourselves to stationary convection in our subsequent analyses and take $\omega=0$ in Eqs. (17) and (18) to arrive at the following stability equations:

$$\left\{ \Lambda_c (D^2 - a^2)^2 - (D^2 - a^2) \right\} \left(D^2 - a^2 \right) W = -\left(R_t + R_{ea} \right) a^2\Theta + R_{ea} a^2 D\Phi \tag{31}$$

$$\left(D^2 - a^2 \right) \Theta = -W \tag{32}$$

$$\left(D^2 - a^2 \right) \Phi = D\Theta. \tag{33}$$

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4. Solution

Equations (31)-(33) together with the chosen boundary conditions constitute an eigenvalue problem for R_{ea} or R_t as functions of the remaining parameters. The instability characteristics are explored for two types of velocity boundary conditions namely, both boundaries free (free-free) and both boundaries rigid (rigid-rigid).

4.1 Both boundaries free

Although the free boundaries are admittedly artificial, it is of importance because the solution to the eigenvalue problem can be obtained exactly and thereby the essential features of the problem can be disclosed. Eliminating Θ and Φ between Eqs. (31)-(33), we obtain an equation in W in the form

$$\left\{ \Lambda_c (D^2 - a^2)^2 - (D^2 - a^2) \right\} \left(D^2 - a^2 \right)^3 W = R_t a^2 \left(D^2 - a^2 \right) W - R_{ea} a^4 W . \tag{34}$$

The boundary conditions suggest that the appropriate solution for W belonging to the lowest mode is

$$W = A \sin(\pi z) \tag{35}$$

where A is a constant. Substituting Eq.(35) in Eq. (34) yields an expression for the electric Rayleigh number in the form

$$R_{ea} = \frac{\delta^8 \left(1 + \Lambda_c \delta^2 \right)}{a^4} - \frac{\delta^2}{a^2} R_t \tag{36}$$

where $\delta^2 = \pi^2 + a^2$.

From Eq. (36) it is observed that $dR_{ea} / d\Lambda_c = \delta^{10} / a^4 > 0$ and hence increasing the value of couple stress parameter has a stabilizing effect on the system. But, it is noted that $dR_{ea} / dR_t = -\delta^2 / a^2 < 0$ indicating the system gets destabilized with increasing value of R_t (i.e., when the layer is heated from below), while opposite is the trend when the fluid layer is heated from above.

To find the critical value of R_{ea} , Eq. (36) is differentiated with respect to a^2 and equated to zero to get a polynomial in (a_c^2) in the form

$$3\Lambda_c \left(a_c^2 \right)^5 + 2\left(1 + 5\pi^2 \Lambda_c \right)\left(a_c^2 \right)^4 + 2\pi^2 \left(2 + 5\pi^2 \Lambda_c \right)\left(a_c^2 \right)^3$$
$$- \pi^2 \left(4\pi^4 + 5\pi^6 \Lambda_c - R_t \right)\left(a_c^2 \right) - 2\pi^8 \left(1 + \pi^2 \Lambda_c \right) = 0. \tag{37}$$

In the absence of couple stress (i.e., $\mathsf{L}_c = 0$), Eqs. (36) and (37) respectively reduce to

$$R_{ea} = \frac{\delta^2}{a^2}\left(\frac{\delta^6}{a^2} - R_t\right) \tag{38}$$

$$2\left(a_c^2\right)^4 + 4\pi^2\left(a_c^2\right)^3 - \pi^2\left(4\pi^4 - R_t\right)\left(a_c^2\right) - 2\pi^8 = 0. \tag{39}$$

The above results coincide with those of Roberts (1969).

4.2 Both boundaries rigid

As in the case of free boundaries, an exact solution is not possible for the rigid boundaries and one has to resort to numerical methods to extract the critical stability parameters. For this, the Galerkin method is adopted to solve the resulting eigenvalue problem. To approximate the perturbed variables, the sets of spatial basis functions with constant coefficients are used:

$$W = \sum A_i W_i , \; \Theta = \sum B_i \Theta_i , \; \Phi = \sum C_i \Phi_i \tag{40}$$

where A_i, B_i and C_i are constants and the basis functions W_i, Θ_i and Φ_i will be represented by the power series satisfying the respective boundary conditions. Substituting Eq. (40) into Eqs. (31)–(33), multiplying the resulting momentum equation by $W_j(z)$, energy equation by $\Theta_j(z)$, electric potential equation by $\Phi_j(z)$; performing the integration by parts with respect to z between z = 0 and z = 1 and using the boundary conditions, leads to the following system of linear homogeneous algebraic equations:

$$E_{ji}A_i + F_{ji}B_i + G_{ji}C_i = 0$$
$$H_{ji}A_i + I_{ji}B_i = 0 \tag{38}$$
$$J_{ji}B_i + K_{ji}C_i = 0$$

where

$$E_{ji} = < \Lambda_c D^3 W_j D^3 W_i + \left(1 + 3a^2\Lambda_c\right) D^2 W_j D^2 W_i >$$

$$+ < a^2\left(2 + 3a^2\Lambda_c\right) DW_j DW_i + a^4\left(1 + a^2\Lambda_c\right) W_j W_i >$$

$$F_{ji} = -\left(R_t + R_{ea}\right)a^2 < W_j\Theta_i > , \quad G_{ji} = R_{ea}a^2 < W_j D\Phi_i > , \quad H_{ji} = < \Theta_j W_i >$$

$$I_{ji} = - < D\Theta_j D\Theta_i + a^2\Theta_j\Theta_i > , \quad J_{ji} = - < \Phi_j D\Theta_i > , \quad K_{ji} = - < D\Phi_j D\Phi_i + a^2\Phi_j\Phi_i > .$$

Here the inner product is defined as $< f_1 f_2 > = \int_0^1 f_1 f_2 \, dz$.

The above set of homogeneous algebraic equations can have a non-trivial solution if and only if

$$\begin{vmatrix} E_{ji} & F_{ji} & G_{ji} \\ H_{ji} & I_{ji} & 0 \\ J_{ji} & 0 & K_{ji} \end{vmatrix} = 0. \tag{39}$$

We select trial functions satisfying the appropriate boundary conditions as follows:

$$W_i = \left(z^2 - 5z^4 + 6z^5 - 2z^6\right)T_{i-1}^*, \; \Phi_i = \left(z - z^2\right)T_{i-1}^*, \; \Theta_i = \left(z - z^2\right)T_{i-1}^* \tag{40}$$

where T_i^* ($i = 1, 2, \ldots\ldots n$) is the modified Chebyshev polynomial of ith order. Substituting the above trial functions in Eq. (39) and expanding the determinant leads to the characteristic equation giving the AC electric Rayleigh number R_{ea} or the thermal Rayleigh number R_t as a function of wave number a as well as other parameters. The inner products involved in the determinant are evaluated analytically rather than numerically in order to avoid errors in the numerical integration. Numerical computations carried out reveal that the convergence in finding R_{eac} with respect to the wave number crucially depends on the value of Λ_c, and for higher values of Λ_c more number of terms are found to be required in the Galerkin expansion. The results presented here are for $i = j = 6$ the order at which the convergence is achieved, in general.

5. Results and discussion

The onset of convection in a layer of couple stress dielectric fluid heated uniformly either from below or from above is investigated under the influence of a uniform vertical AC electric field. The isothermal boundaries of the fluid layer are considered to be either free-free or rigid-rigid. It is shown that the principle of exchange of stability holds even in the presence of a vertical AC electric field. The eigenvalue problem is solved exactly in the case of free boundaries, while for rigid boundaries the eigenvalues are extracted numerically using the Galerkin technique. To validate the numerical procedure employed, first the test computations were carried out for rigid boundaries under the limiting case of $\Lambda_c = 0$ (i.e., Newtonian case) and compared with the earlier published results. The critical AC electric Rayleigh number R_{eac} and the corresponding wave number a_c obtained for different values

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of R_t are compared with those of Roberts (1969) in Table 1. The comparison shows excellent agreement of the numerical results and verifies the accuracy of the numerical method used.

Figures 2 and 3 respectively demonstrate the neutral curves in the (R_{ea}, a) plane for various values of Λ_c when $R_t = -500$ (heating from above) and for different values of R_t when $\Lambda_c = 0.1$. It is seen that the neutral curves exhibit single but different minimum with respect to the wave number for various values of physical parameters and also for two kinds of velocity boundary conditions. The region above each neutral curve corresponds to the unstable state of the system. The presence of couple stresses shows a significant effect on the stability characteristics of the system. It is observed that even small variations in the couple stress parameter amounts to significant change in the AC electric Rayleigh number. This is more so in the case of rigid boundaries (Fig. 2). As can be seen from Fig. 2, increasing Λ_c is to increase the region of stability. Figure 3 illustrates that the region of stability increases for negative values of R_t but the trend is reversed if the system is heated from below (i.e., $R_t > 0$). From the figures it is also evident that, except for a quantitative change, the stability characteristics are identical for free-free and rigid-rigid boundaries.

To know the impact of physical parameters on the onset criterion, the critical AC electric Rayleigh number R_{eac} is obtained with respect to the wave number a for different values of physical parameters as well as for two types of velocity boundary conditions. The variations of R_{eac} and a_c as a function of R_t are presented for different values of couple stress parameter Λ_c in Figs. 4 (a) and (b), respectively. For any fixed value of Λ_c, increasing the value of thermal Rayleigh number amounts to decrease in the critical AC electric Raleigh number monotonically (Fig. 4a) indicating its effect is destabilizing. In other words, the presence of AC electric field is to augment heat transfer and to hasten the onset of convection in a layer of couple stress dielectric fluid. This is so irrespective of the velocity boundary conditions considered. This is due to the destabilizing electrical body force induced by the gradient of dielectric constant due to variations in temperature under the action of electric field which eventually drive an upward fluid motion. Nonetheless, increasing Λ_c is to delay the onset of electrothermal convection due to the presence of couple stress. That is, the effect of suspended particles is to increase the viscosity of the fluid and thereby exhibits stabilizing influence on the system with increasing Λ_c. Moreover, the deviation in the critical values of

R_{eac} for different values of Λ_c is found to be significant in the case of rigid boundaries when compared to free boundaries. From Fig. 4(a), it is also seen that the rigid-rigid boundaries are more stabilizing compared to free-free boundaries due to the increased suppression of disturbances in the case of rigid boundaries. Figure 4(b) shows that the critical wave number decreases with increasing R_t and this trend is found to be insignificant in the case of rigid-rigid boundaries. Furthermore, a_c decreases with increasing Λ_c for rigid boundaries case irrespective of the system heated from below or from above. However, a dual behavior is observed in the case of free boundaries. That is, the critical wave number decreases with increasing Λ_c when the system is heated from above, but an opposite trend is noticed after $R_t (> 0)$ exceeds certain value.

6. Conclusions

The criterion for the onset of convection in a couple stress dielectric fluid layer heated either from below or above subject to a vertical AC electric field has been investigated for free-free and rigid-rigid isothermal boundaries. The principle of exchange of stability is shown to be valid. The effect of increasing AC electric field is to enhance the heat transfer and to hasten the onset of convection, while the presence of couple stress inhibits the onset of electrothermal convection. Therefore, it appears the presence of couple stress enhances the performance of the microfluidic devices. It is observed that there is a qualitative agreement between the results of free-free and rigid-rigid boundaries. However, the presence of couple stresses exhibit significant effect on the stability of the system for the case with rigid boundaries compared to the free boundaries case. The foregoing results give an insight for the coupling effect of electrical body force, buoyancy force and the couple stresses on the stability characteristics of a dielectric fluid layer for two types of velocity boundary conditions.

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Roberts (1969)			Present study	
R_t	R_{eac}	a_c	R_{eac}	a_c
-1000	3370.077	3.2945	3374.4899	3.2924
-500	2749.868	3.2598	2754.1106	3.2581
0	2128.696	3.2260	2132.7933	3.2245
500	1506.573	3.1929	1510.5526	3.1917
1000	883.517	3.1606	887.4040	3.1597
1707.762	0.0	3.1162	0.0	3.1158

Table 1: Comparison of R_{eac} and a_c for different values of R_t for rigid-rigid boundaries when $\Lambda_c = 0$.
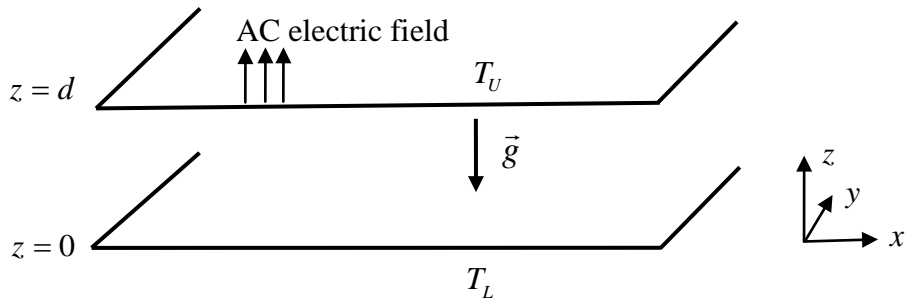
Fig. 1. Physical Configuration
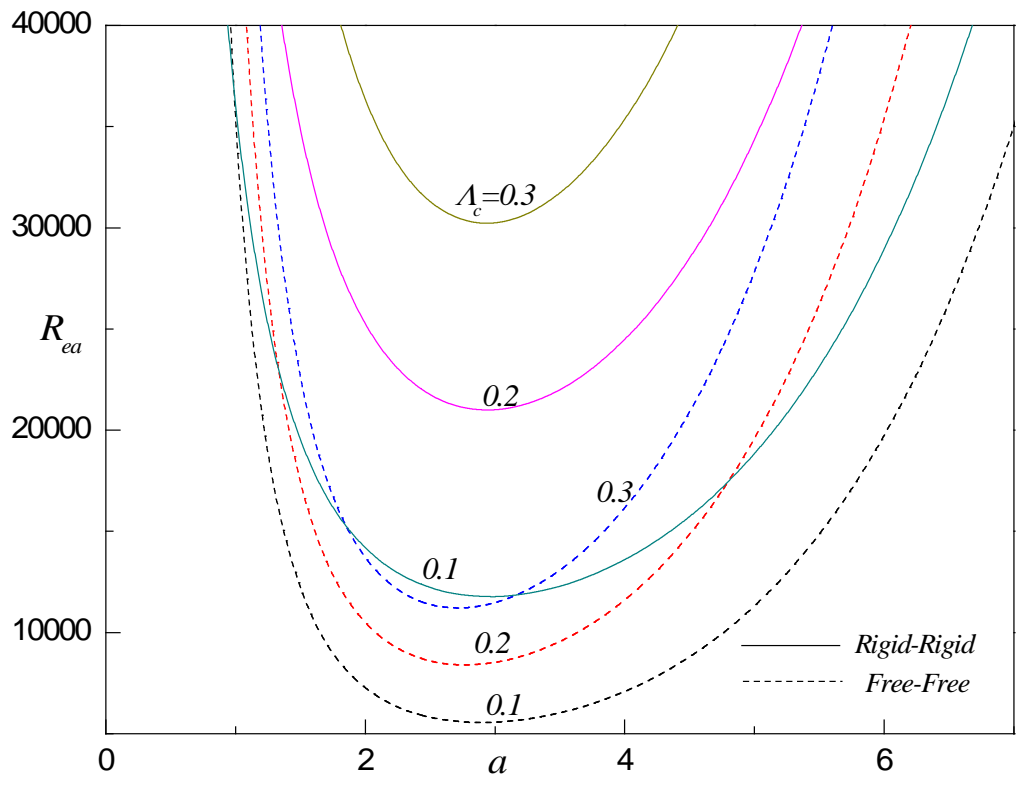
Fig. 2. Neutral curves for different values of Λ_c with $R_t = -500$
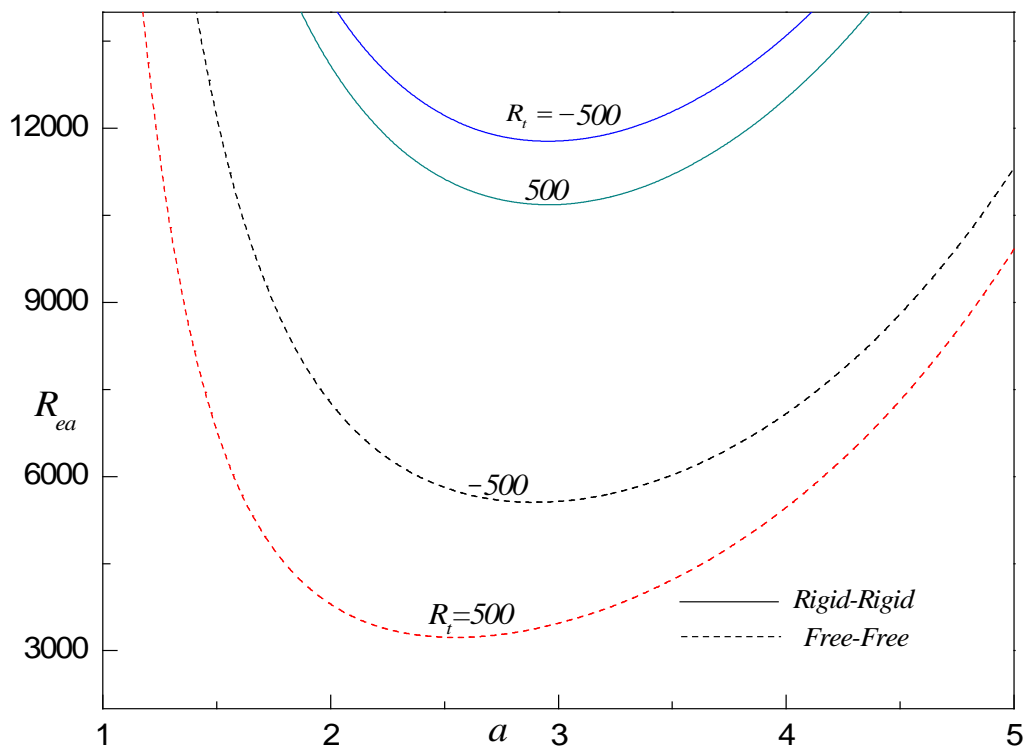

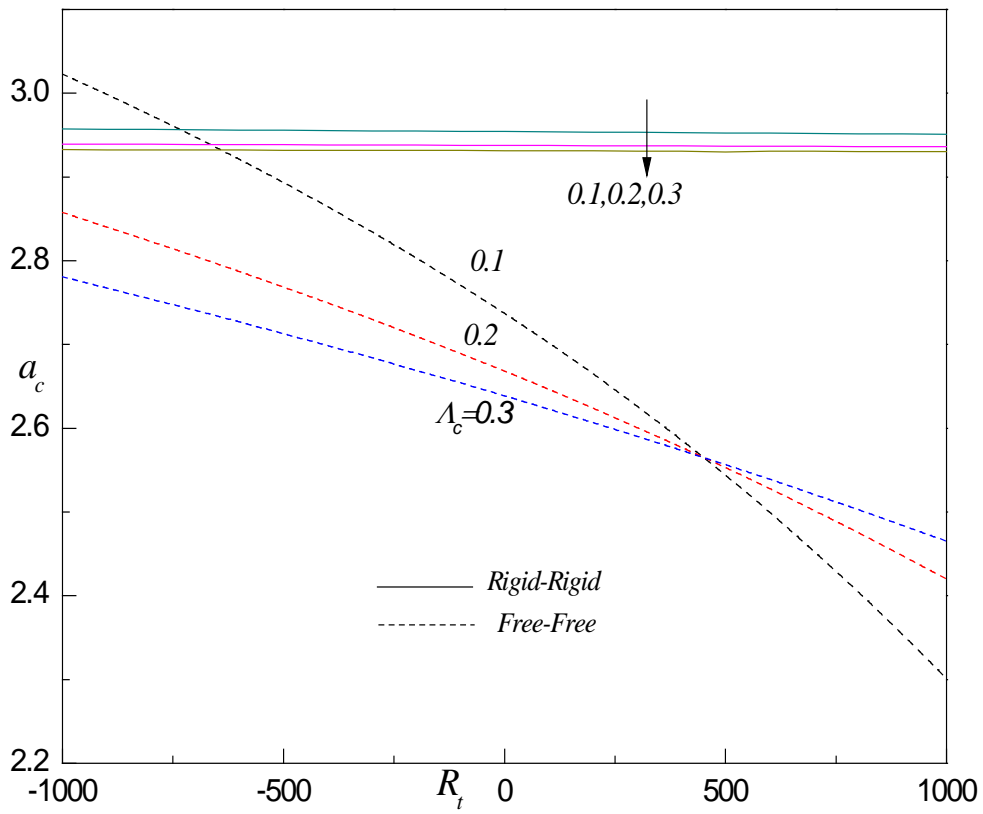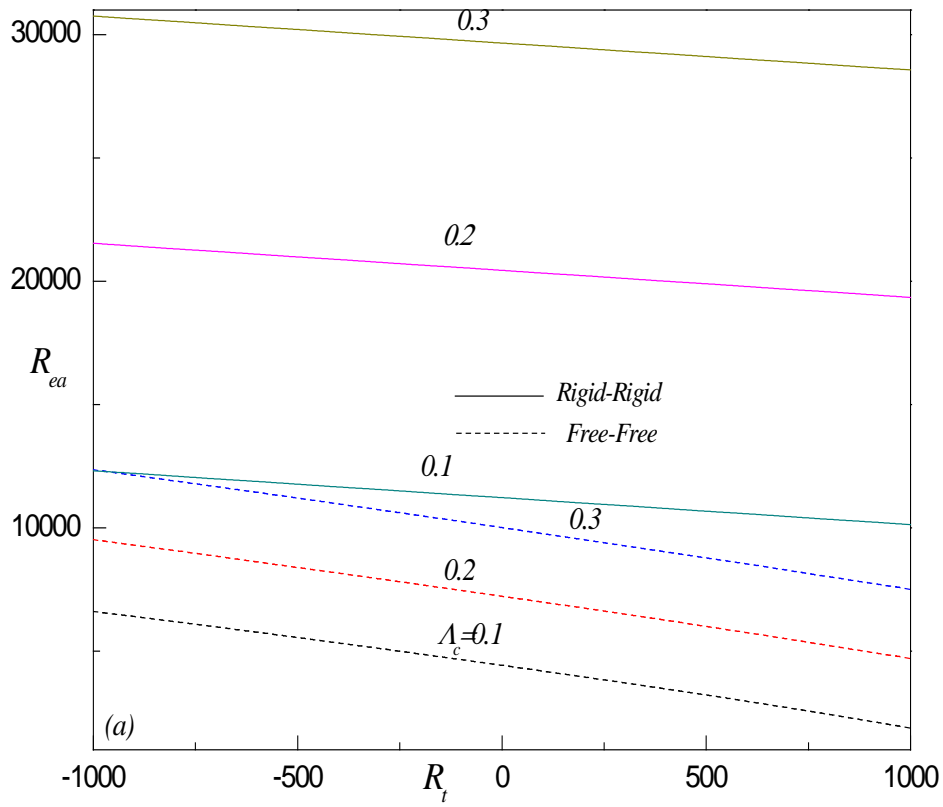**Fig. 3.** Neutral curves for two values of R_t with $\Lambda_c = 0.1$.

Fig. 4. Variation of (a) R_{eac} and (b) a_c as a function of R_t for different values of Λ_c